Stokes Phenomenon and Confluence in Non-autonomous Hamiltonian Systems

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Abstract This article studies a confluence of a pair of regular singular points to an irregular one in a generic family of time-dependent Hamiltonian systems in dimension 2. This is a general setting for the understanding of the degeneration of the sixth Painlevé equation to the fifth one. The main result is a theorem of sectoral normalization of the family to an integrable formal normal form, through which is explained the relation between the local monodromy operators at the two regular singularities and the non-linear Stokes phenomenon at the irregular singularity of the limit system. The problem of analytic classification is also addressed.

1 Introduction

We consider a parametric family of non-autonomous Hamiltonian systems of the form

$$
\begin{aligned}
x(x-\epsilon)\frac{dy_1}{dx} &= \frac{\partial H}{\partial y_2}(y,x,\epsilon) \\
x(x-\epsilon)\frac{dy_2}{dx} &= -\frac{\partial H}{\partial y_1}(y,x,\epsilon),
\end{aligned}
\qquad (y,x,\epsilon)\in(\mathbb{C}^2\times\mathbb{C}\times\mathbb{C},0),
\qquad (1)
$$

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shortly written as

$$x(x - \epsilon)\frac{dy}{dx} = J\,^{\mathrm{t}}(D_y H), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2}$$

with a singular Hamiltonian function $\frac{H(y,x,\epsilon)}{x(x-\epsilon)}$, where $H(y, x, \epsilon)$ is an analytic germ such that $H(y, 0, 0)$ has a non-degenerate critical point (Morse point) at $y = 0$:

$$D_y H(0, 0, 0) = 0, \quad \det D_y^2 H(0, 0, 0) \neq 0.$$

The last condition means that the y-linear terms of the right side of (2) are of the form $A(x, \epsilon)y$ for $A(x, \epsilon) = J D_y^2 H(0, x, \epsilon)$, with $\mathrm{tr}\,A(x, \epsilon) \equiv 0$ and $\det A(0, 0) \neq 0$, i.e. $A(0, 0) \sim \begin{pmatrix} \lambda^{(0)}(0) & 0 \\ 0 & -\lambda^{(0)}(0) \end{pmatrix}$ for some $\lambda^{(0)}(0) \neq 0$.

For $\epsilon \neq 0$ the system (1) has two regular singular points at $x = 0$ and $x = \epsilon$. At each one of them, the local information about the system is carried by a formal invariant and a monodromy (holonomy) operator. On the other hand, for $\epsilon = 0$ the corresponding information about the irregular singularity at $x = 0$ is carried by a formal invariant and by a pair of non-linear Stokes operators. Our main goal is to explain the relation between these two distinct phenomena, and to show how the Stokes operators are related to the monodromy operators. The principal thesis is, that while the monodromy operators diverge when $\epsilon \to 0$, they each accumulate to a 1-parameter family of "wild monodromy operators" which encode the Stokes phenomenon (Theorem 40). It is expected that this "wild monodromy" should have Galoisian interpretation, this question is however outside the reach of this paper.

An important example of a confluent family of systems (1), which in fact motivates this study, is the degeneration of the sixth Painlevé equation to the fifth one, presented in Sect. 6. A more detailed treatment of this confluence will be the subject of an upcoming article [14], which will provide an explicit description of a representation of the "wild monodromy" of the fifth Painlevé equation as an action on the "wild character variety" of the associated isomonodromic problem.

The main tool in our investigation is a theorem on *"sectoral normalization"* of the family (Theorem 17) to its formal normal form which is integrable. The domains on which these transformations exist are of the form introduced in recent years by C. Rousseau and her collaborators [10,15,16,25,26], consisting of two sectors in the ϵ-space and a family of ramified domains in the x-space attached to both singular points $x = 0, \epsilon$ along certain logarithmic spirals. When $\epsilon \to 0$ they tend to a pair of usual sectors attached to the irregular singularity at $x = 0$. This construction allows to define *unfolded Stokes operators* for the family, analogical to those introduced in [10,15] for linear systems, and to *decompose the nonlinear monodromy operators* around 0 and ϵ for $\epsilon \neq 0$ into a convergent "Stokes" part and a wild "exponential torus" part (Theorem 38). We also provide an *analytic classification* for the family (Theorem 36), where the analytic modulus is expressed in terms of a conjugacy class of the unfolded Stokes operators.

In Sect. 5, we illustrate all this on the example of 2-dimensional traceless linear differential systems

$$x(x - \epsilon)\frac{dy}{dx} = A(x, \epsilon)y, \quad A(x, \epsilon) \in \mathfrak{sl}_2(\mathbb{C}), \quad \det A(0, 0) \neq 0, \tag{3}$$

for which our description follows from the more general work of Lambert and Rousseau [10,15]. Here the relation between the monodromy of the unfolded system and the Stokes operators of the limit systems can be summarized as:

Theorem *When $\epsilon \to 0$ the elements of the monodromy group of the system* (3) *accumulate to generators of the wild monodromy group of the limit system (that is the group generated by the Stokes operators and the exponential torus).*

The non-linear analogy of this theorem is Theorem 40. The linear case can be kept in mind as a leading example of which the general non-linear case is a close parallel.

2 The Foliation and its Formal Invariants

The family of systems (1) defines a family of singular foliations in the (y, x)-space, leaves of which are the solutions. We associate to (1) a family of vector fields tangent to the foliation

$$Z_H(y, x, \epsilon) = x(x - \epsilon)\partial_x + X_H(y, x, \epsilon), \tag{4}$$

where

$$X_H(y, x, \epsilon) = \frac{\partial H}{\partial y_2}(y, x, \epsilon)\partial_{y_1} - \frac{\partial H}{\partial y_1}(y, x, \epsilon)\partial_{y_2}, \tag{5}$$

which may also be seen as a single vector field in the (y, x, ϵ)-space. We will denote

$$Z_{H,\epsilon}(y, x), \quad \text{resp. } X_{H,x,\epsilon}(y),$$

the restriction of (4) to a fixed value of ϵ, resp. of (5) to a fixed value of (x, ϵ).

The vector field $Z_{H,0}(y, x)$ has a saddle-node type singularity at $(y, x) = 0$, i.e. its linearization matrix has one zero eigenvalue, corresponding to the x-direction. It follows from the Implicit Function Theorem that, for small $\epsilon \neq 0$, $Z_{H,\epsilon}$ has two singular points $(y_0(0, \epsilon), 0)$ and $(y_0(\epsilon, \epsilon), \epsilon)$ bifurcating from $(y_0(0, 0), 0) = 0$ and depending analytically on ϵ. The aim of this paper is a study of their confluence when $\epsilon \to 0$.

The two singularities of $Z_{H,\epsilon}$ have each a strong invariant manifold $Y_0 = \{(y, x) : x = 0\}$, resp. $Y_\epsilon = \{(y, x) : x = \epsilon\}$. Away of these invariant manifolds the vector field $Z_{H,\epsilon}$ is transverse to the *fibration* with fibers $Y_c = \{(y, x) : x = c\}$. The (y, x)-space is endowed with a Poisson structure associated to the 2-form

$$\omega = dy_1 \wedge dy_2, \tag{6}$$

the restriction of which on each fiber Y_c is symplectic. The vector field $Z_{H,\epsilon}$ is transversely Hamiltonian with respect to this fibration, the form ω, and the Hamiltonian function $H(y, x, \epsilon)$.

2.1 Fibered Changes of Coordinates

We consider the problem of analytic classification of families of systems (1), or orbital analytic classification of vector fields (4), with respect to *fiber-preserving* (shortly *fibered*) *changes of coordinates*

$$(y, x, \epsilon) = (\Phi(u, x, \epsilon), x, \epsilon).$$

Such a change of coordinate transforms a system (2) to a system

$$x(x - \epsilon)\frac{du}{dx} = (D_u\Phi)^{-1}J\,{}^{\text{t}}(D_y H) \circ \Phi - x(x - \epsilon)(D_u\Phi)^{-1}\frac{\partial \Phi}{\partial x}$$
$$= (\det D_u\Phi)^{-1}J\,{}^{\text{t}}(D_u(H \circ \Phi)) - x(x - \epsilon)(D_u\Phi)^{-1}\frac{\partial \Phi}{\partial x}, \quad (7)$$

using the identity

$$P J \,{}^{\text{t}}P = \det P \cdot J \quad \text{for any } 2 \times 2 \text{ matrix } P. \tag{8}$$

Definition 1 We call a fibered transformation Φ *transversely symplectic* if $\det(D_u\Phi) \equiv 1$, i.e. if it preserves the restriction of ω to each fiber Y_x.

Definition 2 Two systems (1) with Hamiltonian functions $\frac{H(y,x,\epsilon)}{x(x-\epsilon)}$ and $\frac{\tilde{H}(u,x,\epsilon)}{x(x-\epsilon)}$ are called *analytically equivalent* if there exists an analytic germ of a transversely symplectic transformation $y = \Phi(u, x, \epsilon)$ that is analytic in (u, x, ϵ) and transforms one system to another: $\Phi^* Z_{H,\epsilon} = Z_{\tilde{H},\epsilon}$.

Lemma 3 *If a transformation $y = \Phi(u, x, \epsilon)$ is transversely symplectic, then the transformed system (7) is transversely Hamiltonian w.r.t. $\omega = du_1 \wedge du_2$, i.e. it is again of the form (1).*

Proof It is enough to show that the system $\frac{du}{dx} = -(D_u\Phi)^{-1}\frac{\partial \Phi}{\partial x}$ is transversely Hamiltonian, that is, denoting $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := -(D_u\Phi)^{-1}\frac{\partial \Phi}{\partial x}$, to show that $\partial_{u_1} f_1 + \partial_{u_2} f_2 = 0$. Using the identity (8), we can express $f_1 = -\frac{1}{\det D_u\Phi}{}^{\text{t}}(\partial_{u_2}\Phi)J\partial_x\Phi$, $f_2 = \frac{1}{\det D_u\Phi}{}^{\text{t}}(\partial_{u_1}\Phi)J\partial_x\Phi$, hence

$$\partial_{u_1} f_1 + \partial_{u_2} f_2 = \frac{-1}{\det D_u\Phi}\left[{}^{\text{t}}(\partial_{u_2}\Phi)J\partial_x\partial_{u_1}\Phi - {}^{\text{t}}(\partial_{u_1}\Phi)J\partial_x\partial_{u_2}\Phi\right]$$
$$= \frac{-1}{\det D_u\Phi}\partial_x\left[{}^{\text{t}}(\partial_{u_2}\Phi)J\partial_{u_1}\Phi\right] = \frac{-1}{\det D_u\Phi}\partial_x \det D_u\Phi = 0.$$

$\square$

2.2 The Formal Invariant $\chi(h, x, \epsilon)$

Theorem 4 (Siegel) *Let $H : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ have a non-degenerate critical point at 0, and let ω be a symplectic volume form. There exists an analytic system of coordinates $u = (u_1, u_2)$ in which*

$$\omega = du_1 \wedge du_2, \quad and \quad H = G_H(u_1 u_2), \quad G_H(0) = 0.$$

The function G_H is uniquely determined by the pair (H, ω) up to the involution

$$G_H(u_1 u_2) \mapsto G_H(-u_1 u_2), \tag{9}$$

induced by the symplectic change of variable $J : (u_1, u_2) \mapsto (u_2, -u_1)$. The pair (G_H, ω) is called the Birkhoff–Siegel normal form *of the pair (H, ω). Moreover, if (H, ω) depend analytically on a parameter, then so does G_H and the change of coordinates.*

Proof While not explicitly stated in this form, the existence part of the theorem is originally proved by Siegel [28, chaps. 16 and 17]. See also [8,36]. The uniqueness can be seen by expressing $G_H(h)$ in terms of a period map of $\frac{\omega}{dH}$ over a vanishing cycle, see Sect. 2.2.1 below. $\square$

Remark 5 • The Theorem 4 provides the existence of an analytic transformation of a Hamiltonian vector field $\dot{y} = J\,^t(D_y H)$ in dimension 2 to its Birkhoff normal form $\dot{u} = J\,^t(D_u G_H)$, with

$$G_H(h) = \lambda h + \cdots,$$

where $\pm \lambda \neq 0$ are the eigenvalues of the linear part $J D_y^2 H(0)$. The involution (9) corresponds to the freedom of choice of the eigenvalue λ.
• The change of coordinates is far from unique. Indeed, the flow of any vector field $\xi = a(u_1 u_2)\left(u_1 \partial_{u_1} - u_2 \partial_{u_2}\right)$ preserves the normal form.

Let $H = H(y, x, \epsilon)$ be our germ. By the implicit function theorem, for each small (x, ϵ), the function $H(\cdot, x, \epsilon)$ has an *isolated non-degenerate critical point* $y_0(x, \epsilon)$, depending analytically on (x, ϵ). Let $y = \Phi(u, x, \epsilon)$ be the transformation to the Birkhoff–Siegel normal form for the function $y \mapsto H(y, x, \epsilon)$ and the form $\omega = dy_1 \wedge dy_2$, depending analytically on (x, ϵ), i.e.

$$H(\cdot, x, \epsilon) \circ \Phi(u, x, \epsilon) = G_H(u_1 u_2, x, \epsilon), \quad \det D_u \Phi(u, x, \epsilon) \equiv 1.$$

By (7), it brings the system (1) to a *prenormal form*

$$x(x - \epsilon)\frac{du}{dx} = J\,^t(D_u G_H) + O(x(x - \epsilon))$$

$$= \chi(u_1 u_2, x, \epsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u + O(x(x - \epsilon)), \tag{10}$$

where

$$\chi(h, x, \epsilon) = \chi^{(0)}(h, \epsilon) + x\chi^{(1)}(h, \epsilon) := \frac{\partial G_H}{\partial h}(h, x, \epsilon) \mod x(x - \epsilon), \tag{11}$$

$h = u_1 u_2$, or equivalently,

$$\chi(h, x, \epsilon) = \begin{cases} \frac{1}{\epsilon}\left[x\frac{\partial G_H}{\partial h}(h, \epsilon, \epsilon) - (x - \epsilon)\frac{\partial G_H}{\partial h}(h, 0, \epsilon) \right], & \epsilon \neq 0, \\ \frac{\partial G_H}{\partial h}(h, 0, 0) + x\frac{\partial^2 G_H}{\partial h \partial x}(h, 0, 0), & \epsilon = 0. \end{cases} \tag{12}$$

Definition 6 The function $\chi(h, x, \epsilon)$ is called the *formal invariant* of the system (1).

For $\epsilon \neq 0$ the formal invariant χ is completely determined by the functions $G_H(\cdot, 0, \epsilon)$ and $G_H(\cdot, \epsilon, \epsilon)$, which are *analytic invariants of the autonomous Hamiltonian systems* $X_{H,0,\epsilon}$, $X_{H,\epsilon,\epsilon}$ (5) *on the strong invariant manifolds* Y_0, Y_ϵ.

Corollary 7 *The formal invariant* $\chi(h, x, \epsilon)$ *is well-defined up to the involution*

$$J^* : \chi(h, x, \epsilon) \mapsto -\chi(-h, x, \epsilon), \tag{13}$$

induced by the symplectic transformation $u \mapsto Ju$. *It is uniquely determined by the polar part of the Hamiltonian* $\frac{H(y,x,\epsilon)}{x(x-\epsilon)}$, *and it is invariant with respect to fibered transversely symplectic changes of coordinates.*

Let

$$\lambda(x, \epsilon) = \chi(0, x, \epsilon).$$

Then $\pm\lambda(x, \epsilon)$ are the eigenvalues of the matrix of the linearized system $A(x, \epsilon) = J \, {}^t D_y^2 H(y_0(x, \epsilon), x, \epsilon)$ modulo $x(x - \epsilon)$, see Example 8, and the involution (13) corresponds to the freedom of choice of the eigenvalue λ.

Example 8 (*Traceless linear systems*) A traceless linear system

$$x(x - \epsilon)\frac{dy}{dx} = A(x, \epsilon)y, \tag{14}$$

with $\mathrm{tr}\, A(x, \epsilon) = 0$ and $A(0, 0) \sim \begin{pmatrix} \lambda^{(0)}(0) & 0 \\ 0 & -\lambda^{(0)}(0) \end{pmatrix}$ for some $\lambda^{(0)}(0) \neq 0$, is of the form (1) for the quadratic form $H(y, x, \epsilon) = \frac{1}{2}\,{}^t y J\,{}^t A(x, \epsilon)y$. Let $\pm\tilde\lambda(x, \epsilon)$ be the eigenvalues of $A(x, \epsilon)$, and let $C(x, \epsilon)$ be a corresponding matrix of eigenvectors of $A(x, \epsilon)$, depending analytically on (x, ϵ) and normalized so that $\det C(x, \epsilon) = 1$. The change of variable $y = C(x, \epsilon)u$ is the one which brings $(H, dy_1 \wedge dy_2)$ to its Birkhoff–Siegel normal form $(G_H = \frac{1}{2}\tilde\lambda(x, \epsilon)h^2, du_1 \wedge du_2)$. It transforms the system (14) to a prenormal form

$$x(x - \epsilon)\frac{du}{dx} = \left[\tilde\lambda(x, \epsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x(x - \epsilon)C^{-1}\frac{dC}{dx} \right]u.$$

Denoting $\lambda(x, \epsilon) := \left(\lambda^{(0)}(\epsilon) + x\lambda^{(1)}(\epsilon)\right) = \tilde\lambda(x, \epsilon) \mod x(x - \epsilon)$, then we have $\chi(h, x, \epsilon) = \lambda(x, \epsilon)$ up to the choice of sign (13). It coincides with the formal invariant of [10,15].

2.2.1 Geometric Interpretation of the Invariant χ.

For each small (x, ϵ), the function $H(\cdot, x, \epsilon)$ has an isolated non-degenerate critical point $y_0(x, \epsilon)$, depending analytically on (x, ϵ), with a critical value $h_0(x, \epsilon)$. For (x, ϵ) fixed, $h \in (\mathbb{C}, h_0)$, consider the germ of the level set $S_h(x, \epsilon) = \{y \in (\mathbb{C}^2, y_0(x, \epsilon)) : H(y, x, \epsilon) = h\} \subset Y_x$. As a basic fact of the Picard–Lefschetz theory [1], we know that if h is a non-critical value for $H(\cdot, x, \epsilon)$, i.e. $h \neq h_0$, then $S_h(x, \epsilon)$ has the homotopy type of a circle. Let $\gamma_h(x, \epsilon)$ depending continuously on (x, ϵ) be a loop generating the first homology group of $S_h(x, \epsilon)$, the so called *vanishing cycle*. And let μ be a 1-form such that $\omega = dH \wedge \mu$; its restriction to a non-critical level $S_h(x, \epsilon)$ is called the *Gelfand–Leray* form of ω and is denoted

$$\mu = \frac{\omega}{dH}.$$

Its period function over the vanishing cycle

$$p(h, x, \epsilon) := \frac{1}{2\pi i} \int_{\gamma_h(x, \epsilon)} \frac{\omega}{dH}, \tag{15}$$

is well-defined up to a sign change (orientation of γ_h), and depends analytically on (x, ϵ) [1, chap. 10]. Let $G_H(\cdot, x, \epsilon)$ be the inverse of the function $h \mapsto \int_{h_0(x, \epsilon)}^{h} p(s, x, \epsilon) \, ds$. Then (G_H, ω) is the Birkhoff–Siegel normal form of (H, ω).

