



High Energy Solutions for p -Kirchhoff Elliptic Problems with Hardy–Littlewood–Sobolev Nonlinearity

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Abstract

This article deals with the study of the following Kirchhoff–Choquard problem:

$$M \left(\int_{\mathbb{R}^N} |\nabla u|^p \right) (-\Delta_p)u + V(x)|u|^{p-2}u = \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u), \quad \text{in } \mathbb{R}^N,$$
$$u > 0, \quad \text{in } \mathbb{R}^N,$$

where M models Kirchhoff-type nonlinear term of the form $M(t) = a + bt^{\theta-1}$, where $a, b > 0$ are given constants; $1 < p < N$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian operator; potential $V \in C^2(\mathbb{R}^N)$; f is monotonic function with suitable growth conditions. We obtain the existence of a positive high energy solution for $\theta \in \left[1, \frac{2N-\mu}{N-p}\right)$ via the Pohožaev manifold and linking theorem. Apart from this, we also studied the radial symmetry of solutions of the associated limit problem.

Keywords p -Laplacian · Hardy–Littlewood–Sobolev inequality · Pohožaev manifold · Radial solution · Kirchhoff equation

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1 Introduction

The classical work by Ding and Ni in [11] and Rabinowitz in [26] opens a new stage in solving partial differential equations. The authors proved the existence of critical points using constrained minimization. The innovative idea was to establish that the mountain-pass min–max level of the functional J , defined on the Hilbert space H and associated with the equation, is equal to the minimum of J restricted to the so-called Nehari manifold $\mathcal{N} = \{u \in H \mid \langle J'(u), u \rangle = 0\}$. Further, this technique has solved numerous problems, given the homogeneous and superlinear condition at infinity of the nonlinearity.

In contrast, if the nonlinear term is not homogeneous and superlinear, then we cannot employ the strategy of minimizing over the Nehari manifold. To overcome this, Jeanjean and Tanaka [17] defined the Pohožaev manifold and obtained solutions as minimizers of the functional over this manifold. Afterward, many authors used this technique to prove the existence of ground-state solutions to elliptic equations. In this article, we are interested in studying the equations for the p -Kirchhoff type operator with nonlocal non-homogeneous nonlinearity and without the superlinear condition at infinity.

Elliptic equations of the Kirchhoff type have been studied extensively in the literature. Consider the following model problem

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x, u) \text{ in } \mathbb{R}^N \quad (1.1)$$

where $f(x, t)$ is subcritical and superlinear at $t = 0$. For this class of problems, extensive literature exists on the existence and multiplicity of results using variational methods. For instance, one can see [1, 8, 12, 14, 24, 27–29, 33] and the references therein. For the energy functional associated to (1.1), Kirchhoff term is homogeneous with degree 4, so to obtain the geometric structure and the boundedness of Palais-Smale sequences for the energy functional, one usually assumes one of the following conditions:

(SL) 4-superlinear condition:

$$\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{t^4} = \infty \text{ uniformly in } x \in \mathbb{R}^N, \text{ where } F(x, t) = \int_0^t f(x, s) ds;$$

(AR) Ambrosetti–Rabinowitz type condition: For all $t \in \mathbb{R}$, there exists $\lambda > 4$ such that

$$f(x, t)t \geq \lambda F(x, t) > 0;$$

(VC) Variant convex condition: $\frac{f(x,t)}{|t|^3}$ is strictly increasing for $t \in \mathbb{R} \setminus \{0\}$.

Even though these conditions seem to be fundamental, they are restrictive in nature. For example, $f(x, t) = |t|^{q-2}t$, when $2 < q \leq 4$, the above conditions do not hold. So the elementary question arises as to whether one can obtain a ground-state solution without these conditions.

In 2014, Li and Ye [20] worked in this direction for the Laplacian and obtained a positive ground-state solution of (1.1) on $H^1(\mathbb{R}^3)$ for $f(x, u) = |u|^{q-2}u$ and $3 < q \leq 4$ with some suitable assumptions on V . They used the minimizing argument on a new manifold called Nehari-Pohožaev manifold, defined as the set of all functions $u \in H^1(\mathbb{R}^3)$ satisfying $\langle I'(u), u \rangle + Q(u) = 0$, where $I(u)$ is the energy functional and $Q(u) := \lim_{n \rightarrow \infty} \langle I'(u), \phi_n(x \cdot \nabla u) \rangle$ is the Pohožaev functional for (1.1) and $\{\phi_n\}$ are suitable test functions.

Subsequently, Guo [13] generalized the results of [20] for a general nonlinearity $f(u)$ with the following assumptions:

$$f \in C^1 \text{ and } \frac{f(s)}{s} \text{ is strictly increasing in } (0, \infty).$$

Here, the author established that (1.1) admits a least energy solution using Jeanjean’s [16] monotonicity approach on a Nehari-Pohožaev type manifold.

However, in [34], Ye, with the superlinear condition at infinity, observed that the ground-state solution for (1.1) does not exist. Hence the author obtained a positive high energy solution by assuming suitable conditions for the potential function V . This was achieved by using the linking theorem on Nehari-Pohožaev type manifold.

For more results, one can refer to [2, 14, 18, 30].

On the other hand, problems that involve two nonlocal terms have been studied by Lü [22] with the potential $V(x) = 1 + \lambda g(x)$. More precisely, he considered the problem

$$-M \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^q}{|x - y|^{3-\alpha}} dy \right) |u|^{q-2}u, \quad u > 0 \text{ in } \mathbb{R}^3$$

where $M(t) = a + bt$, $q \in (2, 3 + \alpha)$, $\lambda > 0$ is a parameter and $g(x)$ is a non-negative steep potential well function. For λ large enough, Lü proved the existence of ground-state solutions using the Nehari manifold and the concentration-compactness principle. Since the associated functional is not bounded below, the assertions used here can not be carried forward for the case of $q \in (1 + \alpha/3, 2]$. But using Nehari-Pohožaev manifold with some strong assumptions on the potential $V(x)$, Chen & Liu [9], obtained a ground-state solution in \mathbb{R}^3 for the complete range, $q \in (1 + \alpha/3, 3 + \alpha)$. For more recent results on ground-state solutions, refer to [10, 35].

Inspired by the works described above, in this paper, we study the existence of solutions for a class of p -Laplace equations with a general Kirchoff terms $M(t) = a + bt^{\theta-1}$ and the nonlinearity that does not satisfy (SL), (AR), and (VC) assumptions.

Precisely, we study the following class of Kirchhoff problems with the general Choquard term:

$$\begin{aligned}
 & -M \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right) \Delta_p u + V(x)|u|^{p-2}u \\
 & = \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u), \quad u > 0 \text{ in } \mathbb{R}^N,
 \end{aligned} \tag{1.2}$$

where $N > \max\{2, p\}$ with $1 < p < N$, a, b, θ are positive parameters and $0 < \mu < N$. Here, Δ_p is the p -Laplacian operator, defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. The function $f \in C(\mathbb{R}, \mathbb{R})$ is such that $f(t) \equiv 0$ for $t \leq 0$ and F is the primitive of f . The general function f satisfies the following growth conditions:

- (\mathcal{F}_1) $\lim_{t \rightarrow 0} \frac{f(t)}{t^{p^*, \mu - 1}} = 0$;
- (\mathcal{F}_2) f is a monotonic function and for some positive constant C and $q \in (p^*, \mu, p_\mu^*)$

$$|f(t)t| \leq C|t|^{p^*, \mu} + C|t|^q;$$

where $p^*, \mu = \frac{p(2N-\mu)}{2N}$ is the lower critical exponent and $p_\mu^* = \frac{p(2N-\mu)}{2(N-p)}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. Moreover, we assume that the potential function $V \in C^2(\mathbb{R}^N, \mathbb{R})$ verifies:

- (\mathcal{V}_1) $\lim_{|x| \rightarrow \infty} V(x) := V_\infty > 0$;
- (\mathcal{V}_2) $\nabla V(x) \cdot x \leq 0$, for all $x \in \mathbb{R}^N$ with the strict inequality on a subset of positive Lebesgue measure;
- (\mathcal{V}_3) $NV(x) + \nabla V(x) \cdot x > NV_\infty$, for all $x \in \mathbb{R}^N$;
- (\mathcal{V}_4) $N\nabla V(x) \cdot x + x \cdot H(x) \cdot x \leq 0$, for all $x \in \mathbb{R}^N$ where H represents the Hessian matrix of the function V .

Remark 1.1 Since we are looking for positive solutions, we assume $f(t) = 0$ for $t \leq 0$, which does not allow the typical case $f(t) = |t|^{\alpha-2}t$, $t \in \mathbb{R}$.

Remark 1.2 For the case when $q = p^*, \mu$, the function would not satisfy the assumption (\mathcal{F}_1). For the case when $q = p_\mu^*$, the upper critical exponent, then the nonlinearity would dominate the Kirchhoff term and we naturally have the mountain-pass geometry, unlike the subcritical case we have explored in this article, which requires a different methodology.

We established the existence of high energy solutions to (1.2) for a larger class of nonlinearity f . Here, high energy solutions are those solutions with energy level greater than the ground-state level (least energy level).

This paper presents a new approach to show the existence results to (1.2) for all $N \geq 3$. Most of the literature mentioned above for Kirchhoff problems is for the case when $N = 3$, as the case $N \geq 3$ is more intricate. The techniques and ideas presented in [34] cannot be extended to higher dimensions using the Nehari-Pohožaev

type manifold. This is primarily because the super quadratic growth in the Kirchhoff operator dominates the nonlinearity.

We study the problem with the Kirchhoff operator $M(t) = a + bt^{\theta-1}$, for all values of $\theta \in \left[1, \frac{2N-\mu}{N-p}\right)$.

We remark that we incorporated the case $\theta = 2$, that is, $M(t) = a + bt$ when $\mu < \min\{N, 2p\}$. To successfully establish the existence of a solution to (1.2), we need first to study the following limit problem

$$-M \left(\int_{\mathbb{R}^N} |\nabla u|^p \right) \Delta_p u + V_\infty |u|^{p-2} u = \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u) \text{ in } \mathbb{R}^N. \tag{1.3}$$

We study the existence of a solution to this limit problem using the concentration-compactness arguments of Lions and the radial symmetry nature of the solution.

Proving the radial symmetry nature of the solutions for this class of equations is an independent topic of interest. In the Laplacian case ($p = 2, M = 1$), the radial nature of the solution was established long back by Li & Ni [19] for general ground-state solutions, under the condition $f'(s) < 0$. However, the same approach cannot be carried forward for the p -Laplacian case ($p \neq 2, M = 1$). In [3], the radial symmetry of solutions for a class of p -Laplacian equations was proved under the assumption that solution v has only one critical point in \mathbb{R}^N . Subsequently, many researchers obtained radial symmetry result by using the *a priori* estimates on the solutions. But in our case, we establish the radial nature of the solution by aptly using the nonlocal nature of nonlinearity.

