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Commutators on Spaces of Homogeneous Type in Generalized Block Spaces

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Abstract

In this paper, we are interested in studying a generalized block space (denoted as $\mathbf{B}_{\varphi}^{p,r}$) on a space of homogeneous type. We show that this space is the predual of certain generalized Morrey–Lorentz space. By duality, we obtain the $\mathbf{B}_{\varphi}^{p,r}$-bound of operators of Calderón–Zygmund type. In addition, we prove a weak Hardy factorization in terms of commutators of integral operator of Calderón–Zygmund type in block spaces. Thanks to the Hardy factorization result, we obtain a characterization of functions in BMO via the boundedness of commutators of homogeneous linear Calderón–Zygmund operators in the generalized block space (resp. the generalized Morrey–Lorentz space). Finally, we study a compactness characterization of commutators of Calderón–Zygmund type in generalized Morrey–Lorentz spaces.

Keywords Block space · Hardy factorization · Commutators · Singular integral operators of Calderón–Zygmund type · Space of homogeneous type

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1 Introduction and Main Results

The theory of Calderón–Zygmund operators is a central part of modern harmonic analysis. Many applications to partial differential equations are among the motivations to study Calderón–Zygmund operators on spaces which are beyond the Lebesgue spaces L^p of the Euclidean spaces. This research direction has been studied extensively and lead to a successful theory of function spaces including Hardy spaces, BMO spaces, Campanato spaces, Morrey–Lorentz spaces on the Euclidean space $\mathbb{R}^n$ or more general, on a space of homogeneous type X (see for example [7]) in the last fifty years.

In this paper, we study certain aspects of operators of Calderón–Zygmund type on various function spaces. The aim of this paper is threefold.

(i) Firstly, we study generalized block space $\mathbf{B}_\varphi^{p,r}(X)$ on space of homogeneous type, where we assume that $p \in (1, \infty)$, $r \in [1, \infty]$, and function $\varphi(t)$ satisfies (1.5) below. Then, we prove that $\mathbf{B}_\varphi^{p',r'}(X)$ is the predual of certain generalized Morrey–Lorentz space $\mathbf{M}_\varphi^{p,r}(X)$.

(ii) Secondly, we investigate the $\mathbf{M}_\varphi^{p,r}$-bound (resp. $\mathbf{B}_\varphi^{p',r'}$-bound) of operators of Calderón–Zygmund type, and prove a weak Hardy factorization in terms of commutators of Calderón–Zygmund type in $\mathbf{B}_\varphi^{p',r'}(X)$ and $\mathbf{M}_\varphi^{p,r}(X)$. As a result, we obtain a characterization of functions in BMO(X) via the $\mathbf{M}_\varphi^{p,r}(X)$ (resp. $\mathbf{B}_\varphi^{p',r'}(X)$) boundedness of commutators of Calderón–Zygmund type.

(iii) Thirdly, we prove a compactness characterization of commutators of Calderón–Zygmund type in $\mathbf{M}_\varphi^{p,r}(X)$.

Notation: In this paper, we denote by $\mathcal{C}_0^\infty(X)$ and $\mathcal{D}'(X)$, the space of infinitely differentiable functions with compact support and the space of distributions respectively. For any $q \in [1, \infty]$, we denote q' the conjugate exponent, $\frac{1}{q} + \frac{1}{q'} = 1$. Moreover, we denote B_t by a ball in X with radius $t > 0$.

As usual, we denote a constant by C, which may depend on p, r, n and may change at different lines. We also denote $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$. Finally, we denote $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Let us recall the definition of a space of homogeneous type, introduced by Coifman and Weiss [7]. Then (X, d, μ) is a space of homogeneous type if d is a quasi-metric on X and μ is a nonzero measure satisfying the doubling condition. A quasi-metric d on a set X is a function $d : X \times X \to [0, \infty)$ satisfying

(i) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(ii) $d(x, y) = 0$ if and only if $x = y$; and
(iii) the quasi-triangle inequality: there is a constant $A_0 \in [1, \infty)$ such that for all $x, y, z \in X$,

$$d(x, y) \le A_0 \left(d(x, z) + d(z, y)\right) . \tag{1.1}$$

We say that a nonzero measure μ satisfies the doubling condition if there is a constant C_μ only depending on μ, such that for all $x \in X$ and $t > 0$,

$$\mu(B(x, 2t)) \le C_\mu \, \mu(B(x, t)) < \infty , \tag{1.2}$$

where $B(x,t)$ is the quasi-metric ball by $B(x,t) = \{y \in X : d(x,y) < t\}$ for $x \in X$ and $t > 0$. We note that the doubling condition (1.2) implies that there exists a positive constant n (the upper dimension of μ) such that for all $x \in X$, $\lambda \geq 1$, and $t > 0$,

$$\mu\left(B(x,\lambda t)\right) \leq A_1 \lambda^n \mu\left(B(x,t)\right), \tag{1.3}$$

for some constant $A_1 > 0$.

We emphasize that the ball $B(x,t)$ may not be an open set. However, Macías–Segovia, [30] constructed a quasi-metric d', which is equivalent to d in the sense that

$$A_2^{-1} d(x,y) \leq d'(x,y) \leq A_2 d(x,y)$$

for some constant $A_2 > 0$ and for all $x, y \in X$. In addition, d' satisfies a regularity estimate of the type

$$|d'(x,z) - d'(y,z)| \leq A_3 d'(x,y)^\theta [d'(x,z) + d'(y,z)]^{1-\theta} \tag{1.4}$$

for some constant $A_3 > 0$, and for some $\theta \in (0,1]$. Then, the balls associated to d' are open sets. This fact allows us to work on the quasi-metric d having the same topology with d'.

Throughout this paper, we assume that $\mu(X) = \infty$ and that $\mu(\{x_0\}) = 0$ for every $x_0 \in X$. In addition, we also assume that the function $\varphi(t) : (0,\infty) \to (0,\infty)$ satisfies the following conditions:

$$\begin{cases} i)\ \varphi(t) \text{ is nonincreasing,} \\ ii)\ \mu\left(B_t\right) \varphi^p(t) \text{ is nondecreasing,}\ \text{ for any ball } B_t \subset X, \\ iii)\ \varphi(2t) \leq D\varphi(t),\ \ \forall t > 0, \end{cases} \tag{1.5}$$

for some constant $0 < D < 1$. Note that the last condition implies that $\varphi(t)$ cannot be a constant function.

Now, we define the generalized Morrey–Lorentz space. A real-valued function f is said to belong to the generalized Morrey–Lorentz space $\mathbf{M}_\varphi^{p,r}(X)$ provided the following norm is finite:

$$\|f\|_{\mathbf{M}_\varphi^{p,r}} = \sup_{B(x,t)} \frac{\|f\|_{L^{p,r}(B(x,t))}}{\mu\left(B(x,t)\right)^{\frac{1}{p}} \varphi(t)}, \tag{1.6}$$

where the *supremum* is taken over all the balls $B(x,t)$ in X, and $\|f\|_{L^{p,r}((B(x,t))}$ denotes the Lorentz norm of f on $B(x,t)$ (see e.g. [11] for more details of Lorentz spaces).

Remark 1.1 When $r = p$, we denote $\mathbf{M}_\varphi^{p,r}(X)$ as $\mathbf{M}_\varphi^{p}(X)$.

A canonical example is the following case: $X = \mathbb{R}^n$ equipped with the Lebesgue measure, and $\varphi(t) = t^{-\alpha}$, $\alpha \in (0, \frac{n}{p}]$. In this case, $\mathbf{M}_\varphi^{p}(\mathbb{R}^n)$ is the classical Morrey space.

It is known that Morrey spaces are generalizations of L^p-spaces, and they play a crucial role in studying the calculus of variations and the theory of elliptic PDE's (see e.g. [1, 5, 6, 12, 14–16, 31–36], and the references therein). Later, Campanato [5] extended the classical Morrey spaces by using the modified mean oscillation. As a matter of fact, this family of spaces includes the Morrey spaces, BMO spaces (the spaces of functions with bounded mean oscillation), and the Lipschitz spaces.

In [42], Zorko studied the generalized Campanato spaces $\mathcal{L}^p_\varphi(X)$. We say that $f \in \mathcal{L}^p_\varphi(X)$ if there is a constant $C_0 > 0$ such that for every ball $B(x,t)$ in X, we have

$$\inf_{P\in\mathcal{P}_k}\left(\frac{1}{\mu\left(B(x,t)\right)\varphi^p(t)}\int_{B(x,t)}|f(y)-P(y)|^p\,d\mu(y)\right)^{1/p}\le C_0\,, \tag{1.7}$$

where $\mathcal{P}_k$ is the class of polynomials of degree $\le k$.

If φ satisfies condition 1.5, then the author showed that the space $\mathcal{L}^p_\varphi(X)$ is independent of k. Furthermore, $\mathcal{L}^p_\varphi(\mathbb{R}^n)$ and $\mathbf{M}^p_\varphi(\mathbb{R}^n)$ describe the same space, and $\mathbf{M}^p_\varphi(\mathbb{R}^n)$ is the dual of certain atomic space $\mathbf{H}^p_\varphi(\mathbb{R}^n)$ for $p \in (1,\infty)$. We note that a celebrated result by Fefferman–Stein [20] is that the dual of $\mathbf{H}^1(\mathbb{R}^n)$ (the classical Hardy space) is $\mathrm{BMO}(\mathbb{R}^n)$. Thus, the Morrey space exhibits some similarity to $\mathrm{BMO}(\mathbb{R}^n)$, concerning duality.

For convenience, we recall here the definition of $\mathrm{BMO}(X)$.

Definition 1.2 A function $b \in L^1_{\mathrm{loc}}(X)$ belongs to $\mathrm{BMO}(X)$ if

$$\|b\|_{\mathrm{BMO}} := \sup_B \frac{1}{\mu(B)}\int_B |b(x)-b_B|\,d\mu(x) < \infty,$$

where

$$b_B = \frac{1}{\mu(B)}\int_B b(x)\,d\mu(x),$$

and the *supremum* is taken over all balls $B \subset X$.

Note that Alvarez, [2] defined $\mathbf{H}^p_\varphi(\mathbb{R}^n)$ in terms of suitable molecules, and used this result to prove the $\mathbf{H}^p_\varphi$-bound of linear Calderón–Zygmund operators satisfying the cancellation condition. By duality, he also obtained the boundedness of linear Calderón–Zygmund operators on $\mathbf{M}^p_\varphi(\mathbb{R}^n)$ space. It is known that such a linear operator of Calderón–Zygmund type does not map atoms into atoms, but it maps molecules into molecules. That is a reason why the author introduced the suitable molecules. We emphasize that his proof relies on the cancellation condition, so it cannot be applied to a general linear Calderón–Zygmund operators, such as the Cauchy integrals associated to the Lipschitz curves (see [12, 29, 38]). Then, one of the main purposes in this paper is to extend the boundedness result by Alvarez, [2] to the linear Calderón–Zygmund operators on $\mathbf{M}^{p,r}_\varphi(X)$ (see Theorem 1.11).

For convenience, let us recall the definition of linear Calderón–Zygmund operators.

Definition 1.3 We say that T is a Calderón–Zygmund operator on (X, d, μ) if T is bounded on $L^2(X)$ and has the associated kernel $K(x, y)$ such that

$$T(f)(x) = p.v. \int_X K(x, y) f(y)\, d\mu(y),$$

for any $x \notin \operatorname{supp}(f)$, and $K(x, y)$ satisfies the following estimates: for all $x \neq y$,

$$|K(x, y)| \leq \frac{C}{V(x, y)},$$

and for $d(x, z) \leq (2A_0)^{-1} d(x, y)$,

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq \frac{C}{V(x, y)} \left(\frac{d(x, z)}{d(x, y)}\right)^{\eta}, \qquad (1.8)$$

for some $\eta > 0$, where $V(x, y) = \mu(B(x, d(x, y)))$.

Note that by the doubling condition we have that $V(x, y) \approx V(y, x)$.

Definition 1.4 We say that T is homogeneous if for any ball $B(y, t)$ in X, T satisfies

$$\left|T\left(\mathbf{1}_{B(y,t)}\right)(x)\right| \geq \frac{\mu(B(y, t))}{\mu(B(x, Mt))}, \qquad (1.9)$$

for all $d(x, y) = Mt$, $M > 10A_0$.

A typical example of such operator is either the Riesz transforms, or the Cauchy integral operators associated with Lipschitz curves.

Next, we define a generalized block space $\mathbf{B}_{\varphi}^{p,r}(X)$.

Definition 1.5 Let $p \in (1, \infty)$, $r \in [1, \infty]$, and let $\varphi(t) : (0, \infty) \to (0, \infty)$. A function $b(x)$ is called a (p, r, φ)-block, if there exists a ball B_t in X such that

$$(i)\ \operatorname{supp}(b) \subset B_t,$$

$$(ii)\ \|b\|_{L^{p,r}(B_t)} \leq \frac{1}{\mu(B_t)^{\frac{1}{p'}} \varphi(t)}.$$

Next, we define space $\mathbf{B}_{\varphi}^{p,r}(X)$ via (p, r, φ)-block.

Definition 1.6 Let $p \in (1, \infty)$, $r \in [1, \infty]$, and let $\varphi(t)$ satisfy (1.5). We denote, by $\mathbf{B}_{\varphi}^{p',r'}(X)$, the family of distributions f that, in the sense of distributions, can be written as

$$f = \sum_{k=1}^{\infty} \lambda_k b_k,$$

where b_k is a (p, r, φ)-block and $\{\lambda_k\}_{k \geq 1} \in l^1$.

It is clear that $\mathbf{B}_{\varphi}^{p',r'}(X)$ is a vector space. In addition, we denote

$$\|f\|_{\mathbf{B}_{\varphi}^{p',r'}} = \inf \sum_{k=1}^{\infty} |\lambda_k|$$

where *infimum* is taken over all possible decompositions of f as above.

Then, $\left(\mathbf{B}_{\varphi}^{p',r'}(X), \|\cdot\|_{\mathbf{B}_{\varphi}^{p',r'}}\right)$ becomes a norm space.

For short, we denote $\mathbf{B}_{\varphi}^{p,p}(X)$ by $\mathbf{B}_{\varphi}^{p}(X)$ if $r = p$.

Remark 1.7 In $\mathbf{B}_{\varphi}^{p',r'}(X)$, the series $\sum_{k=1}^{\infty} \lambda_k b_k$ is convergent in $L^1(B_t)$ for any ball B_t in X, see Lemma 2.1.

This space was introduced by Blasco et al., [4] when $X = \mathbb{R}^n$, $r = p$, and $\varphi(t) = t^{-\alpha}$, $\alpha \in (0, \frac{n}{p}]$ in order to prove a non-interpolation result. Moreover, the authors also showed that $\mathbf{B}_{\varphi}^{p'}(\mathbb{R}^n)$ is a predual of $\mathbf{M}_{\varphi}^{p}(\mathbb{R}^n)$ for $p \in (1, \infty)$.

Our first result is the duality between $\mathbf{M}_{\varphi}^{p,r}(X)$ and $\mathbf{B}_{\varphi}^{p',r'}(X)$.

