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RESEARCH ARTICLE

Shapley-based risk rankings: some theoretical considerations

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Abstract
In this note, we enhance the analysis done by Auer and Hiller (Int J Finance Econ 24(2):884–889, 2019; Manag Decis Econ 42(4):876–884, 2021). Whereas their articles uses several simulation settings to illustrate that cooperative game theory may have the potential to solve the low-risk puzzle, we calculate for the three-asset case the conditions for partial ranking corrections between assets. Hence, our note could be interpreted as theoretical counterpart to Auer and Hiller (Int J Finance Econ 24(2):884–889, 2019; Manag Decis Econ 42(4):876–884, 2021).

Keywords Low-risk puzzle · Cooperative game theory · Shapley value

JEL Classification C71 · G10 · G11

1 Introduction

In a recent article, Auer and Hiller (2019) illustrated by several simulation settings that cooperative game theory—in particular the Shapley value (Shapley 1953)—may have the potential to solve the so called low-risk puzzle. According to this phenomenon, investment opportunities with low risk consistently tend to outperform their high-risk counterparts (Ang et al. 2006, 2009; Dutt and Humphery-Jenner 2013; Frazzini and Pedersen 2014; Auer and Schuhmacher 2021). Because it holds across different markets and asset classes and seriously challenges asset pricing theory's traditional notion of a positive risk-return trade-off, the low-risk puzzle is considered to be one of the most important capital market anomalies discovered so far.

One rationale for this phenomenon might be that researchers are simply using "the wrong measure of risk" (Baker et al. 2011). Previous studies have relied on the standard deviation of returns (Dutt and Humphery-Jenner 2013), idiosyncratic volatility (Ang et al. 2006) or beta (Frazzini and Pedersen 2014) to document the puzzle

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and to point out its relevance for investors.[1] However, all these measures have serious limitations because they do not fully capture the risk that is actually relevant for typical investment decisions and/or because they are bound to restrictive theoretical assumptions. For example, popular measures of systematic risk (such as beta) capture how an asset contributes to the risk of a portfolio which is not relevant from a practical perspective.

To show a new research approach to solve the low-risk puzzle, Auer and Hiller (2019) model the asset market by a cooperative game where the assets are the players that form coalitions to reduce risk (Kadan 2016; Colini-Baldeschi et al. 2018; Balog et al. 2017). The authors argued that ranking assets based on their Shapley payoffs to the risk of an investment portfolio has the potential to solve the low-risk puzzle, since the Shapley value considers marginal contributions to risk to all possible asset coalitions. Hence, in contrast to standalone risk measures like single assets variance, Shapley payoffs can capture additional information (on favorable correlation). In their model, the authors look at an investor who is interested in combining individual assets in a portfolio and evaluates the riskiness of each asset based on the different payoffs with respect to the portfolio variance and assume that the low-risk puzzle exists in the individual asset variances. In their three-player simulation, Auer and Hiller (2019) provide simulation evidence that ranking assets based on their Shapley values instead of standard risk measures can provide asset orders consistent with the classic notion of a risk-return trade-off. The analysis by Auer and Hiller (2019) is limited by a focus on one specific risk allocation method: the Shapley value. In Auer and Hiller (2021), two additional methods are considered into analysis: the cost gap method (Tijs and Driessen 1986) and the nucleolus (Schmeidler 1969). A detailed analysis ranks the Shapley method on top of the other two approaches. Hence in our note, we focus on the Shapley value and calculate for the three-player case the mathematical conditions for partial ranking corrections between assets. Afterward, we interpret our results with respect to portfolio theory. The three-player case is sufficient to demonstrate the possibilities of cooperative game theory, i.e., take into account one more player would not change the basic results.

Underscoring the importance of cooperative game theory, a growing body of literature has emerged that uses cooperative game theory, especially the Shapley value (Shapley 1953), the Nucleolus (Schmeidler 1969) and the τ value—also known as cost gap method (Tijs and Driessen 1986; Tijs 1987), for risk allocation (Mussard and Terraza 2008; Ortmann 2016, 2018; Balog et al. 2017; Shalit 2020). Hiller (2022) used coalition structure games of cooperative game theory to analyze the low-risk puzzle. Hiller (2023) applied cooperation structures of cooperative game theory to this analysis.[2]

[1] Others have extended the traditional capital asset pricing model by using partial moments (Bawa and Lindenberg 1977), the conditional value at risk (Kaplanski 2004) or drawdowns (Zabarankin et al. 2014) as alternative concepts to measure risk.