Our arguments are based on the Euler-Lagrange equation satisfied by polarizations and equality cases in polarization inequalities. We also establish some inequalities in the frame of polarization which we did not find explicitly in the literature. We dedicate Sects. 3 and 4 to establish that the ground state solution of (1.3) is radially decreasing. We develop a unified approach in this paper to study both (1.2) and (1.3). In the next section, we give some preliminary framework of the problem and state the main results of the article.

2 Preliminaries and Main Results

This section of the article is intended to provide the variational setting. Refer to [21, 36] for the complete and rigid details. Further in this section, we state the main results of the current article with a short sketch of the proof. The functional space associated to this problem is

$$W^{1,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p < \infty \right\}$$

with the norm,

$$\|u\|^p = a \int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} V(x)|u|^p dx.$$

It is easy to see from the condition $(V_2) - (V_3)$, $V_\infty \leq V(x)$ and from (V_1) we get $\sup_{\mathbb{R}^N} V(x) < \infty$, which infers that the norm is well-defined.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality) *Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t, r, \mu, N)$ independent of f, h such that*

$$\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x - y|^\mu} dx dy \leq C(t, r, \mu, N) \|f\|_{L^t(\mathbb{R}^N)} \|h\|_{L^r(\mathbb{R}^N)}.$$

By Hardy–Littlewood–Sobolev inequality and condition (F_2) , there exists constant $C > 0$ such that for each $u \in W^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x - y|^\mu} dx dy &\leq C \left[\int_{\mathbb{R}^N} |u|^p + |u|^{\frac{2Nq}{2N-\mu}} dx \right]^{\frac{2p^*,\mu}{p}} \\ &\leq C \|u\|_{L^p}^{2p^*,\mu} + C \|u\|_{L^{\frac{2Nq}{2N-\mu}}}^{2q}. \end{aligned} \tag{2.1}$$

Lemma 2.1 *Let $1 < p \leq \infty, 1 \leq q < \infty$, with $q \neq \frac{Np}{N-p}$ if $p < N$. Assume that $\{v_n\}$ is bounded in $L^q(\mathbb{R}^N)$, $\{|\nabla v_n|\}$ is bounded in $L^p(\mathbb{R}^N)$ and $\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^q dx \rightarrow 0$, for some $R > 0$ as $n \rightarrow \infty$. Then $v_n \rightarrow 0$ in $L^\alpha(\mathbb{R}^N)$, for $\alpha \in (q, \frac{Np}{N-p})$.*

The energy functionals associated with the problems (1.2) and (1.3) are $J, J_\infty : W^{1,p}(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} J(u) &= \frac{a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx \\ &\quad - \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x - y|^\mu} dx dy, \\ J_\infty(u) &= \frac{a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta + \frac{V_\infty}{p} \int_{\mathbb{R}^N} |u|^p dx \end{aligned}$$

$$-\frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu} dx dy.$$

Observe that both the functionals $J, J_\infty \in C^1$, that is, for any $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned} \langle J'(u), \phi \rangle &= a \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx + b \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\theta-1} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \\ &\quad + \int_{\mathbb{R}^N} V(x) |u|^{p-2} u \phi \, dx - \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\phi(x)}{|x-y|^\mu} dx dy, \end{aligned}$$

and

$$\begin{aligned} \langle J'_\infty(u), \phi \rangle &= a \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx + b \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\theta-1} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \\ &\quad + V_\infty \int_{\mathbb{R}^N} |u|^{p-2} u \phi \, dx - \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\phi(x)}{|x-y|^\mu} dx dy. \end{aligned}$$

We also define the following Pohožaev manifolds associated with problems (1.2) and (1.3)

$$\begin{aligned} \mathcal{P} &= \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : P(u) = 0\} \text{ and} \\ \mathcal{P}_\infty &= \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : P_\infty(u) = 0\}. \end{aligned}$$

Here, $P(u)$ and $P_\infty(u)$ are the Pohožaev identities associated with problems (1.2) and (1.3) respectively, (see [23, Theorem 3])

$$\begin{aligned} P(u) &:= \frac{(N-p)a}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{(N-p)b}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^\theta \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^N} (NV(x) + \nabla V(x) \cdot x) |u|^p \\ &\quad - \frac{2N-\mu}{2} \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu} dx dy, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
 P_\infty(u) := & \frac{(N-p)a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N-p)b}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\
 & + \frac{NV_\infty}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{2N-\mu}{2} \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu} dx dy.
 \end{aligned}
 \tag{2.3}$$

With these preliminaries, we state our main results

Theorem 2.1 *Assume that f and V satisfy $(\mathcal{F}_1) - (\mathcal{F}_2)$ and $(\mathcal{V}_1) - (\mathcal{V}_4)$. Then, problem (1.3) has a positive ground-state solution that is radially symmetric.*

Theorem 2.2 *Assume that f and V satisfy $(\mathcal{F}_1) - (\mathcal{F}_2)$ and $(\mathcal{V}_1) - (\mathcal{V}_4)$ along with (\mathcal{V}_5) there exists a constant $T > 1$ such that*

$$\sup_{x \in \mathbb{R}^N} V(x) \leq V_\infty + \frac{pJ_\infty(\bar{u})}{T^N \int_{\mathbb{R}^N} |\bar{u}|^p},$$

where \bar{u} is the ground-state solution of the functional J_∞ obtained in Theorem 2.1. Then there exists at least one positive solution to problem (1.2), which is a high energy solution.

A standard methodology to obtain solutions using variational techniques consists of looking for minimizers of the functional. Here, using the (SL) and (AR) conditions, one can prove the boundedness of the minimizing sequences, which will subsequently give us solutions. But in the above-mentioned results, due to the absence of conditions (SL) or (AR), verifying the mountain-pass geometry and obtaining the boundedness of the Palais-Smale sequence for the functionals J and J_∞ is a tedious job. Moreover, the lack of condition (VC) makes the classical method of the Nehari manifold inappropriate. Taking into account all these points, we observed that J_∞ satisfies the mountain-pass geometry of the functional with the scaling function u_t defined as

$$u_t(x) := u(t^{-1}x),$$

for $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. Keeping this in mind, we take minimum over the Pohožaev manifold \mathcal{P}_∞ .

Using the assumptions on f and V_∞ , we conclude that

- (1) \mathcal{P}_∞ is a natural constraint, in the sense that J_∞ is coercive and bounded below on \mathcal{P}_∞
- (2) Every critical point of J_∞ restricted to \mathcal{P}_∞ is a critical point of J_∞
- (3) $p_\infty := \inf_{\mathcal{P}_\infty} J_\infty(u) = c_\infty$ where c_∞ is the mountain-pass level.

Further, to construct a bounded Palais-Smale sequence for J_∞ , we employ Jeanjean’s [15] technique and prove the existence of a nontrivial critical point of functional J_∞ by the concentration-compactness lemma. This motivates us to use the same type of approach for the existence of a solution to (1.2). That is, we look for the solution in the Pohožaev manifold \mathcal{P} and with the same type of scaling, one can establish that the functional J satisfies the mountain-pass geometry. However, we encounter that the functional J could not attain the mountain-pass level, thus, indicating the absence of a ground-state solution for equation (1.2). To overcome this obstacle, we establish that J satisfies the Palais-Smale condition above the min-max level, leading to solutions with higher energy. To address the issue of compactness, we employ the splitting lemma (refer to Lemma 5.9). Additionally, we utilize the linking theorem in conjunction with a Barycenter mapping to achieve the goal in Theorem 2.2. To successfully apply the machinery of Barycenter mapping, we need the radial nature of ground-state solution to (1.3). The salient feature of the approach is the existence of the radial solution to (1.3). To the best of our knowledge, there has been no attempt till now on the existence of high energy solutions to a generalized class of Kirchhoff-Choquard equation driven by the p -Laplacian operator.

Remark 2.1 Conditions $(\mathcal{V}_1), (\mathcal{V}_2)$ and (\mathcal{V}_3) imply that

$$\nabla V(x) \cdot x \rightarrow 0, \text{ if } |x| \rightarrow \infty.$$

Remark 2.2 There are a number of functions that satisfy $(\mathcal{V}_1) - (\mathcal{V}_5)$. For example, $V(x) = V_\infty + \frac{C}{|x|+1}$, where $C = \frac{pJ_\infty(\bar{u})}{T^N \int_{\mathbb{R}^N} |\bar{u}|^p}$.

Throughout the paper, we make use of the following notations:

- $u^\pm := \max\{\pm u, 0\}$ and $(H^*, \|\cdot\|_*)$ denotes the dual space of $(H, \|\cdot\|)$.
- For any $u \in W^{1,p}(\mathbb{R}^N)$

$$\|F(u)\|_0 := \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu} dx dy.$$

- For any $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, $u_t(x) := u(t^{-1}x)$ for $t > 0$.
- The letters C , and C_i denote various positive constants possibly different in different places.

We organize the rest of the paper as follows. In Sect. 3, we give some technical lemmas that will help prove our Theorem 2.1. In Sect. 4, we study the limit problem and prove it has a positive radial solution. In Sect. 5, we give some background material for Theorem 2.2, and in the last section, we present its proof.

3 Technical Lemmas

In this section, we will first collect some variational frameworks and results that form the background material. Then we prove the limiting case problem (1.3), using a couple of critical results.

First, we look at the Mountain-pass geometry of the functional J_∞ .

Lemma 3.1 *The functional J_∞ satisfies the following conditions:*

- (i) *There exist $\alpha, \rho > 0$ such that $J_\infty(u) > \alpha$ for $\|u\| = \rho$.*
- (ii) *$J_\infty(0) = 0$ and there exists $e \in W^{1,p}(\mathbb{R}^N)$ with $\|e\| > \rho$ and $J_\infty(e) < 0$.*

Proof (i) From (2.1) and Sobolev embedding theorem, there exist constants $C_1, C_2 > 0$ such that for each $u \in W^{1,p}(\mathbb{R}^N)$,

$$\|F(u)\|_0 \leq C_1 \|u\|^{2 \cdot p^*, \mu} + C_2 \|u\|^{2q}.$$

Then it follows that

$$J_\infty(u) \geq C \left(\|u\|^p - \|u\|^{2 \cdot p^*, \mu} - \|u\|^{2q} \right).$$

It is easy to see that there exists $\alpha > 0$ such that $J_\infty(u) > \alpha$, for $\|u\| = \rho$, provided that ρ is sufficiently small.