Theorem 1.8 *Let $p \in (1, \infty)$, $r \in [1, \infty]$, and let $\varphi(t)$ satisfy (1.5). Then, we have*

$$\left(\mathbf{B}_{\varphi}^{p',r'}(X)\right)' = \mathbf{M}_{\varphi}^{p,r}(X), \tag{1.10}$$

and

$$\mathbf{B}_{\varphi}^{p',r'}(X) = \mathbf{M}_{\varphi}^{p,r}(X)'. \tag{1.11}$$

Remark 1.9 As a consequence of Theorem 1.8, we observe that $\mathbf{B}_{\varphi}^{p',r'}(X)$ is a reflexive Banach space. Moreover, $\mathbf{M}_{\varphi}^{p,r}(X)$ is a Banach space. This fact will be used to deduce a compactness criterion in $\mathbf{M}_{\varphi}^{p,r}(X)$ under certain assumptions on X, see Lemma 5.1.

Next, we define a commutator of linear operator of Calderón–Zygmund type.

Definition 1.10 Let T be a linear Calderón–Zygmund operator. Suppose that $b \in L^1_{\text{loc}}(X)$. Then, the commutator $[b, T]$ is defined by

$$[b, T]f(x) := b(x)T(f)(x) - T(bf)(x)$$

for suitable functions f.

It is known that the theory of commutators has been generalized to other contexts, and it has many important applications to some nonlinear partial differential equations (see e.g. [8, 16, 23] and the references therein). When T is a linear Calderón–Zygmund operator, the L^p boundedness of $[b, T]$ was first proved by Coifman–Rochberg–Weiss [9]. After that this result has been developed by many authors in [3, 12, 13, 15, 17–19, 24, 26, 29, 38, 40] and the references therein. In [40], Uchiyama obtained the compactness of operators of Hankel type. Furthermore, Beatrous–Li [3] proved a boundedness and compactness characterization for $[b, T]$ on $L^p(X)$, where X is a space of homogeneous type, and some applications to Hankel type operators on Bergman spaces were given by the authors in [10–12, 26].

When T is the Cauchy integral operator associated with the Lipschitz curves, the L^p boundedness and compactness characterization of $[b, T]$ was obtained in [29]. Moreover, Tao et al. [38] extended this result to Morrey spaces (see also [13] for the Lorentz boundedness and compactness characterization of $[b, T]$). It is obvious that such an operator mentioned above is associated with a kernel K of Calderón–Zygmund type satisfying the homogeneity, i.e:

There exist positive constants c_0 and $\tilde{C}$ such that for every $x \in X$ and $r > 0$, there exists $y \in B(x, \tilde{C}r)\backslash B(x, r)$ satisfying

$$|K(x, y)| \geq \frac{c_0}{\mu\,(B(x, r))}\,. \tag{1.12}$$

In this case, Duong et al. [19] established the two weight commutator theorem of Calderón–Zygmund operators in the sense of Coifman–Weiss on spaces of homogeneous type. As applications, they can obtain a two weight commutator theorem for the following Calderón–Zygmund operators: Cauchy integral operator on $\mathbb{R}$, Cauchy–Szegö projection operator on Heisenberg groups, Szegö projection operators on a family of unbounded weakly pseudoconvex domains, the Riesz transform associated with the sub-Laplacian on stratified Lie groups, as well as the Bessel Riesz transforms.

Next, we discuss the Hardy factorization in terms of the commutators. A famous result of Coifman–Rochberg–Weiss [9] is that every $f \in \mathbf{H}^1(\mathbb{R}^n)$ can be written as

$$f = \sum_{k=1}^{\infty}\sum_{j=1}^{n}\left(h_{k,j}\mathcal{R}_j(g_{k,j}) + g_{k,j}\mathcal{R}_j(h_{k,j})\right)$$

with

$$\sum_{k=1}^{\infty}\sum_{j=1}^{n}\left\|g_{k,j}\right\|_{L^2(\mathbb{R}^n)}\left\|h_{k,j}\right\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{\mathbf{H}^1(\mathbb{R}^n)},$$

where $\mathcal{R}_j$ are the Riesz transform for $j = 1, \ldots, n$.

As a consequence, the authors obtained a characterization of functions b in $\mathrm{BMO}(\mathbb{R}^n)$ through the L^2 boundedness of $[b, \mathcal{R}_j]$, $j = 1, \ldots, n$. This theory has been studied by many authors in [13, 14, 18, 25, 28, 41], and the references cited therein. For instance, Uchiyama [41] extended the Hardy factorization to $\mathbf{H}^p$ on the space of homogeneous type. In addition, Komori–Mizuhara [25] proved the weak $\mathbf{H}^1$ factorization in terms of the commutators of Calderoón–Zygmund type in generalized Morrey spaces. We do not forget to mention that a weak Hardy factorization for the Bessel operators was obtained by the authors in [18]. Recently, the first author and Wick [14] proved a weak Hardy factorization in terms of multi-linear operator in Morrey spaces.

Inspired by the above results, we would like to generalize the theory of commutators to the generalized Morrey–Lorentz spaces, and the Block spaces. Concerning the $\mathbf{M}_{\varphi}^{p,r}(X)$ boundedness of commutators in Definition 1.10, we have the following theorem.

Theorem 1.11 *Assume the same hypotheses as in Theorem 1.8. If* $b \in \mathrm{BMO}(X)$, *and* T *is a linear Calderón–Zygmund operator in Definition 1.3, then* $[b, T]$ *maps* $\mathbf{M}_{\varphi}^{p,r}(X)$ *into itself continuously. Moreover, we have*

$$\|[b, T](f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|b\|_{\mathrm{BMO}} \|f\|_{\mathbf{M}_{\varphi}^{p,r}}, \quad \forall f \in \mathbf{M}_{\varphi}^{p,r}(X).$$

Remark 1.12 By duality in Theorem 1.8, we have that $[b, T]$ also maps $\mathbf{B}_{\varphi}^{p',r'}(X) \to \mathbf{B}_{\varphi}^{p',r'}(X)$ continuously.

Our next result is the weak Hardy factorization in terms of commutators in $\mathbf{B}_{\varphi}^{p',r'}(X)$ and $\mathbf{M}_{\varphi}^{p,r}(X)$.

Theorem 1.13 *Let* $p \in (1, \infty)$, $r \in [1, \infty]$, *and let* $\varphi(t)$ *satisfy* (1.5). *Suppose that* T *is a homogeneous operator of Calderón–Zygmund type. Then, for every function* $f \in \mathbf{H}^1(X)$, *there exist sequences* $\{\lambda_{k,j}\} \in l^1$ *and functions* $\{g_{k,j}\}, \{h_{k,j}\} \subset L_c^{\infty}(X)$ *(the space of bounded functions with compact support), such that*

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{k,j} \left(h_{k,j} T^*(g_{k,j}) - g_{k,j} T(h_{k,j}) \right) \tag{1.13}$$

in the sense of $\mathbf{H}^1(X)$. *In addition, we have that*

$$\|f\|_{\mathbf{H}^1} \approx \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_{k,j}| \left\| g_{k,j} \right\|_{\mathbf{B}_{\varphi}^{p',r'}} \left\| h_{k,j} \right\|_{\mathbf{M}_{\varphi}^{p,r}} \right\},$$

where the infimum above is taken over all possible representations of f *that satisfy* (1.13).

We prove this result in Sect. 4.

Remark 1.14 Note that our assumption on the homogeneity of operator T in Theorem 1.13 is weaker than (1.12) used in [19, 25].

As a consequence of Theorem 1.13, we obtain a characterization of functions in $\mathrm{BMO}(X)$ via the $\mathbf{M}_{\varphi}^{p,r}$ (resp. $\mathbf{B}_{\varphi}^{p',r'}$) boundedness of commutators of Calderón–Zygmund types.

Corollary 1.15 *Let* $p \in (1, \infty)$, $r \in [1, \infty]$, *and let* $\varphi(t)$ *satisfy* (1.5). *Suppose that* T *is a linear Calderón–Zygmund operator. If* $b \in \mathrm{BMO}(X)$, *then the commutator* $[b, T]$ *maps* $\mathbf{M}_{\varphi}^{p,r}(X)$ *into* $\mathbf{M}_{\varphi}^{p,r}(X)$ *continuously. Moreover, it holds true that*

$$\|[b, T]\|_{\mathbf{M}_{\varphi}^{p,r} \to \mathbf{M}_{\varphi}^{p,r}} \leq C \|b\|_{\mathrm{BMO}}.$$

Conversely, for $b \in L^1_{\mathrm{loc}}(X)$*, if* T *is homogeneous, and* $[b, T]$ *maps* $\mathbf{M}^{p,r}_{\varphi}(X) \to \mathbf{M}^{p,r}_{\varphi}(X)$ *continuously, then* $b \in \mathrm{BMO}(X)$*, and*

$$\|b\|_{\mathrm{BMO}} \le C\,\|[b, T]\|_{\mathbf{M}^{p,r}_{\varphi} \to \mathbf{M}^{p,r}_{\varphi}}\,.$$

Remark 1.16 By duality, the result of Corollary 1.15 also holds for $\mathbf{B}^{p',r'}_{\varphi}(X)$ in place of $\mathbf{M}^{p,r}_{\varphi}(X)$.

The last result is a $\mathbf{M}^{p,r}_{\varphi}$-compactness characterization in terms of $[b, T]$. Since our assumptions on the homogeneous space (X, d, μ) are quite general, then we have to make some additional assumptions to (X, d, μ). Concerning the compactness, we suppose that the homogeneous space (X, d, μ) is a vector space, and is a locally compact space such that

$$d(x+z, y) \approx d(z, y-x), \quad \text{for all } x, y, z \in X\,, \tag{1.14}$$

and

$$\mu\,(B(x,t)) \approx t^n, \quad \forall x \in X\,. \tag{1.15}$$

A typical example of such space is the Euclidean space, equipped with the Lebesgue measure. In addition, we also note that any Ahlfors n-regular metric measure space (X, d, μ) satisfies (1.15).

Then, we have the following theorem.

Theorem 1.17 *Let* $p \in (1, \infty)$*,* $r \in [1, \infty]$*, and let* $\varphi(t)$ *satisfy* (1.5)*. Assume that* (X, d, μ) *is a locally compact space such that* (1.14) *and* (1.15) *hold. Then the following statements hold true.*

If $b \in \mathrm{CMO}(X)$*, and* T *is a linear Calderón–Zygmund operator, then* $[b, T]$ *is compact on* $\mathbf{M}^{p,r}_{\varphi}(X)$*.*

Conversely, for $b \in L^1_{\mathrm{loc}}(X)$*, if* T *is homogeneous, and* $[b, T]$ *is a compact operator on* $\mathbf{M}^{p,r}_{\varphi}(X)$*, then* $b \in \mathrm{CMO}(X)$*.*

To end this section, we list some operators of Calderón–Zygmund type that our results are applicable to: the Cauchy integral operators, the Cauchy-Szegö projection operator on the Heisenberg group $\mathbb{H}^n$, the Szegö projection operator on a family of unbounded weakly pseudo-convex domains, the Riesz transforms associated with sub-Laplacian on stratified nilpotent Lie groups, the Riesz transform associated with the Bessel operator on $\mathbb{R}_+$, the Riesz transforms associated with Bessel operators on $\mathbb{R}^{n+1}_+$. We refer to [19] for the details of these operators.

Finally, we emphasize that our results extend the boundedness and compactness characterization of linear Calderón–Zygmund operators to the generalized Morrey–Lorentz spaces, and Block spaces.

2 Generalied Morrey–Lorentz Space as Dual of Block Space

In this part, we study some properties of the Morrey–Lorentz spaces and the block spaces.

Lemma 2.1 *Let $p \in (1, \infty)$, $r \in [1, \infty]$, and let $\varphi(t)$ satisfy (1.5). Then, we have*

$$\mathbf{B}_{\varphi}^{p',r'}(X) \hookrightarrow L^1_{\text{loc}}(X).$$

Proof of Lemma 2.1 It suffices to show that for any ball $B(x,t) \subset X$, we have

$$\|f\|_{L^1(B(x,t))} \leq C\|f\|_{\mathbf{B}_{\varphi}^{p',r'}}, \quad \forall f \in \mathbf{B}_{\varphi}^{p',r'}(X)\,, \tag{2.1}$$

where constant $C > 0$ is independent of f.

Indeed, let b be a (p', r', φ)-block. Suppose that $\operatorname{supp}(b) \subset B(z,\tau)$, for some ball $B(z,\tau)$ in X. Then, applying Hölder's inequality in Lorentz spaces yields

$$\|b\|_{L^1(B(x,t))} \lesssim \|b\|_{L^{p',r'}}\mu\left(B(z,\tau)\cap B(x,t)\right)^{\frac{1}{p}} \leq \frac{\mu\left(B(z,\tau)\cap B(x,t)\right)^{\frac{1}{p}}}{\mu(B(z,\tau))^{\frac{1}{p}}\varphi(\tau)}\,. \tag{2.2}$$

If $\tau \geq t$, then since $\mu(B(z,\tau))^{\frac{1}{p}}\varphi(\tau)$ is nondecreasing, then we deduce from (2.2) that

$$\|b\|_{L^1(B(x,t))} \lesssim \frac{\mu\left(B(z,\tau)\cap B(x,t)\right)^{\frac{1}{p}}}{\mu(B(z,t))^{\frac{1}{p}}\varphi(t)} \leq \frac{1}{\varphi(t)}.$$

Otherwise, we have $\varphi(\tau) \geq \varphi(t)$. Therefore

$$\|b\|_{L^1(B(x,t))} \lesssim \frac{\mu\left(B(z,\tau)\cap B(x,t)\right)^{\frac{1}{p}}}{\mu(B(z,\tau))^{\frac{1}{p}}\varphi(t)} \leq \frac{1}{\varphi(t)}.$$

As a result, we obtain from (2.2) that

$$\|b\|_{L^1(B(x,t))} \lesssim \frac{1}{\varphi(t)}\,. \tag{2.3}$$

Now, for any $f \in \mathbf{B}_{\varphi}^{p',r'}(X)$, we can write

$$f = \sum_{k\geq 1}\lambda_k b_k,$$

where $\{b_k\}_{k\geq 1}$ is a sequence of (p', r', φ)-blocks, and $\sum_{k\geq 1}|\lambda_k| < \infty$.

Thanks to (2.3), we get

$$\|f\|_{L^1(B(x,t))} \leq \sum_{k\geq 1}|\lambda_k|\|b_k\|_{L^1(B(x,t))} \lesssim \sum_{k\geq 1}|\lambda_k|\frac{1}{\varphi(t)} \leq \frac{\|f\|_{\mathbf{B}_{\varphi}^{p',r'}}}{\varphi(t)}\,.$$

This yields (2.1).

Hence, we obtain Lemma 2.1. □

Remark 2.2 As a consequence of Lemma 2.1, for every ball $B(x,t)\subset X$ the series $\sum_{j=1}^{\infty}\lambda_j b_j$ is convergent in $L^1(B(x,t))$ whenever $\sum_{j=1}^{\infty}|\lambda_j|<\infty$, and $\{b_j\}_{j\geq 1}$ is a sequence of (p',r',φ)-blocks.