Indeed, the above formula for G_H is invariant with respect to analytic transversely symplectic changes of coordinates: Supposing that $H = g(y_1 y_2, x, \epsilon)$ is in its Birkhoff–Siegel normal form, then the level sets are written as $S_h = \{y_1 \neq 0, \ y_2 = \frac{g^{\circ(-1)}(h, x, \epsilon)}{y_1}\}$, and $\frac{\omega}{dH} = \frac{dy_1}{y_1 \cdot \frac{\partial g}{\partial(y_1 y_2)} \circ g^{\circ(-1)}(h, x, \epsilon)}$, and therefore $p(h) = \left(\frac{\partial g}{\partial(y_1 y_2)}\right)^{-1} \circ g^{\circ(-1)}(h, x, \epsilon)$, i.e. $G_H = g$.

The above formula for the Birkhoff–Siegel normal form and hence for the formal invariant χ involves a double inversion which makes it difficult to calculate. The following proposition, which will be proved in Sect. 7.4, allows to determine it in some special cases. This will be useful in the case of the fifth Painlevé equation (Sect. 6).

Proposition 9 (Birkhoff–Siegel normal form of an autonomous Hamiltonian system) *Let $H(y)$ be of the form*

$$H(y) = G(h) + y_i \Delta(y_i, h), \quad h = y_1 y_2,$$

for some $i \in \{1, 2\}$, with G, Δ analytic germs, and

$$G(h) = \lambda h + O(h^2), \quad \lambda \neq 0.$$

Then (G, ω) is the Birkhoff–Siegel normal form for the pair (H, ω), $\omega = dy_1 \wedge dy_2$.

Corollary 10 (Invariant χ for $\epsilon = 0$) *For $\epsilon = 0$, suppose that*

$$H(y, x, 0) = G^{(0)}(h, 0) + x H^{(1)}(y, 0) + O(x^2), \quad h = y_1 y_2,$$

where $G^{(0)}(h, 0) = \lambda^{(0)}(0)h + O(h^2)$, $\lambda^{(0)}(0) \neq 0$. Write

$$H^{(1)}(y, 0) = G^{(1)}(h, 0) + y_1 \Delta_1(y_1, h, 0) + y_2 \Delta_2(y_2, h, 0),$$

with $\Delta_i(y_i, h, 0) = O(y_i)$. Then

$$\chi(h, x, 0) = \frac{\partial}{\partial h}\left(G^{(0)}(h, 0) + x\, G^{(1)}(h, 0)\right)$$

is the formal invariant of the vector field $Z_{H,0} = x^2 \partial_x + X_H$ associated to H.

Proof Consider a deformation

$$\begin{aligned} H(y, x, \epsilon) = G^{(0)}(h, 0) &+ xG^{(1)}(h, 0) + xy_1 \Delta_1(y_1, h, 0) \\ &+ (x - \epsilon)y_2 \Delta_2(y_2, h, 0) + O(x(x - \epsilon)), \end{aligned}$$

and calculate the Birkhoff–Siegel invariants for $H(y, 0, \epsilon)$, $H(y, \epsilon, \epsilon)$, using Proposition 9. $\qquad\square$

2.3 Model System (Formal Normal Form)

Definition 11 (*Model family*) Let $\chi(h, x, \epsilon)$ be the formal invariant of the system (1). The *model family* (*formal normal form*) for the the system (1) is the family of systems

$$\begin{aligned} x(x - \epsilon)\frac{du_1}{dx} &= \chi(u_1 u_2, x, \epsilon) \cdot u_1 \\ x(x - \epsilon)\frac{du_2}{dx} &= -\chi(u_1 u_2, x, \epsilon) \cdot u_2, \end{aligned} \tag{16}$$

which is Hamiltonian with respect to the Hamiltonian function $\frac{G(u_1 u_2, x, \epsilon)}{x(x-\epsilon)}$,

$$G(h, x, \epsilon) = \int_0^h \chi(s, x, \epsilon)\, ds = G_H(h, x, \epsilon) \mod x(x - \epsilon). \tag{17}$$

The formal normal form of the family Z_H is the associated family of vector fields

$$Z_G(u, x, \epsilon) = x(x - \epsilon)\partial_x + \chi(u_1 u_2, x, \epsilon)\left(u_1 \partial_{u_1} - u_2 \partial_{u_2}\right). \tag{18}$$

The system (16) is *integrable* with the function $h(u) = u_1 u_2$ being its first integral, $Z_G \cdot h = 0$. The general solutions of (16) are of the form

$$\begin{aligned} u_1(x, \epsilon; c) &= c_1 E_\chi(c_1 c_2, x, \epsilon), \\ u_2(x, \epsilon; c) &= c_2 E_\chi(c_1 c_2, x, \epsilon)^{-1}, \end{aligned} \quad c = (c_1, c_2) \in \mathbb{C}^2, \tag{19}$$

where $E_\chi(h, x, \epsilon) = \exp \int_\infty^x \frac{\chi(h,x,\epsilon)}{x(x-\epsilon)} dx$, namely

$$E_\chi(h, x, \epsilon) = \begin{cases} x^{-\frac{\chi^{(0)}(h,\epsilon)}{\epsilon}} (x - \epsilon)^{\frac{\chi^{(0)}(h,\epsilon)}{\epsilon} + \chi^{(1)}(h,\epsilon)}, & \text{for } \epsilon \neq 0, \\ e^{-\frac{\chi^{(0)}(h,0)}{x}} x^{\chi^{(1)}(h,0)}, & \text{for } \epsilon = 0. \end{cases} \tag{20}$$

In particular, $E_\chi(h, x, \epsilon)$ is analytic in ϵ near the origin for $x \neq 0$.

3 Formal and Sectoral Normalization Theorem

Throughout the text we will denote

$$\mathsf{Y} = \{|y| < \delta_y\}, \quad \mathsf{U} = \{|u| < \delta_u\}, \quad \mathsf{X} = \{|x| < \delta_x\}, \quad \mathsf{E} = \{|\epsilon| < \delta_\epsilon\}, \tag{21}$$

for some $\delta_y, \delta_u, \delta_x, \delta_\epsilon > 0$ sufficiently small,[1] and consider the systems (1), resp. (16), as defined on $\mathsf{Y} \times \mathsf{X} \times \mathsf{E}$, resp. $\mathsf{U} \times \mathsf{X} \times \mathsf{E}$. We will implicitly suppose that $\delta_\epsilon \ll \delta_x$ so that the singular points $(y, x) = (y_0(0, \epsilon), 0)$ and $(y_0(\epsilon, \epsilon), \epsilon)$ of the original foliation are both well inside $\mathsf{Y} \times \mathsf{X}$ for all $\epsilon \in \mathsf{E}_\pm$.

We will now define "sectoral" domains in the ϵ- and x-space on which the system can be normalized. They are of the same type as those introduced in [10, 13, 15, 16, 25].

Definition 12 (*Family of spiraling sectoral domains* $\mathsf{X}_\pm(\epsilon), \epsilon \in \mathsf{E}_\pm$) Let $\eta > 0$ be an arbitrarily small constant, and let $\delta_x \gg \delta_\epsilon > 0$ be radii of small discs at 0 in the x-and ϵ-space. Let $\lambda(x, \epsilon) = \chi(0, x, \epsilon)$, and let

$$\mathsf{E}_\pm := \{|\epsilon| < \delta_\epsilon, \ |\arg\left(\pm \frac{\epsilon}{\lambda(0,0)}\right)| < \pi - 2\eta\} \cup \{0\}, \tag{22}$$

be two sectors in the ϵ-space. For $\epsilon \in \mathsf{E}_\pm$ define a domain

$$\mathsf{X}_\pm(\epsilon)$$

in the x-space as a simply connected ramified domain spanned by the complete real trajectories of the vector fields

$$e^{i\theta_\pm} \cdot \frac{x(x-\epsilon)}{\lambda(x,\epsilon)} \partial_x \tag{23}$$

that never leave the disc of radius δ_x, where the phase $\theta_\pm$ varies continuously in the interval

$$\begin{cases} \max\left\{0, \arg\left(\frac{\pm\epsilon}{\lambda(0,0)}\right)\right\} - \frac{\pi}{2} + \eta < \theta_\pm < \min\left\{0, \arg\left(\frac{\pm\epsilon}{\lambda(0,0)}\right)\right\} + \frac{\pi}{2} - \eta, & \text{for } \epsilon \neq 0, \\ |\theta_\pm| < \frac{\pi}{2} - \eta, & \text{for } \epsilon = 0. \end{cases} \tag{24}$$

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[1] The constants $\delta_u, \delta_x, \delta_\epsilon$ will appear in Theorem 17 in purely existential way; they're "small enough".
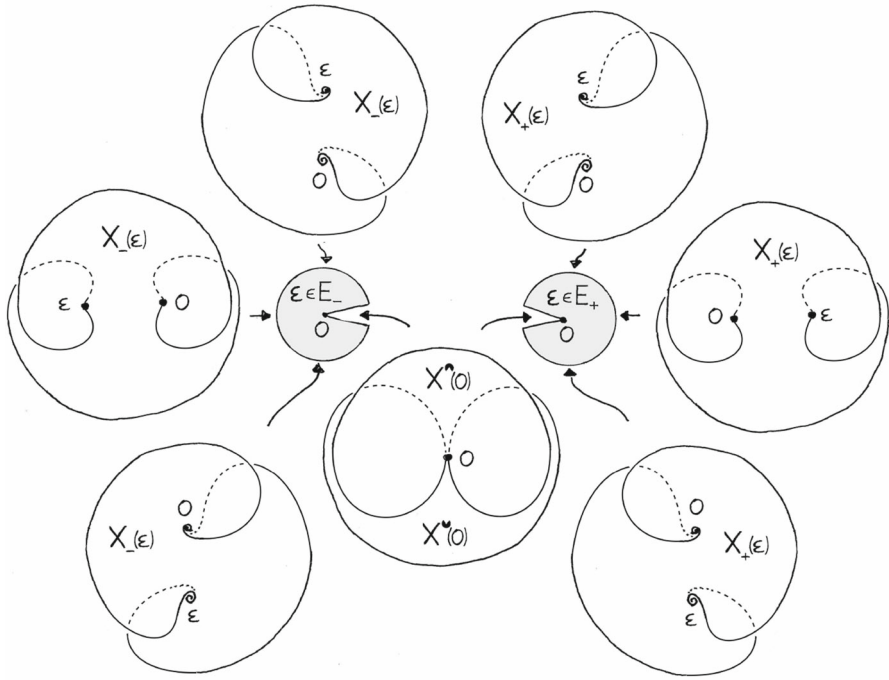
Fig. 1 The domains $X_\pm(\epsilon)$ in dependence on $\epsilon \in E_\pm$. (Picture with $\lambda^{(0)}(0) = 1$)

The constraints (24) on the variation of $\theta_\pm$ are such that the real dynamics of the vector field (23) and the asymptotic behavior of the solutions (19) would not change drastically depending on $\theta_\pm$. Namely, for $\epsilon \neq 0$:

- The point $x = \epsilon$ is repulsive when $|\arg \frac{\epsilon}{\lambda(x,\epsilon)} - \theta_\pm| < \frac{\pi}{2}$ and attractive when $|\arg \frac{-\epsilon}{\lambda(x,\epsilon)} - \theta_\pm| < \frac{\pi}{2}$, and vice-versa for the point $x = 0$.
- The u_1-component of the solution (19) tends to 0 along a negative real trajectory of (23) and to ∞ along a positive real trajectory for $|\theta_\pm| < \frac{\pi}{2}$, and vice-versa for the u_2-component.

For $\epsilon = 0$, the domain $X_+(0) = X_-(0)$ consists of a pair of overlapping sectoral domains[2] $X^\spadesuit(0)$, $X^\heartsuit(0)$ of opening $2\pi - 2\eta$ with a common point at $x = 0$. See Fig. 1. Moreover, it follows from the construction that when $\epsilon \to 0$ radially with some fixed $\arg \epsilon = \beta$, then $X_\pm(\epsilon)$ tends to a sectoral subdomain of $X_\pm(0)$ corresponding to the constraints (24) on the variation of the phase $\theta_\pm$.

Note that by our definition *the points $x = 0, \epsilon$ are included in $X_\pm(\epsilon)$*.

We denote $XE_\pm$ the union of the domains in the (x, ϵ)-space

$$XE_\pm := \{(x, \epsilon) \mid x \in X_\pm(\epsilon)\}. \tag{25}$$

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[2] Our notion of sectoral domains deviates slightly from the usual one by including also the vertex point 0.

Remark 13 Denoting $x_{1,\pm}(\epsilon)$ the *attractive* equilibrium point of (23) and $x_{2,\pm}(\epsilon)$ the *repulsive* one,

$$x_{1,+}(\epsilon) = x_{2,-}(\epsilon) = 0, \quad x_{1,-}(\epsilon) = x_{2,+}(\epsilon) = \epsilon, \tag{26}$$

then

$$u_i(x, \epsilon) \to \infty, \ u_j(x, \epsilon) \to 0, \ (j = 3 - i), \ \text{when} \ x \to x_{i,\pm}(\epsilon)$$

along a real trajectory of (23).

Remark 14 For each $\epsilon \in \mathsf{E}_\pm \smallsetminus \{0\}$, $\mathsf{X}_\pm(\epsilon)$ is a simply connected ramified domain. Strictly speaking, the domain $\mathsf{X}_\pm(\epsilon)$ is defined on the Riemann surface of $t(x, \epsilon) = \int_\infty^x \frac{\lambda(x,\epsilon)}{x(x-\epsilon)} dx$, on which it corresponds to a simply connected domain in the coordinate $t(x, \epsilon)$ obtained as a union of infinite stripes of varying directions $\theta_\pm$ for $\epsilon \neq 0$, resp. half-planes with boundaries in directions $\theta_\pm$ for $\epsilon = 0$. See [13] for more details.

Remark 15 In the variable $x = \epsilon z$, the system (2) takes the form of a singularly perturbed system

$$\epsilon z(z - 1)\frac{dy}{dz} = J^{\,\mathrm{t}}(D_y H)(y, \epsilon z, \epsilon).$$

The domains $z \in \frac{1}{\epsilon}\mathsf{X}_\pm(\epsilon)$, $\epsilon \in \mathsf{E}_\pm$, then roughly correspond to the Stokes domains in the sense of exact WKB analysis [12], where the Stokes curves would be the real separatrices of the point $z = \infty$ of the vector field (23) $e^{i(\theta_\pm + \arg \epsilon)}\frac{z(z-1)}{\lambda(0,0)}\partial_z$ with a fixed phase $\theta_\pm$ (24).

Before giving a general theorem on sectoral normalization for the parametric family (1), let us first state it for the limit system with $\epsilon = 0$ which has an irregular singularity of Poincaré rank 1 at $x = 0$.

Theorem 16 (Formal and sectoral normalization at $\epsilon = 0$) *The system* (1) *with* $\epsilon = 0$ *can be brought to its formal normal form* (16) *through a formal transversely symplectic change of coordinates*

$$(y, x) = (\hat{\Psi}(u, x, 0), x), \quad \hat{\Psi}(u, x, 0) = \sum_{k \geq 0} \psi^{(k,0)}(u)x^k, \tag{27}$$

where $\psi^{(k,0)}(u)$ are analytic in u on a fixed neighborhood U of 0. This formal series is generally divergent, but it is Borel 1-summable, with a pair of Borel sums $\Psi^{\spadesuit}(u, x, 0)$ and $\Psi^{\heartsuit}(u, x, 0)$ defined respectively above the sectors $x \in \mathsf{X}^{\spadesuit}(0)$, $\mathsf{X}^{\heartsuit}(0)$ of Definition 12 (for some $0 < \eta < \frac{\pi}{2}$ arbitrarily small and some $\delta_x > 0$ depending on η), and $u \in \mathsf{U}$. The fibered sectoral transformations $(y, x) = (\Psi^{\bullet}(u, x, 0), x)$, $\bullet = \spadesuit, \heartsuit$, are transversely symplectic and bring the system (1) *with* $\epsilon = 0$ *to its formal normal form.*

The Theorem 16 is originally due to Takano [32] for systems (1) whose formal invariant is of the form $\chi(h, x) = \lambda^{(0)} + x\chi^{(1)}(h)$. In the case of the irregular singularity of the fifth Painlevé equation it was proved earlier by Takano [30]. Some similar

and closely related theorems are due to Shimomura [27], Yoshida [34], and recently by Bittmann [3,4], which apply to doubly resonant systems $x^2 \frac{dy}{dx} = F(y, x)$, with $D_y F(0, 0) = \lambda^{(0)}(0) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ under the condition that $\Re \left(\operatorname{tr} \frac{\partial}{\partial x} D_y F(0, 0) \right) > 0$. This condition is not satisfied for Hamiltonian systems (1) but nevertheless allows to treat Painlevé equations.

Theorem 17 (Formal and sectoral normalization) *Let $Z_H(y, x, \epsilon)$ be a family of vector fields (4) and let $\chi(h, x, \epsilon)$ be their formal invariant.*
(i) There exists a formal transversely symplectic change of coordinates $(y, x, \epsilon) = (\hat{\Psi}(u, x, \epsilon), x, \epsilon)$ which brings Z_H to its formal normal form (18). It is written as a formal power series

$$\hat{\Psi}(u, x, \epsilon) = \sum_{k,l \geq 0} \psi^{(k,l)}(u) x^k \epsilon^l, \tag{28}$$

with $\psi^{(k,l)}(u)$ analytic in u on the same fixed neighborhood $\mathsf{U} = \{|u_1|, |u_2| < \delta_u\}$ of 0 for some $\delta_u > 0$, and $\hat{\Psi}(u, 0, \epsilon) = \sum_{l \geq 0} \psi^{(0,l)}(u) \epsilon^l$ is convergent on $\mathsf{U} \times \mathsf{E}$ for some $\delta_\epsilon > 0$. The coefficients $\psi^{(k,l)}$ grow at most factorially in $k + l$:

$$\sup_{u \in \mathsf{U}} |\psi^{(k,l)}(u)| \leq C r^{-(k+l)} (k + l)! \quad \text{for some} \quad C, r > 0.$$

(ii) There exists a transversely symplectic fibered change of coordinates $(y, x, \epsilon) = (\Psi_\pm(u, x, \epsilon), x, \epsilon)$, with $\Psi_\pm(u, 0, \epsilon) = \hat{\Psi}(u, 0, \epsilon)$, defined for x in the spiraling domain $\mathsf{X}_\pm(\epsilon)$, $\epsilon \in \mathsf{E}_\pm$, of Definition 12 (for some $0 < \eta < \frac{\pi}{2}$ arbitrarily small and some $\delta_x, \delta_\epsilon > 0$ depending on η), and for $u \in \mathsf{U}$, which brings Z_H to its formal normal form (18). It is uniformly continuous on $\mathsf{U} \times \mathsf{XE}_\pm$ (25) and analytic on its interior. When ϵ tends radially to 0 with $\arg \epsilon = \beta$, then $\Psi_\pm(u, x, \epsilon)$ converges to $\Psi_\pm(u, x, 0)$ uniformly on compact sets of the sub-domain $\lim_{\substack{\epsilon \to 0 \\ \arg \epsilon = \beta}} \mathsf{X}_\pm(\epsilon) \subseteq \mathsf{X}_\pm(0)$. Note that in our notation $\Psi_\pm(u, x, 0)$ consists of a pair of sectoral transformations $\Psi^{\spadesuit}(u, x, 0)$ and $\Psi^{\heartsuit}(u, x, 0)$; it is a functional cochain using the terminology of [11,18].
(iii) Let $\widetilde{\Psi}(u, x, \epsilon)$ be an analytic extension of the function given by the convergent series

$$\widetilde{\Psi}(u, x, \epsilon) = \sum_{k,l \geq 0} \frac{\psi^{(k,l)}(u)}{(k + l)!} x^k \epsilon^l, \quad \psi^{(k,l)} \text{ as in (28)}.$$

For each point (x, ϵ), for which there is $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ such that $\mathbb{B}_\theta \cdot (x, \epsilon) \subseteq \mathsf{XE}_\pm$, with $\mathbb{B}_\theta \subset \mathbb{C}$ denoting the closed disc with center on $e^{i\theta} \mathbb{R}^+$ and 0 and 1 on its circumference, we can express $\Psi_\pm(u, x, \epsilon)$ through the following Laplace transform of $\widetilde{\Psi}$:

$$\Psi_\pm(u, x, \epsilon) = \int_0^{+\infty e^{i\theta}} \widetilde{\Psi}(u, sx, s\epsilon) e^{-s} \, ds. \tag{29}$$

In particular, $\Psi_\pm(u, x, 0)$ is the pair of sectoral Borel sums $\Psi^{\spadesuit}(u, x, 0)$, $\Psi^{\heartsuit}(u, x, 0)$ of the formal series $\hat{\Psi}(u, x, 0)$.

