The Journal of Geometric Analysis (2024) 34:150
https://doi.org/10.1007/s12220-024-01594-w

Covariant Derivatives on Homogeneous Spaces: Horizontal Lifts and Parallel Transport

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Received: 28 November 2023 / Accepted: 18 February 2024 / Published online: 27 March 2024

Abstract
We consider invariant covariant derivatives on reductive homogeneous spaces corresponding to the well-known invariant affine connections. These invariant covariant derivatives are expressed in terms of horizontally lifted vector fields on the Lie group. This point of view allows for a characterization of parallel vector fields along curves. Moreover, metric invariant covariant derivatives on a reductive homogeneous space equipped with an invariant pseudo-Riemannian metric are characterized. As a by-product, a new proof for the existence of invariant covariant derivatives on reductive homogeneous spaces and their the one-to-one correspondence to certain bilinear maps is obtained.

Keywords Geodesic equation · Horizontal lifts · Invariant covariant derivatives · Parallel vector fields along curves · Reductive homogeneous spaces

Mathematics Subject Classification 53B05 · 53C22 · 53C30

1 Introduction

Reductive homogeneous spaces play a role in a wide range of applications from mathematical physics to an engineering context. Without going into details, geodesics and parallel transport are certainly of interest. These notions can be defined with respect to invariant covariant derivatives which correspond to the well-known invariant affine connections from the literature. In fact, the existence of invariant affine connections on a reductive homogeneous space G/H with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and their one-to-one-correspondence to $\mathrm{Ad}(H)$-invariant bilinear maps $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ were proven in [1].

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The initial motivation for this text was to derive a characterization of parallel vector fields along curves generalizing [2, Lem. 1] to an arbitrary reductive homogeneous space equipped with some invariant covariant derivative. In order to obtain such a characterization, given in Corollary 4.27 below, we express an arbitrary invariant covariant derivative on G/H associated to an $\mathrm{Ad}(H)$-invariant bilinear map $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ in terms of horizontally lifted vector fields on G. This expression generalizes formulas for the Levi-Civita covariant derivative on G/H in terms of horizontally lifted vector fields from the literature, where G is equipped with a bi-invariant metric and G/H is endowed with a pseudo-Riemannian metric such that $G \to G/H$ is a pseudo-Riemannian submersion. Indeed, in the proof of [2, Lem. 1], a formula for the Levi-Civita covariant derivative on a pseudo-Riemannian symmetric space in terms of horizontally lifted vector fields is obtained. Moreover, a formula for the Levi-Civita covariant derivative in terms of horizontal vector fields is derived in [3, Sec. 4.2] for certain homogeneous spaces of compact Lie groups equipped with bi-invariant metrics. Here we also mention the recent work [4], where similar questions are independently discussed in the context of spray geometry.

We now give an overview of this text. We start with introducing some notations in Sect. 2. Moreover, in Sect. 3, we recall some facts on reductive homogeneous spaces and discuss the principal connections defined by reductive decompositions. After this preparation, we come to Sect. 4, where invariant covariant derivatives are investigated in detail. In Sect. 4.1, we show that an invariant covariant derivative is uniquely determined by evaluating it on certain fundamental vector fields of the left action $G \times G/H \ni (g, g' \cdot H) \mapsto (gg') \cdot H \in G/H$. Afterwards, we express an invariant covariant derivative ∇^α corresponding to an $\mathrm{Ad}(H)$-invariant bilinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ in terms of horizontally lifted vector fields on G as follows. For two vector fields X and Y on G/H whose horizontal lifts are the vector fields on G denoted by $\overline{X}$ and $\overline{Y}$, respectively, we express the horizontal lift $\overline{\nabla^\alpha_X Y}$ of $\nabla^\alpha_X Y$ in terms of $\overline{X}$ and $\overline{Y}$. The exact expression for $\overline{\nabla^\alpha_X Y}$ is obtained in Theorem 4.15. As a by-product, a new proof for the existence of invariant covariant derivatives associated to $\mathrm{Ad}(H)$-invariant bilinear maps $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is obtained. Moreover, the formula from Theorem 4.15 is used to derive the curvature of ∇^α in Sect. 4.2 In addition, we characterize invariant metric covariant derivatives if G/H is equipped with an invariant pseudo-Riemannian metric. In Sect. 4.4, we turn our attention to vector fields along curves. In particular, the expression of ∇^α in terms of horizontally lifted vector fields from Theorem 4.15 allows for characterizing parallel vector fields along curves on G/H in terms of an ODE on $\mathfrak{m}$. In addition, we obtain a geodesic equation for the reductive homogeneous space G/H equipped with an invariant covariant derivative. If this geodesic equation is specialized to a Lie group endowed with some left-invariant metric, the well-known geodesic equation from [5, Ap. 2] is obtained. Finally, we discuss the canonical invariant covariant derivatives of first and second kind which correspond to the canonical affine connections of first and second kind from [1, Sec. 10].