Proposition 2.3 *Let $p\in(1,\infty)$, $r\in[1,\infty]$, and let $\varphi(t)$ satisfy (1.5). Then, for every ball $B(z,t)$ in X we have*

$$\left\|\mathbf{1}_{B(z,t)}f\right\|_{\mathbf{B}_{\varphi}^{p',r'}}\leq\mu(B(z,t))^{\frac{1}{p}}\varphi(t)\left\|\mathbf{1}_{B(z,t)}f\right\|_{L^{p',r'}}\tag{2.4}$$

for $f\in L_{\rm loc}^{p',r'}(X)$.

In addition, we have

$$\left\|\mathbf{1}_{B(z,t)}\right\|_{\mathbf{M}_{\varphi}^{p,r}}\approx\frac{1}{\varphi(t)},\tag{2.5}$$

and

$$\left\|\mathbf{1}_{B(z,t)}\right\|_{\mathbf{B}_{\varphi}^{p',r'}}\approx\mu\left(B(z,t)\right)\varphi(t).\tag{2.6}$$

Proof of Proposition 2.3 The proof of (2.4) is done by letting

$$b(x)=\frac{\mathbf{1}_{B(z,t)}f(x)}{\mu(B(z,t))^{\frac{1}{p}}\varphi(t)\left\|\mathbf{1}_{B(z,t)}f\right\|_{L^{p',r'}}}.$$

Next, we prove (2.5). By (1.6), we can mimic the proof of a) in Lemma 2.1 in order to obtain

$$\left\|\mathbf{1}_{B(z,t)}\right\|_{\mathbf{M}_{\varphi}^{p,r}}=\sup_{B(x,\tau)\subset X}\frac{\|\mathbf{1}_{B(z,t)}\|_{L^{p,r}(B(x,\tau))}}{\mu\left(B(x,\tau)\right)^{\frac{1}{p}}\varphi(\tau)}\lesssim\frac{1}{\varphi(t)}.$$

On the other hand, it is obvious that

$$\sup_{B(x,\tau)\subset X}\frac{\|\mathbf{1}_{B(z,t)}\|_{L^{p,r}(B(x,\tau))}}{\mu\left(B(x,\tau)\right)^{\frac{1}{p}}\varphi(\tau)}\gtrsim\frac{1}{\varphi(t)}.$$

Thus, we obtain the desired result.

Concerning (2.6), it follows from duality and (2.5) that

$$\begin{aligned}\left\|\mathbf{1}_{B(z,t)}\right\|_{\mathbf{B}_{\varphi}^{p',r'}}&=\sup_{\|g\|_{\mathbf{M}_{\varphi}^{p,r}}\leq 1}\left|\int_X\mathbf{1}_{B(z,t)}g(x)\,d\mu(x)\right|\\&\geq\left|\int_{B(z,t)}g_0(x)\,d\mu(x)\right|\approx\mu\left(B(z,t)\right)\varphi(t),\end{aligned}$$

with $g_0=c_0\mathbf{1}_{B(z,t)}\varphi(t)$, and c_0 is a normalized constant such that $\|g\|_{\mathbf{M}_{\varphi}^{p,r}}=1$.

With the last inequality noted, and by applying (2.4) with $f\equiv 1$, we obtain (2.6). □

The next result is a dual inequality.

Proposition 2.4 *Same hypotheses as in Proposition 2.3. If $f \in \mathbf{M}_{\varphi}^{p,r}(X)$, and $g \in \mathbf{B}_{\varphi}^{p',r'}(X)$, then*

$$\left|\int_X f(x)g(x)\,d\mu(x)\right| \leq \|f\|_{\mathbf{M}_{\varphi}^{p,r}} \|g\|_{\mathbf{B}_{\varphi}^{p',r'}}.$$

Proof of Proposition 2.4 Since $g \in \mathbf{B}_{\varphi}^{p',r'}(X)$, then we can write

$$g(x) = \sum_{j=1}^{\infty} \lambda_j b_j(x),$$

where $\{\lambda_j\}_{j\geq 1} \in l^1$, and $\{b_j\}_{j\geq 1}$ are (p', r', φ)-blocks.

Assume that $\operatorname{supp}(b_j) \subset B_j$ with its radius R_j, $j \geq 1$. Then, applying Hölder's inequality yields

$$\begin{aligned}
\left|\int_X f(x)g(x)\,d\mu(x)\right| &= \left|\sum_{j=1}^{\infty} \lambda_j \int_{B_j} f(x)b_j(x)\,d\mu(x)\right| \\
&\lesssim \sum_{j=1}^{\infty} |\lambda_j|\, \|f\|_{L^{p,r}(B_j)} \|b_j\|_{L^{p',r'}(B_j)} \\
&= \sum_{j=1}^{\infty} |\lambda_j| \frac{\|f\|_{L^{p,r}(B_j)}}{\mu\left(B_j\right)^{1/p} \varphi(R_j)} \left[\|b_j\|_{L^{p',r'}(B_j)}\, \mu(B_j)^{1/p} \varphi(R_j)\right] \\
&\leq \left(\sum_{j=1}^{\infty} |\lambda_j|\right) \|f\|_{\mathbf{M}_{\varphi}^{p,r}}.
\end{aligned}$$

This yields the proof of Proposition 2.4. □

Next, we study the Fatou property of block spaces $\mathbf{B}_{\varphi}^{p',r'}(X)$. Such a result was obtained by the authors, [37] for $\mathbf{B}_{\varphi}^{p'}(X)$, $\varphi(t) = t^{-n+\alpha}$, $\alpha \in (0, \frac{n}{p})$.

Lemma 2.5 *Let $1 < p < \infty$, $r \in [1, \infty]$, and let φ satisfy (1.5). Suppose that f and f_k, $k \geq 1$, are nonnegative, $\|f_k\|_{\mathbf{B}_{\varphi}^{p',r'}} \leq 1$, and $f_k(x) \uparrow f(x)$ for a.e. $x \in X$. Then $f \in \mathbf{B}_{\varphi}^{p',r'}(X)$ and $\|f\|_{\mathbf{B}_{\varphi}^{p',r'}} \leq 1$.*

Proof Note that the dyadic cubes were constructed by the authors, [22]. Thus, the proof of Lemma 2.5 follows by using the same argument as in the proof of Theorem 1.2, [37] in that one can replace the L^p-norm by the $L^{p,r}$-norm. □

Now we have the tools to prove Theorem 1.8.

Proof of Theorem 1.8 We first show that

$$\mathbf{M}_\varphi^{p,r}(X) \hookrightarrow \mathbf{B}_\varphi^{p',r'}(X)'. \tag{2.7}$$

In fact, thanks to Proposition 2.4, we observe that operator $\mathcal{T}_f : \mathbf{B}_\varphi^{p',r'}(X) \to \mathbb{R}$ defined by

$$\mathcal{T}_f(g) = \int_X f(x)g(x)\,dx$$

is linear and continuous.

Now, we define $\mathcal{T}(f) = \mathcal{T}_f$ for $f \in \mathbf{M}_\varphi^{p,r}(X)$. It is obvious that $\mathcal{T} : \mathbf{M}_\varphi^{p,r}(X) \to \mathbf{B}_\varphi^{p',r'}(X)'$ is a linear operator. We claim that $\mathcal{T}$ is injective.

To obtain the result, it suffices to show that if $\mathcal{T}(f) = 0$, then $f(x) = 0$ for a.e $x \in X$. We argue with a contradiction that there is $R_0 > 0$ such that $f(x) \neq 0$ for a.e. $x \in B_{R_0} = B(0, R_0)$.

Since $f \in \mathbf{M}_\varphi^{p,r}(X)$, then we have $f \in L^{p,r}(B_{R_0})$. By duality, there exits a function $\bar{g} \in L^{p',r'}(B_{R_0})$, $\bar{g} \neq 0$ such that

$$0 < \|f\|_{L^{p',r'}(B_{R_0})} = \left| \int_{B_{R_0}} f(x)\bar{g}(x)\,dx \right|. \tag{2.8}$$

Put

$$g(x) = \frac{\bar{g}(x)\mathbf{1}_{B_{R_0}}}{\mu(B_{R_0})^{\frac{1}{p}}\varphi(R_0)} \|\bar{g}\|_{L^{p',r'}(B_{R_0})}.$$

It is clear that g is a (p', r', φ)-block. By this fact and (2.8), we obtain

$$|\mathcal{T}(f)(g)| = \left| \int_{B_{R_0}} f(x)g(x)\,dx \right| > 0,$$

which contradict to $\mathcal{T}(f) = 0$ in $\mathbf{B}_\varphi^{p',r'}(X)'$.

Thus, we conclude that linear operator $\mathcal{T} : \mathbf{M}_\varphi^{p,r}(X) \to \mathbf{B}_\varphi^{p',r'}(X)'$ is injective. As a result, (2.7) follows.

Therefore, it remains to prove that

$$\left(\mathbf{B}_\varphi^{p',r'}(X)\right)' \hookrightarrow \mathbf{M}_\varphi^{p,r}(X). \tag{2.9}$$

Indeed, let $F \in \left(\mathbf{B}_\varphi^{p',r'}(X)\right)'$, and let B be a ball in X. By Proposition 2.3, we have $F\mathbf{1}_B \in L^{p',r'}(X)$. Thus, by the duality, there exists $f_B \in L^{p,r}(X)$ such that

$$\langle F\mathbf{1}_B, g\rangle_{L^{p',r'},L^{p,r}} = \int_X f_B(x)g(x)\,d\mu(x),\ \forall g \in L^{p',r'}(X)\,. \tag{2.10}$$

Let $X = \bigcup_{k\geq 1} B_k$, with $B_k \Subset B_{k+1}$ for all $k \geq 1$. Then, we define $f(x) = f_{B_k}$ if $x \in B_k$, which makes sense by (2.10). This implies that $f_{B_k}(x) = f_{B_{k+1}}(x)$ for a.e. $x \in B_k$.

Thus, it suffices to prove that $f \in \mathbf{M}_\varphi^{p,r}(X)$. Indeed, for any ball $B_t \subset X$, there exists $k_0 \geq 1$ such that $B_t \subset B_{k_0}$; and by the duality argument we have

$$\begin{aligned}\frac{\|f\|_{L^{p,r}(B_t)}}{\mu(B_t)^{1/p}\varphi(t)} &= \frac{1}{\mu(B_t)^{1/p}\varphi(t)} \sup_{\|g\|_{L^{p',r'}(B_t)}=1} \left|\int_{B_t} f(x)g(x)\,d\mu(x)\right| \\ &= \sup_{\|g\|_{L^{p',r'}(B_t)}=1} \int_{B_{k_0}} f_{B_{k_0}} \frac{g\mathbf{1}_{B_t}}{\mu(B_t)^{1/p}\varphi(t)}\,d\mu(x)\,.\end{aligned}$$

Obviously, $\frac{g\mathbf{1}_{B_t}}{\mu(B_t)^{1/p}\varphi(t)}$ is a (p', r', φ)-block. Thus, it follows from the last inequality that

$$\frac{\|f\|_{L^{p,r}(B_t)}}{\mu(B_t)^{1/p}\varphi(t)} \leq \|F\|_{(\mathbf{B}_\varphi^{p',r'})'} \left\|\frac{g\mathbf{1}_{B_t}}{\mu(B_t)^{1/p}\varphi(t)}\right\|_{\mathbf{B}_\varphi^{p',r'}} \leq \|F\|_{(\mathbf{B}_\varphi^{p',r'})'}\,.$$

Therefore,

$$\|f\|_{\mathbf{M}_\varphi^{p,r}} \leq \|F\|_{(\mathbf{B}_\varphi^{p',r'})'}\,.$$

This implies that $\mathbf{M}_\varphi^{p,r}(X) = \mathbf{B}_\varphi^{p',r'}(X)'$.

Next, we prove (1.11).

It follows from (1.10) that $\mathbf{B}_\varphi^{p',r'}(X) \hookrightarrow \mathbf{M}_\varphi^{p,r}(X)' = \mathbf{B}_\varphi^{p',r'}(X)''$. Thus, it is enough to show that

$$\mathbf{M}_\varphi^{p,r}(X)' \hookrightarrow \mathbf{B}_\varphi^{p',r'}(X)\,. \tag{2.11}$$

To obtain (2.11), we mimic the proof of Theorem 4.1, [37]. Assume that a measurable function f on X satisfies

$$\sup\left\{\left|\int_X f(x)g(x)\,dx\right| : \|g\|_{\mathbf{M}_\varphi^{p,r}} \leq 1\right\} \leq 1\,. \tag{2.12}$$

It is obvious that $|f(x)| < \infty$ for a.e. $x \in X$. Assume without loss of generality that $f \geq 0$ in X, if not we write $f = f_+ - f_-$, with $f_+ = \max\{f, 0\}$, and $f_- = \max\{-f, 0\}$; and we treat each of them.

For every $k \geq 1$, let us set $B_k = B(z_0, k)$ for some $z_0 \in X$, and let $f_k(x) := \min\{f(x), k\}\mathbf{1}_{B_k}(x)$. Note that $f_k(x) \uparrow f(x)$ for a.e. $x \in X$. Since $f_k \in L_c^\infty(X)$

(the space of bounded functions with compact support in X), then it is clear that $f_k \in \mathbf{B}_\varphi^{p',r'}(X)$.

Since $\mathbf{M}_\varphi^{p,r}(X) = \mathbf{B}_\varphi^{p',r'}(X)'$, and by (2.12), we apply the Hahn–Banach theorem to obtain

$$\|f_k\|_{\mathbf{B}_\varphi^{p',r'}} = \sup_{\|g\|_{\mathbf{M}_\varphi^{p,r}} \le 1} \left| \int_X f_k(x) g(x)\, dx \right| \le \sup_{\|g\|_{\mathbf{M}_\varphi^{p,r}} \le 1} \left| \int_X f(x) g(x)\, dx \right| \le 1\,.$$

Thanks to Lemma 2.5, we deduce $\|f\|_{\mathbf{B}_\varphi^{p',r'}} \le 1$.

By duality and (2.12), we obtain $\|f\|_{(\mathbf{M}_\varphi^{p,r})'} \ge \|f\|_{\mathbf{B}_\varphi^{p',r'}}$, which yields (2.11).

Hence, we have completed the proof of Theorem 1.8. □

3 Several Lemmas

This part is devoted to the study of the $\mathbf{M}_\varphi^{p,r}(X)$-bounds, and the $\mathbf{B}_\varphi^{p,r}(X)$-bounds of the maximal function, the sharp function, and the linear Calderón–Zygmund operators.

Let us first recall the definition of the Hardy–Littlewood maximal function.

For any $q > 0$, we define

$$\mathcal{M}_q f(x) = \sup_{x \in B} \left\{ \frac{1}{|B|^{\frac{1}{q}}} \|f\|_{L^q(B)} \right\},$$

for any $x \in X$, where the *supremum* is taken over all balls B containing x.

In brief, we denote $\mathcal{M} = \mathcal{M}_1$.