- (ii) For any $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$ we define $u_t(x) := u(\frac{x}{t})$.

$$\begin{aligned} J_\infty(u_t) &= \frac{at^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{bt^{(N-p)\theta}}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ &\quad + \frac{V_\infty t^N}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{t^{2N-\mu}}{2} \|F(u)\|_0. \end{aligned}$$

Since, $\max\{N - p, (N - p)\theta, N\} < 2N - \mu$, therefore, as $t \rightarrow \infty$ $J_\infty(u_t) \rightarrow -\infty$. Then for sufficiently large t , we have $e = u_t \in W^{1,p}(\mathbb{R}^N)$, with $\|e\| > \rho$, such that $J_\infty(e) < 0$. This completes the proof. □

By the classical Mountain-pass theorem, we have the minimax value defined as

$$c_\infty := \inf_{h \in \Gamma_\infty} \max_{t \in [0,1]} J_\infty(h(t)), \tag{3.1}$$

where

$$\Gamma_\infty = \{h \in C([0, 1], W^{1,p}(\mathbb{R}^N)) : h(0) = 0 \text{ and } J_\infty(h(1)) < 0\}.$$

By Lemma 3.1, there exists a $(PS)_{c_\infty}$ sequence $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)$ at the level c_∞ , that is,

$$J_\infty(u_n) \rightarrow c_\infty \quad \text{and} \quad J'_\infty(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, we have established the existence of the Palais-Smale sequence. Now, to ensure that the Pohožaev identity acting on the Palais-Smale sequence approaches zero, we will use Jeanjean’s technique [15].

Lemma 3.2 *There exists a sequence $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)$ such that, as $n \rightarrow \infty$,*

$$J_\infty(u_n) \rightarrow c_\infty, \quad J'_\infty(u_n) \rightarrow 0 \quad \text{and} \quad P_\infty(u_n) \rightarrow 0. \tag{3.2}$$

Moreover, $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$.

Proof Define the map $\phi : W^{1,p}(\mathbb{R}^N) \times \mathbb{R} \rightarrow W^{1,p}(\mathbb{R}^N)$ by

$$\phi(u, t) = u(e^{-t}x), \quad \text{for all } x \in \mathbb{R}^N.$$

For any $(u, t) \in W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$, we introduce the composite functional $J_\infty \circ \phi$ computed as

$$\begin{aligned} (J_\infty \circ \phi)(u, t) = & \frac{ae^{t(N-p)}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{be^{t(N-p)\theta}}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ & + \frac{V_\infty e^{tN}}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{e^{t(2N-\mu)}}{2} \|F(u)\|_0, \end{aligned}$$

then $J_\infty \circ \phi$ is a C^1 -functional. Define the family of paths

$$\tilde{\Gamma}_\infty = \{ \tilde{\gamma} \in C([0, 1], W^{1,p}(\mathbb{R}^N) \times \mathbb{R}) : \tilde{\gamma}(0) = (0, 0) \text{ and } (J_\infty \circ \phi)(\tilde{\gamma}(1)) < 0 \},$$

and the corresponding minimax level

$$\tilde{c}_\infty = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_\infty} \max_{t \in [0,1]} (J_\infty \circ \phi)(\tilde{\gamma}(t)).$$

Since $\{\phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}_\infty\} \subset \Gamma_\infty$ and $\Gamma_\infty \times \{0\} \subset \tilde{\Gamma}_\infty$, the mountain-pass levels of the functional J_∞ and $J_\infty \circ \phi$ coincide, that is

$$c_\infty = \tilde{c}_\infty. \tag{3.3}$$

For each $n \in \mathbb{N}$, by (3.1), there exists $\{g_n\} \subset \Gamma_\infty$ such that $\max_{t \in [0,1]} J_\infty(g_n(t)) \leq c_\infty + \frac{1}{n}$.

Setting $\tilde{g}_n(t) := (g_n(t), 0)$ for $t \in [0, 1]$, we see that $\tilde{g}_n \in \tilde{\Gamma}_\infty$ and using (3.3), we

get

$$\max_{t \in [0,1]} (J_\infty \circ \phi)(\tilde{g}_n(t)) \leq \tilde{c}_\infty + \frac{1}{n}.$$

By Ekeland’s variational principle [32, Theorem 8.5], there exists a sequence $\{(v_n, t_n)\} \subset W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$ such that, as $n \rightarrow \infty$,

$$\begin{aligned} (J_\infty \circ \phi)(v_n, t_n) &\rightarrow \tilde{c}_\infty, \quad (J_\infty \circ \phi)'(v_n, t_n) \rightarrow 0, \\ \min_{t \in [0,1]} \|(v_n, t_n) - \tilde{g}_n(t)\|_{W^{1,p}(\mathbb{R}^N) \times \mathbb{R}} &= \min_{t \in [0,1]} (\|v_n - g_n\|^2 + t_n^2)^{\frac{1}{2}} \rightarrow 0. \end{aligned} \tag{3.4}$$

Note that, $|t_n| \rightarrow 0$ as $n \rightarrow \infty$. Let $u_n := \phi(v_n, t_n)$, from (3.4), we have

$$J_\infty(u_n) \rightarrow c_\infty \text{ as } n \rightarrow \infty.$$

It is easy to verify that for every $(w, s) \in W^{1,p}(\mathbb{R}^N) \times \mathbb{R}$,

$$\begin{aligned} &\langle (J_\infty \circ \phi)'(v_n, t_n), (w, s) \rangle \\ &= ae^{t_n(N-p)} \int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla w dx + be^{t_n(N-p)\theta} \left(\int_{\mathbb{R}^N} |\nabla v_n|^p dx \right)^{\theta-1} \int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla w dx \\ &+ V_\infty e^{t_n N} \int_{\mathbb{R}^N} |v_n|^{p-2} v_n w dx - e^{t_n(2N-\mu)} \iint_{\mathbb{R}^{2N}} \frac{F(v_n(y))f(v_n(x))w(x)}{|x-y|^\mu} dx dy + sP_\infty(u_n) \\ &= \langle J'_\infty(u_n), \phi(w, t_n) \rangle + sP_\infty(u_n). \end{aligned} \tag{3.5}$$

Employing (3.5) and (3.4) with $(w, s) = (0, 1)$, we reach the conclusion that

$$P_\infty(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For any $z \in W^{1,p}(\mathbb{R}^N)$, set $w(x) = \phi(z, -t_n) = z(e^{t_n}x)$ and $s = 0$ in (3.5), we get

$$\langle J'_\infty(u_n), z \rangle = \langle (J_\infty \circ \phi)'(v_n, t_n), (\phi(z, -t_n), 0) \rangle.$$

Since $|t_n| \rightarrow 0$, as $n \rightarrow \infty$ we have that $\{\phi(z, -t_n)\}$ is uniformly bounded. Hence, as $n \rightarrow \infty$

$$\langle J'_\infty(u_n), z \rangle = o(1)\|z\|.$$

That is $J'_\infty(u_n) \rightarrow 0$ in $(W^{1,p}(\mathbb{R}^N))^*$. Hence, we have got a sequence $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)$ that satisfies (3.2).

Further, the boundedness of $(PS)_{c_\infty}$ sequence $\{u_n\}$ follows from

$$\begin{aligned}
 C\|u_n\|^p &\leq J_\infty(u_n) - \frac{P_\infty(u_n)}{2N - \mu} \\
 &= \left(\frac{N - \mu + p}{2N - \mu}\right) \frac{a}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p dx + \left(\frac{N - \mu}{2N - \mu}\right) \frac{V_\infty}{p} \int_{\mathbb{R}^N} |u_n|^p dx \\
 &\quad + \left(\frac{2N - \mu - \theta(N - p)}{\theta(2N - \mu)}\right) \frac{b}{p} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right)^\theta,
 \end{aligned}$$

where $C > 0$. This completes the proof. □

Lemma 3.3 *For any $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $\tilde{t} > 0$ such that $u_{\tilde{t}} \in \mathcal{P}_\infty$, where $u_t(x) = u(t^{-1}x)$. Moreover, $J_\infty(u_{\tilde{t}}) = \max_{t>0} J_\infty(u_t)$.*

Proof For any $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$, we define

$$\begin{aligned}
 h_\infty(t) &:= J_\infty(u_t) \\
 &= \frac{at^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{bt^{(N-p)\theta}}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^\theta \\
 &\quad + \frac{V_\infty t^N}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{t^{2N-\mu}}{2} \|F(u)\|_0.
 \end{aligned}$$

Observe that for t small enough, $h_\infty(t) > 0$ and $h_\infty(t) \rightarrow -\infty$ as $t \rightarrow \infty$, since $2N - \mu > \max\{N - p, (N - p)\theta, N\}$. Thus, h_∞ has at least one critical point, say at $\tilde{t} > 0$. Further, taking the derivative of h_∞ , we obtain that $t \frac{d}{dt} h_\infty(t) = P_\infty(u_t)$, which implies $u_{\tilde{t}} \in \mathcal{P}_\infty$.

Now we claim that \tilde{t} is unique. Let us suppose that t_1 and t_2 are two critical points of h_∞ , such that $0 < t_1 < t_2$. Depending upon the range of θ , we consider the following cases:

Case I: $1 \leq \theta \leq \frac{N}{N - p}$

For $i = 1, 2$ and $t_i > 0$, we have

$$\begin{aligned}
 0 = P_\infty(u_{t_i}) &= \frac{(N - p)at_i^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N - p)bt_i^{(N-p)\theta}}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^\theta \\
 &\quad + \frac{NV_\infty t_i^N}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{(2N - \mu)t_i^{2N-\mu}}{2} \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x - y|^\mu} dx dy.
 \end{aligned}$$

Further, we deduce that

$$\begin{aligned} & \frac{(N-p)a}{p} \left(\frac{1}{t_1^p} - \frac{1}{t_2^p} \right) \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N-p)b}{p} \left(\frac{1}{t_1^{N-(N-p)\theta}} - \frac{1}{t_2^{N-(N-p)\theta}} \right) \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ &= \frac{(2N-\mu)}{2} \left(t_1^{N-\mu} - t_2^{N-\mu} \right) \|F(u)\|_0. \end{aligned}$$

This is not true as $0 < t_1 < t_2$.