As a consequence, $\Psi_\pm$ and $\hat\Psi$ satisfy the same $(\partial_u, \partial_x, \partial_\epsilon)$-differential relations with meromorphic coefficients.

The proof of the theorem will be given in Sect. 7.

The transformations $\Psi_\pm$ and $\hat\Psi$ are unique up to left composition with an analytic symmetry of the model system, see Corollary 32.

Corollary 18 *The system* (1) *possesses:*

(i) *A formal first integral given by* $h \circ \hat\Psi^{\circ(-1)}(y, x, \epsilon)$, *where* $\hat\Psi$ *is as above and* $h(u) = u_1 u_2$ *is a first integral of the model system.*
(ii) *An actual first integral given by* $h \circ \Psi_\pm^{\circ(-1)}(y, x, \epsilon)$ *that is bounded and analytic on the domain* $\mathsf{XE}_\pm$.

Definition 19 The solution $y = \Psi_\pm(0, x, \epsilon)$ is called *ramified center manifold* of the vector field Z_H. It is the unique solution of the corresponding system (1) that is bounded on $\mathsf{X}_\pm(\epsilon)$ (cf. [13]).

Remark 20 More generally, instead of the system (1) one could consider a parametric family of systems unfolding a higher order singularity

$$(x^{k+1} + \epsilon_k x^k + \cdots + \epsilon_0)\tfrac{dy}{dx} = J^{\,t}(D_y H),$$
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (y, x, \epsilon) \in (\mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}^{k+1}, 0),$$

for any $k \geq 1$ under the same non-degeneracy condition on H. The definition of the formal invariant $\chi(h, x, \epsilon) = \chi^{(0)}(h, \epsilon) + \cdots + x^k \chi^{(k)}(h, \epsilon)$ would then be a straightforward generalization of (11), and it is expected that the same should be also true for the assertions (i) and (ii) of Theorem 17 where the "sectoral" domains in the (x, ϵ)-space should be generalized in the spirit of those introduced in [10].

4 Stokes Operators and Accumulation of Monodromy

We will define several operators acting as transversely symplectic fibered isotropies on the three following parametric foliations given by three different vector fields:

- Foliations in the (u, x)-space given by the model vector field $Z_G(u, x, \epsilon)$ (18).
- Foliations in the (c, x)-space, c being the constant of initial condition in (19), given by the rectified vector field $Z_0(c, x, \epsilon) = x(x - \epsilon)\partial_x$. Note that a fibered isotropy of Z_0 is necessarily *independent* of x; it acts on the c-space of *initial conditions* only.
- Foliations in the (y, x)-space given by the original vector field $Z_H(y, x, \epsilon)$ (4).

Notation 21 If $f = f(u, x, \epsilon)$ is a map, we denote $f(\cdot, x, \epsilon) : u \mapsto f(u, x, \epsilon)$ the restricted map. Hence $f(\cdot, x, \epsilon) \circ g(\cdot, x, \epsilon) = f(g(\cdot, x, \epsilon), x, \epsilon)$.

4.1 Symmetries of the Model System: Exponential Torus

A *vertical infinitesimal symplectic symmetry*, or shortly *infinitesimal symmetry*, of the normal form vector field Z_G (18) is an analytic germ of a family of vector fields $\xi(u, x, \epsilon)$ in the (u, x)-space that preserves:

(i) the x-coordinate: $\mathcal{L}_\xi x = \xi \cdot x = 0$,
(ii) the symplectic form $\omega = du_1 \wedge du_2$: $\mathcal{L}_\xi \omega = 0$,
(iii) the vector field Z_G: $\mathcal{L}_\xi Z_G = [\xi, Z_G] = 0$.

Similarly, we can define a *vertical infinitesimal symplectic symmetry* $\xi_\epsilon(u, x)$ of the normal form vector field $Z_{G,\epsilon}$ restricted to a fixed ϵ.

Lemma 22 *A family of vector fields $\xi(u, x, \epsilon)$ is an infinitesimal symmetry of Z_G if and only if there exists an analytic first integral $f(u, x, \epsilon)$ of Z_G, i.e. $Z_G \cdot f = 0$, such that $\xi = X_f(u, x, \epsilon)$ is a Hamiltonian vector field for f with respect to ω. Similarly for an infinitesimal symmetry ξ_ϵ of $Z_{G,\epsilon}$ for any fixed ϵ.*

Proof The conditions *(i)* and *(ii)* say that $\xi = a_1(u, x, \epsilon)\partial_{u_1} + a_2(u, x, \epsilon)\partial_{u_2}$ with $\frac{\partial a_1}{\partial u_1} + \frac{\partial a_2}{\partial u_2} = 0$, i.e. $a_1 = \frac{\partial f}{\partial u_2}$, $a_2 = -\frac{\partial f}{\partial u_1}$ for some f with $f(0, x, \epsilon) = 0$, and $\xi = \frac{\partial f}{\partial u_2}\partial_{u_1} - \frac{\partial f}{\partial u_1}\partial_{u_2} = X_f$. The condition *(iii)* now says that

$$0 = Z_G \cdot D_u f - X_f \cdot D_u G = D_u(Z_G \cdot f),$$

which means that $Z_G \cdot f$ is independent of u, hence by evaluating at $u = 0$: $Z_G \cdot f|_{u=0} = x(x - \epsilon)\frac{\partial f}{\partial x}|_{u=0} = 0$. □

The "initial condition" constant $c \in (\mathbb{C}^2, 0)$ in the general solution $u(x, \epsilon; c)$ (19) is obviously locally constant along the solution, hence it can be seen as a local first integral of the corresponding vector field Z_G. By expressing it as a function of (u, x, ϵ), we obtain

$$c_1(u, x, \epsilon) = u_1 \cdot E_\chi(h, x, \epsilon)^{-1}, \quad c_2(u, x, \epsilon) = u_2 \cdot E_\chi(h, x, \epsilon), \qquad (30)$$

and

$$h(u) = u_1 u_2 = c_1 c_2,$$

where $E_\chi(h, x, \epsilon)$ is as in (20). Clearly, any function of $c = (c_1, c_2)$ and ϵ is again a first integral of Z_G, and since c defines local coordinates on the space of leaves (space of initial conditions), the converse is also true. Note that the map $c : u \mapsto c(u, x, \epsilon)$ conjugates the vector field Z_G to the "rectified" vector field $Z_0 = x(x - \epsilon)\partial_x$ in the (c, x)-space:

$$Z_G = c^*(Z_0).$$

It turns out that analytic first integrals are functions of $h = c_1 c_2$ and ϵ only.

Proposition 23 *If $f(u, x, \epsilon)$ is an analytic first integral of Z_G on some neighborhood $\mathsf{U} \times \mathsf{X} \times \mathsf{E}$ of 0, then $f = F(u_1 u_2, \epsilon)$ with F analytic. Similarly, an analytic first integral $f_\epsilon(u, x)$ of $Z_{G,\epsilon}$ on some neighborhood $\mathsf{U} \times \mathsf{X}$ of 0 is of the form $f_\epsilon = F_\epsilon(u_1 u_2)$ for some analytic F_ϵ.*

Proof Since $c_1(u, x, \epsilon)$, $c_2(u, x, \epsilon)$ are local coordinates on the space of leaves, we can write $f(u, x, \epsilon) = g(c(u, x, \epsilon), \epsilon)$ for some analytic germ g, which can be decomposed uniquely as $g(c, \epsilon) = g_0(h, \epsilon) + c_1 g_1(c_1, h, \epsilon) + c_2 g_2(c_2, h, \epsilon)$ with g_l analytic. When $x \to x_{i,\pm}$ from within $\mathsf{X}_\pm(\epsilon)$, then $c_i(u, x, \epsilon) \to 0$ and $c_j(u, x, \epsilon) \to \infty$ for every $u \in \mathsf{U}$. Since $f(u, x, \epsilon) = g(c(u, x, \epsilon), \epsilon)$ is bounded on $\mathsf{X}_\pm(\epsilon)$, this means that $g_j = 0$. Therefore $f(u, x, \epsilon) = g_0(h, \epsilon)$. $\qquad \square$

Corollary 24 *The Lie algebra* $\mathfrak{tor}$ *of analytic infinitesimal symmetries of Z_G consists of Hamiltonian vector fields*

$$\mathfrak{tor} = \{\xi = a(u_1 u_2, \epsilon)\left(u_1 \partial_{u_1} - u_2 \partial_{u_2}\right) = a(h, \epsilon)X_h \ : \ a(h, \epsilon) \text{ analytic}\}, \quad (31)$$

and is commutative. It is called the infinitesimal torus *of Z_G.*

Similarly, for a fixed $\epsilon \in \mathsf{E}$, the Lie algebra $\mathfrak{tor}_\epsilon$ *of analytic infinitesimal symmetries of $Z_{G,\epsilon}$ consists of Hamiltonian vector fields*

$$\mathfrak{tor}_\epsilon = \{\xi_\epsilon = a_\epsilon(u_1 u_2)\left(u_1 \partial_{u_1} - u_2 \partial_{u_2}\right) = a_\epsilon(h)X_h \ : \ a_\epsilon(h) \text{ analytic}\},$$

and is commutative. It is called the infinitesimal torus *of $Z_{G,\epsilon}$.*

The time-1 flow map of a vector field (31) is given by

$$u \mapsto T_a(u) := \Phi^1_{aX_h}(u) = \begin{pmatrix} e^{a(h,\epsilon)} & 0 \\ 0 & e^{-a(h,\epsilon)} \end{pmatrix} u. \quad (32)$$

Definition 25 We define the *exponential torus of Z_G* as the connected commutative Lie group

$$\mathbf{Tor} = \exp \mathfrak{tor} = \{T_a \ : \ a(h, \epsilon) \text{ analytic}\}, \quad (33)$$

and likewise, for a fixed $\epsilon \in \mathsf{E}$, the *exponential torus of $Z_{G,\epsilon}$* as the commutative Lie group

$$\mathbf{Tor}_\epsilon = \exp \mathfrak{tor}_\epsilon = \{T_{a_\epsilon} \ : \ a_\epsilon(h) \text{ analytic}\}. \quad (34)$$

Definition 26 A *transversely symplectic fibered symmetry*, or shortly *symmetry*, of a family of vector fields Z_G is a germ of analytic symplectic transformation $(u, x, \epsilon) \mapsto (\phi(u, x, \epsilon), x, \epsilon)$, such that $\phi^* Z_G = Z_G$. Similarly, for a fixed $\epsilon \in \mathsf{E}$, a *symmetry* of a vector field $Z_{G,\epsilon}$ is a germ of analytic symplectic transformation $(u, x) \mapsto (\phi_\epsilon(u, x), x)$, such that $\phi_\epsilon^* Z_{G,\epsilon} = Z_{G,\epsilon}$.

Proposition 27 *The (transversely symplectic fibered) symmetries of Z_G, resp. $Z_{G,\epsilon}$, are exactly the elements of the exponential torus* **Tor***, resp.* **Tor**$_\epsilon$*.*

This follows as an immediate corollary from Proposition 31 below. A characterization of the Lie group of symmetries of a general system (1) will be given in Proposition 34.

Fig. 2 The intersection sectors $X_{i,\pm}^{\cap}(\epsilon)$, $i = 1, 2$, for some fixed value of ϵ. The two arrows symbolize the Stokes operators (40) which change the branch of the transformation $\Psi_\pm$ above the intersection sectors
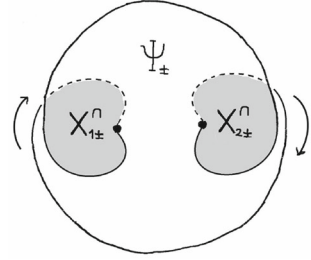
Definition 28 (*Intersection sectors*) For $\epsilon \in \mathsf{E}_\pm \setminus \{0\}$ define the *left and right intersection sectors*

$$X_{i,\pm}^{\cap}(\epsilon) = \{x \in X_\pm(\epsilon) : x_{i,\pm} + e^{2\pi i}(x - x_{i,\pm}) \in X_\pm(\epsilon)\},$$

and for $\epsilon = 0$ let $X_{i\pm}^{\cap}(0)$ be their limits (see Fig. 2). They are the domains of self-intersection of $X_\pm(\epsilon)$ attached to the points $x_{i,\pm}(\epsilon)$ (26). And they are simply connected; in fact in the coordinate $t(x, \epsilon) = \int_\infty^x \frac{\lambda(x,\epsilon)}{x(x-\epsilon)} dx$ of Remark 14 each intersection sectors corresponds to an "infinite wedge".

Definition 29 An *isotropy* of a vector field $Z_{G,\epsilon}$ is a germ of transversely symplectic transformation $(u, x) \mapsto (\phi_\epsilon(u, x), x)$, such that $\phi_\epsilon^* Z_{G,\epsilon} = Z_{G,\epsilon}$. We will consider isotropies of $Z_{G,\epsilon}$ which are analytic and bounded either on the domain $\mathsf{U} \times X_\pm(\epsilon)$ or on $\mathsf{U} \times X_{i,\pm}^{\cap}(\epsilon)$.

Lemma 30 *Let $\phi_\epsilon(u, x)$ be a sectoral isotropy of the normal form vector field $Z_{G,\epsilon}$, analytic and bounded for $x \in X_{i,\pm}^{\cap}(\epsilon)$, $u \in \mathsf{U}$. Then*

$$c_i \circ \phi_\epsilon = c_i \cdot e^{a_\epsilon(h)} + c_i^2 g_{i,\epsilon}(h, c_i), \quad and \quad c_j \circ \phi_\epsilon = c_j \cdot e^{-a_\epsilon(h)} + g_{j,\epsilon}(h, c_i),$$

for some analytic germs $a_\epsilon, g_{i,\epsilon}, g_{j,\epsilon}$, subject to a transversal symplecticity condition $\det D_c(c \circ \phi_\epsilon) \equiv 1$.

Proof The isotropy $\phi_\epsilon(u, x)$ is analytic in u on some neighborhood of $u = 0$ and bounded when $x \to x_{i,\pm}(\epsilon)$. In particular, the restriction of $c \circ \phi_\epsilon$ to any fiber $\{x = cst \neq 0, \epsilon\}$ is analytic in u, and therefore the function $c \circ \phi_\epsilon(u(x, \epsilon; c), x)$, which is a function only of c, independent of x, is analytic in c on some neighborhood of $c = 0$.

We have $c_k = u_k E_\chi(h, x, \epsilon)^{(-1)^k}$, $k = 1, 2$, (30), and

$$\lim_{\substack{x \to x_{i,\pm} \\ x \in X_\pm(\epsilon)}} E_\chi(h, x, \epsilon)^{(-1)^i} = 0.$$

Writing $\phi_\epsilon = (\phi_{1,\epsilon}, \phi_{2,\epsilon})$, its i-th component is given by

$$\phi_{i,\epsilon} = (c_i \circ \phi_\epsilon) \cdot \left(E^{-(-1)^i} \circ h \circ \phi_\epsilon \right),$$

and we see that the expansion of $c_i \circ \phi_\epsilon$ in powers of c can contain only terms $c_i^{n_i} c_j^{n_j} = u_i^{n_i} u_j^{n_j} E_\chi(h, x, \epsilon)^{(-1)^i (n_i - n_j)}$ with $n_i \geq n_j + 1$, while the expansion of $c_j \circ \phi_\epsilon$ in powers of c can contain only terms $c_i^{n_i} c_j^{n_j} = u_i^{n_i} u_j^{n_j} E_\chi(h, x, \epsilon)^{(-1)^i (n_i - n_j)}$ with $n_i \geq n_j - 1$, i.e.

$$c_i \circ \phi_\epsilon = c_i \cdot e^{a_{i,\epsilon}(h)} + c_i^2 g_{i,\epsilon}(h, c_i), \quad \text{and} \quad c_j \circ \phi_\epsilon = c_j \cdot e^{a_{j,\epsilon}(h)} + g_{j,\epsilon}(h, c_i).$$

By transversal symplecticity $1 = \det D_u \phi_\epsilon = \det D_c(c \circ \phi_\epsilon)$. One can verify that the last determinant is a function of (h, c_i) only; expressing it for $c_i = 0$, we have

$$1 = e^{a_{i,\epsilon} + a_{j,\epsilon}}(1 + h a'_{i,\epsilon} + h a'_{j,\epsilon}) = \tfrac{d}{dh}\left(h e^{a_{i,\epsilon} + a_{j,\epsilon}}\right),$$

which implies that $a_{i,\epsilon} + a_{j,\epsilon} = 0$. □

Note that the hypersurface $\{u_i = 0\} = \{c_i = 0\}$ consists of all leaves of $Z_{G,\epsilon}$ that are bounded when $x \to x_{i,\pm}(\epsilon)$ inside $X_\pm(\epsilon)$, and therefore ϕ_ϵ must preserve it.