2 Notations and Terminology

We start with introducing the notation and terminology that is used throughout this text.

Notation 2.1 *We follow the convention in* [6, Chap. 2]. *A scalar product is defined as a non-degenerated symmetric bilinear form. An inner product is a positive definite symmetric bilinear form.*

Next we introduce some notations concerning differential geometry. Let M be a smooth (finite-dimensional) manifold. We denote by TM and T^*M the tangent and cotangent bundle of M, respectively. For a smooth map $f: M \to N$ between manifolds M and N, the tangent map of f is denoted by $Tf: TM \to TN$. We write $\mathscr{C}^\infty(M)$ for the algebra of smooth real-valued functions on M.

Let $E \to M$ be a vector bundle over M with typical fiber V. The smooth sections of E are denoted by $\Gamma^\infty(E)$. We write $\mathrm{End}(E) \cong E^* \otimes E$ for the endomorphism bundle of E. Moreover, we denote by $E^{\otimes k}$, $\mathrm{S}^k E$ and $\Lambda^k E$ the k-th tensor power, the k-th symmetrized tensor power and the k-th anti-symmetrized tensor power of E. If $T \in \Gamma^\infty\big((T^*M)^{\otimes k} \otimes (TM)^{\otimes \ell}\big)$ is a tensor field on M and $X \in \Gamma^\infty(TM)$ is a vector field on, we write $\mathscr{L}_X T$ for the Lie derivative. The pull-back of a smooth function $x: N \to \mathbb{R}$ by the smooth map $f: M \to N$ is denoted by $f^*x = x \circ f: M \to \mathbb{R}$. More generally, if $\omega \in \Gamma^\infty\big(\Lambda^k(T^*N)\big) \otimes V$ is a differential form taking values in a finite dimensional $\mathbb{R}$-vector space V, its pull-back by f is denoted by $f^*\omega$.

Concerning the regularity of curves on manifolds, we use the following convention.

Notation 2.2 *Whenever $c: I \to M$ denotes a curve in a manifold M defined on an interval $I \subseteq \mathbb{R}$, we assume for simplicity that c is smooth if not indicated otherwise. If I is not open, we assume that c can be extended to smooth curve defined on an open interval $J \subseteq \mathbb{R}$ containing I. Moreover, we implicitly assume that* 0 *is contained in I if we write* $0 \in I$.

Notation 2.3 *If not indicated otherwise, we use Einstein summation convention.*

3 Background on Reductive Homogeneous Spaces

In this section, we introduce some more notations and recall some well-known facts concerning Lie groups and reductive homogeneous spaces. Moreover, the principal connection on the H-principal fiber bundle $G \to G/H$ obtained by a reductive decomposition is discussed in detail.