Lemma 3.1 *Let $p \in (1, \infty)$, $r \in [1, \infty]$, and let $\varphi(t)$ satisfy (1.5). Then, for any $0 < q < p$ there is a positive constant $C = C(q, p, r)$ such that*

$$\left\|\mathcal{M}_q(f)\right\|_{\mathbf{M}_\varphi^{p,r}} \le C \|f\|_{\mathbf{M}_\varphi^{p,r}}\,. \tag{3.1}$$

Proof of Lemma 3.1 Let $B_t = B(x, t)$ be a ball in X, and we write

$$f = f\mathbf{1}_{B_t} + f\mathbf{1}_{B_t^c} := f_1 + f_2.$$

So,

$$\mathcal{M}_q(f)(x) \le \mathcal{M}_q(f_1)(x) + \mathcal{M}_q(f_2)(x), \quad \forall x \in X.$$

We first estimate $\mathcal{M}_q(f_1)$. Since $\mathcal{M}_q$ maps $L^{p,r}(X) \to L^{p,r}(X)$ (see e.g. [11]), then we get

$$\frac{1}{\mu(B_t)^{\frac{1}{p}}\varphi(t)}\|\mathcal{M}_q(f_1)\|_{L^{p,r}(B_t)} \lesssim \frac{1}{\mu(B_t)^{\frac{1}{p}}\varphi(t)}\|f_1\|_{L^{p,r}} = \frac{1}{\mu(B_t)^{\frac{1}{p}}\varphi(t)}\|f\|_{L^{p,r}(B_t)} \leq \|f\|_{\mathbf{M}_\varphi^{p,r}},$$

which yields

$$\|\mathcal{M}_q(f_1)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|f\|_{\mathbf{M}_\varphi^{p,r}}. \tag{3.2}$$

Next, since $f_2 = 0$ in B_t, then we observe that for any $z \in B_{t/2}$,

$$\mathcal{M}_q(f_2)(z) \leq \sup_{z\in B_\delta, \delta>t/8}\left(\frac{1}{\mu\left(B_\delta\right)}\int_{B_\delta}|f(y)|^q\, d\mu(y)\right)^{1/q}.$$

Applying Hölder's inequality in Lorentz spaces yields

$$\begin{aligned}\mathcal{M}_q(f_2)(z) &\lesssim \sup_{z\in B_\delta, \delta>t/8}\left\{\mu\left(B_\delta\right)^{-1/q}\|f\|_{L^{p,r}(B_\delta)}\,\mu\left(B_\delta\right)^{1/q-1/p}\right\}\\ &\lesssim \sup_{z\in B_\delta, \delta>t/8}\left\{\mu\left(B_\delta\right)^{-1/p}\|f\|_{L^{p,r}(B_\delta)}\right\}\\ &\lesssim \sup_{z\in B_\delta, \delta>t/8}\left\{\varphi(\delta)\,\|f\|_{\mathbf{M}_\varphi^{p,r}}\right\} \leq \varphi(t/8)\,\|f\|_{\mathbf{M}_\varphi^{p,r}}.\end{aligned}$$

Note that the last inequality follows from the monotonicity of $\varphi(t)$.

Then, we obtain

$$\left\|\mathcal{M}_q(f_2)\right\|_{L^{p,r}(B_{t/8})} \lesssim \mu\left(B_{t/8}\right)^{\frac{1}{p}}\varphi(t/8)\,\|f\|_{\mathbf{M}_\varphi^{p,r}}.$$

This implies that

$$\left\|\mathcal{M}_q(f_2)\right\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|f\|_{\mathbf{M}_\varphi^{p,r}}. \tag{3.3}$$

By combining (3.2) and (3.3), we obtain Lemma 3.1. □

As a consequence of Lemma 3.1, and duality, we obtain the $\mathbf{B}_\varphi^{p',r'}$-bound for $\mathcal{M}$.

Corollary 3.2 *Assume hypotheses as in Lemma 3.1. Then, $\mathcal{M}_q$ maps $\mathbf{B}_\varphi^{p',r'}(X) \to \mathbf{B}_\varphi^{p',r'}(X)$ continuously.*

Proof of Corollary 3.2 For any $f \in \mathbf{B}_\varphi^{p',r'}(X)$, we can assume that $f \geq 0$.

Thanks to Lemma 3.1, applying the Fefferman–Stein inequality (see [21]), and Hölder's inequality yields

$$\begin{aligned}\int_X \mathcal{M}_q(f)(x)g(x)\, d\mu(x) &\lesssim \int_X f(x)\mathcal{M}_q(g)(x)\, d\mu(x)\\ &\lesssim \|f\|_{\mathbf{B}_\varphi^{p',r'}}\|\mathcal{M}_q(g)\|_{\mathbf{M}_\varphi^{p,r}}\\ &\lesssim \|f\|_{\mathbf{B}_\varphi^{p',r'}}\|g\|_{\mathbf{M}_\varphi^{p,r}}.\end{aligned}$$

By duality, we obtain from the last inequality that

$$\left\|\mathcal{M}_q(f)\right\|_{\mathbf{B}_\varphi^{p',r'}} = \sup_{\|g\|_{\mathbf{M}_\varphi^{p,r}}\le 1} \int_X \mathcal{M}_q(f)(x)g(x)\,d\mu(x)$$
$$\lesssim \sup_{\|g\|_{\mathbf{M}_\varphi^{p,r}}\le 1} \|f\|_{\mathbf{B}_\varphi^{p',r'}} \|g\|_{\mathbf{M}_\varphi^{p,r}} = \|f\|_{\mathbf{B}_\varphi^{p',r'}} .$$

Hence, we get the conclusion. □

Next we prove the $\mathbf{M}_\varphi^{p,r}$-bound for the sharp maximal function, introduced by Fefferman–Stein, [21]

$$\mathbf{M}^\sharp(f)(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y) - f_B|\,d\mu(y),$$

where the *supremum* is taken over all balls B containing x.

Lemma 3.3 *Let $p \in (1,\infty)$, $r \in [1,\infty]$, and let $\varphi(t)$ satisfy (1.5). Then, for any $f \in \mathbf{M}_\varphi^{p,r}(X)$, we have that*

$$\|f\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|\mathbf{M}^\sharp(f)\|_{\mathbf{M}_\varphi^{p,r}} . \tag{3.4}$$

Proof of Lemma 3.3 The conclusion will follow by way of the Fefferman–Stein inequality and duality. Indeed, there is a constant $C = C(X) > 0$ such that, for every $f \in \mathbf{M}_\varphi^{p,r}(X)$, and every function $g \in L^1_{\mathrm{loc}}(X)$, we have (see e.g. [27])

$$\int_X |f(x)g(x)|\,d\mu(x) \le C \int_X \mathbf{M}^\sharp(f)(x)\mathcal{M}(g)(x)\,d\mu(x) .$$

Thanks to duality and Corollary 3.2, we obtain

$$\|f\|_{\mathbf{M}_\varphi^{p,r}} = \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}}=1} \left|\int f(x)g(x)\,d\mu(x)\right| \le \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}}=1} C\int \mathbf{M}^\sharp(f)(x)\mathcal{M}(g)(x)\,d\mu(x)$$
$$\lesssim \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}}=1} \left\|\mathbf{M}^\sharp(f)\right\|_{\mathbf{M}_\varphi^{p,r}} \|\mathcal{M}(g)\|_{\mathbf{B}_\varphi^{p',r'}}$$
$$\lesssim \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}}=1} \left\|\mathbf{M}^\sharp(f)\right\|_{\mathbf{M}_\varphi^{p,r}} \|g\|_{\mathbf{B}_\varphi^{p',r'}} = \left\|\mathbf{M}^\sharp(f)\right\|_{\mathbf{M}_\varphi^{p,r}} .$$

Then, we obtain Lemma 3.3. □

Remark 3.4 By duality, and Fefferman–Stein's inequality, the conclusion of Lemma 3.3 also holds for $\mathbf{B}_\varphi^{p,r}(X)$ in place of $\mathbf{M}_\varphi^{p,r}(X)$.

Thanks to Lemmas 3.1 and 3.3, we prove the $\mathbf{M}_\varphi^{p,r}$-bound for linear Calderón–Zygmund operators as follows.

Lemma 3.5 *Let $p \in (1, \infty)$, $r \in [1, \infty]$, and let $\varphi(t)$ satisfy (1.5). Let T be a linear Calderón–Zygmund operator. Then, T maps $\mathbf{M}_\varphi^{p,r}(X) \to \mathbf{M}_\varphi^{p,r}(X)$. Furthermore, we have*

$$\|T(f)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|f\|_{\mathbf{M}_\varphi^{p,r}} \ . \tag{3.5}$$

Proof of Lemma 3.5 Let us recall the following pointwise estimate (see e.g. [15]).

For any $q > 1$, there exists a constant $C > 0$ such that

$$\mathbf{M}^\sharp (T(f))(x) \le C\mathcal{M}_q(f)(x), \ \text{ for } x \in X \ . \tag{3.6}$$

Thanks to Lemma 3.3, for any $q \in (1, p)$ we obtain

$$\|T(f)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \left\|\mathbf{M}^\sharp (T(f))\right\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \left\|\mathcal{M}_q(f)\right\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|f\|_{\mathbf{M}_\varphi^{p,r}} \ .$$

This yields the proof of Lemma 3.5. □

Remark 3.6 By duality, (3.5) also holds for $\mathbf{B}_\varphi^{p',r'}(X)$ in place of $\mathbf{M}_\varphi^{p,r}(X)$.

Our next result is an inequality of Minkowski type in $\mathbf{M}_\varphi^{p,r}(X)$, used several times in the following.

Lemma 3.7 *Same hypotheses as in Lemma 3.5. Then, we have*

$$\left\|\int f(\cdot, y)\, d\mu(y)\right\|_{\mathbf{M}_\varphi^{p,r}} \le \int \|f(\cdot, y)\|_{\mathbf{M}_\varphi^{p,r}} \ d\mu(y) \, . \tag{3.7}$$

Proof of Lemma 3.7 Applying Theorem 1.8 and Proposition 2.4 yields

$$\begin{aligned}
\left\|\int f(., y)\, d\mu(y)\right\|_{\mathbf{M}_\varphi^{p,r}} &= \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}} \le 1} \left|\int \left(\int f(x, y)\, d\mu(y)\right) g(x)\, d\mu(x)\right| \\
&= \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}} \le 1} \int \int |f(x, y)g(x)| \ d\mu(x)\, d\mu(y) \\
&\le \sup_{\|g\|_{\mathbf{B}_\varphi^{p',r'}} \le 1} \int \|g\|_{\mathbf{B}_\varphi^{p',r'}} \|f(., y)\|_{\mathbf{M}_\varphi^{p,r}} \ d\mu(y) \\
&= \int \|f(., y)\|_{\mathbf{M}_\varphi^{p,r}} \ d\mu(y) \, .
\end{aligned}$$

Hence, we get Lemma 3.7. □

4 Hardy Factorization in Morrey–Lorentz Spaces

Proof of Theorem 1.13 The proof follows by way of the following lemmas.

The first lemma is a fundamental result of $\mathbf{H}^1(X)$ (see e.g. [25, Lemma 4.3]).

Lemma 4.1 *Let* $x_0, y_0 \in X$ *be such that* $d(x_0, y_0) = Mt$, *for some* $t > 0$, *and* $M > 10A_0$. *If*

$$\int_X F(x)\, d\mu(x) = 0, \ \ \textit{and} \ |F(x)| \leq \frac{1}{\mu\left(B(x_0, t)\right)} \left(\mathbf{1}_{B(x_0,t)}(x) + \mathbf{1}_{B(y_0,t)}(x)\right), \ \forall x \in \mathbb{R}^n,$$

then there is a positive constant C *independent of* x_0, y_0, t, M, *such that*

$$\|F\|_{\mathbf{H}^1(X)} \leq C \log M.$$

Next, we have the following result.

Lemma 4.2 *If* $f \in \mathbf{H}^1(X)$ *can be written as*

$$f = \sum_{k \geq 1} \lambda_k a_k$$

then, there exist $\{g_k\}_{k\geq 1}, \{h_k\}_{k \geq 1} \subset L_c^\infty(X)$ *such that*

$$\left\| a_k - \left[g_k T^*(h_k) - h_k T(g_k) \right] \right\|_{\mathbf{H}^1} \leq C \frac{\log M}{M^\eta}, \tag{4.1}$$

and

$$\sum_{k\geq 1} |\lambda_k| \, \|g_k\|_{\mathbf{B}_\varphi^{p',r'}} \|h_k\|_{\mathbf{M}_\varphi^{p,r}} \leq C M^n \|f\|_{\mathbf{H}^1}, \tag{4.2}$$

where $M > 0$ *is sufficiently large.*

Furthermore, we have

$$\left\| f - \sum_{k\geq 1} |\lambda_k| \left[g_k T^*(h_k) - h_k T(g_k) \right] \right\|_{\mathbf{H}^1} \leq \frac{1}{2} \|f\|_{\mathbf{H}^1}. \tag{4.3}$$

Proof of Lemma 4.2 Let a be an atom, supported in $B(x_0, t) \subset X$, such that

$$\|a\|_{L^\infty} \leq \frac{1}{\mu\left(B(x_0, t)\right)}, \quad \text{and} \quad \int_X a(x)\, d\mu(x) = 0.$$

Let $M \geq 10$ be a real number, which will be determined later, and let $y_0 \in X$ be such that $d(x_0, y_0) = Mt$.

Now, we set

$$g(x) = \mathbf{1}_{B(y_0,t)}(x), \quad \text{and} \ \ h(x) = -\frac{a(x)}{T(g)(x_0)}.$$

It is clear that these functions are in $L_c^\infty(X)$. In addition, since T is homogeneous, then we have

$$|T(g)(x_0)| \geq \frac{\mu\left(B(y_0, t)\right)}{\mu\left(B(x_0, Mt)\right)}.$$

Thanks to Proposition 2.3, we obtain

$$\|g\|_{\mathbf{B}_{\varphi}^{p',r'}} \approx \mu\left(B(y_0,t)\right)\varphi(t)\,, \tag{4.4}$$

and

$$\begin{aligned}\|h\|_{\mathbf{M}_{\varphi}^{p,r}} = \frac{\|a\|_{\mathbf{M}_{\varphi}^{p,r}}}{|T(g)(x_0)|} &\leq \frac{\mu\left(B(x_0,Mt)\right)}{\mu\left(B(y_0,t)\right)}\|a\|_{L^\infty}\|\mathbf{1}_{B(x_0,t)}\|_{\mathbf{M}_{\varphi}^{p,r}}\\ &\lesssim \frac{\mu\left(B(x_0,Mt)\right)}{\mu\left(B(y_0,t)\right)}\frac{1}{\mu\left(B(x_0,t)\right)}\frac{1}{\varphi(t)}\,.\end{aligned} \tag{4.5}$$

Combining (1.3), (4.4), and (4.5) yields

$$\|g\|_{\mathbf{B}_{\varphi}^{p',r'}}\|h\|_{\mathbf{M}_{\varphi}^{p,r}} \leq C\frac{\mu\left(B(x_0,Mt)\right)}{\mu\left(B(x_0,t)\right)} \leq CM^n\,, \tag{4.6}$$

where $C>0$ only depends on p.