Proposition 31 *An isotropy of the normal form vector field $Z_{G,\epsilon}$ that is bounded and analytic on $\mathsf{U} \times \mathsf{X}_\pm(\epsilon)$ is an element of the exponential torus $\mathbf{Tor}_\epsilon$ (34).*

Proof An isotropy $\phi_\epsilon(u, x)$ of the model system bounded and analytic on $\mathsf{U} \times \mathsf{X}_\pm(\epsilon)$ is in particular bounded and analytic on $\mathsf{U} \times \mathsf{X}_{i,\pm}^\cap(\epsilon)$, and therefore by Lemma 30, it is such that

$$c \circ \phi_\epsilon = {}^{t}(c_1 e^{a_\epsilon(h)}, c_2 e^{-a_\epsilon(h)}), \quad \text{i.e.} \quad \phi_\epsilon(u, x) = {}^{t}(u_1 e^{a_\epsilon(h)}, u_2 e^{-a_\epsilon(h)})$$

for some analytic germ a_ϵ. □

Corollary 32 *The normalizing transformations $\hat{\Psi}$ and $\Psi_\pm$ of Theorem 17 are unique modulo right composition with elements of the exponential torus $\mathbf{Tor}$. They are uniquely determined by the analytic germ $\hat{\Psi}(u, 0, \epsilon) = \Psi_\pm(u, 0, \epsilon)$.*

Proof If $\Psi_\pm$, $\Phi_\pm$ are two normalizing transformations for Z_H, then $\phi_\pm(u, \epsilon) := \Psi_\pm^{\circ(-1)} \circ \Phi_\pm(u, x, \epsilon)$ is by Proposition 31 independent of x and such that $\phi_\pm(\cdot, \epsilon) \in \mathbf{Tor}_\epsilon$. Hence $\phi_\pm(u, \epsilon) = \hat{\Psi}^{\circ(-1)}(\cdot, 0, \epsilon) \circ \hat{\Phi}(u, 0, \epsilon)$ which is analytic in $\epsilon \in \mathsf{E}$ by Theorem 17 (i). □

4.2 Canonical General Solutions

The model system has a *canonical general solution* $u(x, \epsilon; c)$ (19), depending on an *"initial condition" parameter* $c \in \mathbb{C}^2$, uniquely determined by a choice of a branch of the function $E_\chi(h, x, \epsilon)$ (20). Correspondingly, $y = \Psi_\pm(u(x, \epsilon; c), x, \epsilon)$ is a germ of general solution of the original system on $\mathsf{Y} \times \mathsf{X}_\pm(\epsilon)$. In order for this solution to have a continuous limit when $\epsilon \to 0$, one has to split the domain $\mathsf{X}_\pm(\epsilon)$ in two parts, corresponding to the two parts of $\mathsf{X}_\pm(0)$, by making a cut in between the singular
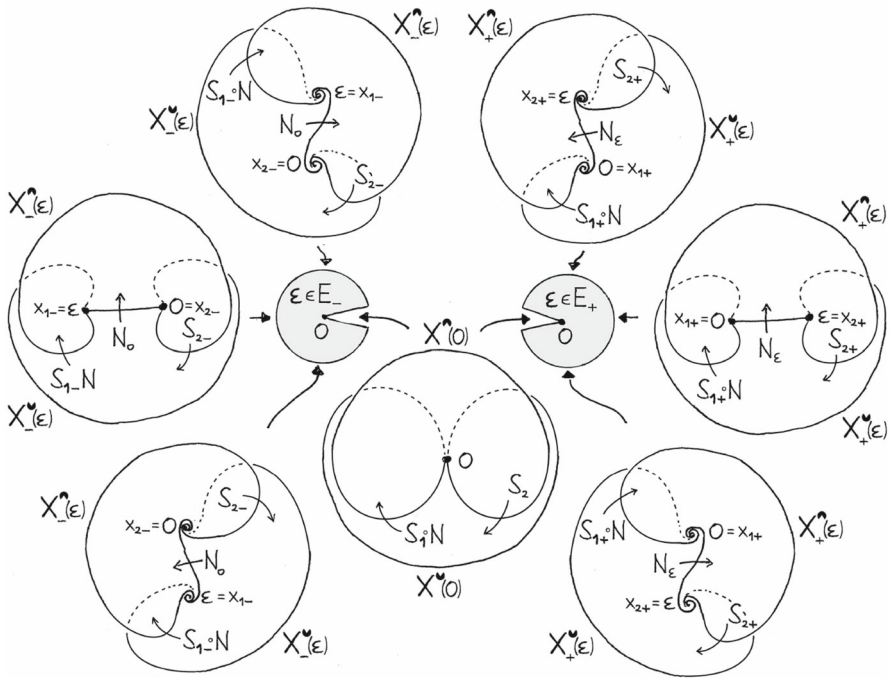
Fig. 3 The sectoral isotropies $S_{1,\pm} \circ N$, $S_{2,\pm}$ and $N_{x_{2,\pm}}$ as connecting transformations between the canonical general solutions $y_\pm^{\wedge}(x, \epsilon; \cdot)$ on $X_\pm^{\wedge}$ and $y_\pm^{\vee}(x, \epsilon; \cdot)$ on $X_\pm^{\vee}$ above the intersection sectors

points $x_{1,\pm}$, $x_{2,\pm}$ along a trajectory of (23) through the mid-point $\frac{\epsilon}{2}$ (see Fig. 3). Let us denote $X_\pm^{\wedge}(\epsilon)$ the upper and $X_\pm^{\vee}(\epsilon)$ the lower part (with respect to the oriented line $\lambda^{(0)}(0)\,\mathbb{R}$) of the cut domain

$$X_\pm(\epsilon) = X_\pm^{\wedge}(\epsilon) \cup X_\pm^{\vee}(\epsilon).$$

The two parts of $X_\pm(\epsilon)$ intersect in the left and right intersection sectors $X_{i,\pm}^{\cap}(\epsilon)$ (Definition 28) attached to $x_{i,\pm}$, $i = 1, 2$, and for $\epsilon \neq 0$ also in a central part along the cut.[3]

Now take two branches $E_\chi^{\wedge}(h, x, \epsilon)$ and $E_\chi^{\vee}(h, x, \epsilon)$ of $E_\chi(h, x, \epsilon)$ on the two parts of the domain, that agree on the right intersection sector $X_{2,\pm}^{\cap}$, and have a limit when $\epsilon \to 0$. Correspondingly they determine a pair of general solutions of the model system

$$u^{\bullet}(x, \epsilon; c), \quad \bullet = \wedge, \vee,$$

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[3] Technically speaking, the domains $X_\pm(\epsilon)$ and $X_{i,\pm}^{\cap}(\epsilon)$ are defined on some Riemann surface as explained in Remark 14, what we mean here is the intersection of the projection of $X_\pm^{\bullet}(\epsilon)$, $\bullet = \wedge, \vee$, to the x-plane.

and a pair of *canonical general solutions* of the original system

$$y^{\bullet}_{\pm}(x, \epsilon; c) := \Psi_{\pm}(u^{\bullet}(x, \epsilon; c), x, \epsilon), \quad \bullet = \blacktriangle, \blacktriangledown. \tag{35}$$

Since the transformation $\Psi_{\pm}$ is unique only modulo right composition with an exponential torus action $T_a(u, \epsilon)$ (32), which acts on $u^{\bullet}(x, \epsilon; c)$ as

$$T_a(\cdot, \epsilon) \circ u^{\bullet}(x, \epsilon; c) = u^{\bullet}(x, \epsilon; \cdot) \circ T_a(c, \epsilon),$$

the solutions $y^{\bullet}_{\pm}$ are determined only up to the same right action of $T_a(c, \epsilon)$.

4.3 Formal Monodromy

The formal monodromy operators are induced by monodromy acting on the solutions $u^{\bullet}(x, \epsilon; c)$, $\bullet = \blacktriangle, \blacktriangledown$, of the model system. For $\epsilon \neq 0$ the induced action of formal monodromies along simple counterclockwise loops around each singular point $x_{i,\pm} = 0, \epsilon$ on the 3 foliations is given by:

- Monodromy operators of the model system, defined by

$$\mathscr{N}_{x_{i,\pm}}(\cdot, x, \epsilon) \circ u(x, \epsilon; c) = u(e^{2\pi i}(x - x_{i,\pm}) + x_{i,\pm}, \epsilon; c), \quad x \in \mathsf{X}_{\pm}(\epsilon), \tag{36}$$

 whose action on the foliation of the normal form vector field $Z_{G,\epsilon}$ is given by

$$\mathscr{N}_0 : u \mapsto \exp\left(-2\pi i \frac{\chi^0(h, \epsilon)}{\epsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot u = T_{-\frac{2\pi i}{\epsilon}\chi\upharpoonright_{x=0}}(u),$$

$$\mathscr{N}_\epsilon : u \mapsto \exp\left(2\pi i \left[\frac{\chi^0(h, \epsilon)}{\epsilon} + \chi^{(1)}(h, \epsilon)\right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot u = T_{\frac{2\pi i}{\epsilon}\chi\upharpoonright_{x=\epsilon}}(u). \tag{37}$$

 They belong to the exponential torus **Tor**. The *outer monodromy* of the model system, that is the monodromy around both $x_{1,\pm}, x_{2,\pm}$, is given by

$$\mathscr{N} = \mathscr{N}_0 \circ \mathscr{N}_\epsilon = \mathscr{N}_\epsilon \circ \mathscr{N}_0 : u \mapsto \exp\left(2\pi i \chi^{(1)}(u_1 u_2, \epsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot u.$$

- Formal monodromy operators

$$N_{x_{i,\pm}}(\cdot, \epsilon) \circ c(u, x, \epsilon) = c(\cdot, x, \epsilon) \circ \mathscr{N}_{i,\pm}(u, x, \epsilon),$$

 acting on the space of initial conditions c *commutatively* by

$$N_0 : c \mapsto \exp\left(-2\pi i \frac{\chi^0(h, \epsilon)}{\epsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot c = T_{-\frac{2\pi i}{\epsilon}\chi\upharpoonright_{x=0}}(c),$$

$$N_\epsilon : c \mapsto \exp\left(2\pi i \left[\frac{\chi^0(h, \epsilon)}{\epsilon} + \chi^{(1)}(h, \epsilon)\right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot c = T_{\frac{2\pi i}{\epsilon}\chi\upharpoonright_{x=\epsilon}}(c), \tag{38}$$

and a formal outer monodromy

$$N = N_0 \circ N_\epsilon = N_\epsilon \circ N_0 : c \mapsto \exp\left(2\pi i \chi^{(1)}(h, \epsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot c.$$

- Formal monodromy operators $\mathfrak{N}_{i,\pm}(y, x, \epsilon)$ acting on the foliation of the original vector field $Z_{H,\epsilon}$:

$$\mathfrak{N}_{i,\pm}(\cdot, x, \epsilon) \circ \Psi_\pm(u, x, \epsilon) = \Psi_\pm(\cdot, x, \epsilon) \circ \mathcal{N}_{i,\pm}(u, x, \epsilon), \quad x \in \mathsf{X}_\pm(\epsilon), \quad (39)$$

and

$$\mathfrak{N}_\pm(\cdot, x, \epsilon) = \mathfrak{N}_{1,\pm}(\cdot, x, \epsilon) \circ \mathfrak{N}_{2,\pm}(\cdot, x, \epsilon) = \mathfrak{N}_{2,\pm}(\cdot, x, \epsilon) \circ \mathfrak{N}_{1,\pm}(\cdot, x, \epsilon).$$

The canonical solutions $u_\pm^\spadesuit$, $u_\pm^\heartsuit$ of the model system on the domains $\mathsf{X}_\pm^\spadesuit$, $\mathsf{X}_\pm^\heartsuit$, are defined such that they agree on the right intersection sector $\mathsf{X}_{2,\pm}^\cap$. Therefore on the left intersection sector they are connected by the formal outer monodromy operator

$$u_\pm^\spadesuit(x, \epsilon; c) = \mathcal{N}(\cdot, x, \epsilon) \circ u_\pm^\heartsuit(x, \epsilon; c) = u_\pm^\heartsuit(x, \epsilon; \cdot) \circ N(c, \epsilon), \quad x \in \mathsf{X}_{1,\pm}^\cap,$$

and by the formal monodromy $\mathcal{N}_{x_i,\pm}$ on the central cut between the two domains for $\epsilon \neq 0$ (cf. Fig. 3).

4.4 Stokes Operators and Sectoral Isotropies

Let $y = \Psi_\pm(u, x, \epsilon)$ be the normalizing transformation on $\mathsf{X}_\pm(\epsilon)$. We call *Stokes operators* the operators that change the determination of the branch of $\Psi_\pm$ over the left or right intersection sectors (see Fig. 2). If $x \in \mathsf{X}_{i,\pm}^\cap(\epsilon)$, then for $\epsilon \neq 0$ we denote

$$\bar{x} = e^{2\pi i}(x - x_{i,\pm}) + x_{i,\pm}$$

the corresponding point in $\mathsf{X}_\pm(\epsilon)$ on the other sheet, and extend this notation by limit to $\epsilon = 0$. Namely,

$$\text{if } x \in \mathsf{X}_{1,\pm}^\cap(\epsilon) \subset \mathsf{X}_\pm^\heartsuit(\epsilon), \quad \text{then } \bar{x} \in \mathsf{X}_\pm^\spadesuit(\epsilon),$$
$$\text{if } x \in \mathsf{X}_{2,\pm}^\cap(\epsilon) \subset \mathsf{X}_\pm^\spadesuit(\epsilon), \quad \text{then } \bar{x} \in \mathsf{X}_\pm^\heartsuit(\epsilon).$$

Then the Stokes operators are the operators

$$\Psi_\pm(u, x, \epsilon) \mapsto \Psi_\pm(u, \bar{x}, \epsilon), \quad x \in \mathsf{X}_{i,\pm}^\cap(\epsilon), \quad (40)$$

which for $\epsilon = 0$ are the Stokes operators in the usual sense that send the Borel sum of the formal x-series $\hat{\Psi}(u, x, 0)$ in one non-singular direction to the Borel sum in a following non-singular direction.

To each of these Stokes operators we associate *sectoral isotropies* of the 3 foliations.

- Sectoral isotropies $\mathscr{S}_{i,\pm}(u, x, \epsilon)$ of the normal form vector field $Z_{G,\epsilon}$:

$$\Psi_\pm(\cdot, x, \epsilon) \circ \mathscr{S}_{i,\pm}(u, x, \epsilon) = \Psi_\pm(u, \bar{x}, \epsilon), \quad x \in \mathsf{X}^\cap_{i,\pm}(\epsilon). \qquad (41)$$

The pair $(\mathscr{S}_{1,\pm}, \mathscr{S}_{2,\pm})$ is an analog of the Martinet–Ramis invariant of saddle-node singularity [17,25].

- Sectoral isotropies $S_{1,\pm}(c, \epsilon)$ and $S_{2,\pm}(c, \epsilon)$ of the rectified vector field $Z_{0,\epsilon} = x(x - \epsilon)\partial_x$ in the c-space:

$$\begin{aligned}
u^\heartsuit(x, \epsilon; \cdot) \circ S_{1,\pm}(c, \epsilon) &= \mathscr{S}_{1,\pm}(\cdot, x, \epsilon) \circ u^\heartsuit(x, \epsilon; c), \quad x \in \mathsf{X}^\cap_{1,\pm}(\epsilon), \\
u^\spadesuit(x, \epsilon; \cdot) \circ S_{2,\pm}(c, \epsilon) &= \mathscr{S}_{2,\pm}(\cdot, x, \epsilon) \circ u^\spadesuit(x, \epsilon; c), \quad x \in \mathsf{X}^\cap_{2,\pm}(\epsilon).
\end{aligned} \qquad (42)$$

- Sectoral isotropies $\mathfrak{S}_{i,\pm}(y, x, \epsilon)$ of the original vector field $Z_{H,\epsilon}$:

$$\mathfrak{S}_{i,\pm}(\cdot, x, \epsilon) \circ \Psi_\pm(u, x, \epsilon) = \Psi_\pm(u, \bar{x}, \epsilon), \quad x \in \mathsf{X}^\cap_{i,\pm}(\epsilon). \qquad (43)$$

Proposition 33 (Form of the Stokes isotropies) *Let* $S_{i,\pm}(c, \epsilon) = {}^t(S_{1,i,\pm}(c, \epsilon), S_{2,i,\pm}(c, \epsilon))$ *be a Stokes sectoral isotropy* (42). *Then*

- $S_{i,i,\pm}(c, \epsilon) = c_i + c_i^2 \cdot \sigma_{i,i,\pm}(h, c_i, \epsilon)$ *for an analytic germ* $\sigma_{i,i,\pm}$,
- $S_{j,i,\pm}(c, \epsilon) = c_j + \sigma_{j,i,\pm}(h, c_i, \epsilon)$ *for an analytic germ* $\sigma_{j,i,\pm}$, $j = 3 - i$,

subject to a condition $\det D_c(S_{i,\pm}) = 1$.

The term $\sigma_{j,i,\pm}(0, 0, \epsilon)$ is responsible for the ramification of the ramified center manifold $y = \Psi_\pm(0, x, \epsilon)$ of the original vector field Z_ϵ at the sector $\mathsf{X}^\cap_{i,\pm}(\epsilon)$.

Proof The isotropy $\mathscr{S}_{i,\pm} = (\mathscr{S}_{1,i,\pm}, \mathscr{S}_{2,i,\pm})$ is analytic in u on some neighborhood of $u = 0$ and bounded in x with $\lim_{x \to x_{i,\pm}} \mathscr{S}_{k,i,\pm}(u, x, \epsilon) = u_k$, $k = 1, 2$. By Lemma 30 $S_{k,i,\pm} \circ c = c_k \circ \mathscr{S}_{i,\pm}$ is of the form

$$\begin{aligned}
c_i \circ \mathscr{S}_{i,\pm} &= c_i \cdot e^{a_{i,\pm}(h,\epsilon)} + c_i^2 \sigma_{i,i,\pm}(h, c_i, \epsilon), \quad \text{and} \\
c_j \circ \mathscr{S}_{i,\pm} &= c_j \cdot e^{-a_{i,\pm}(h,\epsilon)} + \sigma_{j,i,\pm}(h, c_i, \epsilon)
\end{aligned}$$

for some some germs $a_{i,\pm}, \sigma_{i,i,\pm}, \sigma_{j,i,\pm}$. We want to show that $a_{i,\pm} = 0$.

Writing tautologically

$$\mathscr{S}_{k,i,\pm} = \left(c_k \circ \mathscr{S}_{i,\pm}\right) \cdot \left(E^{-(-1)^k} \circ h \circ \mathscr{S}_{i,\pm}\right),$$

and knowing that $\lim_{x \to x_{i,\pm}} h \circ \mathscr{S}_{i,\pm}(u, x, \epsilon) = h$, we have $h \circ \mathscr{S}_{i,\pm} = h + c_i \cdot (\ldots)$ where $(\ldots)$ is an analytic function of (h, c_i, ϵ), hence $\lim_{x \to x_{i,\pm}} E_\chi^{-1} \cdot (E_\chi \circ h \circ \mathscr{S}_{i,\pm}) = 1$, and therefore $\lim_{x \to x_{i,\pm}} \left(c_k \circ \mathscr{S}_{i,\pm}\right) \cdot E^{-(-1)^k} = u_k$, from which we conclude that $a_{i,\pm} = 0$. $\qquad \square$

4.5 Symmetry Group of the System

The following proposition describes the group of analytic symmetries of the original vector field Z_H, resp. $Z_{H,\epsilon}$, where the Stokes operators provide an obstruction for analyticity. It is a direct analogue of similar theorems for e.g. saddle-node singularities [25, Proposition 10.6] or for irregular singularities of linear systems [10, Lemma 6.2].

Proposition 34 *The group of (analytic transversely symplectic fibered) symmetries of the original vector field Z_H, resp. its restriction $Z_{H,\epsilon}$ to a fixed ϵ, is either*

*(1) isomorphic to the exponential torus **Tor**, resp. **Tor**$_\epsilon$: this happens if and only if the system is analytically equivalent to the model (16), or*

(2) isomorphic to a finite cyclic group.

If the symmetry group is non-trivial, then the system has an analytic center manifold (bounded analytic solution on a neighborhood of both singular points).

Proof If $\Phi(y, x, \epsilon)$ is a symmetry of the system (1), then $\Phi(\cdot, x, \epsilon) \circ \Psi_\pm(u, x, \epsilon) = \Psi_\pm(\cdot, x, \epsilon) \circ \phi(u, \epsilon)$ for some germ

$$\phi : u \mapsto \begin{pmatrix} e^{a(h,\epsilon)} & \\ & e^{-a(h,\epsilon)} \end{pmatrix},$$

from the exponential torus, and the analyticity of Φ means that this ϕ must commute with the Stokes operators $S_{i,\pm}$ (42) (note that ϕ acts the same way on c as on u). Using their characterization in Proposition 33, this means that

$$\sigma_{i1,\pm}(h, c_1) = e^a \sigma_{i1,\pm}(h, e^a c_1), \quad \sigma_{i2,\pm}(h, c_2) = e^{-a} \sigma_{i2,\pm}(h, e^{-a} c_2), \quad i = 1, 2.$$

This can be satisfied only if

- either $\sigma_{ij,\pm}(h, c_j) = 0$ for all i, j, i.e. if $S_{1,\pm} = \mathrm{id}$, $S_{2,\pm} = \mathrm{id}$ and the system is analytically equivalent to its formal normal form,
- or there is $k \in \mathbb{N}$ such that $c_j \sigma_{ij,\pm}(h, c_j) = 0$ contains only powers of c_j^k for all i, j, and $e^{ka} = 1$, i.e. $a \in \frac{2\pi i}{k} \mathbb{Z}$.