3.1 Lie Groups

We start with introducing some notations and well-known facts concerning Lie groups and Lie algebras. Let G be a Lie group and denote its Lie algebra by $\mathfrak{g}$. The identity of G is usually denoted by e. We write

$$\ell_g: G \to G, \quad h \mapsto \ell_g(h) = gh \tag{3.1}$$

for the left translation by $g \in G$ and the right translation by $g \in G$ is denoted by

$$r_g \colon G \to G, \quad h \mapsto r_g(h) = hg. \tag{3.2}$$

The conjugation by an element $g \in G$ is given by

$$\mathrm{Conj}_g \colon G \to G, \quad h \mapsto \mathrm{Conj}_g(h) = (\ell_g \circ r_{g^{-1}})(h) = (r_{g^{-1}} \circ \ell_g)(h) = ghg^{-1} \tag{3.3}$$

and the adjoint representation of G is defined as

$$\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g = \big(\xi \mapsto \mathrm{Ad}_g(\xi) = T_e \mathrm{Conj}_g \xi\big). \tag{3.4}$$

Moreover, we denote the adjoint representation of $\mathfrak{g}$ by

$$\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \xi \mapsto \big(\eta \mapsto \mathrm{ad}_\xi(\eta) = [\xi, \eta]\big). \tag{3.5}$$

Next we recall [7, Def. 19.7]. A vector field $X \in \Gamma^\infty(TG)$ is called left-invariant or right-invariant if for all $g, k \in G$

$$T_k \ell_g X(k) = X(\ell_g(k)) \quad \text{or} \quad T_k r_g X(k) = X(r_g(k)), \tag{3.6}$$

respectively, holds. For $\xi \in \mathfrak{g}$, we denote by $\xi^L \in \Gamma^\infty(TG)$ and $\xi^R \in \Gamma^\infty(TG)$ the corresponding left and right-invariant vector fields, respectively, which are given by

$$\xi^L(g) = T_e \ell_g \xi \quad \text{and} \quad \xi^R(g) = T_e r_g \xi, \quad g \in G. \tag{3.7}$$

We write

$$\exp \colon \mathfrak{g} \to G. \tag{3.8}$$

for the exponential map of G.

3.2 Reductive Homogeneous Spaces

Next we recall some well-known facts on reductive homogeneous spaces and introduce the notation that is used throughout this text. We refer to [7, Sec. 23.4] or [6, Chap. 11] for details.

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Moreover, let $H \subseteq G$ a closed subgroup whose Lie algebra is denoted by $\mathfrak{h} \subseteq \mathfrak{g}$. We consider the homogeneous space G/H. Then

$$\tau \colon G \times G/H \to G/H, \quad (g, g' \cdot H) \mapsto (gg') \cdot H \tag{3.9}$$

is a smooth action of G on G/H from the left, where $g \cdot H \in G/H$ denotes the coset defined by $g \in G$. Borrowing the notation from [7, p. 676], for fixed $g \in G$, the associated diffeomorphism is denote by

$$\tau_g \colon G/H \to G/H, \quad g' \cdot H \mapsto \tau_g(g' \cdot H) = (gg') \cdot H. \tag{3.10}$$

In addition, we write

$$\mathrm{pr} \colon G \to G/H, \quad g \mapsto \mathrm{pr}(g) = g \cdot H \tag{3.11}$$

for the canonical projection.

Since reductive homogeneous spaces play a central role in this text, we recall their definition from [7, Def. 23.8], see also [1, Sec. 7] or [6, Def. 21, Chap. 11].

Definition 3.1 Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Moreover, let $H \subseteq G$ be a closed subgroup and denote its Lie algebra by $\mathfrak{h} \subseteq \mathfrak{g}$. Then the homogeneous space G/H is called reductive if there exists a subspace $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is fulfilled and

$$\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m} \tag{3.12}$$

holds for all $h \in H$.

Following [7, Prop. 23.22], we recall a well-known property of the isotropy representation of a reductive homogeneous space. This is the next lemma.