[2] Another interesting application of cooperative game theory in the area of risk is the analysis of terrorist attacks and attacker–defender situations (Cox 2009; Zhang and Reniers 2016; Algaba et al. 2023).

The remainder of our article is organized as follows. In Sect. 2, we present the basic notations of cooperative game theory. Section 3 presents our results. Finally, Sect. 4 concludes and outlines directions for future research.

2 Cooperative game theory

A cooperative game can be defined by a pair(N, v). $N = \{1, 2, ..., n\}$ is the set of players (assets). The coalition function v specifies for every subset K of N a certain worth $v(K)$ reflecting the risk of portfolio K,, i.e., $v : 2^N \to \mathbb{R}$ such that$v(\emptyset) = 0$. The function v is subadditive if for all $K, S \subseteq N$ and $K \cap S = \emptyset, v(K) + v(S) \geq v(K \cup S)$ is fulfilled.

A value is an operator ϕ that assigns (unique) payoff vectors to all games (N, v) (i.e., uniquely determines a payoff for every player in every TU game). We interpret the payoff of player i as i's contribution to the portfolio risk $v(N)$, i.e., we use this payoff as a measure for the "risk" of assets mentioned in the introduction. One important value is the Shapley value. For calculating the player's payoffs, rank orders ρ on N are used. They are written as $(\rho_1, \ldots, \rho_n)$ where ρ_1 is the first player in the order, ρ_2 the second player, etc. The set of these orders is denoted by $RO(N)$; $n!$ rank orders exist. The set of players before i in rank order ρ including i is called $K_i(\rho)$. For player i, the Shapley payoff is determined by (Shapley 1953):

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\rho \in RO(N)} v\big(K_i(\rho)\big) - v\big(K_i(\rho) \backslash \{i\}\big). \tag{1}$$

An alternative formula is given by:

$$Sh_i(N, v) = \sum_{K \subseteq N \backslash \{i\}} \frac{k! \cdot (n - k - 1)!}{n!} \cdot v(K \cup \{i\}) - v(K) \tag{2}$$

The Shapley value has been applied to the problem of risk allocation in papers by Mussard and Terraza (2008), Ortmann (2016), Balog et al. (2017), Auer and Hiller (2019, 2021) and Shalit (2020).[3]

3 Results

In this section, we analyze conditions on which ranking risky assets based on the Shapley values instead of variances corrects the "wrong" ranking. We assume a three-asset scenario, $N = \{1, 2, 3\}$ with

[3] Other value-like solution concepts of cooperative game theory are presented by Banzhaf (1965), Schmeidler (1969), Holler (1982), Tijs (1987), for example.

w_i, asset i's weight in N
μ_i, asset i's mean returns
ρ_{ij} correlation between i and j
σ_i^2, variance of asset i.

with $\sum_{i \in N} w_i = 1$.

The covariance σ_{ij} between two assets $i, j \in N, i \neq j$ is defined by:

$$\sigma_{ij} = \sigma_i \cdot \sigma_j \cdot \rho_{ij}. \tag{3}$$

A "wrong" ranking is defined by $\mu_i > \mu_j > \mu_h$ and $\sigma_i^2 \leq \sigma_j^2 \leq \sigma_h^2$. The coalition function is given by

$$v(\{i\}) = \sigma_i^2, \tag{4}$$

$$v(\{i,j\}) = \left(\frac{w_i}{w_i + w_j}\right)^2 \cdot \sigma_i^2 + \left(\frac{w_j}{w_i + w_j}\right)^2 \cdot \sigma_j^2 + 2 \cdot \left(\frac{w_i}{w_i + w_j}\right) \cdot \left(\frac{w_j}{w_i + w_j}\right) \cdot \sigma_{ij} \tag{5}$$

for $i, j \in N$ and

$$\begin{aligned} v(N) = {} & w_1^2 \cdot \sigma_1^2 + w_2^2 \cdot \sigma_2^2 + w_3^2 \cdot \sigma_3^2 \\ & + 2 \cdot \left(w_1 \cdot w_2 \cdot \sigma_{12} + w_1 \cdot w_3 \cdot \sigma_{13} + w_2 \cdot w_3 \cdot \sigma_{23}\right). \end{aligned} \tag{6}$$

The worth $v(N)$ is the variance of the portfolio of all assets.