Next, we show that

$$\left\|a-\left[gT^*(h)-hT(g)\right]\right\|_{\mathbf{H}^1} \leq C\frac{\log M}{M^\eta}\,. \tag{4.7}$$

Put

$$F = a-\left[gT^*(h)-hT(g)\right].$$

Then, we have

$$\begin{aligned}|F(x)| &\leq |a(x)+hT(g)|+\left|gT^*(h)\right|\\ &= \left|\frac{a(x)\left[T(g)(x_0)-T(g)(x)\right]}{T(g)(x_0)}\right|+\left|gT^*(h)\right| := \mathbf{J}_1(x)+\mathbf{J}_2(x)\,.\end{aligned} \tag{4.8}$$

We first consider $\mathbf{J}_1(x)$. Since $\operatorname{supp}(a)\subset B(x_0,t)$, then $\mathbf{J}_1(x)=0$ if $x\notin B(x_0,t)$. And, for any $x\in B(x_0,t)$ we use the smoothness of K in (1.8) and the homogeneity of T in order to obtain

$$\begin{aligned}|\mathbf{J}_1(x)| &\leq \frac{\mu\left(B(x_0,Mt)\right)}{\mu\left(B(y_0,t)\right)}\|a\|_{L^\infty}\int_{B(y_0,t)}|K(x_0,y)-K(x,y)|\,d\mu(y)\\ &\leq C\frac{\mu\left(B(x_0,Mt)\right)}{\mu\left(B(y_0,t)\right)}\frac{1}{\mu\left(B(x_0,t)\right)}\int_{B(y_0,t)}\frac{1}{V(x_0,y)}\left(\frac{d(x_0,x)}{d(x_0,y)}\right)^\eta d\mu(y)\\ &\leq C\frac{\mu\left(B(x_0,Mt)\right)}{\mu\left(B(y_0,t)\right)}\frac{1}{\mu\left(B(x_0,t)\right)}\frac{\mu\left(B(y_0,t)\right)}{\mu\left(B(x_0,d(x_0,y_0))\right)}M^{-\eta}\end{aligned}$$

$$\leq CM^{-\eta}\frac{\mu\left(B(x_0, Mt)\right)}{\mu\left(B(x_0, \frac{1}{2}Mt)\right)}\frac{1}{\mu\left(B(x_0, t)\right)}$$

$$\leq CM^{-\eta}\frac{1}{\mu\left(B(x_0, t)\right)}. \tag{4.9}$$

Note that the last inequality follows from the doubling property in (1.2).

For $\mathbf{J}_2(x)$, by using the cancellation of $a(x)$, and the fact $\mathrm{supp}(a) \subset B(x_0, t)$, we get that

$$|\mathbf{J}_2| \leq \mathbf{1}_{B(y_0,t)}(x)\left|\int_{B(x_0,t)} K(z, x)\left(\frac{-a(z)}{T(g)(x_0)}\right) d\mu(z)\right|$$

$$= \mathbf{1}_{B(y_0,t)}(x)\left|\int_{B(x_0,t)} [K(x_0, x) - K(z, x)]\left(\frac{a(z)}{T(g)(x_0)}\right) d\mu(z)\right|.$$

Analogously to the proof of $\mathbf{J}_1$, we also obtain

$$|\mathbf{J}_2(x)| \leq \frac{CM^{-\eta}}{\mu\left(B(x_0, t)\right)}\mathbf{1}_{B(y_0,t)}. \tag{4.10}$$

Combining (4.8), (4.9), and (4.10) yields

$$|F(x)| \leq \frac{CM^{-\eta}}{\mu\left(B(x_0, t)\right)}\left(\mathbf{1}_{B(x_0,t)}(x) + \mathbf{1}_{B(y_0,t)}(x)\right). \tag{4.11}$$

Thus, we obtain (4.7) by applying Lemma 4.1 to $F(x)$.

Next, we apply (4.7) to $a = a_k$, for $k \geq 1$. Then, there exist functions $\{g_k\}_{k\geq 1}, \{h_k\}_{k\geq 1} \subset L_c^\infty(X)$, such that

$$\left\|a_k - \left[g_k T^*(h_k) - h_k T(g_k)\right]\right\|_{\mathbf{H}^1} \leq C\frac{\log M}{M^\eta}.$$

With this inequality noted, we get

$$\left\|f - \sum_{k\geq 1}\lambda_k\left[g_k T^*(h_k) - h_k T(g_k)\right]\right\|_{\mathbf{H}^1} \leq \sum_{k\geq 1}|\lambda_k|\left\|a_k - \left[g_k T^*(h_k) - h_k T(g_k)\right]\right\|_{\mathbf{H}^1}$$

$$\leq C\frac{\log M}{M^\eta}\sum_{k\geq 1}|\lambda_k| \leq \frac{1}{2}\|f\|_{\mathbf{H}^1}, \tag{4.12}$$

provided that M is large enough.

This yields the proof of Lemma 4.2. □

Now we can suppose that $f \in \mathbf{H}^1(X)$ can be written

$$f = \sum_{k\geq 1}\lambda_{k,1}a_{k,1},$$

with $\{\lambda_{k,1}\}_{k\geq 1} \in l^1$, and $\{a_{k,1}\}_{k\geq 1}$ are atoms.

Thanks to Lemma 4.2 and (4.6), there exist functions $\{g_{k,1}\}, \{h_{k,1}\} \subset L_c^\infty(X)$ such that

$$\begin{cases} \sum_{k\geq 1} \left|\lambda_{k,1}\right| \, \left\|g_{k,1}\right\|_{\mathbf{B}_\varphi^{p',r'}} \, \left\|h_{k,1}\right\|_{\mathbf{M}_\varphi^{p,r}} \leq CM^n \|f\|_{\mathbf{H}^1}\,, \\ \left\| f - \sum_{k\geq 1} \lambda_{k,1} \left[g_{k,1}T^*(h_{k,1}) - h_{k,1}T(g_{k,1})\right] \right\|_{\mathbf{H}^1} \leq \frac{1}{2}\|f\|_{\mathbf{H}^1}\,. \end{cases} \tag{4.13}$$

Put

$$f_1 = f - \sum_{k\geq 1} \lambda_{k,1} \left[g_{k,1}T^*(h_{k,1}) - h_{k,1}T(g_{k,1})\right].$$

Since $f_1 \in \mathbf{H}^1(X)$, then we can decompose

$$f_1 = \sum_{k\geq 1} \lambda_{k,2} a_{k,2},$$

where $\{\lambda_{k,2}\}_{k\geq 1} \in l^1$, and $\{a_{k,2}\}_{k\geq 1}$ are atoms.

By (4.13), and by applying Lemma 4.2 to f_1, there exist $\{g_{k,2}\}_{k\geq 1}, \{h_{k,2}\}_{k\geq 1} \subset L_c^\infty(X)$, such that

$$\begin{cases} \sum_{k\geq 1} \left|\lambda_{k,2}\right| \, \left\|g_{k,2}\right\|_{\mathbf{B}_\varphi^{p',r'}} \, \left\|h_{k,2}\right\|_{\mathbf{M}_\varphi^{p,r}} \leq CM^n \|f_1\|_{\mathbf{H}^1} \leq CM^n \frac{1}{2}\|f\|_{\mathbf{H}^1} \\ \left\| f_1 - \sum_{k\geq 1} \lambda_{k,2} \left[g_{k,2}T^*(h_{k,2}) - h_{k,2}T(g_{k,2})\right] \right\|_{\mathbf{H}^1} \leq \frac{1}{2}\|f_1\|_{\mathbf{H}^1} \leq \frac{1}{2^2}\|f\|_{\mathbf{H}^1}\,. \end{cases}$$

Similarly, we can apply the above argument to

$$\begin{aligned} f_2 &= f_1 - \sum_{k\geq 1} \lambda_{k,2} \left[g_{k,2}T^*(h_{k,2}) - h_{k,2}T(g_{k,2})\right] \\ &= f - \sum_{k\geq 1} \lambda_{k,1} \left[g_{k,1}T^*(h_{k,1}) - h_{k,1}T(g_{k,1})\right] - \sum_{k\geq 1} \lambda_{k,2} \left[g_{k,2}T^*(h_{k,2}) - h_{k,2}T(g_{k,2})\right]. \end{aligned}$$

By induction, we can construct sequence $\{\lambda_{k,j}\} \in l^1$, and functions $\{g_{k,j}\}, \{h_{k,j}\} \subset L_c^\infty(X)$, such that

$$\begin{cases} f = \sum_{j=1}^{N} \sum_{k \geq 1} \lambda_{k,j} \left[g_{k,j} T^*(h_{k,j}) - h_{k,j} T(g_{k,j}) \right] + f_N \,, \\ \sum_{j=1}^{N} \sum_{k \geq 1} \left| \lambda_{k,j} \right| \, \| g_{k,j} \|_{\mathbf{B}_{\varphi}^{p',r'}} \, \| h_{k,j} \|_{\mathbf{M}_{\varphi}^{p,r}} \leq C M^n \sum_{j=1}^{N} \frac{1}{2^{j-1}} \| f \|_{\mathbf{H}^1} \,, \\ \| f_N \|_{\mathbf{H}^1} \leq \frac{1}{2^N} \| f \|_{\mathbf{H}^1} \,, \end{cases}$$

which yields the desired result when $N \to \infty$.

Thus, we complete the proof of Theorem 1.13. □

Proof of Corollary 1.15 To obtain the upper bound of $[b, T]$, we recall the following result (see e.g. [15, Lemma 1]).

Lemma 4.3 *Let $b \in \mathrm{BMO}(X)$. Then, for any $1 < q < p$, there exists a positive constant C such that*

$$\mathbf{M}^{\sharp} \left([b, T](f) \right) (x) \leq C \|b\|_{\mathrm{BMO}} \left(\mathcal{M}_q(f)(x) + \mathcal{M}_q \left(T(f) \right) (x) \right) \,, \; for \; x \in X \,. \tag{4.14}$$

By Lemmas 3.1, 3.3, 3.5, and 4.3, we obtain

$$\begin{aligned} \|[b, T](f)\|_{\mathbf{M}_{\varphi}^{p,r}} &\lesssim \left\| \mathbf{M}^{\sharp} \left([b, T](f) \right) \right\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|b\|_{\mathrm{BMO}} \left\| \mathcal{M}_q(f) + \mathcal{M}_q \left(T(f) \right) \right\|_{\mathbf{M}_{\varphi}^{p,r}} \\ &\lesssim \|b\|_{\mathrm{BMO}} \left(\left\| \mathcal{M}_q(f) \right\|_{\mathbf{M}_{\varphi}^{p,r}} + \left\| \mathcal{M}_q \left(T(f) \right) \right\|_{\mathbf{M}_{\varphi}^{p,r}} \right) \\ &\lesssim \|b\|_{\mathrm{BMO}} \left(\|f\|_{\mathbf{M}_{\varphi}^{p,r}} + \|T(f)\|_{\mathbf{M}_{\varphi}^{p,r}} \right) \lesssim \|b\|_{\mathrm{BMO}} \|f\|_{\mathbf{M}_{\varphi}^{p,r}} \,. \end{aligned} \tag{4.15}$$

Hence, we get the desired result.

Now, we prove the lower bound of $[b, T]$. To obtain the result, we utilize the Hardy factorization in Theorem 1.13, and the duality between $\mathrm{BMO}(X)$ and $\mathbf{H}^1(X)$.

As a matter of fact, $\mathbf{H}^1(X) \cap L_c^{\infty}(X)$ is dense in $\mathbf{H}^1(X)$.

Next, for every $L > 0$, let us put

$$b_L(x) = b(x) \mathbf{1}_{B(x_0, L)}(x).$$

For every $f \in \mathbf{H}^1(X) \cap L_c^{\infty}(X)$, it follows from Theorem 1.13 that there exist sequences $\{\lambda_{k,j}\} \in l^1$ and functions $g_{k,j}, h_{k,j} \in L_c^{\infty}(X)$, such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{k,j} \left(g_{k,j} T^*(h_{k,j}) - h_{k,j} T(g_{k,j}) \right) \,.$$

Furthermore, we have

$$\|f\|_{\mathbf{H}^1} \approx \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \lambda_{k,j} \right| \, \| g_{k,j} \|_{\mathbf{B}_{\varphi}^{p',r'}} \, \| h_{k,j} \|_{\mathbf{M}_{\varphi}^{p,r}} \,.$$

Now, since $b_L \to b$ in $L^1_{\mathrm{loc}}(X)$ as $L \to \infty$, and $f \in \mathbf{H}^1(X) \cap L^\infty_c(X)$, then we have

$$\lim_{L\to\infty} \langle b_L, f\rangle = \langle b, f\rangle,$$

where we denote $\langle f, g\rangle = \int f(x)g(x)\,d\mu(x)$.

Thanks to the facts $g_{k,j}T^*(h_{k,j}) - h_{k,j}T(g_{k,j}) \in \mathbf{H}^1(X)$ and $\mathrm{supp}(g_{k,j}T^*(h_{k,j}) - h_{k,j}T(g_{k,j})) \Subset X$, then we get

$$\begin{aligned}
\langle b, f\rangle = \lim_{L\to\infty}\langle b_L, f\rangle &= \lim_{L\to\infty}\left\langle b_L, \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_{k,j}\left(g_{k,j}T^*(h_{k,j}) - h_{k,j}T(g_{k,j})\right)\right\rangle\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_{k,j}\lim_{L\to\infty}\left\langle b_L, g_{k,j}T^*(h_{k,j}) - h_{k,j}T(g_{k,j})\right\rangle\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_{k,j}\left\langle b, g_{k,j}T^*(h_{k,j}) - h_{k,j}T(g_{k,j})\right\rangle\\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_{k,j}\left\langle [b, T](g_{k,j}), h_{k,j}\right\rangle. \qquad (4.16)
\end{aligned}$$

By Proposition 2.4, since $[b, T]$ maps $\mathbf{B}^{p',r'}_\varphi(X) \to \mathbf{B}^{p',r'}_\varphi(X)$ (see Corollary 3.2), then we obtain

$$\begin{aligned}
|\langle b, f\rangle| &\le \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\lambda_{k,j}\right| \left\|[b, T](g_{k,j})\right\|_{\mathbf{B}^{p',r'}_\varphi} \left\|h_{k,j}\right\|_{\mathbf{M}^{p,r}_\varphi}\\
&\lesssim \|[b, T]\|_{\mathbf{B}^{p',r'}_\varphi \to \mathbf{B}^{p',r'}_\varphi} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\lambda_{k,j}\right| \left\|g_{k,j}\right\|_{\mathbf{B}^{p',r'}_\varphi} \left\|h_{k,j}\right\|_{\mathbf{M}^{p,r}_\varphi}\\
&\lesssim \|[b, T]\|_{\mathbf{B}^{p',r'}_\varphi \to \mathbf{B}^{p',r'}_\varphi} \|f\|_{\mathbf{H}^1}.
\end{aligned}$$

Therefore,

$$\|b\|_{\mathrm{BMO}} \lesssim \|[b, T]\|_{\mathbf{B}^{p',r'}_\varphi \to \mathbf{B}^{p',r'}_\varphi}.$$

This ends the proof of Corollary 1.15. □

5 Compactness Characterization of $[b, T]$ in $\mathbf{M}^{p,r}_\varphi(X)$

In the last section, we study the compactness of $[b, T]$ in $\mathbf{M}^{p,r}_\varphi(X)$. Then, we point out a compactness criterion in $\mathbf{M}^{p,r}_\varphi(X)$.