Case II: $\frac{N}{N-p} < \theta < \frac{2N-\mu}{N-p}$

Following the same argument as in Case I, as $P_\infty(u_{t_i}) = 0$, for $i = 1, 2$ we deduce that

$$\begin{aligned} & \frac{1}{p} \left(\frac{(N-p)a}{t_1^{(N-p)(\theta-1)}} - \frac{(N-p)a}{t_2^{(N-p)(\theta-1)}} \right) \int_{\mathbb{R}^N} |\nabla u|^p dx \\ &+ \frac{NV_\infty}{p} \left(\frac{1}{t_1^{(N-p)\theta-N}} - \frac{1}{t_2^{(N-p)\theta-N}} \right) \int_{\mathbb{R}^N} |u|^p dx \\ &= \frac{(2N-\mu)}{2} \left(t_1^{2N-\mu-(N-p)\theta} - t_2^{2N-\mu-(N-p)\theta} \right) \|F(u)\|_0, \end{aligned}$$

which contradicts $0 < t_1 < t_2$. This completes the proof. □

Set

$$p_\infty = \inf_{\mathcal{P}_\infty} J_\infty(u), \tag{*}$$

which is well defined, since for any $u \in \mathcal{P}_\infty$,

$$J_\infty(u) = J_\infty(u) - \frac{P_\infty(u)}{2N-\mu} \geq 0.$$

Lemma 3.4 For c_∞ as defined in (3.1) and p_∞ as in (*), we have $p_\infty = c_\infty$.

Proof For any $\gamma \in \Gamma_\infty$, we claim that $\gamma([0, 1]) \cap \mathcal{P}_\infty \neq \emptyset$. Indeed, there exists $\rho > 0$, small enough such that for $\|u\| \leq \rho$

$$P_\infty(u) \geq 0$$

and

$$J_\infty(u) - \frac{P_\infty(u)}{2N-\mu} \geq 0, \text{ for any } u \in W^{1,p}(\mathbb{R}^N).$$

Hence, γ crosses the manifold \mathcal{P}_∞ , since $J_\infty(\gamma(1)) < 0$. Therefore,

$$p_\infty \leq \max_{t \in [0,1]} J_\infty(\gamma(t)),$$

which implies that $p_\infty \leq c_\infty$.

On the other hand, for t_0 large enough we see that $J_\infty(u_{t_0}) < 0$, for $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$. Define

$$\bar{\gamma}(t) = \begin{cases} u_{t_0 t} & \text{for } t > 0; \\ 0 & \text{for } t = 0, \end{cases}$$

then $\bar{\gamma} \in \Gamma_\infty$. So

$$c_\infty \leq \max_{t \in [0,1]} J_\infty(u_{t_0 t}) \leq \max_{t > 0} J_\infty(u_{t_0 t}),$$

which gives $c_\infty \leq p_\infty$. □

4 Radial Ground States of Limiting Problem

This section is devoted to the proof of Theorem 2.1. First, we will revisit the polarization technique and derive some necessary lemma to prove the radial symmetry of a given ground-state solution. Then, we deploy the concentration-compactness principle to obtain the existence of a positive radial ground-state solution.

Let $H \subset \mathbb{R}^N$ be a closed half-space and that σ_H denotes the reflection concerning its boundary ∂H . Define $x_H := \sigma_H(x)$ for $x \in \mathbb{R}^N$.

Definition 4.1 The polarization $u^H : \mathbb{R}^N \rightarrow \mathbb{R}$ of $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is defined for $x \in \mathbb{R}^N$ by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\} & \text{if } x \in H \\ \min\{u(x), u(x_H)\} & \text{if } x \notin H \end{cases}$$

Lemma 4.1 [7, Lemma 5.3] *If $u \in W^{1,p}(\mathbb{R}^N)$, then $u^H \in W^{1,p}(\mathbb{R}^N)$. In particular, we have*

$$\|\nabla u\|_{L^p} = \|\nabla u^H\|_{L^p} \text{ and } \|u\|_{L^p} = \|u^H\|_{L^p}.$$

Lemma 4.2 *Let $u \in W^{1,p}(\mathbb{R}^N)$, then*

$$\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x - y|^\mu} dx dy \leq \iint_{\mathbb{R}^{2N}} \frac{F(u^H(x))F(u^H(y))}{|x - y|^\mu} dx dy. \tag{4.1}$$

Proof As F is a non-decreasing function, we get that $F(u^H) = F(u)^H$. Observe that for any $x, y \in \mathbb{R}^N$

$$|x - y| = |x_H - y_H| \leq |x_H - y| = |x - y_H| \tag{4.2}$$

As the space \mathbb{R}^{2N} can be written as

$$\mathbb{R}^{2N} = (H \times H) \cup (\mathbb{R}^N \setminus H \times \mathbb{R}^N \setminus H) \cup (\mathbb{R}^N \setminus H \times H) \cup (H \times \mathbb{R}^N \setminus H).$$

Further, using the fact that if $x \in \mathbb{R}^N \setminus H$, then $x_H \in H$ and also from (4.2), we obtain

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{F(u^H(x))F(u^H(y))}{|x - y|^\mu} dx dy \\ &= \iint_{H \times H} \frac{F(u^H(x))F(u^H(y)) + F(u^H(x_H))F(u^H(y_H))}{|x - y|^\mu} dx dy \\ &+ \iint_{H \times H} \frac{F(u^H(x_H))F(u^H(y)) + F(u^H(x))F(u^H(y_H))}{|x_H - y|^\mu} dx dy. \end{aligned} \tag{4.3}$$

We divide the proof into four cases:

Case I: When $u(x_H) \leq u(x)$ and $u(y_H) \leq u(y)$

From the definition of polarization and (4.3), we deduce that

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{F(u^H(x))F(u^H(y))}{|x - y|^\mu} dx dy = \iint_{H \times H} \frac{F(u(x))F(u(y)) + F(u(x_H))F(u(y_H))}{|x - y|^\mu} dx dy \\ &+ \iint_{H \times H} \frac{F(u(x_H))F(u(y)) + F(u(x))F(u(y_H))}{|x_H - y|^\mu} dx dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x - y|^\mu} dx dy. \end{aligned}$$

Case II: When $u(x) \leq u(x_H)$ and $u(y) \leq u(y_H)$

One can prove the desired result using the arguments as in Case I.

Case III: When $u(x) \leq u(x_H)$ and $u(y_H) \leq u(y)$

Employing the definition of polarization, (4.3) and (4.2), we have

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{F(u^H(x))F(u^H(y))}{|x - y|^\mu} dx dy \\ &= \iint_{H \times H} \left[\frac{1}{|x - y|^\mu} - \frac{1}{|x_H - y|^\mu} \right] (F(u(x_H))F(u(y)) + F(u(x))F(u(y_H))) dx dy \end{aligned}$$

$$\begin{aligned}
 & - \iint_{H \times H} \left[\frac{1}{|x - y|^\mu} - \frac{1}{|x_H - y|^\mu} \right] (F(u(x))F(u(y)) + F(u(x_H))F(u(y_H))) \, dx dy \\
 & + \iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x - y|^\mu} \, dx dy.
 \end{aligned} \tag{4.4}$$

Furthermore, by direct calculation and the assumption of the case, we see that

$$F(u(x_H))F(u(y)) + F(u(x))F(u(y_H)) \geq F(u(x))F(u(y)) + F(u(x_H))F(u(y_H)).$$

Thus, we get the desired claim from (4.4) and above inequality.

Case IV: When $u(x_H) \leq u(x)$ and $u(y) \leq u(y_H)$

Similar to Case III.

Hence the result holds. □

Lemma 4.3 [23, Lemma 5.5] *Let $u \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ and $H \subset \mathbb{R}^N$ be a closed half-space. If $u \geq 0$, then*

$$\iint_{\mathbb{R}^{2N}} \frac{u(x)u(y)}{|x - y|^\mu} \, dx dy \leq \iint_{\mathbb{R}^{2N}} \frac{u^H(x)u^H(y)}{|x - y|^\mu} \, dx dy,$$

with equality if and only if either $u^H = u$ or $u^H = u \circ \sigma_H$.

Lemma 4.4 [23, Lemma 5.6] *Assume that $u \in L^2(\mathbb{R}^N)$ is a non-negative. There exist $x_0 \in \mathbb{R}^N$ and a non increasing function $v : (0, \infty) \rightarrow \mathbb{R}$ such that for almost every $x \in \mathbb{R}^N$, $u(x) = v(|x - x_0|)$ if and only if for every closed half-space $H \subset \mathbb{R}^N$, $u^H = u$ or $u^H = u \circ \sigma_H$.*

Lemma 4.5 *Let $u \in W^{1,p}(\mathbb{R}^N)$ be a positive ground-state solution to problem (1.3). Then u is radially symmetric to any point $x \in \mathbb{R}^N$.*

Proof Using the strategy of Tanaka and Jeanjean [17], we can easily lift the ground-state solution to the path $h \in C([0, 1], W^{1,p}(\mathbb{R}^N))$ such that $h(0) = 0$, $J_\infty(h(t)) < J_\infty(u)$, $J_\infty(h(1)) < 0$. The proof directly follows from the local problem considered in Tanaka and Jeanjean [17, Lemma 2.1]. We define the path $\gamma : [0, \infty) \rightarrow W^{1,p}(\mathbb{R}^N)$ by

$$\gamma(t)(x) = \begin{cases} u\left(\frac{x}{t}\right) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

From the Pohožaev identity, we get for every $t > 0$

$$\begin{aligned}
 J_\infty(\gamma(t)) &= \frac{at^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{bt^{(N-p)\theta}}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\
 &\quad + \frac{V_\infty t^N}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{t^{2N-\mu}}{2} \|F(u)\|_0 \\
 &= \frac{a}{p} \left(t^{N-p} - \frac{N-p}{2N-\mu} t^{2N-\mu} \right) \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b}{p} \left(\frac{t^{(N-p)\theta}}{\theta} - \frac{N-p}{2N-\mu} t^{2N-\mu} \right) \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\
 &\quad + \frac{V_\infty}{p} \left(t^N - \frac{N-p}{2N-\mu} t^{2N-\mu} \right) \int_{\mathbb{R}^N} |u|^p dx.
 \end{aligned}$$

The functional $J_\infty(\gamma)$ achieves strict global maximum at $t = 1$. Thus, for every $t \in [0, \infty) \setminus \{1\}$, $J_\infty(\gamma(t)) < J(u)$. Since $\lim_{t \rightarrow \infty} J(\gamma(t)) = -\infty$, the path γ can then be defined by a suitable change of variable, such that $h(0) = 0$, $J_\infty(h(t)) < J_\infty(u)$, $J_\infty(h(1)) < 0$. Now applying the polarization on the path h , we can achieve that u^H is also a critical point (for a more details, we refer to [23]). Consequently, applying Lemma 4.1, and the fact that $u \in W^{1,p}(\mathbb{R}^N)$ is a positive ground-state solution we deduce that

$$\iint_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu} dx dy \geq \iint_{\mathbb{R}^{2N}} \frac{F(u^H(x))F(u^H(y))}{|x-y|^\mu} dx dy.$$

Employing (4.1) and Lemma (4.3), we get either $F(u^H) = F(u)$ or $F(u^H) = F(u \circ \sigma_H)$. Let us assume that $F(u^H) = F(u)$ in \mathbb{R}^N . Moreover, since F is non-decreasing

$$\int_{u(x_H)}^{u(x)} f(t) dt = F(u(x)) - F(u(x_H)) \begin{cases} \geq 0 & \text{if } x \in H; \\ \leq 0 & \text{if } x \notin H; \end{cases}$$

this implies that $u^H = u$. Similarly in the case of $F(u^H) = F(u \circ \sigma_H)$, we can conclude that $u^H = u \circ \sigma_H$. By Lemma 4.4, we get the desired result. \square

Proof of Theorem 2.1 Let $\{u_n\}$ be a $(PS)_{c_\infty}$ sequence for J_∞ , by Lemma 3.2, we have $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and $P_\infty(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then there exists $A \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \rightarrow A^p.$$

Define $\delta := \limsup_{n \rightarrow \infty} \int_{B_1(y)} |u_n|^p$. Now we divide the proof into the following parts:

Case I: If $\delta = 0$

From Lemma 2.1, we see that $u_n \rightarrow 0$ in $L^r(\mathbb{R}^N)$, for all $r \in (p, p^*)$; hence $\|F(u_n)\|_0 \rightarrow 0$ as $n \rightarrow \infty$. Also, as $P_\infty(u_n) \rightarrow 0$, we have $u_n \rightarrow 0$ in $W^{1,p}(\mathbb{R}^N)$. This implies that $0 = \lim_{n \rightarrow \infty} J_\infty(u_n) = c_\infty$, which is not true.