In our analyses, we will use the term *partial ranking correction*:

Definition 1 A partial ranking correction occurs if for at least two assets $i, j \in N$ we have $\sigma_i^2 \geq \sigma_j^2$ and $Sh_i(N, v) < Sh_j(N, v)$.

For first insights, we assume equal weights for the assets of our portfolio. The condition for partial ranking corrections between assets is:

Theorem 2 *Let* $N = \{1, 2, 3\}$ *be a portfolio with* $w_1 = w_2 = w_3$. *For* $\sigma_i^2 > \sigma_j^2$, $i, j, h \in N, i \neq j \neq h$, *we have* $Sh_i(N, v) < Sh_j(N, v)$ *iff*

$$\frac{2}{3} \cdot \left(\sigma_{ih} - \sigma_{jh}\right) < \sigma_j^2 - \sigma_i^2. \tag{7}$$

The proof is given in the "Appendix." Since subadditivity is one of the desirable properties of risk measures in risk management, we show, by example, that the conditions of Eq. 7 are compatible with subadditivity of $v(N)$.

Example 3 The portfolio $N = \{1, 2, 3\}$ with $w_1 = w_2 = w_3$ and

σ_1^2	σ_2^2	σ_3^2	ρ_{12}	ρ_{13}	ρ_{23}
23.44	14.91	31.13	5.61	−12.70	−9.48

is subadditive and using the Shapley-risk measures yields a partial ranking correction.

From Theorem 2, we deduce:

Corollary 4 *Let* $N = \{1, 2, 3\}$ *be a portfolio with* $w_i = w_j = w_h$, $\sigma_{ij} = \sigma_{ih} = \sigma_{jh}$. *For* $\sigma_i^2 > \sigma_j^2, i, j \in N, i \neq j$, *we have* $Sh_i(N, v) > Sh_j(N, v)$.

To gain some more insights, we assume as limit case $\sigma_i^2 = \sigma_j^2 = \sigma_h^2 = \sigma^2$. Again, we are interested on the conditions for a ranking correction between two assets i and j. From Theorem 2, we deduce:

Corollary 5 *Let* $N = \{1, 2, 3\}$ *be a portfolio with* $w_i = w_j = w_h$ *and* $\sigma_i^2 = \sigma_j^2 = \sigma_h^2 = \sigma^2$. *We have* $Sh_i(N, v) < Sh_j(N, v)$ *iff* $\sigma_{jh} > \sigma_{ih}, i, j, h \in N, i \neq j \neq h$.

Hence, a partial ranking correction occurs in the limit case $\sigma_i^2 = \sigma_j^2 = \sigma_h^2 = \sigma^2$, if:

- Asset i contributes a negative covariance with h to portfolio risk, whereas j contributes a positive covariance with h.
- The positive covariance of i with h is lower than the covariance between j and h.

for example. With respect to portfolio literature, this is a desirable property of the Shapley value, since one aim when structuring portfolios is to reduce portfolio risk. The Shapley value honors assets with negative marginal contributions to portfolio risk in all possible asset subsets.

In the last step, we analyze the impact of changes of asset i's share of the portfolio. Therefore, we assume $w_j = w_h = \frac{1-w_i}{2}$. Hence, reducing/increasing w_i affects the weights of assets j and h in the same way. Again, we assume equal variance of assets.

Theorem 6 *Let* $N = \{1, 2, 3\}$ *be a portfolio with* $w_j = w_h = \frac{1-w_i}{2}, \sigma_i^2 = \sigma_j^2 = \sigma_h^2 = \sigma^2$, $i, j, h \in N, i \neq j \neq h$. *An increase in* w_i *could not cause a partial ranking correction, i.e.,* $Sh_i(N, v) < Sh_j(N, v)$, *between assets i and j if* $w_i > \frac{1}{6}$.

The proof is given in the "Appendix." Hence, only a small starting weight w_i could cause a ranking correction while increasing w_i depending on the constellation of correlations. This result is also in line with portfolio theory. The effect of diversification of an additional asset is greatest at lower weights. Since Shapley value captures these effects on risk, in this interval of weights a partial ranking correction is most likely. In addition, our limit w_i is in range of assets optimal weights in minimum variance portfolios (Jorion 1985).

4 Conclusion

In our note, we enhance the articles by Auer and Hiller (2019, 2021) in a theoretical way. We calculate for the three-asset case the conditions for partial ranking corrections between rankings based on assets variance and rankings based on assets Shapley payoffs. We interpret our results and fit it into current portfolio theory literature. The results refer to the three-asset case. Already with this, the possibilities of cooperative game theory can be demonstrated. Considering more assets would entail stronger assumptions regarding the variable of the assets. This would also limit the findings.