$\square$

4.6 Analytic Classification

Definition 35 (*Analytic invariants*) The collection $(\chi, \{\mathscr{S}_{1,+}, \mathscr{S}_{2,+}, \mathscr{S}_{1,-}, \mathscr{S}_{2,-}\})$ is called an *analytic invariant* of a system (1). Two analytic invariants $(\chi, \{\mathscr{S}_{i,\pm}\})$, $(\tilde{\chi}, \{\tilde{\mathscr{S}}_{i,\pm}\})$ are equivalent if

- either $\chi = \tilde{\chi}$ and there is an element $\phi(u, \epsilon) \in \mathbf{Tor}$ of the exponential torus such that: $\mathscr{S}_{i,\pm} = \phi \circ \tilde{\mathscr{S}}_{i,\pm} \circ \phi^{\circ(-1)}$, $i = 1, 2$,
- or $\chi(h, x, \epsilon) = -\tilde{\chi}(-h, x, \epsilon)$ and there is an element $\phi(u, \epsilon) \in \mathbf{Tor}$ of the exponential torus such that: $\mathscr{S}_{i,\pm} = J\phi \circ \tilde{\mathscr{S}}_{j,\mp} \circ (J\phi)^{\circ(-1)}$, $i = 1, 2$, $j = $

$3 - i$, where $J : (u_1, u_2) \mapsto (u_2, -u_1)$. Note that the definition of $\mathsf{E}_\pm$, $\mathsf{X}_\pm$ and $x_{i,\pm}$ depends on $\lambda(x, \epsilon) = \chi(0, x, \epsilon)$, therefore the relation $\tilde{\lambda} = -\lambda$ entails the renaming

$$\tilde{\mathsf{E}}_\pm = \mathsf{E}_\mp, \quad \tilde{\mathsf{X}}_\pm^\bullet = \mathsf{X}_\mp^\bullet, \quad \tilde{x}_{i,\pm} = x_{j,\mp}.$$

By the construction, an *analytic invariant* of a system (1) is uniquely defined up to the equivalence.

Theorem 36 (Analytic classification) *Two systems* (1) *are analytically equivalent (in the sense of Definition 2) if and only if their analytic invariants are equivalent.*

Proof If $y = \Phi(\tilde{y}, x, \epsilon)$ is analytic transformation from one system to another, then the sectoral normalizations $y = \Psi_\pm(u, x, \epsilon)$ and $\tilde{y} = \tilde{\Psi}_\pm(u, x, \epsilon) = \Phi \circ \Psi_\pm$ provide the same analytic invariant. Conversely, if the analytic invariants are equivalent, then up to modifying one of the normalizing transformation, one can suppose that they are in fact equal, in which case $\Phi_\pm = \tilde{\Psi}_\pm \circ \Psi_\pm^{\circ(-1)}$ are analytic transformations between the systems on E_+ and E_-. In fact $\Phi_+ = \Phi_-$ is analytic on the whole ϵ-neighborhood E. Indeed, the composition $\Phi_+ \circ \Phi_-^{\circ-1}$ is a symmetry of the second system on the intersection $\mathsf{E}_+ \cap \mathsf{E}_-$, and as such it is determined by its value at $x = 0$; but since by Theorem 17(i) $\tilde{\Psi}_+(u, 0, \epsilon) = \tilde{\Psi}_-(u, 0, \epsilon)$ and $\Psi_+(u, 0, \epsilon) = \Psi_-(u, 0, \epsilon)$ are analytic in ϵ (29), this means that $\Phi_+ \circ \Phi_-^{\circ-1} = \mathrm{id}$. □

Example 37 In the case of linear systems (14), the Stokes operators $S_{i,\pm}$ are given by two unipotent matrices

$$S_{1,\pm} : c \mapsto \begin{pmatrix} 1 & 0 \\ s_{1,\pm}(\epsilon) & 1 \end{pmatrix} c, \quad S_{2,\pm} : c \mapsto \begin{pmatrix} 1 & s_{2,\pm}(\epsilon) \\ 0 & 1 \end{pmatrix} c.$$

In this particular case, the conjugacy class of the outer monodromy $M = S_{2,\pm} S_{1,\pm} N$ is given by $\det M = 1$, $\mathrm{tr} M = 2\cos(2\pi\lambda^{(1)}) + e^{2\pi i \lambda^{(1)}} s_{1,\pm} s_{2,\pm}$, and therefore it determines completely, together with the formal invariant $\lambda(x, \epsilon)$, the analytic invariant of the system.

4.7 Decomposition of Monodromy Operators

For $\epsilon \neq 0$, let $x_0 \in \mathsf{X}_\pm(\epsilon) \smallsetminus \{0, \epsilon\}$ be a base-point, and let two counterclockwise simple loops around the singular points $x_{i,\pm}$, $i = 1, 2$, be as in Fig. 4. Correspondingly, we have two monodromy operators $\mathfrak{M}_{x_{i,\pm}}$ acting on the foliation by the solutions of the original system (1) by analytic continuation along the loops. Since the monodromy operators $\mathfrak{M}_{x_{i,\pm}}$ act on the foliation, they are independent of the choice of the two-parameter general solution on which they act on the left (a different general solution is related to it by a change of c independent of x, i.e. by right action on the general solution). In particular

$$\mathfrak{M}_{x_{1,+}} = \mathfrak{M}_{x_{2,-}}, \quad \mathfrak{M}_{x_{2,+}} = \mathfrak{M}_{x_{1,-}}.$$

Fig. 4 Simple loops along which the monodromy operators $\mathfrak{M}_0$ and $\mathfrak{M}_\epsilon$ are defined
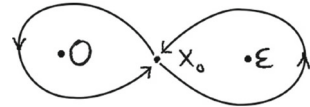
Theorem 38 *For $\epsilon \neq 0$, the monodromy operators $\mathfrak{M}_{x_{i,\pm}}$ of the original foliation are well defined on some open neighborhood of the ramified center manifold $y = \Psi_\pm(0, x, \epsilon)$ in $\mathsf{Y} \times \mathsf{X}_\pm(\epsilon)$. Their (left) action is given by*

$$\mathfrak{M}_{x_{i,\pm}} = \mathfrak{S}_{i,\pm} \circ \mathfrak{N}_{i,\pm}, \tag{44}$$

where $\mathfrak{S}_{i,\pm}$ are the Stokes operators (43) *and $\mathfrak{N}_{i,\pm}$ are the formal monodromy operators* (39). *Hence*

$$\mathfrak{M}_0 = \mathfrak{S}_{1,+} \circ \mathfrak{N}_{1,+} = \mathfrak{S}_{2,-} \circ \mathfrak{N}_{2,-},$$
$$\mathfrak{M}_\epsilon = \mathfrak{S}_{2,+} \circ \mathfrak{N}_{2,+} = \mathfrak{S}_{1,-} \circ \mathfrak{N}_{1,-}.$$

Their right action on analytic extension of the canonical general solutions $y_\pm^\bullet$ (35) *to the whole $\mathsf{X}_\pm(\epsilon)$ is given by*

$$\mathfrak{M}_{x_{i,\pm}}(y_\pm^\bullet(x, \epsilon; c), x, \epsilon) = y_\pm^\bullet(x, \epsilon; \cdot) \circ M_{i,\pm}^\bullet(c, \epsilon), \qquad \bullet = \spadesuit, \heartsuit,$$

where

$$M_{1,\pm}^\spadesuit = N_{x_{2,\pm}}^{\circ(-1)} \circ S_{1,\pm} \circ N, \quad M_{1,\pm}^\heartsuit = S_{1,\pm} \circ N_{x_{1,\pm}},$$
$$M_{2,\pm}^\spadesuit = S_{2,\pm} \circ N_{x_{2,\pm}}, \qquad M_{2,\pm}^\heartsuit = N_{x_{2,\pm}} \circ S_{2,\pm}, \tag{45}$$

cf. Fig. 3.

Proof of Theorem 38 The demonstration of the given formulas is straightforward, but we include it here for the sake of completeness. Let $\bar{x} := x_{i,\pm} + e^{2\pi i}(x - x_{i,\pm})$, then

$$\begin{aligned}
\mathfrak{M}_{x_{i,\pm}}(\cdot, x, \epsilon) \circ y_\pm(x, \epsilon; c) &= y_\pm(\bar{x}, \epsilon; c) = \Psi_\pm(\cdot, \bar{x}, \epsilon) \circ u(\bar{x}, \epsilon; c) \\
&= \left(\mathfrak{S}_{i,\pm} \circ \Psi_\pm\right)(\cdot, x, \epsilon) \circ u(\bar{x}, \epsilon; c) \\
&= \left(\mathfrak{S}_{i,\pm} \circ \Psi_\pm \circ \mathcal{N}_{i,\pm}\right)(\cdot, x, \epsilon) \circ u(x, \epsilon; c) \\
&= \left(\mathfrak{S}_{i,\pm} \circ \mathfrak{N}_{i,\pm} \circ \Psi_\pm\right)(\cdot, x, \epsilon) \circ u(x, \epsilon; c) \\
&= \left(\mathfrak{S}_{i,\pm} \circ \mathfrak{N}_{i,\pm} \circ y_\pm\right)(x, \epsilon; c),
\end{aligned}$$

using (36), (39), (43). Similarly, to calculate $M_{1,\pm}^\spadesuit$ for example,

$$\begin{aligned}
y_\pm^\spadesuit(\bar{x}, \epsilon; c) &= \Psi_\pm(\cdot, \bar{x}, \epsilon) \circ u^\spadesuit(\bar{x}, \epsilon; c) \\
&= \Psi_\pm(\cdot, \bar{x}, \epsilon) \circ u^\heartsuit(\bar{x}, \epsilon; \cdot) \circ N_{x_{2,\pm}}(c, \epsilon) \\
&= \Psi_\pm(\cdot, \bar{x}, \epsilon) \circ u^\heartsuit(x, \epsilon; \cdot) \circ N_{x_{1,\pm}} \circ N_{x_{2,\pm}}(c, \epsilon)
\end{aligned}$$

$$= \Psi_\pm(\cdot, x, \epsilon) \circ u^\heartsuit(x, \epsilon; \cdot) \circ S_{1,\pm} \circ N_{x_{1,\pm}} \circ N_{x_{2,\pm}}(c, \epsilon)$$

$$= \Psi_\pm(\cdot, x, \epsilon) \circ u^\spadesuit(x, \epsilon; \cdot) \circ N_{x_{2,\pm}}^{\circ(-1)} \circ S_{1,\pm} \circ N_{x_{1,\pm}} \circ N_{x_{2,\pm}}(c, \epsilon)$$

$$= y_\pm^\spadesuit(x, \epsilon; \cdot) \circ N_{x_{2,\pm}}^{\circ(-1)} \circ S_{1,\pm} \circ N(c, \epsilon),$$

see Fig. 3. The general solutions $y_\pm^\spadesuit$, $y_\pm^\heartsuit$ are related by

$$y_\pm^\spadesuit(x, \epsilon; c) = y_\pm^\heartsuit(x, \epsilon; \cdot) \circ N_{x_{2,\pm}}(c, \epsilon).$$

$\square$

Note that in general, a composition of the two monodromies may not be defined if the image of the first does not intersect the domain of definition the second. This is due to the ramification of the center manifold.

4.8 Accumulation of Monodromy

Definition 39 (*Monodromy pseudogroup*)

(1) For $\epsilon \neq 0$, the pseudogroup generated by the monodromy operators

$$\langle \mathfrak{M}_0(\cdot, x, \epsilon), \ \mathfrak{M}_\epsilon(\cdot, x, \epsilon) \rangle$$

is called the (local) *monodromy pseudogroup*. The pseudogroup generated by the corresponding action on the initial condition c

$$\langle M_{1,\pm}^\bullet(\cdot, \epsilon), \ M_{2,\pm}^\bullet(\cdot, \epsilon) \rangle$$

is its representation with respect to the general solution $y_\pm^\bullet(x, \epsilon; c)$.

(2) For $\epsilon = 0$, the pseudogroup generated by the Stokes operators and by the elements of the exponential torus (pushed-forward by the sectoral transformations $\Psi^\bullet$):

$$\langle \mathfrak{S}_{1,\pm}(\cdot, x, 0), \ \mathfrak{S}_{2,\pm}(\cdot, x, 0), \ \{\mathfrak{T}_a^\bullet(\cdot, x, 0)\}_a \rangle,$$

where $\mathfrak{T}_a^\bullet(\cdot, x, 0) = \Psi_\pm^\bullet(\cdot, x, 0)_* T_a = \Phi_{(\Psi_\pm^\bullet)_*(a(h)X_h)}^1$, is called the (local) *wild monodromy pseudogroup*. The pseudogroup generated by the corresponding action on the initial condition c

$$\langle S_{1,\pm}(\cdot, 0), \ S_{2,\pm}(\cdot, 0), \ \{T_a(\cdot)\}_a \rangle, \tag{46}$$

is its representation with respect to the formal transseries solution $\hat{y}(x, 0; c) = \hat{\Psi}(u(x, 0; c), x, 0)$.

Note that the pseudogroup (46) is independent of the freedom of choice of the sectoral normalizations $\Psi_\pm^\bullet$ of Theorem 16.

One of the main goals of this paper is to understand the relation between the monodromy pseudogroup for $\epsilon \neq 0$ and the wild monodromy pseudogroup for $\epsilon = 0$.

Suppose that the formal invariant $\chi(h, x, \epsilon)$ is such that

$$\chi(h, x, 0) = \lambda^{(0)}(0) + x\chi^{(1)}(h, 0), \tag{47}$$

and therefore

$$\chi(h, x, \epsilon) = \lambda^{(0)}(0) + \epsilon \frac{\partial \chi^{(0)}}{\partial \epsilon}(h, 0) + x\chi^{(1)}(h, 0) + O(x\epsilon) + O(\epsilon^2).$$

Let $\{\epsilon_n\}_{n \in \pm\mathbb{N}}$ be sequence in $\mathsf{E}_\pm \smallsetminus \{0\}$ defined by

$$\frac{\lambda^{(0)}(0)}{\epsilon_n} = \frac{\lambda^{(0)}(0)}{\epsilon_0} + n, \qquad \epsilon_0 \in \mathsf{E}_\pm \smallsetminus \{0\} \tag{48}$$

along which the exponential factor $e^{\frac{2\pi i \lambda^{(0)}}{\epsilon}}$ in the formal monodromy (38) stays constant, and denote

$$\kappa := e^{\frac{2\pi i \lambda^{(0)}}{\epsilon_0}}. \tag{49}$$

Then the formal monodromy operators $\mathcal{N}_0(u, x, \epsilon)$, $\mathcal{N}_\epsilon(u, x, \epsilon)$, resp. $N_0(u, x, \epsilon)$, $N_\epsilon(u, x, \epsilon)$, converge along each such sequence to a symmetry of the model system (element of the exponential torus)

$$\tilde{N}_0(\kappa; u, x) := \lim_{n \to \pm\infty} N_0(u, x, \epsilon_n) : c \mapsto \begin{pmatrix} \kappa^{-1} e^{-2\pi i \frac{\partial \chi^{(0)}}{\partial \epsilon}(h,0)} & 0 \\ 0 & \kappa e^{2\pi i \frac{\partial \chi^{(0)}}{\partial \epsilon}(h,0)} \end{pmatrix} c,$$

$$\tilde{N}_\epsilon(\kappa; u, x)$$
$$:= \lim_{n \to \pm\infty} N_\epsilon(u, x, \epsilon_n) : c \mapsto \begin{pmatrix} \kappa e^{2\pi i [\frac{\partial \chi^{(0)}}{\partial \epsilon}(h,0) + \chi^{(1)}(h,0)]} & 0 \\ 0 & \kappa^{-1} e^{-2\pi i [\frac{\partial \chi^{(0)}}{\partial \epsilon}(h,0) + \chi^{(1)}(h,0)]} \end{pmatrix} c, \tag{50}$$

$\kappa \in \mathbb{C}^*$. This implies that also the monodromy operators $M_{i,\pm}(c, \epsilon)$, resp. $\mathfrak{M}_{x_{i,\pm}}(y, x, \epsilon)$, converge along such sequences $\{\epsilon_n\}_{n \in \pm\mathbb{N}} \subset \mathsf{E}_\pm \smallsetminus \{0\}$. Denote

$$\tilde{\mathfrak{N}}^\bullet_{i,\pm}(\kappa; y, x) := \lim_{n \to \pm\infty} \mathfrak{N}_{i,\pm}(y, x, \epsilon_n), \qquad x \in \mathsf{X}^\bullet_\pm, \quad \bullet = \spadesuit, \heartsuit.$$

Theorem 40 *Suppose that the formal invariant of the form* (47). *Then the monodromy operators of the system* (1) *for $\epsilon \neq 0$ accumulate along the sequences $\{\epsilon_n\}_{n \in \pm\mathbb{N}}$* (48) *to a 1-parameter family of wild monodromy operators*

$$\tilde{\mathfrak{M}}_{1,\pm}(\kappa; y, x) := \lim_{n \to \pm\infty} \mathfrak{M}_{x_{1,\pm}}(y, x, \epsilon_n)$$

$$= \mathfrak{S}_{1,\pm}(\cdot, x, 0) \circ \tilde{\mathfrak{N}}^{\heartsuit}_{x_{1,\pm}}(\kappa; y, x), \quad x \in \mathsf{X}^{\cap}_{1,\pm}(0),$$

$$\tilde{\mathfrak{M}}_{2,\pm}(\kappa; y, x) := \lim_{n \to \pm\infty} \mathfrak{M}_{x_{2,\pm}}(y, x, \epsilon_n)$$

$$= \mathfrak{S}_{2,\pm}(\cdot, x, 0) \circ \tilde{\mathfrak{N}}^{\spadesuit}_{x_{2,\pm}}(\kappa; y, x), \quad x \in \mathsf{X}^{\cap}_{2,\pm}(0).$$

In particular, if we replace κ by $e^{-2\pi i \left[\frac{\partial \chi^{(0)}}{\partial \epsilon}(h,0) + \delta_{i,\pm}(0)\chi^{(1)}(h,0) \right]}$, $\delta_{i,\pm}(\epsilon) = \frac{x_{i,\pm}(\epsilon)}{\epsilon}$, so that $\tilde{\mathfrak{N}}^{\bullet}_{x_{i,\pm}}(\kappa; y, x)$ becomes an identity, we obtain the Stokes operators

$$\mathfrak{S}_{i,\pm}(y, x, 0) = \tilde{\mathfrak{M}}_{i,\pm}(e^{-2\pi i \left[\frac{\partial \chi^{(0)}}{\partial \epsilon}(h,0) + \delta_{i,\pm}(0)\chi^{(1)}(h,0) \right]}; y, x). \tag{51}$$

The vector field corresponding to the differential system

$$\dot{y} = \pm(-1)^i \left(\kappa \frac{\partial}{\partial \kappa} \tilde{\mathfrak{M}}_{i,\pm}(\kappa; y, x)^{\circ(-1)} \right) \circ \tilde{\mathfrak{M}}_{i,\pm}(\kappa; y, x) \tag{52}$$

equals to the push-forward $\Psi^{\bullet}_{\pm}(\cdot, x, 0)_* (X_h)$ of the vector field $X_h = u_1 \partial_{u_1} - u_2 \partial_{u_2}$, where $\bullet = \heartsuit$ if $i = 1$ and $\bullet = \spadesuit$ if $i = 2$, which "generates" the commutative Lie algebra of bounded infinitesimal symmetries on the sector $\mathsf{X}^{\bullet}(0)$.