Lemma 3.2 *The isotropy representation of a reductive homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$*

$$H \ni h \mapsto T_{\mathrm{pr}(e)}\tau_h \in \mathrm{GL}\big(T_{\mathrm{pr}(e)} G/H\big) \tag{3.13}$$

is equivalent to the representation

$$H \to \mathrm{GL}(\mathfrak{m}), \quad h \mapsto \mathrm{Ad}_h\big|_{\mathfrak{m}} = \big(X \mapsto \mathrm{Ad}_h(X)\big), \tag{3.14}$$

i.e.

$$T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr}\big|_{\mathfrak{m}} = T_e\mathrm{pr} \circ \mathrm{Ad}_h\big|_{\mathfrak{m}} \tag{3.15}$$

is fulfilled for all $h \in H$.

Notation 3.3 *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$. Then the projection onto $\mathfrak{m}$ whose kernel is given by $\mathfrak{h}$ is denoted by $\mathrm{pr}_{\mathfrak{m}} \colon \mathfrak{g} \to \mathfrak{m}$. We write $\mathrm{pr}_{\mathfrak{h}} \colon \mathfrak{g} \to \mathfrak{h}$ for the projection whose kernel is given by $\mathfrak{m}$. Moreover, we write for $\xi \in \mathfrak{g}$*

$$\xi_{\mathfrak{m}} = \mathrm{pr}_{\mathfrak{m}}(\xi) \quad \text{and} \quad \xi_{\mathfrak{h}} = \mathrm{pr}_{\mathfrak{h}}(\xi). \tag{3.16}$$

A scalar product $\langle \cdot, \cdot \rangle \colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ is called $\mathrm{Ad}(H)$-invariant if

$$\big\langle \mathrm{Ad}_h(X), \mathrm{Ad}_h(Y) \big\rangle = \big\langle X, Y \big\rangle \tag{3.17}$$

holds for all $h \in H$ and $X, Y \in \mathfrak{m}$, see e.g [6, p. 301] or [7, Sec. 23.4] for the positive definite case. Reformulating and adapting [7, Def. 23.5], we call a pseudo-Riemannian metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle \in \Gamma^\infty\big(S^2 T^*(G/H)\big)$ invariant if

$$\langle\!\langle v_p, w_p \rangle\!\rangle_p = \langle\!\langle T_p \tau_g v_p, T_p \tau_g w_p \rangle\!\rangle_{\tau_g(p)}, \quad p \in G/H, \quad v_p, w_p \in T_p(G/H) \tag{3.18}$$

holds for all $g \in G$. In the next lemma which is taken from [6, Chap. 11, Prop. 22], see also [7, Prop. 23.22] for the Riemannian case, invariant metrics on G/H are related to $\mathrm{Ad}(H)$-invariant scalar products on $\mathfrak{m}$.

Lemma 3.4 *By requiring the linear isomorphism $T_e \mathrm{pr}\big|_m : \mathfrak{m} \to T_{\mathrm{pr}(e)}(G/H)$ to be an isometry, there is a one-to-one correspondence between $\mathrm{Ad}(H)$-invariant scalar products on $\mathfrak{m}$ and invariant pseudo-Riemannian metrics on G/H.*

Naturally reductive homogeneous spaces are special reductive homogeneous spaces. We recall their definition from [6, Chap. 11, Def. 23].

Definition 3.5 Let G/H be a reductive homogeneous space equipped with an invariant pseudo-Riemannian metric corresponding to the $\mathrm{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$. Then G/H is called a naturally reductive homogeneous space if

$$\big\langle [X, Y]_{\mathfrak{m}}, Z \big\rangle = \big\langle X, [Y, Z]_{\mathfrak{m}} \big\rangle \tag{3.19}$$

holds for all $X, Y, Z \in \mathfrak{m}$.

The following lemma can be considered as a generalization of [7, Prop. 23.29 (1)-(2)] to pseudo-Riemannian metrics and Lie groups which are not necessarily connected.