Case II: If $\delta > 0$

Then there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_n|^p \geq \frac{\delta}{2} > 0.$$

Define $\bar{u}_n(x) := u_n(x + y_n)$, $\{\bar{u}_n\}$ is bounded $(PS)_{c_\infty}$ for J_∞ with $P_\infty(\bar{u}_n) \rightarrow 0$ as $n \rightarrow \infty$. Then there exists $\bar{u} \in W^{1,p}(\mathbb{R}^N)$ such that up to a sub-sequence

$$\begin{aligned} \bar{u}_n &\rightharpoonup \bar{u} \text{ weakly in } W^{1,p}(\mathbb{R}^N); \\ \int_{\mathbb{R}^N} |\nabla \bar{u}_n|^p &\rightarrow A^p, \end{aligned} \tag{4.5}$$

and by Fatou’s Lemma, we conclude that

$$\int_{\mathbb{R}^N} |\nabla \bar{u}|^p \leq A^p.$$

Let us assume that

$$\int_{\mathbb{R}^N} |\nabla \bar{u}|^p < A^p. \tag{4.6}$$

Set

$$J_{A,\infty}(u) = \frac{a + bA^{p(\theta-1)}}{p} \int_{\mathbb{R}^N} |\nabla u|^p + \frac{V_\infty}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{1}{2} \|F(u)\|_0.$$

Employing (4.5), we deduce that $J'_{A,\infty}(\bar{u}) = 0$, also \bar{u} satisfies the Pohožaev identity associated with $J_{A,\infty}$

$$\begin{aligned} P_{A,\infty}(\bar{u}) &= \left(a + bA^{p(\theta-1)}\right) \frac{(N-p)}{p} \int_{\mathbb{R}^N} |\nabla \bar{u}|^p dx \\ &+ \frac{NV_\infty}{p} \int_{\mathbb{R}^N} |\bar{u}|^p dx - \frac{2N-\mu}{2} \|F(\bar{u})\|_0 = 0. \end{aligned} \tag{4.7}$$

Putting together (4.6) and (4.7), we get

$$0 = P_{A,\infty}(\bar{u}) > P_\infty(\bar{u}).$$

Hence by Lemma 3.3, there exists $t_0 < 1$ such that $\bar{u}_{t_0} \in \mathcal{P}_\infty$. Using this information, we get

$$\begin{aligned} c_\infty = \mathfrak{p}_\infty &\leq J_\infty(\bar{u}_{t_0}) - \frac{P_\infty(\bar{u}_{t_0})}{2N - \mu} \\ &< \left(\frac{N - \mu + p}{2N - \mu}\right) \frac{a}{p} \int_{\mathbb{R}^N} |\nabla \bar{u}|^p dx \\ &\quad + \left(\frac{1}{\theta} - \frac{N - p}{2N - \mu}\right) \frac{b}{p} \left(\int_{\mathbb{R}^N} |\nabla \bar{u}|^p dx\right)^\theta \\ &\quad + \left(\frac{N - \mu}{2N - \mu}\right) \frac{V_\infty}{p} \int_{\mathbb{R}^N} |\bar{u}|^p dx \\ &\leq \liminf_{n \rightarrow \infty} \left[J_\infty(\bar{u}_n) - \frac{P_\infty(\bar{u}_n)}{2N - \mu} \right] = c_\infty, \end{aligned} \tag{4.8}$$

which is not possible. So our assumption (4.6) is not true, that is

$$\int_{\mathbb{R}^N} |\nabla \bar{u}|^p = A^p = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla \bar{u}_n|^p,$$

which further implies that $J'_\infty(\bar{u}) = 0$ and $\bar{u} \in \mathcal{P}_\infty$. Following the same argument as in (4.8) for $t_0 = 1$, we get $J_\infty(\bar{u}) = c_\infty$. Thus, \bar{u} is a nontrivial solution of (1.3). Moreover, we claim that \bar{u} is a ground state solution.

Let

$$d = \inf\{J_\infty(u) : u \text{ is a solution of problem (1.3)}\}.$$

Then $d \leq J_\infty(\bar{u}) = c_\infty = \mathfrak{p}_\infty \leq d$, where the last inequality holds by the definition of \mathcal{P}_∞ . From the assumption $f(t) = 0$ for $t \leq 0$, we have

$$0 = \langle J'_\infty(\bar{u}), \bar{u}^- \rangle \geq \|\bar{u}^-\|^2,$$

that is, \bar{u} is a non-negative solution. Using the similar Moser’s iteration as in [6], we get $u \in L^\infty(\mathbb{R}^N)$. Now employing the standard elliptic regularity and maximum principle (see [31, Theorem 5], [25]), we can achieve that u is a positive solution. At this moment using Lemma 4.5 we have the required.

5 Compactness of High Energy (PS)-Sequences

In this section, we will obtain all the technical results that form the background material for the proof of Theorem 2.2. We will first invoke a few elementary results associated with the Pohožaev manifold to achieve this. With the help of (2.2), we define the Pohožaev manifold as follows:

$$\mathcal{P} = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : P(u) = 0\}.$$

Note that \mathcal{P} contains all the critical points of J .

Lemma 5.1 *The following holds*

- (i) *There exist $\sigma_1, \sigma_2 > 0$ such that for any $u \in \mathcal{P}$, $\|u\| > \sigma_1$ and $J(u) \geq \sigma_2 > 0$.*
- (ii) *For any $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $\tilde{t} > 0$ such that $u_{\tilde{t}} \in \mathcal{P}$, where $u_t(x) = u(t^{-1}x)$. Moreover, $J(u_{\tilde{t}}) = \max_{t>0} J(u_t)$.*
- (iii) *$\{u \equiv 0\}$ is an isolated point of $P^{-1}(\{0\})$.*
- (iv) *\mathcal{P} is a closed set.*

Proof (i) By the Hardy-Littlewood-Sobolev inequality and condition (\mathcal{V}_3) , there exists $C > 0$, such that

$$\begin{aligned} 0 = P(u) &\geq \frac{(N-p)a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{NV_\infty}{p} \int_{\mathbb{R}^N} |u|^p dx - C\|u\|^{2-p_{*,\mu}} - C\|u\|^{2q} \\ &\geq \frac{N-p}{p} \min\{a, V_\infty\} \|u\|^p - C\|u\|^{2-p_{*,\mu}} - C\|u\|^{2q}. \end{aligned}$$

Since $2q > 2 \cdot p_{*,\mu} > p$, there exists $\sigma_1 > 0$ such that $\|u\| > \sigma_1$, for any $u \in \mathcal{P}$. Using this information, we deduce for some $C > 0$ and $\sigma_2 > 0$

$$J(u) = J(u) - \frac{P(u)}{2N-\mu} \geq C\|u\|^p > \sigma_2.$$

(ii) It follows the same analysis as in Lemma 3.3. For the uniqueness, we assume that t_1 and t_2 are two critical points of function $h(t) := J(u_t)$, such that $0 < t_1 < t_2$. Depending on the range of θ , we consider the following cases:

Case I: $1 \leq \theta \leq \frac{N}{N-p}$

Set $\phi(t) := \int_{\mathbb{R}^N} (NV(tx) + \nabla V(tx) \cdot tx) |u|^p dx$, with the help of condition (\mathcal{V}_4) and some elementary calculations, we get that ϕ is a decreasing function. Since $P(u_{t_i}) = 0$, for $i = 1, 2$ and $t_i > 0$, we deduce that

$$\begin{aligned} & \frac{(N-p)a}{p} \left(\frac{1}{t_1^p} - \frac{1}{t_2^p} \right) \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N-p)b}{p} \left(\frac{1}{t_1^{N-(N-p)\theta}} - \frac{1}{t_2^{N-(N-p)\theta}} \right) \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ & + \frac{1}{p} (\phi(t_1) - \phi(t_2)) \\ & = \frac{(2N-\mu)}{2} (t_1^{N-\mu} - t_2^{N-\mu}) \|F(u)\|_0, \end{aligned}$$

which gives us a contradiction as $0 < t_1 < t_2$.

Case II: $\frac{N}{N-p} < \theta < \frac{2N-\mu}{N-p}$

Set $\psi(t) := \int_{\mathbb{R}^N} (NV(tx) + \nabla V(tx) \cdot tx) |u|^p dx$, using (\mathcal{V}_2) , (\mathcal{V}_3)

and (\mathcal{V}_4) , we get that $\psi' \leq 0$, and hence it is a decreasing function. By similar reasoning as in Case I, we have

$$\begin{aligned} & \frac{(N-p)a}{p} \left(\frac{1}{t_1^{(N-p)(\theta-1)}} - \frac{1}{t_2^{(N-p)(\theta-1)}} \right) \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p} (\psi(t_1) - \psi(t_2)) \\ & = \frac{(2N-\mu)}{2} (t_1^{N-\mu} - t_2^{N-\mu}) \|F(u)\|_0, \end{aligned}$$

which is not true. Consequently, from both cases, we infer the uniqueness.

(iii) From (i), we see that there exists a ρ small enough such that $P(u) > 0$ for $0 < \|u\| \leq \rho$.

(iv) Since P is a C^1 functional, $P^{-1}(\{0\}) = \mathcal{P} \cup \{0\}$ is a closed set. Now applying (iii), \mathcal{P} is a closed set. □

Lemma 5.2 Assume $(\mathcal{V}_1) - (\mathcal{V}_3)$ and $(\mathcal{F}_1) - (\mathcal{F}_2)$. Define $\Gamma = \{\gamma \in C([0, 1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}$ and $c = \min_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$. Then c is well-defined and $c = c_\infty$.