This note is a starting point for a great range of further research. For example, dynamic models of portfolio construction—e.g., rebalancing assets weights—could be modeled and analyzed using dynamic/evolutionary cooperative game theory (see Newton 2018 for an overview and Casajus et al. 2020 for some new insights). Second, our analysis may be extended to Shapley alternatives like the nucleolus (Schmeidler 1969), for example. After this theoretical research is done, a new asset pricing model may be developed based on the findings (Ortmann 2016). Another line of research could be the empirical testing of Shapley-based asset pricing formulations. In other words, we have to answer the question whether our results hold for typical test assets (like size, value, momentum and industry portfolios).

Appendix

Theorem 2 *For i's Shapley payoffs, we have*:

$$Sh_i(N,v) = \frac{1}{3}\cdot\sigma_i^2 + \frac{1}{6}\cdot\left(v(\{i,j\}) - \sigma_j^2\right) + \frac{1}{6}\cdot\left(v(\{i,h\}) - \sigma_h^2\right) + \frac{1}{3}\cdot\left(v(N) - v(\{j,h\})\right). \tag{8}$$

The payoffs for the other assets are calculated analogous.

We show for i,j by contradiction that $Sh_i(N,v) > Sh_j(N,v)$ and $\frac{2}{3}\cdot\left(\sigma_{ih} - \sigma_{jh}\right) < \sigma_j^2 - \sigma_i^2$ are not possible:

$$\begin{aligned} &Sh_i(N,v) > Sh_j(N,v)\\ &\sigma_i^2 - \sigma_j^2 > v(\{j,h\}) - v(\{i,h\})\\ &\frac{3}{2}\sigma_i^2 - \frac{3}{2}\sigma_j^2 > \sigma_{jh} - \sigma_{ih}\\ &\frac{2}{3}\cdot\left(\sigma_{ih} - \sigma_{jh}\right) > \sigma_j^2 - \sigma_i^2. \end{aligned} \tag{9}$$

Theorem 6 *We have* $Sh_i(N,v) < Sh_j(N,v)$ *if*

$$
\begin{aligned}
&\frac{1}{3}\cdot\sigma^2+\frac{1}{6}\cdot\left(v(\{i,j\})-\sigma^2\right)+\frac{1}{6}\cdot\left(v(\{i,h\})-\sigma^2\right)+\frac{1}{3}\cdot(v(N)-v(\{j,h\}))\\
&\quad<\frac{1}{3}\cdot\sigma^2+\frac{1}{6}\cdot\left(v(\{i,j\})-\sigma^2\right)+\frac{1}{6}\cdot\left(v(\{j,h\})-\sigma^2\right)+\frac{1}{3}\cdot(v(N)-v(\{i,h\}))\\
&\quad v(\{i,h\})<v(\{j,h\})\\
&\quad 0>w_i^2+w_i\cdot\left(\frac{1-w_i}{2}\right)\cdot\rho_{ih}-\left(\frac{1-w_i}{2}\right)^2-\left(\frac{1-w_i}{2}\right)^2\cdot\rho_{jh}
\end{aligned}
\tag{10}
$$

Partial derivation the right side of the Equation with respect to w_i yields:

$$
\frac{1}{2}+\frac{3}{2}w_i+\rho_{ih}\left(\frac{1}{2}-w_i\right)+\rho_{jh}\left(\frac{1}{2}-\frac{1}{2}w_i\right). \tag{11}
$$

Since Inequality 10 is negative, an increase in w_i could only induce a ranking correction, if Eq. 11 is negative (negative influence of w_i).

First, we assume $\frac{1}{2}>w_i$. For a negative Eq. 11, ρ_{ih} and ρ_{jh} must be negative. The lowest correlations (limit case) are $\rho_{ih}=\rho_{jh}=-1$. For this case, we obtain $w_i=\frac{1}{6}$. Only to that limit, a negative influence of an increasing w_i is possible and a ranking correction may occur.

In the case of $\frac{1}{2}\leq w_i$, the first two terms of Eq. 11 are at least $\frac{5}{4}$ ($w_i=\frac{1}{2}$). Even $\rho_{ih}=1$ and $\rho_{jh}=-1$ could not exceed $\frac{5}{4}$ in a negative way. An increasing w_i enhances this finding. Hence, in the case of $\frac{1}{2}\leq w_i$, the influence of an increase in w_i is always positive.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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