Lemma 3.6 *Let G be a Lie group and denote by $\mathfrak{g}$ its Lie algebra. Moreover, let G be equipped with a bi-invariant metric and let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the corresponding $\mathrm{Ad}(G)$-invariant scalar product. Moreover, let $H \subseteq G$ be a closed subgroup such that its Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is non-degenerated with respect to $\langle \cdot, \cdot \rangle$. Then G/H is a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$. Moreover, if G/H is equipped with the invariant metric corresponding to the scalar product on $\mathfrak{m}$ that is obtained by restricting $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}$, the reductive homogeneous space G/H is naturally reductive.*

Proof The claim can be proven analogously to the proof of [7, Prop. 23.29 (1)-(2)] by taking the assumption $\mathfrak{h} \oplus \mathfrak{h}^\perp = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{g}$ into account. □

Remark 3.7 Inspired by the terminology in [7, Sec. 23.6, p. 710], we refer to the naturally reductive homogeneous spaces from Lemma 3.6 as normal naturally reductive homogeneous spaces.

We end this subsection with considering another special class of reductive homogeneous spaces. To this end, we state the following definition which can be found in [8, p. 209].

Definition 3.8 Let G be a connected Lie group and let H be a closed subgroup. Then (G, H) is called a symmetric pair if there exists a smooth involutive automorphism $\sigma\colon G \to G$, i.e. an automorphism of Lie groups fulfilling $\sigma^2 = \sigma$, such that $(H_\sigma)_0 \subseteq H \subseteq H_\sigma$ holds. Here H_σ denotes the set of fixed points of σ and $(H_\sigma)_0$ denotes the connected component of H_σ containing the identity $e \in G$.

Inspired by the terminology used in [7, Def. 23.13], we refer to the triple (G, H, σ) as symmetric pair, as well, where (G, H) is a symmetric pair with respect to the involutive automorphism $\sigma\colon G \to G$. These symmetric pairs lead to reductive homogeneous spaces which are called symmetric homogeneous spaces if a certain "canonical" reductive decomposition is chosen, see e.g. [1, Sec. 14]. Note that the definition in [1, Sec. 14] does not require an invariant pseudo-Riemannian metric on G/H.

The next lemma, see e.g. [1, Sec. 14], shows that a symmetric homogeneous space is a reductive homogeneous space with respect to the so-called canonical reductive decomposition. Here we also refer to [7, Prop. 23.33] for a proof.

Lemma 3.9 *Let (G, H, σ) be a symmetric pair and define the subspaces of $\mathfrak{g}$ by*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid T_e\sigma X = X\} \subseteq \mathfrak{g} \quad \textit{and} \quad \mathfrak{m} = \{X \in \mathfrak{g} \mid T_e\sigma X = -X\} \subseteq \mathfrak{g}. \tag{3.20}$$

Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition of $\mathfrak{g}$ turning G/H into a reductive homogeneous space. Moreover, the inclusion

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h} \tag{3.21}$$

is fulfilled.

Next we define symmetric homogeneous spaces and canonical reductive decompositions following [1, Sec. 14].

Definition 3.10 Let (G, H, σ) be a symmetric pair. Then the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ from Lemma 3.9 is called canonical reductive decomposition. Moreover, the reductive homogeneous space G/H with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called symmetric homogeneous space.

For pseudo-Riemannian symmetric spaces we state the next remark following [6, Chap. 11, p. 317], see also [7, Sec. 23.8] for the Riemannian case.

Remark 3.11 Let (G, H, σ) be symmetric pair and let G/H be the associated symmetric homogeneous space with canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let G/H be equipped with an invariant pseudo-Riemannian metric and let $\langle \cdot, \cdot \rangle\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ be the associated $\mathrm{Ad}(H)$-invariant scalar product. Then G/H is a naturally reductive homogeneous space since $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$ implies that the condition on the scalar product $\langle \cdot, \cdot \rangle$ from Definition 3.5 is always satisfied. In the sequel, we refer to symmetric homogeneous spaces equipped with an invariant pseudo-Riemannian metric as pseudo-Riemannian symmetric homogeneous space or pseudo-Riemannian symmetric spaces, for short.