Proof First, we will show that Γ is non-empty. Similar to Lemma 3.1(i), there exists $\rho > 0$ small enough, such that $J(u) \geq \alpha > 0$ for $\|u\| = \rho$. Additionally for any

$u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, $t > 0$ and $\tau \in \mathbb{R}^N$, we set $u_{t,\tau}(x) = u\left(\frac{x - \tau}{t}\right)$,

$$J(u_{t,\tau}) = \frac{at^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{bt^{(N-p)\theta}}{p\theta} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta + \frac{t^N}{p} \int_{\mathbb{R}^N} V(xt + \tau)|u|^p dx - \frac{t^{2N-\mu}}{2} \|F(u)\|_0.$$

For fixed $t > 0$ and $|\tau| \rightarrow \infty$, we see from \mathcal{V}_1

$$\lim_{|\tau| \rightarrow \infty} J(u_{t,\tau}) = J_\infty(u_t).$$

Hence for t and $|\tau|$ large enough, we have $J(u_{t,\tau}) < 0$, and thus, Γ is non-empty and c is well defined.

From (\mathcal{V}_3) , we get $J_\infty(u) \leq J(u)$, which implies that $\Gamma \subset \Gamma_\infty$, then

$$c_\infty \leq \inf_{h \in \Gamma} \max_{t \in [0,1]} J_\infty(h(t)) \leq \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t)) = c$$

and thus, $c_\infty \leq c$.

To show that $c \leq c_\infty$, let $\epsilon > 0$ be arbitrary, and take $\gamma \in \Gamma_\infty$ such that $\max_{t \in [0,1]} J_\infty(\gamma(t)) \leq c_\infty + \frac{\epsilon}{2}$. Set $\gamma(t)_\tau(\cdot) := \gamma(t)(\cdot - \tau)$, then $J(\gamma(t)_\tau) \rightarrow J_\infty(\gamma(t))$ as $|\tau| \rightarrow \infty$ for fixed $t > 0$. This implies for $|\tau|$ large enough, $\gamma(t)_\tau \in \Gamma$ and

$$J(\gamma(t)_\tau) \leq J_\infty(\gamma(t)) + \frac{\epsilon}{2} \text{ for all } t \in [0, 1].$$

So we get

$$c \leq \max_{t \in [0,1]} J(\gamma(t)_\tau) \leq \max_{t \in [0,1]} J_\infty(\gamma(t)) + \frac{\epsilon}{2} \leq c_\infty + \epsilon.$$

This gives $c \leq c_\infty$ and hence the equality is proved. □

Set

$$p = \inf_{\mathcal{P}} J(u) > 0,$$

it is well defined, by Lemma 5.1(i).

Lemma 5.3 $c = p$, where $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$.

Proof The proof is similar to the proof of Lemma 3.4. □

Lemma 5.4 For any $u \in \mathcal{P}$, there exists a unique $t \in (0, 1)$ such that $u_t \in \mathcal{P}_\infty$, where $u_t(x) = u(t^{-1}x)$.

Proof Using (\mathcal{V}_3) , we get $P_\infty(u) < P(u)$, which implies that

$$P_\infty(u) < 0, \text{ if } u \in \mathcal{P}. \tag{5.1}$$

By Lemma 3.3, we see that there exists unique \tilde{t} such that $P_\infty(u_{\tilde{t}}) = 0$. Hence from (5.1), we conclude that $\tilde{t} \in (0, 1)$. □

Lemma 5.5 c is not a critical value of the functional J .

Proof Let us suppose that c is a critical value of the functional J ; that is, there exists $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$J(u) = c, \quad J'(u) = 0.$$

As a consequence of Lemmas 5.3 and 5.4, we see that $J(u) = \mathfrak{p}$ and $u_t \in \mathcal{P}_\infty$, for some unique $t \in (0, 1)$. Further employing assumption (\mathcal{V}_2) , Lemmas 3.4 and 5.2, we obtain

$$\begin{aligned} \mathfrak{p} = J(u) &= J(u) - \frac{P(u)}{2N - \mu} \\ &> \left(\frac{N - \mu + p}{2N - \mu} \right) \frac{at^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx \\ &\quad + \left(\frac{2N - \mu - \theta(N - p)}{\theta(2N - \mu)} \right) \frac{bt^{(N-p)\theta}}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ &\quad + \left(\frac{N - \mu}{2N - \mu} \right) \frac{V_\infty t^N}{p} \int_{\mathbb{R}^N} |u|^p dx \\ &= J_\infty(u_t) - \frac{P_\infty(u_t)}{2N - \mu} \geq \mathfrak{p}, \end{aligned}$$

which is not possible. □

Lemma 5.6 The following holds

- (i) \mathcal{P} is a C^1 manifold.
- (ii) Each critical point of $J|_{\mathcal{P}}$ is a critical point of J in $W^{1,p}(\mathbb{R}^N)$.

Proof (i) We claim that $P'(u) \neq 0$, for all $u \in \mathcal{P}$. Indeed, suppose for the sake of contradiction, there exists $u \in \mathcal{P}$ such that $P'(u) = 0$, namely, u is a solution to the problem

$$\begin{aligned}
 & - (N - p) \left[a + b\theta \left(\int_{\mathbb{R}^N} |\nabla u|^p \right)^{(\theta-1)} \right] \Delta_p u + (NV(x) + \nabla V(x) \cdot x) |u|^{p-2}u \\
 & = (2N - \mu) \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x - y|^\mu} dy \right) f(u).
 \end{aligned} \tag{5.2}$$

As a consequence, u satisfies the Pohožaev identity related to problem (5.2), that is,

$$\begin{aligned}
 0 = P_1(u) &= \frac{(N - p)^2 a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N - p)^2 b\theta}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\
 &+ \frac{N^2}{p} \int_{\mathbb{R}^N} V(x) |u|^p dx + \frac{2N + 1}{p} \int_{\mathbb{R}^N} \nabla V(x) \cdot x |u|^p dx \\
 &+ \frac{1}{p} \int_{\mathbb{R}^N} \nabla x \cdot H(x) \cdot x |u|^p dx - \frac{(2N - \mu)^2}{2} \|F(u)\|_0.
 \end{aligned} \tag{5.3}$$

In view of (5.3) and $P(u) = 0$, we get

$$\begin{aligned}
 & \frac{(N - \mu + p)(N - p)a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx \\
 & + \frac{[(2N - \mu) - (N - p)\theta](N - p)b}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\
 & + \frac{(N - \mu)}{p} \int_{\mathbb{R}^N} (NV(x) + \nabla V(x) \cdot x) |u|^p dx \\
 & = \frac{N + 1}{p} \int_{\mathbb{R}^N} \nabla V(x) \cdot x |u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} x \cdot H(x) \cdot x |u|^p dx.
 \end{aligned}$$

Using hypothesis $(\mathcal{V}_2) - (\mathcal{V}_4)$, we obtain that

$$\begin{aligned} & \frac{(N - \mu + p)(N - p)a}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx \\ & + \frac{[(2N - \mu) - (N - p)\theta](N - p)b}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ & + \frac{(N - \mu)NV_\infty}{p} \int_{\mathbb{R}^N} |u|^p dx < 0, \end{aligned}$$

which is absurd. Thus, \mathcal{P} is a C^1 manifold.

(ii) Let us suppose that $u \in \mathcal{P}$ is a critical point of the functional $J|_{\mathcal{P}}$. Then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$J'(u) + \lambda P'(u) = 0. \tag{5.4}$$

We will show that $\lambda = 0$. From (5.4), we see that u weakly satisfies the following problem:

$$\begin{aligned} & - \left[a(1 + \lambda(N - p)) + b(1 + \lambda\theta(N - p)) \left(\int_{\mathbb{R}^N} |\nabla u|^p \right)^{(\theta-1)} \right] \Delta_p u + (1 + \lambda N)V(x)|u|^{p-2}u \\ & + \lambda \nabla V(x) \cdot x |u|^{p-2}u = (1 + \lambda(2N - \mu)) \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x - y|^\mu} dy \right) f(u). \end{aligned}$$

Consequently, u satisfies the following Pohožaev identity:

$$P_2(u) = P(u) + \lambda P_1(u) = 0, \tag{5.5}$$

where $P_1(u)$ is defined in (5.3). Using the fact that $P(u) = 0$ in (5.5), we infer that

$$\begin{aligned} & \lambda(N - \mu + p) \frac{a(N - p)}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \lambda[(2N - \mu) - (N - p)\theta] \frac{b(N - p)}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ & + \lambda \frac{(N - \mu)}{p} \int_{\mathbb{R}^N} (NV(x) + \nabla V(x) \cdot x) |u|^p dx \\ & = \frac{\lambda}{p} \int_{\mathbb{R}^N} \nabla V(x) \cdot x |u|^p dx + \frac{\lambda}{p} \int_{\mathbb{R}^N} (N \nabla V(x) \cdot x + x \cdot H(x) \cdot x) |u|^p dx. \end{aligned}$$

From $(\mathcal{V}_2), (\mathcal{V}_3)$ and (\mathcal{V}_4) , we see that it is only possible when $\lambda = 0$. □

Lemma 5.7 For any $u \in \mathcal{P}_\infty$,

- (i) there exists a unique $t > 1$, such that $u_t = u \left(\frac{x}{t} \right) \in \mathcal{P}$.
- (ii) For any $y \in \mathbb{R}^N$, there exists a unique $t_y > 1$, such that $u_{y,t_y} = u \left(\frac{x - y}{t_y} \right) \in \mathcal{P}$ and
 - (a) $t_y \rightarrow 1$ as $|y| \rightarrow \infty$;
 - (b) $\sup_{y \in \mathbb{R}^N} t_y = T \in (1, \infty)$;
 - (c) $t_y \rightarrow t_0$ as $y \rightarrow y_0$.

Proof (i) Since $P(u_t) = t \frac{dJ(u_t)}{dt}$, and in view of assumption (\mathcal{V}_3) , we get

$$P(u) = \left. \frac{dJ(u_t)}{dt} \right|_{t=1} > 0, \text{ for all } u \in \mathcal{P}_\infty.$$

Therefore, by the geometry of the functional P and by Lemma 5.1 (ii), we see that there exists a unique $t > 1$ such that $u_t \in \mathcal{P}$.