Lemma 5.1 *Let* $p \in (1,\infty)$*,* $r \in [1,\infty]$*, and let* $\varphi(t)$ *satisfy* (1.5)*. Assume that* (X, d, μ) *is a locally compact space such that* (1.15) *holds, and the set* $\mathcal{G}$ *in* $\mathbf{M}_\varphi^{p,r}(X)$ *satisfies the following conditions:*

$$i)\quad \sup_{f\in\mathcal{G}} \|f\|_{\mathbf{M}_\varphi^{p,r}} < \infty\,,$$

$$ii)\quad \lim_{y\to 0} \|f(\cdot + y) - f(\cdot)\|_{\mathbf{M}_\varphi^{p,r}} = 0 \text{ uniformly in } f \in \mathcal{G},$$

$$iii)\quad \lim_{R\to\infty} \|f\mathbf{1}_{B_R^c}\|_{\mathbf{M}_\varphi^{p,r}} = 0 \text{ uniformly in } f \in \mathcal{G}.$$

Then, $\mathcal{G}$ *is strongly precompact set in* $\mathbf{M}_\varphi^{p,r}(X)$*.*

Proof of Lemma 5.1 We can assume without loss of generality that $0 \in X$.

For any $\tau > 0$, let us define

$$\overline{f}_\tau(x) = \frac{1}{\mu(B(0,\tau))}\int_{B(0,\tau)} f(x+y)\,d\mu(y).$$

Fix $\tau > 0$, we first claim that the set $\{\overline{f}_\tau : f \in \mathcal{G}\}$ is a precompact set in $\mathcal{C}\left(\overline{B_R}\right)$. Thanks to the Ascoli–Arzelà theorem, it is enough to show that $\{\overline{f}_\tau : f \in \mathcal{G}\}$ is bounded and equicontinuous in $\mathcal{C}\left(\overline{B_R}\right)$.

Indeed, we have from Hölder's inequality that

$$\begin{aligned}
|\overline{f}_\tau(x)| &\le \frac{1}{\mu(B(0,\tau))}\int_{B(x,\tau)} |f(z)|\,d\mu(z)\\
&\lesssim \frac{1}{\mu(B(0,\tau))}\|f\|_{L^{p,r}(B(x,\tau))}\mu(B(x,\tau))^{\frac{1}{p'}}\\
&\lesssim \frac{\varphi(\tau)}{\mu(B(x,\tau))^{\frac{1}{p}}\varphi(\tau)}\|f\|_{L^{p,r}(B(x,\tau))}\\
&\le \varphi(\tau)\|f\|_{\mathbf{M}_\varphi^{p,r}} \le C\varphi(\tau)\,,
\end{aligned}\tag{5.1}$$

uniformly in $f \in \mathcal{G}$.

Concerning the equicontinuity, we have

$$\begin{aligned}
|\overline{f}_\tau(x_1) - \overline{f}_\tau(x_2)| &\le \frac{1}{\mu(B(0,\tau))}\int_{B(0,\tau)} |f(x_1+y) - f(x_2+y)|\,d\mu(y)\\
&= \frac{1}{\mu(B(0,\tau))}\int_{B(x_2,\tau)} |f(z+x_1-x_2) - f(z)|\,d\mu(z)\\
&\lesssim \frac{1}{\mu(B(0,\tau))}\|f(x_1-x_2+\cdot) - f(\cdot)\|_{L^{p,r}(B(x_2,\tau))}\mu(B(x_2,\tau))^{\frac{1}{p'}}\\
&\lesssim \frac{1}{\mu(B(x_2,\tau))^{\frac{1}{p}}}\|f(x_1-x_2+\cdot) - f(\cdot)\|_{L^{p,r}(B(x_2,\tau))}\\
&\le \varphi(\tau)\|f(x_1-x_2+\cdot) - f(\cdot)\|_{\mathbf{M}_\varphi^{p,r}}\,.
\end{aligned}\tag{5.2}$$

By ii), we conclude that $\{\overline{f}_\tau : f \in \mathcal{G}\}$ is equicontinuous in $\mathcal{C}\left(\overline{B_R}\right)$, so the above claim follows.

Next, we show that

$$\lim_{\tau\to 0} \|\overline{f}_\tau - f\|_{\mathbf{M}_\varphi^{p,r}} = 0\,, \tag{5.3}$$

uniformly in $f \in \mathcal{G}$.

Indeed, applying Minkowski's inequality yields

$$\begin{aligned}\frac{\|\overline{f}_\tau - f\|_{L^{p,r}(B(z,t))}}{\mu(B(z,t))^{\frac{1}{p}}\varphi(t)} &\leq \frac{1}{\mu(B(0,\tau))}\int_{B(0,\tau)} \frac{\|f(\cdot+y)-f(\cdot)\|_{L^{p,r}(B(z,t))}}{\mu(B(z,t))^{\frac{1}{p}}\varphi(t)}d\mu(y)\\ &\leq \frac{1}{\mu(B(0,\tau))}\int_{B(0,\tau)} \|f(\cdot+y)-f(\cdot)\|_{\mathbf{M}_\varphi^{p,r}}\,d\mu(y)\\ &\leq \sup_{d(y,0)<\tau} \|f(\cdot+y)-f(\cdot)\|_{\mathbf{M}_\varphi^{p,r}}\,.\end{aligned}$$

This implies that

$$\|\overline{f}_\tau - f\|_{\mathbf{M}_\varphi^{p,r}} \leq \sup_{d(y,0)<\tau} \|f(\cdot+y)-f(\cdot)\|_{\mathbf{M}_\varphi^{p,r}}.$$

With this inequality noted, (5.3) follows from ii).

Now, we prove that $\{\overline{f}_\tau : f \in \mathcal{G}\}$ is relatively compact in $\mathbf{M}_\varphi^{p,r}(X)$.

By iii), for any $0 < \varepsilon < 1$, there exist $R_\varepsilon > 0$ such that for every $f \in \mathcal{G}$

$$\|f\mathbf{1}_{B^c_{R_\varepsilon}}\|_{\mathbf{M}_\varphi^{p,r}} < \varepsilon\,. \tag{5.4}$$

Since $\{\overline{f_\tau} : f \in \mathcal{G}\}$ is strongly precompact in $\mathcal{C}\left(\overline{B_R}\right)$, then for every $\varepsilon > 0$, there exist $f^1, f^2, \ldots, f^m$ in $\mathcal{G}$, with $m = m(\varepsilon) \in \mathbb{N}$ such that $\{\overline{f^1_\tau}, \overline{f^2_\tau}, \ldots, \overline{f^m_\tau}\}$ is a finite $\varepsilon\varphi(R_\varepsilon)$-net in $\{\overline{f_\tau} : f \in \mathcal{G}\}$ with respect to the norm of $\mathcal{C}\left(\overline{B_R}\right)$.

As a result, for any $f \in \mathcal{G}$, there exists $j \in \{1, \ldots, m\}$ such that

$$\left\|\overline{f_\tau} - \overline{f^j_\tau}\right\|_{L^\infty(B_R)} < \varepsilon\varphi(R_\varepsilon)\,. \tag{5.5}$$

Next, we prove that $\{\overline{f^1_\tau}, \overline{f^2_\tau}, \ldots, \overline{f^m_\tau}\}$ is a finite ε-net of $\{\overline{f_\tau} : f \in \mathcal{G}\}$ with respect to the norm of $\mathbf{M}_\varphi^{p,r}(X)$. It is equivalent to show that

$$\left\|\overline{f_\tau} - \overline{f^j_\tau}\right\|_{\mathbf{M}_\varphi^{p,r}} < \varepsilon\,, \tag{5.6}$$

where $\tau > 0$ is small enough.

In fact, we write

$$\overline{f_\tau} - \overline{f^j_\tau} = (\overline{f_\tau} - \overline{f^j_\tau})\mathbf{1}_{B_{R_\varepsilon}} + (\overline{f_\tau} - \overline{f^j_\tau})\mathbf{1}_{B^c_{R_\varepsilon}}\,.$$

We first estimate $(\overline{f_\tau} - \overline{f_\tau^j})\mathbf{1}_{B_{R_\varepsilon}}$ in the $\mathbf{M}_\varphi^{p,r}$-norm. Then, for any ball $B(z,t)$ in X, we have

$$\frac{\left\|(\overline{f_\tau} - \overline{f_\tau^j})\mathbf{1}_{B_{R_\varepsilon}}\right\|_{L^{p,r}(B(z,t))}}{\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(t)} \le C\frac{\left\|\overline{f_\tau} - \overline{f_\tau^j}\right\|_{L^\infty(B(z,t))}\mu\left(B(z,t)\cap B_{R_\varepsilon}\right)^{\frac{1}{p}}}{\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(t)} \le C\frac{\varepsilon\varphi(R_\varepsilon)\,\mu\left(B(z,t)\cap B_{R_\varepsilon}\right)^{\frac{1}{p}}}{\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(t)}. \tag{5.7}$$

If $t \ge R_\varepsilon$ then it follows from (5.7) and the monotonicity of $\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(t)$ that

$$\frac{\left\|(\overline{f_\tau} - \overline{f_\tau^j})\mathbf{1}_{B_{R_\varepsilon}}\right\|_{L^{p,r}(B(z,t))}}{\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(t)} \le C\frac{\varepsilon\varphi(R_\varepsilon)\,\mu\left(B_{R_\varepsilon}\right)^{\frac{1}{p}}}{\mu\left(B_{R_\varepsilon}\right)^{\frac{1}{p}}\varphi(R_\varepsilon)} \le C\varepsilon.$$

Otherwise, we use the monotonicity of $\varphi(t)$ to obtain

$$\frac{\left\|(\overline{f_\tau} - \overline{f_\tau^j})\mathbf{1}_{B_{R_\varepsilon}}\right\|_{L^{p,r}(B(z,t))}}{\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(t)} \le C\frac{\varepsilon\varphi(R_\varepsilon)\,\mu\left(B(z,t)\right)^{\frac{1}{p}}}{\mu\left(B(z,t)\right)^{\frac{1}{p}}\varphi(R_\varepsilon)} = C\varepsilon.$$

By combining the two cases, we get

$$\left\|(\overline{f_\tau} - \overline{f_\tau^j})\mathbf{1}_{B_{R_\varepsilon}}\right\|_{\mathbf{M}_\varphi^{p,r}} \le C\varepsilon\,. \tag{5.8}$$

Concerning the term $(\overline{f_\tau} - \overline{f_\tau^j})\mathbf{1}_{B_{R_\varepsilon}^c}$, by (5.3) and (5.4), we get

$$\begin{aligned}\left\|\left(\overline{f_\tau} - \overline{f_\tau^j}\right)\mathbf{1}_{B_{R_\varepsilon}^c}\right\|_{\mathbf{M}_\varphi^{p,r}} &\le \left\|\left(\overline{f_\tau} - f\right)\mathbf{1}_{B_{R_\varepsilon}^c}\right\|_{\mathbf{M}_\varphi^{p,r}} + \left\|\left(f - f^j\right)\mathbf{1}_{B_{R_\varepsilon}^c}\right\|_{\mathbf{M}_\varphi^{p,r}} \\ &\quad + \left\|\left(f^j - \overline{f_\tau^j}\right)\mathbf{1}_{B_{R_\varepsilon}^c}\right\|_{\mathbf{M}_\varphi^{p,r}} \\ &\le 2\varepsilon + \|f\mathbf{1}_{B_{R_\varepsilon}^c}\|_{\mathbf{M}_\varphi^{p,r}} + \|f^j\mathbf{1}_{B_{R_\varepsilon}^c}\|_{\mathbf{M}_\varphi^{p,r}} \le 4\varepsilon\,,\end{aligned} \tag{5.9}$$

as $\tau > 0$ is small enough.

Thus, (5.6) follows from (5.8) and (5.9).

As a result, $\{\overline{f_\tau} : f \in \mathcal{G}\}$ is relatively compact in $\mathbf{M}_\varphi^{p,r}(X)$.

It suffices to show that $\mathcal{G}$ is relatively compact in $\mathbf{M}_\varphi^{p,r}(X)$. Let $\{f^k\}_{k\ge1} \subset \mathcal{G}$. Since $\{\overline{f_\tau} : f \in \mathcal{G}\}$ is strongly compact in $\mathbf{M}_\varphi^{p,r}(X)$, then there is a subsequence of $\{f^k\}_{k\ge1}$ (still denoted as $\{f^k\}_{k\ge1}$) such that $\overline{f_\tau^k}$ converges in $\mathbf{M}_\varphi^{p,r}(X)$ as $k \to \infty$.

Then, it follows from (5.3) that

$$\begin{aligned}\left\| f^k - f^{k'} \right\|_{\mathbf{M}_\varphi^{p,r}} &\le \left\| f^k - \overline{f_\tau^k} \right\|_{\mathbf{M}_\varphi^{p,r}} + \left\| \overline{f_\tau^k} - \overline{f_\tau^{k'}} \right\|_{\mathbf{M}_\varphi^{p,r}} + \left\| \overline{f_\tau^{k'}} - f^{k'} \right\|_{\mathbf{M}_\varphi^{p,r}} \\ &\le C\varepsilon + \left\| \overline{f_\tau^k} - \overline{f_\tau^{k'}} \right\|_{\mathbf{M}_\varphi^{p,r}} .\end{aligned}$$

Therefore, $\{f^k\}_{k\ge 1}$ is a Cauchy sequence in $\mathbf{M}_\varphi^{p,r}(X)$. Since $\mathbf{M}_\varphi^{p,r}(X)$ is complete, then $\{f^k\}_{k\ge 1}$ converges to a function in $\mathbf{M}_\varphi^{p,r}(X)$.

This puts an end to the proof of Lemma 5.1. □

Now, we are ready to prove Theorem 1.17.

Proof of Theorem 1.17 **a) Necessity:** Assume that $b \in \mathrm{CMO}(X)$. Let $\mathcal{G}$ be a bounded set in $\mathbf{M}_\varphi^{p,r}(X)$. It is enough to show that $[b, T](\mathcal{G})$ is relatively compact in $\mathbf{M}_\varphi^{p,r}(X)$.

Indeed, since $b \in \mathrm{CMO}(X)$, then for every $\varepsilon > 0$ there exists a function $b_\varepsilon \in \mathcal{C}_c^\infty(X)$ such that

$$\|b - b_\varepsilon\|_{\mathrm{BMO}} < \varepsilon .$$

By the triangle inequality and Corollary 1.15, we have for every $f \in \mathcal{G}$

$$\begin{aligned}\|[b, T](f)\|_{\mathbf{M}_\varphi^{p,r}} &\le \|[b - b_\varepsilon, T](f)\|_{\mathbf{M}_\varphi^{p,r}} + \|[b_\varepsilon, T](f)\|_{\mathbf{M}_\varphi^{p,r}} \\ &\lesssim \|b - b_\varepsilon\|_{\mathrm{BMO}} \|f\|_{\mathbf{M}_\varphi^{p,r}} + \|[b_\varepsilon, T](f)\|_{\mathbf{M}_\varphi^{p,r}} \\ &\le C\varepsilon + \|[b_\varepsilon, T](f)\|_{\mathbf{M}_\varphi^{p,r}} .\end{aligned}$$

With this inequality noted, it suffices to show that $[b_\varepsilon, T](\mathcal{G})$ is relatively compact in $\mathbf{M}_\varphi^{p,r}(X)$ for a given $\varepsilon > 0$ small enough.