3.3 Reductive Decompositions and Principal Connections

In this section, we consider G as a H-principal fiber bundle over G/H and discuss certain principal connections on $\mathrm{pr}\colon G \to G/H$. For general properties of principal fiber bundles and connections, we refer to [9, Sec. 18–19] and [10, Sec. 1.1–1.3].

Let G be a Lie group and $H \subseteq G$ be a closed subgroup. It is well-known that $\mathrm{pr}\colon G \to G/H$ is a H-principle fiber bundle, see e.g. [9, Sec. 18.15], where the base is the homogeneous space G/H. The H-principal action on G is denoted by

$$\triangleleft\colon G \times H \to G, \quad (g, h) \mapsto g \triangleleft h = r_h(g) = \ell_g(h) = gh, \tag{3.22}$$

if not indicated otherwise. We now assume that G/H is a reductive homogeneous space and the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is fixed. This reductive decomposition can be used to obtain a principal connection on $\mathrm{pr}\colon G \to G/H$, see [11, Thm. 11.1]. Although this fact is well-known, we provide a detailed proof in order to keep this text more self-contained. To this end, we recall a well-known fiber-wise expression for the vertical bundle of $\mathrm{pr}\colon G \to G/H$ which follows for example from [9, Sec. 18.18]. We have for fixed $g \in G$ by [10, Lem. 1.3.1], see also [9, Sec. 18.8]

$$\mathrm{Ver}(G)_g = \left\{ \tfrac{\mathrm{d}}{\mathrm{d}t}\big(g \triangleleft \exp(t\eta)\big)\big|_{t=0} \mid \eta \in \mathfrak{h} \right\} = (T_e\ell_g)\mathfrak{h}. \tag{3.23}$$

The next proposition provides explicit formulas for the principal connection and the associated principal connection one-form on $G \to G/H$ defined by a reductive decomposition.

Proposition 3.12 *Consider* $\mathrm{pr}\colon G \to G/H$ *as a* H*-principal fiber bundle, where* G/H *is a reductive homogeneous space with a fixed reductive decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *and define* $\mathrm{Hor}(G) \subseteq TG$ *fiber-wise by*

$$\mathrm{Hor}(G)_g = (T_e\ell_g)\mathfrak{m}, \quad g \in G. \tag{3.24}$$

Then $\mathrm{Hor}(G)$ *is a subbundle of* TG *defining a horizontal bundle on* TG*, i.e. a complement of the vertical bundle* $\mathrm{Ver}(G) = \ker(T\mathrm{pr}) \subseteq TG$ *which yields a principal connection on* $\mathrm{pr}\colon G \to G/H$*. This principal connection* $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$ *corresponding to* $\mathrm{Hor}(G)$ *is given by*

$$\mathcal{P}\big|_g(v_g) = T_e\ell_g \circ \mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_g)^{-1} v_g, \quad g \in G, \quad v_g \in T_gG. \tag{3.25}$$

The corresponding connection one-form $\omega \in \Gamma^\infty(T^*G) \otimes \mathfrak{h}$ *reads*

$$\omega\big|_g(v_g) = \mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_g)^{-1} v_g \tag{3.26}$$

for $g \in G$ *and* $v_g \in T_gG$*.*

Proof Although, this statement is well-known, see e.g. [11, Thm. 11.1], we provide a proof, nevertheless. Indeed, $\mathrm{Hor}(G)$ is a complement of the vertical bundle $\mathrm{Ver}(G) =$

$\ker(T\mathrm{pr}) \subseteq TG$ due to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ implying $TG = \mathrm{Ver}(G) \oplus \mathrm{Hor}(G)$ as desired. Moreover, $\mathcal{P}$ defined by (3.25) is clearly a smooth endomorphism of the vector bundle $TG \to G$, i.e. $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$. In addition, $\mathcal{P}^2 = \mathcal{P}$ is obviously fulfilled. Moreover, one has $\mathrm{im}(\mathcal{P}) = \ker(T\mathrm{pr}) = \mathrm{Ver}(G)$ and $\ker(\mathcal{P}) = \mathrm{Hor}(G)$ showing that $\mathcal{P}$ is the connection corresponding to the horizontal bundle $\mathrm{Hor}(G)$.