(ii) As $u \in \mathcal{P}_\infty$, by the translation invariance of \mathcal{P}_∞ , we see that $u(\cdot - y) \in \mathcal{P}_\infty$, for all $y \in \mathbb{R}^N$. So from (i), there exists a unique $t_y > 1$, such that $u_{y,t_y} \in \mathcal{P}$, that is,

$$\begin{aligned} 0 &= P(u_{y,t_y}) \\ &= \frac{a(N-p)t_y^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b(N-p)t_y^{(N-p)\theta}}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ &\quad + \frac{Nt_y^N}{p} \int_{\mathbb{R}^N} V(xt_y + y)|u|^p dx \\ &\quad + \frac{t_y^N}{p} \int_{\mathbb{R}^N} \nabla V(xt_y + y) \cdot (xt_y + y)|u|^p dx - \frac{(2N-\mu)t_y^{2N-\mu}}{2} \|F(u)\|_0. \end{aligned} \tag{5.6}$$

(ii)(a) Let us suppose that $t_y \rightarrow \infty$ as $|y| \rightarrow \infty$. Using \mathcal{V}_1 and the fact that $2N - \mu > \max\{N - p, (N - p)\theta, N\}$, in (5.6), we conclude that this supposition is not possible. So $\lim_{|y| \rightarrow \infty} t_y = C \in [1, \infty)$. By (5.6) and $u \in \mathcal{P}_\infty$, we deduce that

$$\begin{aligned} & \frac{a(N-p)}{p} \left(\frac{1}{C^{N-\mu+p}} - 1 \right) \int_{\mathbb{R}^N} |\nabla u|^p dx \\ & + \frac{b(N-p)}{p} \left(\frac{1}{C^{2N-\mu-(N-p)\theta}} - 1 \right) \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^\theta \\ & + \frac{NV_\infty}{p} \left(\frac{1}{C^{N-\mu}} - 1 \right) \int_{\mathbb{R}^N} |u|^p dx = 0, \end{aligned}$$

which is only possible when $C = 1$.

(ii)(b) By (a), we say that for any $\epsilon > 0$ there exists $R > 0$ such that $t_y < 1 + \epsilon$, whenever $|y| > R$. We are left to check for the interval $0 \leq |y| \leq R$, let us assume that there exist $\{y_n\} \subset \mathbb{R}^N$ with $0 \leq |y_n| \leq R$ such that $|t_{y_n}| \rightarrow \infty$. We get a contradiction by the same argument as in (a). Thus, $\sup_{y \in \mathbb{R}^N} t_y = T \in (1, \infty)$.

(ii)(c) From (b), we get that $\lim_{y \rightarrow y_0} t_y = \tau \geq 1$. Thus, from (5.6) we get that $u_{y_0, \tau} \in \mathcal{P}$ and by uniqueness $\tau = t_{y_0}$. □

Lemma 5.8 *Let $\{u_n\} \subset \mathcal{P}$ such that $J(u_n) \rightarrow c > 0$, as $n \rightarrow \infty$, then $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$.*

Proof By (\mathcal{V}_2) and (\mathcal{V}_3) , it follows that $J(u_n) = J(u_n) - \frac{P(u_n)}{2N-\mu} \geq C\|u_n\|^p$, for some $C > 0$. □

Lemma 5.9 *Let $\{u_n\}$ be a bounded $(PS)_c$ sequence for J with $c > 0$, then there exists $u \in W^{1,p}(\mathbb{R}^N)$ and $A \in \mathbb{R}$ such that $J'_A(u) = 0$, where*

$$J_A(u) = \frac{a + bA^{p(\theta-1)}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{1}{2} \|F(u)\|_0,$$

and either

- (i) $u_n \rightarrow u$ in $W^{1,p}(\mathbb{R}^N)$ or
- (ii) there exists $l \in \mathbb{N}$ and $\{y_n^k\} \subset \mathbb{R}^N$ with $\{y_n^k\} \rightarrow \infty$ as $n \rightarrow \infty$ for each $1 \leq k \leq l$, nontrivial solutions w^1, \dots, w^l of the following problem

$$-(a + bA^{p(\theta-1)})\Delta_p(u) + V_\infty|u|^{p-2}u = \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u),$$

with its related functional

$$J_{A,\infty}(u) = \frac{a + bA^{p(\theta-1)}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{V_\infty}{p} \int_{\mathbb{R}^N} |u|^p dx - \frac{1}{2} \|F(u)\|_0,$$

such that

- (a) $c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta} = J_A(u) + \sum_{k=1}^l J_{A,\infty}(w^k);$
- (b) $\|u_n - u - \sum_{k=1}^l w^k(\cdot - y_n^k)\| \rightarrow 0$ as $n \rightarrow \infty;$

where

$$A^p = \int_{\mathbb{R}^N} |\nabla u|^p + \sum_{k=1}^l \int_{\mathbb{R}^N} |\nabla w^k|^p, \tag{5.7}$$

Moreover,

$$J_{A,\infty}(w^1) = c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p(\theta-1)}. \tag{5.8}$$

Proof The proof of (i), (ii) is similar to that of [9, Proposition 3.1]. It is enough to prove (5.8). Let $\{u_n\} \subset \mathcal{P}$ be a minimizing sequence for p , then by Ekeland’s variational principle [32, Theorem 8.5], there exists a sequence $\{v_n\} \subset \mathcal{P}$ such that

$$J(v_n) \rightarrow p; \quad J'(v_n)|_{\mathcal{P}} \rightarrow 0; \quad \|v_n - u_n\| \rightarrow 0.$$

By Lemmas 5.6(ii), 5.3 and 5.8, we conclude that $\{v_n\}$ is a bounded $(PS)_c$. Then either (i) or (ii) holds, but by Lemma 5.5, as c is not the critical value of the functional J , so (i) cannot hold. As $\{v_n\}$ is a bounded $(PS)_c$ sequence, thus, there exists a function $v \in W^{1,p}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$. We make the following cases:

Case I: When $v \neq 0$

We see that $J'_A(v) = 0$, and v satisfies the following Pohožaev identity:

$$\begin{aligned} 0 = P_A(v) &= \left(a + bA^{p(\theta-1)} \right) \frac{(N-p)}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{N}{p} \int_{\mathbb{R}^N} V(x)|v|^p dx \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^N} \nabla V(x) \cdot x|v|^p dx - \frac{2N-\mu}{2} \|F(v)\|_0. \end{aligned}$$

Also from (5.7), we get that

$$0 = P_A(v) \geq P(v). \tag{5.9}$$

From Lemma 5.1(ii) and (5.9), there exists a unique $t \in (0, 1]$ such that $v_t \in \mathcal{P}$. Further using (5.7), we obtain that

$$\begin{aligned} \left[1 - \frac{N-p}{2N-\mu}\right] \frac{b}{p} A^{p(\theta-1)} &\geq \left[\frac{1}{\theta} - \frac{N-p}{2N-\mu}\right] \frac{bt^{(N-p)\theta}}{p} \left(\int_{\mathbb{R}^N} |\nabla v|^p dx\right)^{\theta-1} \\ &+ b \left[\frac{1}{p} - \frac{1}{p\theta}\right] A^{p(\theta-1)}. \end{aligned} \tag{5.10}$$

Define $g(t) := \frac{t^N}{p(2N-\mu)} \int_{\mathbb{R}^N} [(N-\mu)V(tx) - \nabla V(tx) \cdot tx] |v|^p dx$, we get

$$\begin{aligned} g'(t) &= \frac{t^{(N-1)}(N-\mu)}{p(2N-\mu)} \int_{\mathbb{R}^N} (NV(tx) + \nabla V(tx) \cdot tx) |v|^p dx \\ &- \frac{t^{(N-1)}}{p(2N-\mu)} \int_{\mathbb{R}^N} [\nabla V(tx) \cdot tx - (NV(tx) \cdot tx + tx \cdot H(tx) \cdot tx)] |v|^p dx \geq 0. \end{aligned}$$

The last inequality is attained by using the (\mathcal{V}_2) , (\mathcal{V}_3) and (\mathcal{V}_4) , and it follows that for all $0 < t \leq 1$, $g(t) \leq g(1)$. Along with this observation, we use (5.9) and (5.10), to obtain

$$\begin{aligned} J_A(v) &= J_A(v) - \frac{P_A(v)}{2N-\mu} \\ &\geq \left[1 - \frac{N-p}{2N-\mu}\right] \frac{at^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \left[\frac{1}{\theta} - \frac{N-p}{2N-\mu}\right] \frac{bt^{(N-p)\theta}}{p} \int_{\mathbb{R}^N} |\nabla v|^p \\ &\quad + \left[\frac{1}{p} - \frac{1}{p\theta}\right] bA^{p(\theta-1)} \int_{\mathbb{R}^N} |\nabla v|^p + g(t) \\ &= J(v_t) - \frac{P(v_t)}{2N-\mu} + \left[\frac{1}{p} - \frac{1}{p\theta}\right] bA^{p(\theta-1)} \int_{\mathbb{R}^N} |\nabla v|^p \\ &= c + \left[\frac{1}{p} - \frac{1}{p\theta}\right] bA^{p(\theta-1)} \int_{\mathbb{R}^N} |\nabla v|^p. \end{aligned} \tag{5.11}$$

Since $J'_{A,\infty}(w^i) = 0$, for all $i = 1, \dots, l$, arguing as in (5.11), we claim that

$$J_{A,\infty}(w^i) \geq c + \left[\frac{1}{p} - \frac{1}{p\theta}\right] bA^{p(\theta-1)} \int_{\mathbb{R}^N} |\nabla w^i|^p. \tag{5.12}$$

Using (ii)(a), (5.11), (5.12) and (5.7), we get that

$$\begin{aligned}
 c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta} &\geq (1+l)c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p(\theta-1)} \left(\int_{\mathbb{R}^N} |\nabla v|^p + \sum_{k=1}^l \int_{\mathbb{R}^N} |\nabla w^k|^p \right) \\
 &\geq 2c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta}
 \end{aligned}
 \tag{5.13}$$

which is a contradiction. Hence $v \equiv 0$. Again from (ii)(a), we get that

$$c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta} \geq lc + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta}.
 \tag{5.14}$$

This implies $l = 1$, and thus, we have proved (5.8). □

Lemma 5.10 *Let $\{u_n\} \subset \mathcal{P}$ be a $(PS)_\tau$ sequence for J with $\tau \in (c, 2c)$, then there exists $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ such that up to a sub-sequence, $u_n \rightarrow u$ in $W^{1,p}(\mathbb{R}^N)$.*

Proof By Lemma 5.8, $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Employing Lemma 5.9, we see that either (i) or (ii) holds. We consider the following two cases.

Case I: When $u \equiv 0$

Then from Lemma 5.9, we see that (i) cannot hold since $\tau > 0$. Thus, (ii) holds true with $u \equiv 0$, from (5.14), we get that

$$\tau + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta} = \sum_{k=1}^l J_{A,\infty}(w^k) \geq lc + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta}.$$

Since $\tau < 2c$, we get $l = 1$, when combined with (5.8) gives us a contradiction as $\tau > c$.