Conclusion. The knowledge of the limits $\tilde{\mathfrak{M}}_{i,\pm}(\kappa; y, x)$, $\kappa \in \mathbb{C}^*$, allows to recover the infinitesimal symmetry $\Psi^{\bullet}_{\pm}(\cdot, x, 0)_* (X_h)$ (52), and hence its Hamiltonian, the bounded first integral $\Psi^{\bullet}_{\pm}(\cdot, x, 0)_*(h)$ which vanishes at the singular points $(y_0(x_{i,\pm}, \epsilon), x_{i,\pm})$, and therefore, knowing the formal invariant χ, also the formal monodromy operators $\tilde{\mathfrak{N}}^{\bullet}_{x_{1,\pm}}(\kappa; y, x)$, and finally the Stokes isotropies $\mathfrak{S}_{i,\pm}(y, x, 0)$.

Remark 41 In the case when the assumption (47) is not met, one can nevertheless get a similar "accumulation" result by replacing in (48) $\lambda^{(0)}(0)$ by $\chi^{(0)}(h, 0)$ and defining $\kappa(h) = e^{\frac{2\pi i \chi^{(0)}(h,0)}{\epsilon_0(h)}}$ (49). In fact, one can interpret the limit $\tilde{\mathfrak{M}}_{i,\pm}$ of $\mathfrak{M}_{x_i,\pm}$ along the special sequence $\{\epsilon_n(h)\}_{n \in \pm\mathbb{N}}$ as being obtained by replacing the divergent term $e^{\frac{2\pi i \chi^{(0)}(h,0)}{\epsilon}}$ in an expression of $\mathfrak{M}_{x_i,\pm}$ by a new function $\kappa(h)$ and taking the limit $\epsilon \to 0$ of all the other terms bounded in ϵ.

5 Confluence in 2×2 Traceless Linear Systems and Their Differential Galois Group

To illustrate the matter of the previous section, let us consider a confluence of two regular singular points to a non-resonant irregular singular point in a family of linear systems

$$x(x - \epsilon)\frac{dy}{dx} = A(x, \epsilon)y, \quad y \in \mathbb{C}^2, \tag{53}$$

where A is a 2×2 traceless complex matrix depending analytically on $(x, \epsilon) \in (\mathbb{C} \times \mathbb{C}, 0)$, such that $A(0, 0) \neq 0$ has two distinct eigenvalues $\pm \lambda^{(0)}(0)$.

The Theorem 17 in this case can be found in the thesis of Parise [22] and in the work of Lambert and Rousseau [15] (see also [10]). It provides us with a canonical fundamental solution matrices

$$Y_{\pm}^{\bullet}(x, \epsilon) = \Psi_{\pm}(x, \epsilon) \cdot U_{\pm}^{\bullet}(x, \epsilon), \quad \bullet = \spadesuit, \heartsuit, \tag{54}$$

where the transformation matrix $\Psi_{\pm}(x, \epsilon)$ is bounded on $\mathsf{XE}_{\pm}$, and

$$U_{\pm}^{\bullet}(x, \epsilon) = \begin{pmatrix} E_{\lambda}^{\bullet}(x, \epsilon) & 0 \\ 0 & E_{\lambda}^{\bullet}(x, \epsilon)^{-1} \end{pmatrix}, \tag{55}$$

is a solution to the diagonal model system

$$x(x - \epsilon) \frac{dy}{dx} = \begin{pmatrix} \lambda(x, \epsilon) & \\ & -\lambda(x, \epsilon) \end{pmatrix} y.$$

The solution basis $Y_{\pm}^{\bullet}(x, \epsilon)$ is also called a *mixed basis*: the first (resp. second) column spans the subspace of solutions that asymptotically vanish when $x \to x_{1,\pm}(\epsilon)$ (resp. when $x \to x_{2,\pm}(\epsilon)$), and it is an eigensolution with respect to the corresponding monodromy operator $\mathfrak{M}_{x_{1,\pm}}$ (resp. $\mathfrak{M}_{x_{2,\pm}}$) associated to its eigenvalue $e^{\pm 2\pi i \frac{\lambda(x_{1,\pm}, \epsilon)}{\epsilon}}$ (resp. $e^{\pm 2\pi i \frac{\lambda(x_{2,\pm}, \epsilon)}{\epsilon}}$). A general solution is a linear combination

$$y_{\pm}^{\bullet}(x, \epsilon; c) = Y_{\pm}^{\bullet}(x, \epsilon) \cdot c, \quad \bullet = \spadesuit, \heartsuit.$$

Let $\mathbb{K}$ be the field of meromorphic functions of the variable x on a fixed small neighborhood of 0, equipped with the differentiation $\frac{d}{dx}$. For a fixed small ϵ, the local *differential Galois group* (also called the *Picard–Vessiot group*) of the system (53) is the group of $(\mathbb{K}, \frac{d}{dx})$-automorphisms of the differential field $\mathbb{K}\langle Y(\cdot, \epsilon)\rangle$, generated by the components of any fundamental matrix solution $Y(x, \epsilon)$. The differential Galois group acts on the foliation associated to the system by left multiplication of solutions of the system. Fixing a fundamental solution matrix $Y = Y_{\pm}^{\bullet}$, then each automorphism is represented by a right multiplication of $Y_{\pm}^{\bullet}$ by a constant invertible matrix, hence the differential Galois group is represented by an (algebraic) subgroup of $\mathrm{SL}_2(\mathbb{C})$ acting on the right.

It is well known [19,29] that the differential Galois group is the Zariski closure of

$\epsilon \neq 0$: *the monodromy group* generated by the two monodromy operators around the singular points 0 and ϵ,

$\epsilon = 0$: *the wild monodromy group* [4] generated by the Stokes operators and the linear exponential torus [5] which acts on the fundamental solutions $Y^{\bullet}_{\pm}$ as

$$\mathfrak{T}^{\bullet}_a : Y^{\bullet}_{\pm}(x, 0) \mapsto Y^{\bullet}_{\pm}(x, 0) \cdot T_a, \quad \text{where} \quad T_a = \begin{pmatrix} e^a & \\ & e^{-a} \end{pmatrix}, \quad a \in \mathbb{C}. \quad (56)$$

The question is how are these two different descriptions related?

The monodromy matrices of $Y^{\text{A}}_{\pm}$, $Y^{\text{V}}_{\pm}$, around the points $x_{1,\pm}(\epsilon)$, $x_{2,\pm}(\epsilon)$, $\epsilon \in E_{\pm} \smallsetminus \{0\}$, are given respectively by

$$M^{\text{A}}_{1,\pm} = N^{(-1)}_{x_{2,\pm}} S_{1,\pm} N, \qquad\qquad M^{\text{A}}_{2,\pm} = S_{2,\pm} N_{x_{2,\pm}}$$

$$M^{\text{V}}_{1,\pm} = S_{1,\pm} N_{x_{1,\pm}}, \qquad\qquad M^{\text{V}}_{2,\pm} = N_{x_{2,\pm}} S_{2,\pm},$$

where $S_{i,\pm}$ are of the form

$$S_{1,\pm} = \begin{pmatrix} 1 & 0 \\ s_{1,\pm} & 1 \end{pmatrix}, \qquad S_{2,\pm} = \begin{pmatrix} 1 & s_{2,\pm} \\ 0 & 1 \end{pmatrix}.$$

In particular $M^{\bullet}_{1,\pm}$ is lower-triangular and $M^{\bullet}_{2,\pm}$ is upper-triangular.

When $\epsilon \to 0$ along a sequence [6] $\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + \frac{n}{\lambda^{(0)}}$, $n \in \pm\mathbb{N}$, $\epsilon_0 \in E_{\pm} \smallsetminus \{0\}$, these monodromy converge respectively to $\tilde{M}^{\bullet}_{i,\pm}(\kappa) = \lim_{n \to \pm\infty} M^{\bullet}_{i,\pm}(\epsilon_n)$ given by

$$\tilde{M}^{\text{A}}_{1,+}(\kappa) = \tilde{N}_{\epsilon}(\kappa)^{-1} S_1(0) N(0), \qquad\qquad \tilde{M}^{\text{V}}_{1,+}(\kappa) = S_1(0) \tilde{N}_0(\kappa),$$

$$\tilde{M}^{\text{A}}_{2,+}(\kappa) = S_2(0) \tilde{N}_{\epsilon}(\kappa), \qquad\qquad \tilde{M}^{\text{V}}_{2,+}(\kappa) = \tilde{N}_{\epsilon}(\kappa) S_2(0),$$

$$\tilde{M}^{\text{A}}_{1,-}(\kappa) = \tilde{N}_0(\kappa)^{-1} S_1(0) N(0), \qquad\qquad \tilde{M}^{\text{V}}_{1,-}(\kappa) = S_1(0) \tilde{N}_{\epsilon}(\kappa),$$

$$\tilde{M}^{\text{A}}_{2,-}(\kappa) = S_2(0) \tilde{N}_0(\kappa), \qquad\qquad \tilde{M}^{\text{V}}_{2,-}(\kappa) = \tilde{N}_0(\kappa) S_2(0),$$

with

$$\tilde{N}_0(\kappa) = \begin{pmatrix} \kappa^{-1} & \\ & \kappa \end{pmatrix} T_{-2\pi i \frac{d\lambda(0)}{d\epsilon}(0)}, \quad \tilde{N}_{\epsilon}(\kappa) = \begin{pmatrix} \kappa & \\ & \kappa^{-1} \end{pmatrix} T_{2\pi i [\frac{d\lambda(0)}{d\epsilon}(0) + \lambda^{(1)}(0)]},$$

$$N(0) = T_{2\pi i \lambda^{(1)}(0)}.$$

We call them *wild monodromy matrices*. Their family

$$\{\tilde{M}^{\bullet}_{1,\pm}(\kappa), \ \tilde{M}^{\bullet}_{2,\pm}(\kappa) \mid \kappa \in \mathbb{C}^*\}$$

[4] The name "wild monodromy" is borrowed from [19].

[5] For general linear systems one would need to add also the formal outer monodromy $N(0) = T_{2\pi i \lambda^{(1)}(0)}$, which in our case already belongs to the exponential torus.

[6] The idea of taking limits of monodromy along such sequences can be found in the works of Ramis [23] or Duval [7].

generates the same group, the representation of the *wild monodromy group* with respect to the formal solution $\hat{Y}(x, \epsilon)$, as does the collection of the Stokes matrices and the linear exponential torus

$$\{S_1(0), S_2(0)\} \cup \{T_a \mid a \in \mathbb{C}\}.$$

Hence we have the following theorem, whose general idea was suggested by Ramis [23]:

Theorem 42 *When $\epsilon \to 0$ the elements of the monodromy group of the system* (53) *accumulate to generators of the wild monodromy group of the limit system.*

6 Confluent Degeneration of the Sixth Painlevé Equation to the Fifth

The sixth Painlevé equation is

$$
\begin{aligned}
P_{VI}: \quad q'' = {} & \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)(q')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)q' \\
& + \frac{q(q-1)(q-t)}{2\,t^2(t-1)^2}\left[(\vartheta_\infty - 1)^2 - \vartheta_0^2\frac{t}{q^2} + \vartheta_1^2\frac{(t-1)}{(q-1)^2}\right. \\
& \left. + (1-\vartheta_t^2)\frac{t(t-1)}{(q-t)^2}\right],
\end{aligned}
$$

where $\vartheta = (\vartheta_0, \vartheta_t, \vartheta_1, \vartheta_\infty) \in \mathbb{C}^4$ are complex constants. It is a reduction to the q-variable of a time dependent Hamiltonian system [21]

$$\frac{dq}{dt} = \frac{\partial H_{VI}(q, p, t)}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{VI}(q, p, t)}{\partial q}, \tag{57}$$

with a polynomial Hamiltonian function

$$
\begin{aligned}
H_{VI} = {} & \tfrac{1}{t(t-1)}\Big[q(q-1)(q-t)p^2 \\
& - \left(\vartheta_0(q-1)(q-t) + \vartheta_1 q(q-t) + (\vartheta_t - 1)q(q-1)\right)p \\
& + \tfrac{(\vartheta_0+\vartheta_1+\vartheta_t-1)^2-\vartheta_\infty^2}{4}(q-t)\Big].
\end{aligned}
$$

It has three simple (regular) singular points on the Riemann sphere $\mathbb{CP}^1$ at $t = 0, 1, \infty$.

The fifth Painlevé equation P_V [7]

$$P_V: \quad q'' = \left(\frac{1}{2q} + \frac{1}{q-1}\right)(q')^2 - \frac{1}{\tilde{t}}q' + \frac{(q-1)^2}{2\tilde{t}^2}\left((\vartheta_\infty - 1)^2 q - \frac{\vartheta_0^2}{q}\right)$$

---

[7] The Eq. (6) is the fifth equation of Painlevé with a parameter $\eta_1 = -1$. A general form of this equation would be obtained by a further change of variable $\tilde{t} \mapsto -\eta_1\tilde{t}$. The degenerate case P_V^{deg} with $\eta_1 = 0$ which has only a regular singular point at ∞ is not considered here.

$$+ (1 + \tilde{\vartheta}_1)\frac{q}{\tilde{t}} - \frac{q(q+1)}{2(q-1)},$$

is obtained from P_{VI} as a limit $\epsilon \to 0$ after the change of the independent variable

$$t = 1 + \epsilon\tilde{t}, \qquad \vartheta_t = \frac{1}{\epsilon}, \qquad \vartheta_1 = -\frac{1}{\epsilon} + \tilde{\vartheta}_1 + 1, \tag{58}$$

which sends the three singularities to $\tilde{t} = -\frac{1}{\epsilon}, 0, \infty$. At the limit, the two simple singular points $-\frac{1}{\epsilon}$ and ∞ merge into a double (irregular) singularity at the infinity.

The change of variables (58), changes the function $\epsilon \cdot H_{VI}$ to

$$H_{VI,\epsilon} = \tfrac{1}{\tilde{t}(1+\epsilon\tilde{t})}\Big[q(q-1)(q-1-\epsilon\tilde{t})p^2$$
$$- \Big(\vartheta_0(q-1)(q-1-\epsilon\tilde{t}) + \tilde{\vartheta}_1 q(q-1-\epsilon\tilde{t}) + \tilde{t}q - \epsilon\tilde{t}q\Big)p$$
$$+ \tfrac{(\vartheta_0+\tilde{\vartheta}_1)^2-\vartheta_\infty^2}{4}(q-1-\epsilon\tilde{t})\Big],$$

and the Hamiltonian system to

$$\frac{dq}{d\tilde{t}} = \frac{\partial H_{VI,\epsilon}(q,p,\tilde{t})}{\partial p}, \quad \frac{dp}{d\tilde{t}} = -\frac{\partial H_{VI,\epsilon}(q,p,\tilde{t})}{\partial q},$$

whose limit $\epsilon \to 0$ is a Hamiltonian system of P_V. In the coordinate $x = \frac{1}{\tilde{t}} + \epsilon$, the above system is written as

$$x(x - \epsilon)\frac{dq}{dx} = \frac{\partial H(q,p,x,\epsilon)}{\partial p}, \quad x(x - \epsilon)\frac{dp}{dx} = -\frac{\partial H(q,p,x,\epsilon)}{\partial q}, \tag{59}$$

with

$$H(q, p, x, \epsilon) = -(1 + \epsilon\tilde{t})H_{VI,\epsilon}(q, p, \tilde{t})$$
$$= -\tfrac{(\vartheta_0+\tilde{\vartheta}_1)^2-\vartheta_\infty^2}{4}((x - \epsilon)q - x)$$
$$+ x\vartheta_0 p + \big(1 - \epsilon - (x - \epsilon)\vartheta_0 - x(\vartheta_0 + \tilde{\vartheta}_1)\big)qp$$
$$+ (2x - \epsilon)(qp)^2 + (x - \epsilon)(\vartheta_0 + \tilde{\vartheta}_1)q^2 p - xqp^2 - (x - \epsilon)q^3 p^2,$$

and Theorem 17 can be applied.

Theorem 43 *The formal invariant χ of the system* (59) *is*

$$\chi(h, x, \epsilon) = 1 - \epsilon - (x - \epsilon)\vartheta_0 - x(\vartheta_0 + \tilde{\vartheta}_1) + 2(2x - \epsilon)h. \tag{60}$$

Proof Let

$$\tilde{q} = q - xA, \qquad\qquad A = \frac{\vartheta_0}{1+\epsilon(1+\vartheta_0+\tilde{\vartheta}_1)},$$

$$\tilde{p} = p + (x - \epsilon)B, \qquad\qquad B = \frac{(\vartheta_0 + \tilde{\vartheta}_1)^2 - \vartheta_\infty^2}{4(1 + \epsilon(1 - \vartheta_0))},$$

and let

$$\tilde{H}(\tilde{q}, \tilde{p}, x, \epsilon) = H(q, p, x, \epsilon) - \frac{(\vartheta_0 + \tilde{\vartheta}_1)^2 + \vartheta_\infty^2}{4t} x.$$

Then for $\epsilon \neq 0$,

$$\tilde{H}(\tilde{q}, \tilde{p}, \epsilon, \epsilon) = \left(1 - \epsilon(1 + \vartheta_0 + \tilde{\vartheta}_1)\right)\tilde{q}\tilde{p} + \epsilon(\tilde{q} + \epsilon A)^2 \tilde{p}^2 - \epsilon(\tilde{q} + \epsilon A)\tilde{p}^2,$$

hence by Proposition 9 the Birkhoff–Siegel invariant of $H(q, p, \epsilon, \epsilon)$ is

$$G(h, \epsilon, \epsilon) = \left(1 - \epsilon(1 + \vartheta_0 + \tilde{\vartheta}_1)\right)h + \epsilon h^2,$$

and

$$\tilde{H}(\tilde{q}, \tilde{p}, 0, \epsilon) = \left(1 - \epsilon(1 - \vartheta_0)\right)\tilde{q}\tilde{p} - \epsilon\tilde{q}^2(\tilde{p} + \epsilon B)^2 - \epsilon(\vartheta_0 + \tilde{\vartheta}_1)\tilde{q}^2(\tilde{p} + \epsilon B)$$
$$+ \epsilon\tilde{q}^3(\tilde{p} + \epsilon B)^2,$$

hence by Proposition 9 the Birkhoff–Siegel invariant of $H(q, p, 0, \epsilon)$ is

$$G(h, 0, \epsilon) = \left(1 - \epsilon(1 - \vartheta_0)\right)h - \epsilon h^2,$$

i.e.

$$G(h, x, \epsilon) = \left(1 - \epsilon - (x - \epsilon)\vartheta_0 - x(\vartheta_0 + \tilde{\vartheta}_1)\right)h + (2x - \epsilon)h^2.$$

$$\square$$

The Theorem 16 for the limit system $\epsilon = 0$ is in this case due to Takano [30], see also [27,35]. A separate paper [14] will be devoted to a more detailed study of the confluence $P_{VI} \to P_V$ and of the non-linear Stokes phenomenon in P_V through the Riemann-Hilbert correspondence.

7 Proof of Theorem 17 and of Proposition 9

The proof of Theorem 17 is loosely based on the ideas of Siegel's proof of Theorem 4 [28, chaps. 16 and 17], using similar tools as [4]. We construct the normalizing transformation $y = \Phi_\pm(u, x, \epsilon)$ in a couple of steps as a formal power series in the u-variable with coefficients depending analytically on $(x, \epsilon) \in \mathsf{XE}_\pm$, and then show that the series is convergent. The main tool to prove the convergence is the Lemma 44 below for series with coefficients in a Banach algebra.