Since $\mathcal{G}$ is a bounded set in $\mathbf{M}_\varphi^{p,r}(X)$, and by Theorem 1.13, then it is clear that $[b_\varepsilon, T](\mathcal{G})$ satisfies (i).

Next, we show that $[b_\varepsilon, T](\mathcal{G})$ also satisfies (ii). Indeed, suppose that $\mathrm{supp}(b_\varepsilon) \subset B_{R_\varepsilon}$, for some $R_\varepsilon > 10$. Then, for any $f \in \mathcal{G}$, and for $x \in B_R^c$, with $R > 10A_0R_\varepsilon$, we observe that $d(x, y) \approx d(x, 0)$ for any $y \in B_{R_\varepsilon}$.

Thus, for any $x \in B_R^c$ we have

$$\begin{aligned}|[b_\varepsilon, T](f)(x)| = |T(bf)(x)| &\le C_0 \|b_\varepsilon\|_{L^\infty} \int_{B(y,R_\varepsilon)} \frac{|f(y)|}{V(x,y)}\, d\mu(y) \\ &\lesssim \frac{\|b_\varepsilon\|_{L^\infty}}{\mu\left(B(x, R - R_\varepsilon)\right)} \int_{B(y,R_\varepsilon)} |f(y)|\, d\mu(y) \\ &\le \frac{\|b_\varepsilon\|_{L^\infty}}{\mu\left(B(x, \frac{1}{2}R)\right)} \int_{B(x,R_\varepsilon)} |f(x - w)|\, d\mu(w) .\end{aligned} \tag{5.10}$$

For every ball $B_t = B(x_0, t)$ in X, by (5.10) and Minkowski's inequality, we obtain

$$
\begin{aligned}
\frac{\left\|[b_\varepsilon, T](f)\mathbf{1}_{B_R^c}\right\|_{L^{p,r}(B_t)}}{\mu(B_t)^{\frac{1}{p}}\varphi(t)} &\lesssim \frac{\|b_\varepsilon\|_{L^\infty}}{\mu(B_t)^{\frac{1}{p}}\varphi(t)} \frac{1}{\mu\left(B(x, \frac{1}{2}R)\right)} \int_{B(x,R_\varepsilon)} \|f(\cdot - w)\|_{L^{p,r}(B_t)}\, d\mu(w) \\
&\lesssim \frac{\|b_\varepsilon\|_{L^\infty}}{\mu\left(B(x, \frac{1}{2}R)\right)} \int_{B(x,R_\varepsilon)} \frac{\|f\|_{L^{p,r}(B(x_0 - w, t))}}{\mu\left(B(x_0 - w, t)\right)^{\frac{1}{p}}\varphi(t)}\, d\mu(w) \\
&\lesssim \frac{\|b_\varepsilon\|_{L^\infty}}{\mu\left(B(x, \frac{1}{2}R)\right)} \int_{B(x,R_\varepsilon)} \|f\|_{\mathbf{M}_\varphi^{p,r}}\, d\mu(w) \\
&\lesssim \|b_\varepsilon\|_{L^\infty} \frac{\mu\left(B(x, R_\varepsilon)\right)}{\mu\left(B(x, \frac{1}{2}R)\right)} \|f\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|b_\varepsilon\|_{L^\infty} \frac{\mu\left(B(x, R_\varepsilon)\right)}{\mu\left(B(x, \frac{1}{2}R)\right)},
\end{aligned}
$$

uniformly in $f \in \mathcal{G}$.

This implies that

$$
\left\|[b_\varepsilon, T](f)\mathbf{1}_{B_R^c}\right\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|b_\varepsilon\|_{L^\infty} \frac{\mu\left(B(x, R_\varepsilon)\right)}{\mu\left(B(x, \frac{1}{2}R)\right)}, \quad \forall f \in \mathcal{G}.
$$

Thus, $\left\|[b_\varepsilon, T](f)\mathbf{1}_{B_R^c}\right\|_{\mathbf{M}_\varphi^{p,r}} \to 0$ when $R \to \infty$ uniformly in $f \in \mathcal{G}$. In other words, $[b_\varepsilon, T](\mathcal{G})$ verifies (iii).

It remains to prove the equicontinuity of $[b_\varepsilon, T]$. In fact, we show that for every $\delta > 0$, if $d(z, 0)$ is sufficiently small (merely depending on δ), then

$$
\|[b_\varepsilon, T](f)(\cdot + z) - [b_\varepsilon, T](f)(\cdot)\|_{\mathbf{M}_\varphi^{p,r}} \le C\delta^\eta, \tag{5.11}
$$

uniformly in $f \in \mathcal{G}$, where the constant $C > 0$ is independent of $f, \delta, d(z, 0)$.

To obtain the desired result, we recall the maximal operator of T, defined by

$$
\mathcal{T}(f)(x) = \sup_{\tau > 0} |T_\tau(f)(x)|, \tag{5.12}
$$

where T_τ, the truncated operator of T, is

$$
T_\tau(f)(x) = \int_{\{d(x,y) > \tau\}} K(x, y) f(y)\, d\mu(y). \tag{5.13}
$$

For convenience, we recall here Cotlar's inequality (see [39, Lemma 6.1]). That is for all $l > 0$,

$$
\mathcal{T}(f)(x) \le C\left[\mathcal{M}_l\left(T(f)\right)(x) + \mathcal{M}(f)(x)\right]. \tag{5.14}
$$

Now, for any $x \in X$ we express

$$
\begin{aligned}
[b_\varepsilon, T](f)(x+z) - [b_\varepsilon, T](f)(x) &= \int_X (b_\varepsilon(y) - b_\varepsilon(x+z))\, K(x+z, y) f(y)\, d\mu(y) \\
&\quad - \int_X (b_\varepsilon(y) - b_\varepsilon(x))\, K(x, y) f(y)\, d\mu(y) \\
&= \int_{d(x,y)>\delta^{-1}d(z,0)} (b_\varepsilon(x) - b_\varepsilon(x+z))\, K(x, y) f(y)\, d\mu(y) \\
&\quad + \int_{d(x,y)>\delta^{-1}d(z,0)} (b_\varepsilon(y) - b_\varepsilon(x+z))\, [K(x+z, y) - K(x, y)]\, f(y)\, d\mu(y) \\
&\quad + \int_{d(x,y)\le\delta^{-1}d(z,0)} (b_\varepsilon(x) - b_\varepsilon(y))\, K(x, y) f(y)\, d\mu(y) \\
&\quad + \int_{d(x,y)\le\delta^{-1}d(z,0)} (b_\varepsilon(y) - b_\varepsilon(x+z))\, K(x+z, y) f(y)\, d\mu(y) \\
&:= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4.
\end{aligned}
$$

We first estimate $\mathbf{I}_1$.

$$
\begin{aligned}
|\mathbf{I}_1| &\le |b_\varepsilon(x+z) - b_\varepsilon(x)| \left| \int_{d(x,y)>\delta^{-1}d(z,0)} K(x, y) f(y)\, d\mu(y) \right| \\
&\le |b_\varepsilon(x+z) - b_\varepsilon(x)|\, \mathcal{T}(f)(x)\,.
\end{aligned}
$$

Since b_ε is uniformly continuous on X, then we deduce from the last inequality that

$$
|\mathbf{I}_1| \le \delta \mathcal{T}(f)(x),
$$

as $d(z, 0) \to 0$.

Applying Cotlar's inequality yields

$$
\|\mathbf{I}_1\|_{\mathbf{M}_\varphi^{p,r}} \le \delta\, \|\mathcal{T}(f)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta\, \|f\|_{\mathbf{M}_\varphi^{p,r}}\ . \tag{5.15}
$$

For $\mathbf{I}_2$, we use the smoothness of kernel K, (1.14), and the doubling property of μ in order to get

$$
\begin{aligned}
|\mathbf{I}_2| &\lesssim \|b_\varepsilon\|_{L^\infty} \int_{d(x,y)>\delta^{-1}d(z,0)} \frac{1}{V(x, y)} \left(\frac{d(x+z, x)}{d(x, y)} \right)^\eta |f(y)|\, d\mu(y) \\
&= d(x+z, x)^\eta \|b_\varepsilon\|_{L^\infty} \int_{d(x,y)>\delta^{-1}d(z,0)} \frac{1}{\mu\left(B(x, d(x, y))\right) d(x, y)^\eta} |f(y)|\, d\mu(y) \\
&\le d(x+z, x)^\eta \|b_\varepsilon\|_{L^\infty} \sum_{k\ge0} 2^{-k} \delta^\eta d(z, 0)^{-\eta} \int_{D_{k+1}\setminus D_k} \frac{1}{\mu\left(D_k\right)} |f(y)|\, d\mu(y) \\
&\le \frac{d(x+z, x)^\eta}{d(z, 0)^\eta} \|b_\varepsilon\|_{L^\infty} \sum_{k\ge0} 2^{-k} \delta^\eta \frac{\mu\left(D_{k+1}\right)}{\mu\left(D_k\right)} \frac{1}{\mu\left(D_{k+1}\right)} \int_{D_{k+1}} |f(y)|\, d\mu(y)
\end{aligned}
$$

$$\lesssim \|b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} 2^{-k}\delta^\eta C_\mu \mathcal{M}(f)(x)$$
$$\lesssim \delta^\eta \|b_\varepsilon\|_{L^\infty} \mathcal{M}(f)(x),$$

where $D_k = B\left(x, 2^k\delta^{-1}d(z,0)\right)$, $k \geq 0$.

Then, we get from the last inequality that

$$\|\mathbf{I}_2\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta^\eta \|b_\varepsilon\|_{L^\infty} \|\mathcal{M}(f)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta^\eta \|b_\varepsilon\|_{L^\infty} \|f\|_{\mathbf{M}_\varphi^{p,r}}. \tag{5.16}$$

Next, we estimate $\mathbf{I}_3$. For any $k \geq 0$, let us set $B_k = B\left(x, \delta^{-1}2^{-k}d(z,0)\right)$. Then, it follows from the size condition of K that

$$\begin{aligned}
|\mathbf{I}_3| &\leq C\|\nabla b_\varepsilon\|_{L^\infty} \int_{d(x,y)\leq\delta^{-1}d(z,0)} \frac{d(x,y)}{\mu\left(B(x,d(x,y))\right)} |f(y)|\, d\mu(y) \\
&\lesssim \|\nabla b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} \frac{2^{-k}\delta^{-1}d(z,0)}{\mu\left(B_{k+1}\right)} \int_{B_k\setminus B_{k+1}} |f(y)|\, d\mu(y) \\
&\leq \delta^{-1}d(z,0)\|\nabla b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} 2^{-k} \frac{\mu\left(B_k\right)}{\mu\left(B_{k+1}\right)} \frac{1}{\mu\left(B_k\right)} \int_{B_k} |f(y)|\, d\mu(y) \\
&\leq \delta^{-1}d(z,0)\|\nabla b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} C_\mu 2^{-k} \mathcal{M}(f)(x) \\
&\lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty} \mathcal{M}(f)(x),
\end{aligned}$$

provided that $d(z,0) < \delta^2$.

With the last inequality noted, it follows from the $\mathbf{M}_\varphi^{p,r}$-bound of operator $\mathcal{M}$ that

$$\|\mathbf{I}_3\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty} \|\mathcal{M}(f)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty} \|f\|_{\mathbf{M}_\varphi^{p,r}}. \tag{5.17}$$

Finally, we treat $\mathbf{I}_4$. Since $\mathrm{supp}(b_\varepsilon) \subset B(0, R_\varepsilon)$, then it is sufficient to consider $x \in B(0, 2R_\varepsilon)$ when $z \to 0$.

Thanks to the quasi-triangle inequality (1.1), we get

$$d(x+z, y) \leq A_0\left(d(x+z, x) + d(x,y)\right) \leq 2\delta,$$

when $z \to 0$, for all $d(x,y) < \delta$.

Then,

$$\begin{aligned}
|\mathbf{I}_4| &\leq C\|\nabla b_\varepsilon\|_{L^\infty} \int_{d(x,y)<\delta^{-1}d(z,0)} \frac{d(x+z,y)}{\mu\left(B(x+z, d(x+z,y))\right)} |f(y)|\, d\mu(y) \\
&\leq C\|\nabla b_\varepsilon\|_{L^\infty} \int_{\{d(x+z,y)<2\delta\}} \frac{d(x+z,x)}{\mu\left(B(x+z, d(x+z,y))\right)} |f(y)|\, d\mu(y).
\end{aligned}$$

By arguing as in $\mathbf{I}_3$, we also obtain

$$\|\mathbf{I}_4\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta \|\nabla b_\varepsilon\|_{L^\infty} \|f\|_{\mathbf{M}_\varphi^{p,r}} . \tag{5.18}$$

Combining (5.15), (5.16), (5.17), and (5.18) yields

$$\|[b_\varepsilon, T](f)(x+z) - [b_\varepsilon, T](f)(x)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \delta \|\nabla b_\varepsilon\|_{L^\infty} \|f\|_{\mathbf{M}_\varphi^{p,r}}$$

uniformly in $f \in \mathcal{G}$. Therefore, $[b, T]$ satisfies $ii)$.

Thanks to Lemma 5.1, we conclude that $[b, T]$ is a compact operator on $\mathbf{M}_\varphi^{p,r}(X)$.

b) Sufficiency: Suppose that T is homogeneous, and $[b, T]$ is a compact operator on $\mathbf{M}_\varphi^{p,r}(X)$. Thanks to Corollary 1.15, we have that $b \in \mathrm{BMO}(X)$.

Next, we show that $b \in \mathrm{CMO}(X)$. To obtain the result, we need a characterization of a function in $\mathrm{CMO}(X)$ (see, e.g., [40]).

Lemma 5.2 *A function $b \in \mathrm{CMO}(X)$ if and only if b satisfies the following three conditions.*

(i) $\lim_{R\to\infty} \sup_{B_l,\, l>R} \frac{1}{\mu(B_l)} \int_{B_l} |b(z) - b_{B_l}|\, d\mu(z) = 0,$

(ii) $\lim_{R\to\infty} \sup_{\{B_l, B_l \subset B(0,R)^c\}} \frac{1}{\mu(B_l)} \int_{B_l} |b(z) - b_{B_l}|\, d\mu(z) = 0,$

(iii) $\lim_{\delta\to 0} \sup_{B_l,\, l<\delta} \frac{1}{\mu(B_l)} \int_{B_l} |b(z) - b_{B_l}|\, d\mu(z) = 0.$

We also need the following result for technical reasons.