Case II: When $u \neq 0$

Let us assume that (ii) holds, then by the same argument as in (5.13), we have

$$\tau + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta} = J_A(v) + \sum_{k=1}^l J_{A,\infty}(w^k) \geq 2c + \left[\frac{1}{p} - \frac{1}{p\theta} \right] bA^{p\theta},$$

which is not possible. Thus, there exists $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ such that $u_n \rightarrow u$ in $W^{1,p}(\mathbb{R}^N)$. □

6 Proof of Theorem 2.2

Here our target is to prove the existence of a positive solution to problem (1.2). More precisely, we show that there exists a critical point of the functional J of high

energy. For that purpose, we will use the linking theorem with the Barycenter mapping restricted to the Pohožaev manifold \mathcal{P} .

Definition 6.1 Let S be a closed subset of a Banach space X , and $Q \subset X$ be a submanifold with relative boundary ∂Q . We say that S and ∂Q link if:

- (i) $S \cap \partial Q = \emptyset$;
- (ii) for any $h \in C(X, X)$, such that $h|_{\partial Q} = id$, then $h(Q) \cap S \neq \emptyset$.

Theorem 6.1 ([4], Linking theorem) *Let S be a closed subset of a Banach space X and $Q \subset X$ be a submanifold with relative boundary ∂Q . Suppose that $\phi \in C^1(X, \mathbb{R})$, satisfies the (PS) condition and*

- (i) S and ∂Q link;
- (ii) $\inf_{u \in S} \phi(u) > \sup_{u \in \partial Q} \phi(u)$;
- (iii) $\sup_{u \in Q} \phi(u) < \infty$.

Define

$$\tau = \inf_{h \in H} \sup_{u \in Q} \phi(h(u)),$$

where $H = \{h \in C(X, X) : h|_{\partial Q} = id\}$. Then τ is a critical value of ϕ and $\tau \geq \inf_{u \in S} \phi(u)$.

Next, we introduce the barycenter function and its properties, which is going to be crucial for proving the existence of a solution of problem (1.2).

Definition 6.2 [5] The barycenter map $\beta : W^{1,p}(\mathbb{R}^N) \setminus \{0\} \rightarrow \mathbb{R}^N$, is defined as

$$\beta(u) = \frac{1}{\|u\|_{L^1}} \int_{\mathbb{R}^N} x \tilde{u}(x) \, dx,$$

where $\tilde{u}(x) = \left(\mu(u)(x) - \frac{1}{2} \max_{x \in \mathbb{R}^N} \mu(u)(x) \right)^+$ and $\mu(u)(x) = \frac{1}{|B_1|} \int_{B_1(x)} |u(y)| \, dy$.

It follows that $\mu(u) \in L^\infty(\mathbb{R}^N)$ is a continuous function and $\tilde{u} \in C_0(\mathbb{R}^N)$. As \tilde{u} has a compact support, we provide a sketch of its proof for the reader's convenience. Define

$$\mathcal{A} := \overline{\{x \in \mathbb{R}^N : \tilde{u}(x) \neq 0\}} = \overline{\{x \in \mathbb{R}^N : \mu(u)(x) > \frac{1}{2} \max_{x \in \mathbb{R}^N} \mu(u)(x)\}}.$$

Let if possible \mathcal{A} is unbounded, that is for each $n \in \mathbb{N}$, there exists $x_n \in \mathcal{A}$ such that $x_n > n$. Since $u \in L^1_{loc}(\mathbb{R}^N)$, for large x_n

$$\mu(u)(x_n) = \frac{1}{|B_1|} \int_{B_1(x_n)} |u(y)| \, dy < \epsilon,$$

where we choose $\epsilon < \frac{K}{4}$ and $K = \max_{x \in \mathbb{R}^N} \mu(u)(x)$. However, as $x_n \in \mathcal{A}$, we have

$$\frac{K}{4} > \epsilon > \mu(u)(x_n) > \frac{K}{2},$$

which gives us a contradiction. Thus, \mathcal{A} is compact.

Thus, the barycenter map $\beta(u)$ is well defined and has the following properties:

- (i) β is continuous function in $W^{1,p}(\mathbb{R}^N) \setminus \{0\}$;
- (ii) If u is radial, then $\beta(u) = 0$;
- (iii) $\beta(u(\cdot - y)) = \beta(u) + y$, for any $y \in \mathbb{R}^N$;
- (iv) If u is radial, then for any $y \in \mathbb{R}^N$ and $t > 0$, $\beta(u(\frac{\cdot - y}{t})) = y$.

Set

$$\tilde{\mathcal{P}} := \{u \in \mathcal{P} : \beta(u) = 0\}.$$

Clearly, $\tilde{\mathcal{P}}$ is non-empty. Indeed, let \bar{u} be the positive radial ground-state solution of problem (1.3), where $\bar{u} \in \mathcal{P}_\infty$. Further, using Lemma 5.7 (i), we get that $\bar{u}_t \in \mathcal{P}$, for some $t > 1$, and by using by Barycenter map property (ii), we deduce that $\beta(\bar{u}_t) = 0$. Hence

$$\tilde{p} = \inf_{u \in \tilde{\mathcal{P}}} J(u),$$

is well defined, it is easy to see that $\tilde{p} \geq c$. Moreover, we have the following result:

Lemma 6.1 $\tilde{p} > c$.

Proof We assume that $c = p = \tilde{p}$. Let $\{u_n\} \subset \tilde{\mathcal{P}}$ be a minimizing sequence for \tilde{p} . Using the same assertions as in Lemma 5.9, there exists a bounded $(PS)_c$ sequence $\{v_n\} \subset \mathcal{P}$ such that $\|v_n - u_n\| \rightarrow 0$ and from the continuity of function β , we have

$$\beta(v_n) = o(1), \text{ as } n \rightarrow \infty. \tag{6.1}$$

Also from (5.14), we get $l = 1$, which subsequently gives

$$\|v_n - w^1(\cdot - y_n^1)\| \rightarrow 0 \text{ in } W^{1,p}(\mathbb{R}^N) \tag{6.2}$$

where $y_n^1 \in \mathbb{R}^N$ with $|y_n^1| \rightarrow \infty$. Thus, for $\beta(w^1)$ fixed, using (6.1) and (6.2), we obtain that

$$0 = \lim_{n \rightarrow \infty} |\beta(u_n)| = \lim_{n \rightarrow \infty} |\beta(v_n)| = \lim_{n \rightarrow \infty} |\beta(w^1) + y_n^1| = \infty,$$

which is impossible. Therefore, we must have $\tilde{p} > c$. □

Let us consider again the positive, radially symmetric, ground-state solution \bar{u} of problem (1.3). We define a continuous operator $K : \mathbb{R}^N \rightarrow \mathcal{P}$ as

$$K(y)(x) := \bar{u}_{y,t_y}(x) = \bar{u}\left(\frac{x-y}{t_y}\right),$$

using Lemma 5.7(ii), we see that the map is well defined for all y and some unique $t_y > 1$. Further, using the barycenter property (iv), we get that

$$\beta(K(y)) = y. \tag*{\square}$$

Lemma 6.2 *Assume that there exists a constant $T > 1$, such that the assumption (\mathcal{V}_5) holds true, that is,*

$$\sup_{x \in \mathbb{R}^N} V(x) \leq V_\infty + \frac{pJ_\infty(\bar{u})}{T^N \int_{\mathbb{R}^N} |\bar{u}|^p},$$

where $T = \sup_{y \in \mathbb{R}^N} t_y$, then $J(K(y)) < 2c$.

Proof Since J_∞ is translation invariant and from Lemma 3.3, we see that $J_\infty(\bar{u}) = \max_{t>0} J_\infty(\bar{u}_t)$. Then from (\mathcal{V}_5) , we have

$$\begin{aligned} J(K(y)) &= J_\infty(K(y)) + \frac{1}{p} \int_{\mathbb{R}^N} (V(x) - V_\infty) |K(y)|^p dx \\ &\leq J_\infty(\bar{u}_{y,t_y}) + \frac{t_y^N c}{T^N} < 2c, \end{aligned} \tag{6.3}$$

where c is as defined in Lemma 5.2. □

Proof of Theorem 2.2 From (\mathcal{V}_1) and Lemma 5.2, it follows that

$$\lim_{|y| \rightarrow \infty} J(K(y)) = J_\infty(\bar{u}) = c,$$

by Lemma 6.1, $c < \tilde{p}$, then there exists $R_0 > 0$ such that

$$\max_{|y|=R_0} J(K(y)) < \tilde{p}. \tag{6.4}$$

To apply the linking theorem, we take $Q := K(\overline{B_{R_0}(0)})$ and $S := \tilde{\mathcal{P}}$, we will prove the hypothesis holds true. First, we show that ∂Q and S link. For any $u \in \partial Q$, we see that $\beta(u) = \beta(K(y)) = y$, for any $y \in \mathbb{R}^N$, with $|y| = R_0 > 0$, which implies that $u \notin S$, that is, $\partial Q \cap S = \emptyset$. Next, we show that $h(Q) \cap S \neq \emptyset$, for any $h \in \mathcal{H}$, where

$$\mathcal{H} = \{h \in C(Q, \mathcal{P}) : h|_{\partial Q} = id\}.$$

We define a map $G : \overline{B_{R_0}(0)} \rightarrow \mathbb{R}^N$ as

$$G(y) = (\beta \circ h \circ K)(y).$$

The function G , being a composition of continuous maps, is continuous. Moreover, for any $|y| = R_0$, we have $K(y) \in \partial Q$; thus, $G(y) = y$. By the Brouwer's fixed point theorem, there exists $y_1 \in B_{R_0}(0)$ such that $G(y_1) = 0$, which implies that $h(K(y_1)) \in S$, and thus, $h(K(y_1)) \in h(Q) \cap S$. So S and ∂Q link. We attain the conditions of the linking theorem from (6.3) and (6.4). Thus, there exists a $(PS)_\tau$ sequence $\{u_n\}$ for the functional J , where critical level $\tau = \inf_{h \in H} \sup_{u \in Q} J(h(u)) \geq \inf_{u \in S} J(u) = \tilde{p} > c$.

Whereas by the definition of τ and using (6.3), we get that $\tau \leq \max_{u \in Q} J(u) < 2c$. So $\{u_n\}$ is a $(PS)_\tau$ sequence for J with $\tau \in (c, 2c)$. Lemmas 5.8 and 5.10 guarantee the existence of a nontrivial solution $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ of problem (1.2). Reasoning as in the proof of Theorem 2.1, we conclude that u is a positive solution using the maximum principle. Hence the proof of the theorem is completed. \square

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