Let $(\mathfrak{B}, \|\cdot\|)$ be a Banach algebra, and let

$$\phi(u) = \sum_{|\mathbf{m}| \geq 2} \phi_\mathbf{m} u^\mathbf{m}, \quad \mathbf{m} = (m_1, m_2), \ u^\mathbf{m} = u_1^{m_1} u_2^{m_2}, \ |\mathbf{m}| = m_1 + m_2,$$

be a formal power series in the u-variable with coefficients $\phi_{\mathbf{m}} \in \mathfrak{B}$, and denote

$$\overline{\phi}(u) = \sum_{|\mathbf{m}| \geq 2} \|\phi_{\mathbf{m}}\| u^{\mathbf{m}}$$

its majorant power series. We will use the notation

$$\{\phi\}_{\mathbf{m}} := \phi_{\mathbf{m}},$$

and

$$\overline{\phi}(u) \prec \overline{\psi}(u) \quad \text{if} \quad \|\{\phi\}_{\mathbf{m}}\| \leq \|\{\psi\}_{\mathbf{m}}\| \quad \text{for all } \mathbf{m}.$$

Lemma 44 *Let $(\mathfrak{B}, \| \cdot \|)$ be a Banach algebra, let $\phi = {}^{\mathrm{t}}(\phi_1, \phi_2) = O(u^2)$ be a formal power series in u with coefficients in $\mathfrak{B}^2$, and let $r = {}^{\mathrm{t}}(r_1, r_2) = O(u^2)$ be a power series in u with coefficients in $\mathfrak{B}^2$ convergent w.r.t. the norm $\| \cdot \|$. If*

$$\overline{\phi}_j(u) \prec \overline{r}_j(u + \overline{\phi}(u)), \qquad j = 1, 2,$$

where $\overline{\phi} = {}^{\mathrm{t}}(\overline{\phi}_1, \overline{\phi}_2)$, then ϕ is convergent w.r.t. the norm $\| \cdot \|$.

Proof See [24, Theorem 2.2]—the generalization to Banach algebra setting comes for free with no change in the proof. This lemma can also be implicitly found in [20, p.520] and in [28, chap. 17] as the technique used in Siegel's proof of Theorem 4. □

We will introduce two Banach algebras: (i) an algebra $\mathfrak{B}(\mathsf{XE}_\pm)$ of bounded analytic functions on the domain $\mathsf{XE}_\pm$, and (ii) an algebra $\hat{\mathfrak{B}}(\Omega_\pm^0)$ of certain Borel summable formal power series in (x, ϵ), with the goal to construct the sectoral normalizing transformation $\Psi_\pm(u, x, \epsilon)$, resp. the formal normalizing transformation $\hat{\Psi}(u, x, \epsilon)$, of Theorem 17 as a convergent power series of u in the algebra $\mathfrak{B}(\mathsf{XE}_\pm)$, resp. $\hat{\mathfrak{B}}(\Omega_\pm^0)$.

Banach algebra of bounded analytic functions $\mathfrak{B}(\mathsf{XE}_\pm)$. We denote $\mathfrak{B}(\mathsf{XE}_\pm)$ the space of all such functions $f : \mathsf{XE}_\pm \to \mathbb{C}$ that are bounded and uniformly continuous on the ramified domain $\mathsf{XE}_\pm$ (25) and analytic on its interior. The space $\mathfrak{B}(\mathsf{XE}_\pm)$ endowed with the supremum norm

$$\|f\| := \sup_{(x,\epsilon) \in \mathsf{XE}_\pm} |f(x, \epsilon)|$$

clearly forms a Banach algebra.

Banach algebra of Borel (1,1)-summable formal power series $\hat{\mathfrak{B}}(\Omega_\pm^0)$. Next we introduce a type of Borel summation procedure for certain formal series of (x, ϵ) following [2], and provide a suitable norm which makes the space of them into a Banach algebra. While the construction is a bit technical, the details of it won't be needed beyond this section.

For $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ let us denote

$$\mathbb{B}_\theta := \{\sigma \in \mathbb{C} : \Re\left(\frac{e^{i\theta}}{\sigma}\right) \geq \cos\theta\} \cup \{0\} \tag{61}$$

the closed disc with center on $e^{i\theta}\mathbb{R}^+$ and the points 0 and 1 on its circumference.

Let $r > 0$ be a small constant which will appear in Theorem 47 below. We will consider formal power series

$$\hat{f}(x,\epsilon) = \sum_{k,l\geq 0} f^{(k,l)}x^k\epsilon^l, \quad \text{such that} \quad \sup_{k,l\geq 0} |f^{(k,l)}|\tfrac{r^{k+l}}{(k+l)!} < +\infty,$$

i.e. such that the transformed series

$$\widetilde{f}(x,\epsilon) := \sum_{k,l} \tfrac{f^{(k,l)}}{(k+l)!}x^k\epsilon^l,$$

converges on the polydisc $D_r = \{|x|, |\epsilon| < r\}$. Supposing that $\widetilde{f}$ extends analytically on a ray $(x_0, \epsilon_0) \cdot \mathbb{R}^+$ for some (x_0, ϵ_0) and has at most exponential growth there: $\int_0^{+\infty} |\widetilde{f}(sx_0, s\epsilon_0)|e^{-\Lambda(x_0,\epsilon_0)s}ds < +\infty$ for some $\Lambda(x_0, \epsilon_0) > 0$, then the following Laplace transform of $\widetilde{f}$:

$$\mathcal{L}_0[\widetilde{f}](x,\epsilon) := \int_0^{+\infty} \widetilde{f}(sx, s\epsilon)\, e^{-s}\, ds, \tag{62}$$

is convergent on the segment $(x, \epsilon) \in (x_0, \epsilon_0) \cdot [0, \Lambda(x_0, \epsilon_0)]$. The function (62) extends analytically as a bounded function to the "Borel disc" $(x_0, \epsilon_0) \cdot \Lambda(x_0, \epsilon_0)\mathbb{B}_0$, by

$$\mathcal{L}_\theta[\widetilde{f}](x,\epsilon) := \int_0^{+\infty e^{i\theta}} \widetilde{f}(sx, s\epsilon)\, e^{-s}\, ds,$$

where $\theta \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[$ depending on (x, ϵ) is such that $(x, \epsilon) \cdot e^{i\theta} \in (x_0, \epsilon_0)\mathbb{R}^+$. The function (62) is called the *Borel (1,1)-sum* of $\hat{f}(x, \epsilon)$ in the direction (x_0, ϵ_0) [2], and satisfies the same $(\partial_x, \partial_\epsilon)$-differential relations over the ring of convergent power series as $\hat{f}(x, \epsilon)$.

We shall now define a space $\hat{\mathfrak{B}}(\Omega_\pm^0)$ of thus summable power series, and a suitable norm that would make it into a Banach algebra.

Definition 45 Let

$$\Omega_\pm^0 := \{(x, \epsilon) \in \mathsf{XE}_\pm : (x, \epsilon) \cdot \mathbb{B}_0 \subseteq \mathsf{XE}_\pm\} \subseteq \mathsf{XE}_\pm,$$

where $\mathbb{B}_0$ is the disc (61), and let $r > 0$ as before. Let

$$\Lambda(x, \epsilon) := \max\left\{\tfrac{|x|}{\delta_x}, \tfrac{|\epsilon|}{\delta_\epsilon}\right\},$$

where $\delta_x, \delta_\epsilon > 0$ are such as in Definition 12. Let us consider formal power series $\hat{f}(x, \epsilon) = \sum_{k,l\geq 0} f^{(k,l)}x^k\epsilon^l$ such that the transformed series $\widetilde{f}(x, \epsilon) = \sum_{k,l} \tfrac{f^{(k,l)}}{(k+l)!}x^k\epsilon^l$, is convergent on the polydisc $D_r = \{|x|, |\epsilon| < r\}$ and extends analytically on the cone $\Omega_\pm^0 \cdot \mathbb{R}^+$. We introduce a norm of the formal series $\hat{f}(x, \epsilon)$, based on

similar norms in [5,6], as an exponentially weighted supremum norm of the function $(x\partial_x + \epsilon\partial_\epsilon)\widetilde{f}$:

$$\|\hat{f}\|_\Lambda := \sup_{\substack{(x,\epsilon)\in \\ D_r\cup\Omega_\pm^0\cdot\mathbb{R}^+}} \tfrac{4\pi}{\Lambda(x,\epsilon)}|(x\partial_x + \epsilon\partial_\epsilon)\widetilde{f}(x,\epsilon)|e^{-\Lambda(x,\epsilon)}(1+\Lambda(x,\epsilon)^2), \quad (63)$$

and define $\hat{\mathfrak{B}}(\Omega_\pm^0)$ as the space of such $\hat{f}$ for which $\|\hat{f}\|_\Lambda < +\infty$.

If $\hat{f} \in \hat{\mathfrak{B}}(\Omega_\pm^0)$, then its *Borel (1,1)-sum* $f_\pm(x,\epsilon)$ defined through the Laplace integral (62) converges for every $(x,\epsilon) \in \Omega_\pm^0$ with $\Lambda(x,\epsilon) < 1$ (which is automatically satisfied), and it extends analytically as a bounded function on

$$\Omega_\pm = \bigcup_{(x,\epsilon)\in\Omega_\pm^0} (x,\epsilon)\cdot\mathbb{B}_0 = \{(x,\epsilon)\in \mathsf{XE}_\pm,\ \exists\theta\in\,]-\tfrac{\pi}{2},\tfrac{\pi}{2}[:\ (x,\epsilon)\cdot\mathbb{B}_\theta \subset \mathsf{XE}_\pm\}.$$

$$(64)$$

Note that if a sequence of formal series in $\hat{\mathfrak{B}}(\Omega_\pm^0)$ converges with respect to the norm $\|\cdot\|_\Lambda$ then they also converge in the product topology on $\mathbb{C}[[x,\epsilon]]$, i.e. the coefficients of the power series converge term by term, and moreover the sequence of the corresponding Borel (1,1)-sums converges uniformly on compacts of $\Omega_\pm$. In particular, if $\sum_{\mathbf{m}} \hat{\phi}_{\mathbf{m}}(x,\epsilon)u^{\mathbf{m}}$, $\hat{\phi}_{\mathbf{m}}(x,\epsilon) = \sum_{k,l\geq 0}\phi_{\mathbf{m}}^{(k,l)}x^k\epsilon^l \in \hat{\mathfrak{B}}(\Omega_\pm^0)$, is a convergent power series in the Banach space $(\hat{\mathfrak{B}}(\Omega_\pm^0), \|\cdot\|_\Lambda)$ for $u \in \mathsf{U}$, then its sum is a map $\mathsf{U} \to \hat{\mathfrak{B}}(\Omega_\pm^0)$ given by a formal series $\hat{\phi}(u,x,\epsilon) = \sum_{k,l\geq 0}\phi^{(k,l)}(u)x^k\epsilon^l$, with each $\phi^{(k,l)}(u) = \sum_{\mathbf{m}}\phi_{\mathbf{m}}^{(k,l)}u^{\mathbf{m}}$ convergent on U. Moreover, it follows that the Borel (1,1)-sum of $\hat{\phi}(u,x,\epsilon)$, given by the Laplace integral $\mathcal{L}_0[\widetilde{\phi}(u,\cdot,\cdot)](x,\epsilon)$ (62), with $\widetilde{\phi}(u,x,\epsilon) = \sum_{k,l\geq 0}\frac{\phi^{(k,l)}(u)}{(k+l)!}x^k\epsilon^l$, agrees with the convergent series of the Borel (1,1)-sums $\sum_{\mathbf{m}}\mathcal{L}_0[\widetilde{\phi}_{\mathbf{m}}](x,\epsilon)u^{\mathbf{m}}$, which can be showed by a dominated convergence argument.

Proposition 46 *The space* $(\hat{\mathfrak{B}}(\Omega_\pm^0), \|\cdot\|_\Lambda)$ *is a Banach algebra.*

Proof It is quite clear that $(\hat{\mathfrak{B}}(\Omega_\pm^0), \|\cdot\|_\Lambda)$ is a Banach space. What rests is to show the sub-multiplicativity property of the norm:

$$\|\hat{f}\hat{g}\|_\Lambda \leq \|\hat{f}\|_\Lambda\|\hat{g}\|_\Lambda \quad \text{for all } \hat{f}, \hat{g} \in \hat{\mathfrak{B}}(\Omega_\pm^0).$$

Given $\hat{f}(x,\epsilon) \in \hat{\mathfrak{B}}(\Omega_\pm^0)$, consider the formal series

$$\hat{F}(t,x,\epsilon) = \hat{f}(tx,t\epsilon) = \sum_{k,l\geq 0} f^{(k,l)}x^k\epsilon^l t^{k+l} = \sum_{n\geq 0}\left(\sum_{k+l=n} f^{(k,l)}x^k\epsilon^l\right)t^n$$

of a new variable t with parameters (x, ϵ). The usual Borel transform of $\hat{F}$ with respect to t is given by

$$\mathcal{B}[\hat{F}](s, x, \epsilon) := \sum_{k,l \geq 0} \frac{f^{(k,l)}}{(k+l-1)!} x^k \epsilon^l s^{k+l-1} = \partial_s \widetilde{f}(sx, s\epsilon).$$

Hence

$$\|\hat{f}\|_\Lambda = \sup_{(sx, s\epsilon) \in D_r \cup \Omega_\pm^0 \cdot \mathbb{R}^+} \frac{4\pi}{\Lambda(x, \epsilon)} |\mathcal{B}[\hat{F}](s, x, \epsilon)| e^{-\Lambda(x,\epsilon)|s|} (1 + \Lambda(x, \epsilon)^2 |s|^2),$$

which is a rescaling of the weighted supremum norm introduced in [5, p.312]. The usual Borel transform $\mathcal{B}$ changes products into convolutions,

$$\mathcal{B}[\hat{F}\hat{G}](t, x, \epsilon) = \big(\mathcal{B}[\hat{F}] * \mathcal{B}[\hat{G}]\big)(t, x, \epsilon) := \int_0^t \mathcal{B}[\hat{F}](t - s, x, \epsilon) \mathcal{B}[\hat{G}](s, x, \epsilon) ds,$$

and we can conclude by the estimate of a norm of convolution in [5, Proposition 4] which gives us the result. □

The reason for introducing the two Banach algebras the above way lies in the following theorem adapted from [13] which is the other essential tool in our proof of Theorem 17.

Theorem 47 *Consider a system of the form*

$$x(x - \epsilon)\frac{d\phi}{dx} = M\phi + F(\phi, x, \epsilon), \quad (\phi, x, \epsilon) \in \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}. \qquad (65)$$

with M an invertible $m \times m$-matrix whose eigenvalues all lie on the line $\lambda^{(0)}\mathbb{R}$, and $F(\phi, x, \epsilon)$ analytic germ such that $D_\phi F(0, 0, 0) = 0$, and $F(0, x, \epsilon) = O(x(x-\epsilon))$. Then there exist some constants $r, \eta, \delta_x, \delta_\epsilon > 0$ such that the system (65) possesses:

(1) A unique formal power series solution $\hat{\phi}(x, \epsilon) = \sum_{j,k \geq 0} \phi_{kj} x^k \epsilon^j$. It is formally divisible by $x(x - \epsilon)$, and it belongs to $\big(\hat{\mathfrak{B}}(\Omega_\pm^0)\big)^m$.
(2) A unique bounded analytic solution $\phi_\pm(x, \epsilon) \in \big(\mathfrak{B}(\mathsf{XE}_\pm)\big)^m$ on the domain $\mathsf{XE}_\pm$ of Definition 12. It vanishes at the singular points (it is uniformly $O(x(x - \epsilon))$).

Moreover, the restriction of $\phi_\pm(x, \epsilon)$ to the domain $\Omega_\pm$ agrees with the Borel (1,1)-sum of the formal solution $\hat{\phi}(x, \epsilon)$:

$$\phi_\pm(x, \epsilon) = \int_0^{+\infty e^{i\theta}} \widetilde{\phi}(sx, s\epsilon) e^{-s} ds, \quad for \quad (x, \epsilon) \in \Omega_\pm^0, \qquad (66)$$

where $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ depending on (x, ϵ) is such that $(x, \epsilon) \cdot \mathbb{B}_\theta \subset \mathsf{XE}_\pm$, and $\widetilde{\phi}(x, \epsilon) = \sum_{j,k} \frac{\phi_{kj}}{(k+j)!} x^k \epsilon^j$.

Proof The existence and uniqueness of the bounded solution $\phi_\pm(x, \epsilon)$ is [13, Theorem 2]. The existence and uniqueness of the formal solution $\hat{\phi}(x, \epsilon)$ is [13, Proposition 1], while the Borel (1,1)-summability is [13, Theorem 4]. The finiteness of the norm $\|\hat{\phi}\|_\Lambda$ is just a consequence of its Borel (1,1)-summability, up to perhaps restricting of $\delta_x, \delta_\epsilon > 0$. □

7.1 Step 1: Ramified Straightening of Center Manifold and Diagonalization of the Linear Part

Suppose that the system is in a pre-normal form,

$$x(x - \epsilon)\frac{dy}{dx} = J\,{}^\mathrm{t}D_y F(y, x, \epsilon),$$

$$J\,{}^\mathrm{t}D_y F(y, x, \epsilon) = \chi(y_1 y_2, x, \epsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y + O(x(x - \epsilon)). \qquad (67)$$

We will show that there exists a ramified transversely symplectic change of variable

$$y = T_\pm(x, \epsilon)w + \phi_\pm(x, \epsilon), \quad \det T_\pm(0, \epsilon) = I, \ \phi_\pm(x, \epsilon) = O(x(x - \epsilon)) \quad (68)$$

bounded and analytic on the domain $\mathsf{XE}_\pm$ (25), *i.e. defined over the Banach space* $\mathfrak{B}(\mathsf{XE}_\pm)$, *that brings the system to a form*

$$x(x - \epsilon)\frac{dw}{dx} = \chi(w_1 w_2, x, \epsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w + x(x - \epsilon)f_\pm(w, x, \epsilon), \qquad (69)$$

with $f_\pm(w, x, \epsilon) = O(|w|^2)$, $\frac{\partial f_{1,\pm}}{\partial w_1} + \frac{\partial f_{2,\pm}}{\partial w_2} = 0$.

All this has a formal counterpart over the Banach space $\hat{\mathfrak{B}}(\Omega_\pm^0)$.

The solution $w = 0$ of the transformed system (69), corresponds to a bounded ramified solution $y = \phi_\pm(x, \epsilon)$ of the system (67). The paper [13], see Theorem 47 below, shows that there is a unique such solution on the domain $\mathsf{XE}_\pm$; this it is the *"ramified center manifold"* of the corresponding foliation.

The variable $\tilde{y} = y - \phi_\pm(x, \epsilon)$ then satisfies

$$x(x - \epsilon)\frac{d\tilde{y}}{dx} = J\,{}^\mathrm{t}D_y F(\tilde{y} + \phi_\pm, x, \epsilon) - J\,{}^\mathrm{t}D_y F(\phi_\pm, x, \epsilon),$$

whose linear part is $A_\pm(x, \epsilon) := J(D_y^2 F)(\phi_\pm, x, \epsilon) = \lambda(x, \epsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x(x - \epsilon)R(\phi_\pm, x, \epsilon)$. The transformation matrix $T_\pm$ (68) must then satisfy

$$x(x - \epsilon)\frac{dT_\pm}{dx} = A_\pm T_\pm - \lambda T_\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The existence of such a transformation $T_\pm$ bounded on $\mathsf{XE}_\pm$ is known [10,15] when $A_\pm$ is analytic. In our case the matrix $A_\pm$ is ramified, but their proof would work anyway. We will obtain $T_\pm$ directly using Theorem 47.