Lemma 5.3 *There exists a positive constant $M \geq 10A_0$, such that for any ball $B(x_0, t)$ in X, there is a ball $B(y_0, t)$, $d(x_0, y_0) = Mt$; and for any $x \in B(x_0, t)$, $T\left(\mathbf{1}_{B(y_0,t)}\right)(x)$ does not change sign and*

$$\frac{\mu\left(B(y_0, t)\right)}{\mu\left(B(x_0, Mt)\right)} \lesssim \left|T\left(\mathbf{1}_{B(y_0,t)}\right)(x)\right| . \tag{5.19}$$

Proof of Lemma 5.3 Thanks to the smoothness of K, we have

$$\begin{aligned}
\left|T\left(\mathbf{1}_{B(y_0,t)}\right)(x) - T\left(\mathbf{1}_{B(y_0,t)}\right)(x_0)\right| &\leq \int_{B(y_0,t)} |K(x,y) - K(x_0,y)|\, d\mu(y) \\
&\leq C \int_{B(y_0,t)} \frac{1}{V(x_0,y)} \frac{d(x,x_0)^\eta}{d(x_0,y)^\eta}\, d\mu(y) \\
&\leq C \int_{B(y_0,t)} \frac{1}{\mu\left(B\left(x_0, \frac{Mt}{2}\right)\right)} \frac{t^\eta}{(Mt)^\eta}\, d\mu(y) \\
&\leq C M^{-\eta} \frac{\mu\left(B(y_0,t)\right)}{\mu\left(B\left(x_0, \frac{Mt}{2}\right)\right)}
\end{aligned}$$

$$= CM^{-\eta} \frac{\mu(B(y_0,t))}{\mu(B(x_0,Mt))} \frac{\mu(B(x_0,Mt))}{\mu\left(B\left(x_0,\frac{Mt}{2}\right)\right)}$$
$$\le CC_\mu M^{-\eta} \frac{\mu(B(y_0,t))}{\mu(B(x_0,Mt))}. \tag{5.20}$$

If $T(\mathbf{1}_{B(y_0,t)})(x_0) > 0$, then it follows from the homogeneity of T, the triangle inequality, and (5.20) that

$$\begin{aligned} T(\mathbf{1}_{B(y_0,t)})(x) &\ge T(\mathbf{1}_{B(y_0,t)})(x_0) - CC_\mu M^{-\eta} \frac{\mu(B(y_0,t))}{\mu(B(x_0,Mt))} \\ &\ge \frac{\mu(B(y_0,t))}{\mu(B(x_0,Mt))} - CC_\mu M^{-\eta} \frac{\mu(B(y_0,t))}{\mu(B(x_0,Mt))} \\ &\gtrsim \frac{\mu(B(y_0,t))}{\mu(B(x_0,Mt))} \end{aligned}$$

provided that M is large enough.

By the same argument, we also obtain the conclusion if $T(\mathbf{1}_{B(y_0,R_0)})(x_0) < 0$. This puts an end to the proof of Lemma 5.3. □

Now, we demonstrate that $b \in \mathrm{CMO}(X)$. Seeking a contradiction, we assume that $b \notin \mathrm{CMO}(X)$. Therefore, b violates (i), (ii), and (iii) in Lemma 5.2. We consider these cases orderly.

Case 1. Suppose that b violates (i). Then, there exists a sequence of balls $\{B_k = B(x_k, R_k)\}_{k\ge 1}$ such that $R_k \to \infty$ as $k \to \infty$, and

$$\frac{1}{\mu(B_k)} \int_{B_k} |b(x) - b_{B_k}|\, d\mu(x) \ge c_0 > 0, \quad \text{for every } k \ge 1. \tag{5.21}$$

Since $R_k \to \infty$, we can choose a subsequence of $\{R_k\}_{k\ge1}$ (still denoted by $\{R_k\}_{k\ge1}$) such that

$$R_k \le \frac{1}{C} R_{k+1}, \quad \forall k \ge 1,$$

for some constant $C > 10$.

For technical reason, we denote $m_b(\Omega)$, by the median value of function b on a bounded set $\Omega \subset \mathbb{R}^n$ (possibly non-unique) such that

$$\begin{cases} \mu(\{x \in \Omega : b(x) > m_b(\Omega)\}) \le \frac{1}{2}\mu(\Omega), \\ \mu(\{x \in \Omega : b(x) < m_b(\Omega)\}) \le \frac{1}{2}\mu(\Omega). \end{cases} \tag{5.22}$$

Next, for any $k \ge 1$, let $y_k \in X$ be such that $d(x_k, y_k) = MR_k$, $M > 10A_0$, and put

$$\tilde{B}_k = B(y_k, R_k), \quad \tilde{B}_{k,1} = \left\{y \in \tilde{B}_k : b(y) \le m_b(\tilde{B}_k)\right\},$$
$$\tilde{B}_{k,2} = \left\{y \in \tilde{B}_k : b(y) \ge m_b(\tilde{B}_k)\right\};$$

and

$$B_{k,1} = \left\{x \in B_k : b(x) \geq m_b(\tilde{B}_k)\right\}, \quad B_{k,2} = \left\{x \in B_k : b(x) < m_b(\tilde{B}_k)\right\};$$

also

$$F_{k,1} = \tilde{B}_{k,1} \setminus \bigcup_{j=1}^{k-1} \tilde{B}_j, \quad F_{k,2} = \tilde{B}_{k,2} \setminus \bigcup_{j=1}^{k-1} \tilde{B}_j.$$

Note that $F_{k,1} \cap F_{l,1} = \emptyset$ whenever $j \neq k$. Moreover, we have from the definition of the median value that

$$\mu(F_{k,1}) \geq \mu(\tilde{B}_{k,1}) - \sum_{j=1}^{k-1} \mu(\tilde{B}_j) \geq \frac{1}{2}\mu(\tilde{B}_k) - \sum_{j=1}^{k-1} \mu(\tilde{B}_j) \gtrsim \mu(B_k). \tag{5.23}$$

Similarly, we also obtain

$$\mu(F_{k,2}) \gtrsim \mu(B_k). \tag{5.24}$$

Furthermore, we have from the definition of the median value

$$\left|b(x) - m_b(\tilde{B}_k)\right| \leq |b(x) - b(y)|, \quad \forall (x, y) \in B_{k,j} \times \tilde{B}_{k,j}, \ \text{for } j = 1, 2. \tag{5.25}$$

Next, it follows from (5.21) and the triangle inequality that

$$\begin{aligned} c_0 &\leq \frac{1}{\mu(B_k)} \int_{B_k} |b(x) - b_{B_k}|\, d\mu(x) \leq 2\frac{1}{\mu(B_k)} \int_{B_k} |b(x) - m_b(\tilde{B}_k)|\, d\mu(x) \\ &= \frac{2}{\mu(B_k)} \left(\int_{B_{k,1}} |b(x) - m_b(\tilde{B}_k)|\, d\mu(x) + \int_{B_{k,2}} |b(x) - m_b(\tilde{B}_k)|\, d\mu(x) \right). \end{aligned} \tag{5.26}$$

This implies that there exists a subsequence with respect to k such that either

$$\frac{1}{\mu(B_k)} \int_{B_{k,1}} |b(x) - m_b(\tilde{B}_k)|\, d\mu(x) \geq \frac{c_0}{2}, \tag{5.27}$$

or

$$\frac{1}{\mu(B_k)} \int_{B_{k,2}} |b(x) - m_b(\tilde{B}_k)|\, d\mu(x) \geq \frac{c_0}{2}, \tag{5.28}$$

for any $k \geq 1$. Thus, one can assume without loss of generality that (5.27) occurs.

For any $k \geq 1$, applying Lemma 5.3 and (5.23) yields

$$M^{-n} \lesssim \frac{\mu(B_k)}{\mu(MB_k)} \lesssim \frac{\mu(F_{k,1})}{\mu(MB_k)} \lesssim \left|T\left(\mathbf{1}_{F_{k,1}}\right)(x)\right|, \quad \forall x \in B_k.$$

In addition, $T\left(\mathbf{1}_{F_{k,1}}\right)(x)$ is a constant sign in B_k. Then, it follows from (5.25), (5.27), and Lemma 5.3 that

$$\begin{aligned} M^{-n} &\lesssim \frac{\mu(B_k)}{\mu(MB_k)} \lesssim \frac{\mu(B_k)}{\mu(MB_k)} \frac{1}{\mu(B_k)} \int_{B_{k,1}} \left|b(x) - m_b(\tilde{B}_k)\right| d\mu(x) \\ &\lesssim \frac{1}{\mu(B_k)} \int_{B_{k,1}} \left|b(x) - m_b(\tilde{B}_k)\right| \left|T\left(\mathbf{1}_{F_{k,1}}\right)(x)\right| d\mu(x) \\ &= \frac{1}{\mu(B_k)} \int_{B_{k,1}} \left|\int_{\mathbb{R}^n} \left(b(x) - m_b(\tilde{B}_k)\right) K(x,y)\mathbf{1}_{F_{k,1}}(y)\, d\mu(y)\right| d\mu(x) \\ &\leq \frac{1}{\mu(B_k)} \int_{B_{k,1}} \left|\int_{F_{k,1}} (b(x) - b(y))\, K(x,y)\, d\mu(y)\right| d\mu(x) \\ &= \frac{1}{\mu(B_k)} \int_{B_{k,1}} \left|[b,T]\left(\mathbf{1}_{F_{k,1}}\right)(x)\right| d\mu(x) \\ &\leq \frac{1}{\mu(B_k)\varphi(R_k)} \int_{B_k} |[b,T](\phi_k)(x)|\, d\mu(x)\,, \end{aligned} \tag{5.29}$$

where $\phi_k(x) = \varphi(R_k)\mathbf{1}_{F_{k,1}}(x)$, for $k \geq 1$.

Applying Hölder's inequality in (5.29) yields

$$M^{-n} \lesssim \frac{1}{\mu(B_k)\varphi(R_k)} \|[b,T](\phi_k)\|_{L^{p,r}(B_k)}\, \mu(B_k)^{\frac{1}{p'}} \leq \|[b,T](\phi_k)\|_{\mathbf{M}_\varphi^{p,r}}\,.$$

Since $[b,T]$ maps $\mathbf{M}_\varphi^{p,r}(X) \to \mathbf{M}_\varphi^{p,r}(X)$ continuously, then we deduce from the last inequality that

$$M^{-n} \lesssim \|\phi_k\|_{\mathbf{M}_\varphi^{p,r}}, \quad \forall k \geq 1\,. \tag{5.30}$$

Next, thanks to (2.5) and the definition of $F_{k,1}$, we get

$$\|\phi_k\|_{\mathbf{M}_\varphi^{p,r}} \leq \varphi(R_k) \left\|\mathbf{1}_{\tilde{B}_k}\right\|_{\mathbf{M}_\varphi^{p,r}} \lesssim 1, \quad \forall k \geq 1. \tag{5.31}$$

Combining (5.30) and (5.31) yields

$$\|\phi_k\|_{\mathbf{M}_\varphi^{p,r}} \approx 1\,. \tag{5.32}$$

Thanks to the compactness of $[b,T]$ on $\mathbf{M}_\varphi^{p,r}(X)$, there exists a subsequence of $\{[b,T](\phi_k)\}_{k\geq 1}$ (still denoted as $\{[b,T](\phi_k)\}_{k\geq 1}$) such that

$$[b,T](\phi_k) \to \Phi \quad \text{in } \mathbf{M}_\varphi^{p,r}(X)\,, \tag{5.33}$$

as $k \to \infty$.

By (5.32), we also obtain

$$\|\Phi\|_{\mathbf{M}_\varphi^{p,r}} \approx 1\,. \tag{5.34}$$

Next, for any $q \in (p, \infty)$, since $b \in \mathrm{BMO}(X)$, and T is a linear of Calderón–Zygmund type, then $[b, T]$ maps $L^q(X) \to L^q(X)$ continuously for $q \in (1, \infty)$.

As a result, we obtain

$$\|[b, T](\phi_k)\|_{L^q} \lesssim \|b\|_{\mathrm{BMO}} \|\phi_k\|_{L^q} \leq \|b\|_{\mathrm{BMO}} \varphi(R_k) \left\|\mathbf{1}_{B_k}\right\|_{L^q} = \|b\|_{\mathrm{BMO}} \varphi(R_k) \mu\left(B_k\right)^{\frac{1}{q}} \approx \|b\|_{\mathrm{BMO}} \varphi(R_k) R_k^{n/q}. \tag{5.35}$$

Since $R_k \to \infty$, then we have $2^{\gamma(k)} \leq R_k < 2^{\gamma(k)+1}$, with $\gamma(k) = [\log_2 R_k]$, and $[l]$ denotes by the integer part of a real number l. Thanks to (1.5) and the monotonicity of φ, we achieve

$$\varphi(R_k) \leq \varphi(2^{\gamma(k)}) \leq D^{\gamma(k)} \varphi(1).$$

By inserting this fact into (5.35), we obtain

$$\|[b, T](\phi_k)\|_{L^q} \lesssim \|b\|_{\mathrm{BMO}} D^{\gamma(k)} \varphi(1) 2^{(\gamma(k)+1)n/q} \lesssim \|b\|_{\mathrm{BMO}} \left(2^{\frac{n}{q}} D\right)^{\gamma(k)}. \tag{5.36}$$

To this end, we only take q large enough such that $2^{\frac{n}{q}} D < 1$. Then, the right hand side of (5.36) tends to 0 as $k \to \infty$ since (1.15),.

This implies that $\|[b, T](\phi_k)\|_{L^q} \to 0$. Thus, $\Phi = 0$ a.e. in X. This contradicts (5.32).

Similarly, we also obtain a contradiction if (5.28) holds true. In summary, b cannot violate (i).

Case 2. Assume that b violates (ii). The proof of this case is similar to the one of **Case 1**. Thus, we leave its detail to the reader.

Case 3. The proof of this case is most like that of **Case 1** by considering δ_k in place of R_k, with $\delta_k \to 0$. Since we want to repeat the above proof for δ_k in place of R_k, then it is necessary to make some changes as follows:

Since $\delta_k \to 0$, then for every $C > 10$, there is a subsequence of $\{\delta_k\}_{k \geq 1}$ (still denoted as $\{\delta_k\}_{k \geq 1}$) such that $\delta_{k+1} \leq \frac{1}{C} \delta_k$.

Furthermore, we need to redefine $F_{k,1}$ (resp. $F_{k,2}$):

$$F_{k,1} = \tilde{B}_{k,1} \setminus \bigcup_{j=k+1}^{\infty} \tilde{B}_j, \quad F_{k,2} = \tilde{B}_{k,2} \setminus \bigcup_{j=k+1}^{\infty} \tilde{B}_j.$$

By the definition of the median value, it is not difficult to verify that $\mu(F_{k,1}) \approx \mu(\tilde{B}_k)$, and $\mu(F_{k,2}) \approx \mu(\tilde{B}_k)$ for $k \geq 1$. This enable us to repeat the proof of **Case 1** in order to get (5.33) and (5.34).

Next, for $q \in (1, p)$ we repeat the proof of (5.35) to obtain

$$\|[b, T](\phi_k)\|_{L^q} \lesssim \|b\|_{\mathrm{BMO}} \varphi(\delta_k) \mu(B_k)^{\frac{1}{q}}. \tag{5.37}$$

Since $\varphi(t)\mu(B_t)^{\frac{1}{p}}$ is nondecreasing and $\delta_k \to 0$, then we observe that

$$\varphi(\delta_k)\mu(B_k)^{\frac{1}{q}} = \left(\varphi(\delta_k)\mu(B_k)^{\frac{1}{p}}\right)\mu(B_k)^{\frac{1}{q}-\frac{1}{p}} \to 0$$

as $k \to \infty$.

Again, we get that $[b, T](\phi_k) \to 0$ in $L^q(X)$, when $k \to \infty$. This contradicts (5.34). Therefore, b must satisfy (iii).

From the above cases, we conclude that $b \in \mathrm{CMO}(X)$. Hence, we complete the proof of Theorem 1.17. □

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