Writing $R = (r_{ij})_{i,j}$ and

$$T_\pm = \begin{pmatrix} 1 & t_{1,\pm} \\ t_{2,\pm} & 1 \end{pmatrix} \begin{pmatrix} e^{b_{1,\pm}} & 0 \\ 0 & e^{b_{2,\pm}} \end{pmatrix},$$

then the terms $t_{i,\pm}$, $i = 1, 2$, are solutions to Riccati equations

$$x(x-\epsilon)\frac{dt_{i,\pm}}{dx} = (-1)^{i-1}2\lambda t_{i,\pm} + x(x-\epsilon)\left[r_{ij} + (r_{ii} - r_{jj}) \cdot t_{i,\pm} - r_{ji} \cdot (t_{i,\pm})^2\right], \quad (70)$$

and the terms $b_{i,\pm}$ are solution to $\frac{db_{i,\pm}}{dx} = (r_{ii} + r_{ij}t_{j,\pm})$, i.e.

$$b_{i,\pm} = \int_0^x r_{ii} + r_{ij}t_{j,\pm}dx.$$

Combining together the equations (67) for $\phi_\pm$ and (70) for $t_\pm$, in which $r_{ij} = r_{i,j}(\phi_\pm, x, \epsilon)$, we get an analytic system for which the existence of a unique bounded solution on $\mathsf{XE}_\pm$ is assured by Theorem 47.

Since the trace of the linear part of both systems (67) and (69) is null, then by the Liouville–Ostrogradskii formula $\det T(x, \epsilon)$ is constant in x and equal to $\det T(0, \epsilon) = 1$. Therefore the transformation (68) is transversely symplectic, and by Lemma 3 the transformed system (69) is transversely Hamiltonian.

7.2 Step 2: Normalization

Suppose that the system is in the form (69). *We will show that there exists a ramified change of variable* $w = \Phi_\pm(v, x, \epsilon)$, $\Phi_\pm(\cdot, 0, \epsilon) = \text{id}$, *given as a convergent series of* v *in the Banach space* $\mathfrak{B}(\mathsf{XE}_\pm)$, *that will bring it to an integrable form*

$$x(x-\epsilon)\frac{dv}{dx} = \alpha_\pm(h, x, \epsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}v, \quad h = v_1 v_2, \quad (71)$$

for some germ $\alpha_\pm(h, x, \epsilon)$, *with* $\alpha_\pm(0, x, \epsilon) = \lambda(x, \epsilon)$.

As before, this has a formal counterpart over the Banach space $\hat{\mathfrak{B}}(\Omega_\pm^0)$.

The transformation $\Phi_\pm$ must satisfy

$$x(x-\epsilon)\partial_x\Phi_\pm + \alpha_\pm \cdot (v_1\partial_{v_1} - v_2\partial_{v_2})\Phi_\pm = \chi \circ \Phi_\pm \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_\pm + x(x-\epsilon)f_\pm \circ \Phi_\pm.$$
$$(72)$$

We are looking for $\Phi_\pm$ written as

$$\Phi_\pm(v, x, \epsilon) = v + \Psi_{\pm,\Delta} + x(x-\epsilon)\Psi_{\pm,\star}(v, x, \epsilon), \quad \Psi_{\pm,\Delta} + \Psi_{\pm,\star} =: \Psi_\pm \quad (73)$$

where the power expansion of the j-th coordinate of $\Psi_{\pm,\Delta}$ is equal to

$$\Psi_{j,\pm,\Delta} = \sum_{n \geq 1} \{\Psi_{j,\pm}\}_{(n,n)+\mathbf{e}_j} h^n v_j, \quad \mathbf{e}_j \text{ being the } j\text{-the elementary vector,}$$

while the power expansion of the j-th coordinate of $\{\Psi_{\pm,\star}\}$ does not contains any power $h^n v_j$, $n \geq 0$. In particular, $(v_1 \partial_{v_1} - v_2 \partial_{v_2}) \Psi_{\pm,\Delta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_{\pm,\Delta}$. Therefore (72) becomes

$$\partial_x \Psi_{\pm,\Delta} + \partial_x \left(x(x-\epsilon)\Psi_{\pm,\star} \right) + \lambda \left(v_1 \partial_{v_1} - v_2 \partial_{v_2} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \Psi_{\pm,\star}$$

$$= -\alpha_{\pm}^* \left(v_1 \partial_{v_1} - v_2 \partial_{v_2} \right) \Psi_{\pm,\star} + \frac{\chi^* \circ \Phi_{\pm} - \alpha_{\pm}^*}{x(x-\epsilon)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (v + \Psi_{\pm,\Delta}) + G_{\pm}, \tag{74}$$

where $G_{\pm} = (\chi^* \circ \Phi_{\pm}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_{\pm,\star} + f \circ \Phi_{\pm}$, and $\chi^* = \chi - \lambda$, $\alpha_{\pm}^* = \alpha_{\pm} - \lambda$.

Set

$$\alpha_{\pm}(h, x, \epsilon) = \sum_{n \geq 0} \{\chi \circ \Phi_{\pm}\}_{(n,n)} h^n,$$

and denote

$$K_{\pm} := \frac{\chi^* \circ \Phi_{\pm} - \chi^* \circ (v + \Psi_{\pm,\Delta})}{x(x-\epsilon)},$$

which is an analytic function of $v + \Psi_{\pm,\Delta}$ and $\Psi_{\pm,\star}$ with coefficients depending on x, ϵ. Then $\{\chi^* \circ \Phi_{\pm} - \alpha_{\pm}^*\}_{(n,n)} = 0$ for all $n \geq 0$, and $\{\chi^* \circ \Phi_{\pm} - \alpha_{\pm}^*\}_{\mathbf{n}} = x(x-\epsilon)\{K_{\pm}\}_{\mathbf{n}}$ for all multi-indices $\mathbf{n}$ with $n_1 \neq n_2$, since $\{\alpha_{\pm}^*\}_{\mathbf{n}} = 0 = \{\chi^* \circ (v + \Psi_{\pm,\Delta})\}_{\mathbf{n}}$.

Expanding the j-th coordinate, $j = 1, 2$, of the Eq. (72) in powers of v we get:

- for $\mathbf{m} = (n, n) + \mathbf{e}_j$:

$$\partial_x \{\Psi_{j,\pm}\}_{(n,n)+\mathbf{e}_j} = \{G_{j,\pm}\}_{(n,n)+\mathbf{e}_j}, \tag{75}$$

- for a multi-index $\mathbf{m}$ with $m_1 - m_2 + (-1)^j \neq 0$:

$$\partial_x \{x(x-\epsilon)\Psi_{j,\pm}\}_{\mathbf{m}} + (m_1 - m_2 + (-1)^j)\lambda\{\Psi_{j,\pm}\}_{\mathbf{m}}$$

$$= -(m_1 - m_2)\{\alpha_{\pm}^* \cdot \Psi_{j,\pm}\}_{\mathbf{m}} - (-1)^j\{K_{\pm} \cdot (v_j + \Psi_{j,\pm,\Delta})\}_{\mathbf{m}} + \{G_{j,\pm}\}_{\mathbf{m}}. \tag{76}$$

The right-hand sides of (76) and (75) are functions of $\{\Psi_{\pm}\}_{\mathbf{k}} = {}^t(\{\Psi_{1,\pm}\}_{\mathbf{k}}, \{\Psi_{2,\pm}\}_{\mathbf{k}})$, with $k_1 \leq m_1$, $k_2 \leq m_2$, $|\mathbf{k}| < |\mathbf{m}|$ only, which means that the equations for $\{\Psi_{j,\pm}\}_{\mathbf{m}}$ can be solved recursively.

The Eq. (75) is solved by

$$\{\Psi_{j,\pm}\}_{(n,n)+\mathbf{e}_j}(x, \epsilon) = \int_0^x \{G_{j,\pm}\}_{(n,n)+\mathbf{e}_j} dx. \tag{77}$$

The Eq. (76) has a unique bounded solution $\{\Psi_{j,\pm}\}_{\mathbf{m}}$ given by the integral

$$\{\Psi_{j,\pm}\}_{\mathbf{m}}(x,\epsilon) = \frac{e^{-(m_1-m_2+(-1)^j)t_\lambda}}{x(x-\epsilon)} \int_{x_{i,\pm}}^{x} e^{(m_1-m_2+(-1)^j)t_\lambda} \{F_{j,\pm}\}_{\mathbf{m}} dx, \qquad (78)$$

where $\{F_{j,\pm}\}_{\mathbf{m}}$ is the right side of (76),

$$t_\lambda(x,\epsilon) = \int_\infty^x \frac{\lambda(x,\epsilon)}{x(x-\epsilon)} dx = \begin{cases} -\frac{\lambda^{(0)}}{\epsilon}\log x + \left(\frac{\lambda^{(0)}}{\epsilon}+\lambda^{(1)}\right)\log(x-\epsilon), & \text{for } \epsilon \neq 0, \\ -\frac{\lambda^{(0)}}{x} + \lambda^{(1)}\log x, & \text{for } \epsilon = 0, \end{cases}$$

is a branch of the rectifying coordinate for the vector field

$$\frac{x(x-\epsilon)}{\lambda(x,\epsilon)}\partial_x = \partial_{t_\lambda},$$

on $\mathsf{X}_\pm(\epsilon)$, and the integration follows a real trajectory of the vector field (23) in $\mathsf{X}_\pm(\epsilon)$ from a point

$$x_* = \begin{cases} x_{1,\pm}, & \text{if } m_1-m_2+(-1)^j > 0, \\ x_{2,\pm}, & \text{if } m_1-m_2+(-1)^j < 0, \end{cases} \qquad x_{i,\pm} \text{ is as in (26)},$$

to x, along which the integral is well defined.

Note that the convergence of the constructed formal transformation $\Phi_\pm$ is equivalent to the convergence of $\Psi_\pm$ (73). We prove the convergence of the latter series using Lemma 44. For this we need to estimate the norms of (77) and (78).

Lemma 48 *Let $\{\Psi_{j,\pm}\}_{\mathbf{m}}$ be given by (78). Then*

$$\|\{\Psi_{j,\pm}\}_{\mathbf{m}}\| \leq \frac{c_\star}{m_1-m_2+(-1)^j} \|\{F_{j,\pm}\}_{\mathbf{m}}\|,$$

for some $c_\star > 0$ independent of $\mathbf{m}$.

Proof Let $\tau_k(x,\epsilon) = kt_\lambda(x,\epsilon) + \log(x(x-\epsilon))$. If $k = m_1-m_2+(-1)^j \neq 0$, then for ϵ small enough the integrating path can be deformed so that it corresponds to a ray $\tau_k(\xi,\epsilon) \in \tau_k(x,\epsilon) - e^{i\theta_\pm}[0,+\infty[$, with $\theta_\pm$ as in (24). We have $\frac{d\tau_k}{dx}(x,\epsilon) = \frac{k\lambda(x,\epsilon)+2x-\epsilon}{x(x-\epsilon)}$. Hence

$$\|\{\Psi_{j,\pm}\}_{\mathbf{m}}\| \leq \frac{1}{\cos\theta_\pm} \left\| \frac{\{F_{j,\pm}\}_{\mathbf{m}}(x,\epsilon)}{k\lambda(x,\epsilon)+2x-\epsilon} \right\|,$$

with $\cos\theta_\pm > \sin\eta > 0$, η as in (24). $\qquad \square$

Therefore

- for $\mathbf{m} = (n,n) + \mathbf{e}_j$:

$$\|\{\Psi_{j,\pm}\}_{(n,n)+\mathbf{e}_j}\| \leq c\|\{G_{j,\pm}\}_{(n,n)+\mathbf{e}_j}\|,$$

- for a multi-index **m** with $m_1 - m_2 + (-1)^j \neq 0$:

$$\|\{\Psi_{j,\pm}\}_{\mathbf{m}}\| \leq c \left(\|\{\alpha_{\pm}^* \Psi_{j,\pm}\}_{\mathbf{m}}\| + \|\{K_{\pm} \cdot (v_j + \Psi_{j,\pm,\Delta})\}_{\mathbf{m}}\| + \|\{G_{j,\pm}\}_{\mathbf{m}}\| \right),$$

for some $c > 0$.

Therefore the majorating series satisfies

$$
\begin{aligned}
\overline{\Psi}_{j,\pm} &\prec c\,\overline{\alpha}_{\pm}^* \overline{\Psi}_{j,\pm} + c\,\overline{K}_{\pm} \cdot (v_j + \overline{\Psi}_{j,\pm}) + c\,\overline{G}_{j,\pm} \\
&\prec 2c\,(\overline{\chi}^* \circ \overline{\Phi}_{\pm}) \cdot \overline{\Psi}_{j,\pm} + c\,\overline{k}_{\pm} \cdot (v_j + \overline{\Psi}_{j,\pm}) + c\,\overline{f}_{j,\pm} \circ \overline{\Phi}_{\pm} \\
&=: \overline{R}_{j,\pm}(v + \overline{\Psi}_{\pm,\Delta}, \overline{\Psi}_{\pm,\star}) \\
&\prec \overline{R}_{j,\pm}(v + \overline{\Psi}_{\pm}, v + \overline{\Psi}_{\pm}),
\end{aligned}
$$

where $K_{\pm} = K_{\pm}(v + \Psi_{\pm,\Delta}, \Psi_{\pm,\star}), \overline{k}_{\pm} = \overline{K}_{\pm}(v + \overline{\Psi}_{\pm,\Delta}, \overline{\Psi}_{\pm,\star})$, and $\overline{R}_{j,\pm}(w_1, w_2) = O(|w|^2)$, and we can conclude with Lemma 44.

The same thing is done also in the formal category with the Banach space $\hat{\mathfrak{B}}(\Omega_{\pm}^0)$.

7.3 Step 3: Final Reduction and Transverse Symplecticity of the Transformation

Suppose that the system is in the form (71), and write

$$
\begin{aligned}
\alpha_{\pm}(h, x, \epsilon) &= \tilde{\chi}_{\pm}(h, x, \epsilon) + x(x - \epsilon)\beta_{\pm}(h, x, \epsilon), \\
\tilde{\chi}_{\pm}(h, x, \epsilon) &= \tilde{\chi}_{\pm}^{(0)}(h, \epsilon) + x \tilde{\chi}_{\pm}^{(1)}(h, \epsilon).
\end{aligned}
$$

Then the transformation $v = e^{\int_0^x \beta_{\pm} dx \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} u$ will bring it to the normal form with formal invariant $\tilde{\chi}_{\pm}$.

Let $y = \Psi_{\pm}(u, x, \epsilon)$ be the transformation obtained as a composition of the transformations of Steps 1–3, and $y = \hat{\Psi}(u, x, \epsilon) = \sum_{k,l} \psi^{(k,l)}(u) x^k \epsilon^l$ the corresponding formal transformation. By construction $\Psi_{\pm}(\cdot, 0, \epsilon) = \hat{\Psi}(\cdot, 0, \epsilon) = \mathrm{id}$, in particular it is convergent. The restriction of $\Psi_{\pm}(u, x, \epsilon)$ to $\mathsf{U} \times \Omega_{\pm}$ must agree with the Borel (1,1)-sum of $\hat{\Psi}(u, x, \epsilon)$: indeed one can swap the summation in u and of the Laplace integral by a dominated convergence argument.

Let us now show that $\Psi_{\pm}(u, x, \epsilon)$ is transversely symplectic and therefore $\tilde{\chi}_{\pm} = \chi$. Let $u_{\pm}(x, \epsilon; c)$ be a germ of a general solution of the normal form system with the formal invariant equal to $\tilde{\chi}_{\pm}$, depending on an initial condition parameter $c = (c_1, c_2) \in (\mathbb{C}^2, 0)$, $\det(D_c u_{\pm}) \neq 0$, and let $y_{\pm}(x, \epsilon; c) = \Psi_{\pm}(u_{\pm}(x, \epsilon; c), x, \epsilon)$ be the corresponding solution germ of the system (1). Then $D_c y_{\pm} = D_u \Psi_{\pm} \cdot D_c u_{\pm}$ satisfies the linearized system

$$x(x - \epsilon)\frac{d D_c y_{\pm}}{dx} = J D_y^2 H \cdot D_c y_{\pm},$$

and by the Liouville-Ostrogradskii formula

$$x(x - \epsilon)\frac{d}{dx}\det(D_c y_\pm) = \mathrm{tr}(JD_y^2 H) \cdot \det(D_c y_\pm),$$

but $\mathrm{tr}(JD_y^2 H) = 0$, i.e. $\det(D_c y_\pm) = \det(D_u \Psi_\pm) \cdot \det(D_c u_\pm)$ is constant in x. Similarly, $\det(D_c u_\pm)$ is constant in x. Therefore $\det(D_u \Psi_\pm(u, x, \epsilon))$ is also constant in x, and equal to $\det(D_u \Psi_\pm(u, 0, \epsilon)) = 1$ since $\Psi_\pm(u, 0, \epsilon) = u$.

This terminates the proof of Theorem 17.

7.4 Proof of Proposition 9

We will construct a formal symplectic change of coordinate $\Phi = \Phi(h, u_i)$, written as a formal power series in h and u_i, such that $G = H \circ \Phi$. The transformation Φ is constructed recursively as a formal limit

$$\Phi = \lim_{k \to +\infty} \Phi_{k,1}, \quad \Phi_{k+1,1} = \lim_{l \to +\infty} \Phi_{k,l}, \quad \Phi_{0,1} = \mathrm{id}, \quad H \circ \Phi_{k,l} = G + O\left(h^k u_i^l\right).$$

At each step $(k, l), k \geq 0, l \geq 1$, we want to get rid of the power $h^k u_i^l$ in $H \circ \Phi_{k,l}$. We construct $\Phi_{k,l+1} = \Phi_{k,l} \circ \Phi^1_{f_{k,l}X_{h^k u_i^l}}$ as a composition of $\Phi_{k,l}$ with the time-1 flow of a Hamiltonian vector field $f_{k,l}X_{h^k u_i^l} = -(-1)^i f_{k,l}h^{k-1}u_i^l[ku_i \partial_{u_i} - (k+l)u_j \partial_{u_j}]$ for some $f_{k,l} \in \mathbb{C}$. The flow sends both h and u_i to functions of (h, u_i),

$$\Phi^1_{f_{k,l}X_{h^k u_i^l}} : \quad u_i \mapsto u_i + O\left(h^{k-1}u_i^{l+1}\right), \quad h \mapsto h + (-1)^i l f_{k,l}h^k u_i^l + O\left(h^{2k-1}u_i^{2l}\right),$$

where the terms $O(h^{2k-1}u_i^{2l})$ are null if $k = 0$. If $H \circ \Phi_{k,l} = G + H_{k,l}h^k u_i^l + O(h^k u_i^{l+1})$ for some $H_{k,l} \in \mathbb{C}$, then we want

$$\left(G + H_{k,l}h^k u_i^l\right) \circ \Phi^1_{f_{k,l}X_{h^k u_i^l}} - G(h) = O\left(h^k u_i^{l+1}\right),$$

$$H_{k,l}h^k u_i^l + \lambda \cdot (-1)^i l f_{k,l}h^k u_i^l = O\left(h^k u_i^{l+1}\right),$$

$$f_{k,l} = -(-1)^i \frac{H_{k,l}}{\lambda l}.$$

Now that we have constructed the formal symplectic transformation Φ, we can conclude by the following Proposition.

Proposition 49 *Let $H, \tilde{H} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two germs with a non-degenerate critical point at 0, and let $\omega, \tilde{\omega}$ be germs of symplectic forms. Then the two pairs $(H, \omega), (\tilde{H}, \tilde{\omega})$ are analytically equivalent if and only if they are formally equivalent.*

Proof By Theorem 4, the Birkhoff–Siegel normal form is, up to the involution (13), a complete analytic invariant for each pair. Therefore it is enough to show that it is

also a formal invariant. This can be seen from the invariance of a formalization of the formula (15) of Sect. 2.2.1.   □

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