We now show that ω is the connection one-form corresponding to $\mathcal{P}$ by using the correspondence from [9, Sec. 19.1]. Let $\eta \in \mathfrak{h}$ and denote by η_G the corresponding fundamental vector field, i.e. we have for $g \in G$

$$\eta_G(g) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big(g \triangleleft \exp(t\eta)\big)\big|_{t=0} = \tfrac{\mathrm{d}}{\mathrm{d}t}\ell_g\big(\exp(t\eta)\big)\big|_{t=0} = T_e\ell_g\eta.$$

By this notation, one obtains for $v_g \in T_gG$

$$\big(\omega\big|_g(v_g)\big)_G(g) = T_e\ell_g\big(\omega\big|_g(v_g)\big) = T_e\ell_g\big(\mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_g)^{-1}(v_g)\big) = \mathcal{P}\big|_g(v_g).$$

Moreover, we have

$$\omega\big|_g\big(\eta_G(g)\big) = \big(\mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_g)^{-1}\big)T_e\ell_g\eta = \mathrm{pr}_{\mathfrak{h}}(\eta) = \eta$$

for all $\eta \in \mathfrak{h}$ proving that $\omega \in \Gamma^\infty(T^*G)\otimes\mathfrak{h}$ is the connection one-from corresponding to $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$.

It remains to show that $\mathcal{P}$ is a principal connection. By [9, Sec. 19.1] this is equivalent to showing that ω has the equivariance property

$$\big((\cdot \triangleleft h)^*\omega\big)\big|_g(v_g) = \mathrm{Ad}_{h^{-1}}\big(\omega\big|_g(v_g)\big)$$

for all $h \in H$, $g \in G$ and $v_g \in T_gG$, where $(\cdot \triangleleft h)^*\omega$ denotes the pull-back of ω by $(\cdot \triangleleft h)\colon P \ni p \mapsto p \triangleleft h \in P$. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition, we obtain for $h \in H$ and $\xi \in \mathfrak{g}$

$$(\mathrm{pr}_{\mathfrak{h}} \circ \mathrm{Ad}_h)(\xi) = \mathrm{pr}_{\mathfrak{h}}\big(\mathrm{Ad}_h(\xi_{\mathfrak{h}}) + \mathrm{Ad}_h(\xi_{\mathfrak{m}})\big) = \mathrm{Ad}_h(\xi_{\mathfrak{h}}) = (\mathrm{Ad}_h \circ \mathrm{pr}_{\mathfrak{h}})(\xi) \tag{3.27}$$

Using (3.27) and the chain-rule, we compute for $h \in H$, $g \in G$ and $v_g \in T_gG$

$$\begin{aligned}
\big((\cdot \triangleleft h)^*\omega\big)\big|_g(v_g) &= \omega\big|_{g\triangleleft h}\big(T_g(\cdot \triangleleft h)v_g\big)\\
&= \omega\big|_{gh}\big(T_g r_h v_g\big)\\
&= \big(\mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_{gh})^{-1}\big)T_g r_h v_g\\
&= \mathrm{pr}_{\mathfrak{h}} \circ T_{gh}\ell_{h^{-1}g^{-1}} \circ T_g r_h v_g\\
&= \mathrm{pr}_{\mathfrak{h}} \circ T_g(\ell_h^{-1} \circ \ell_g^{-1} \circ r_h)v_g\\
&= \mathrm{pr}_{\mathfrak{h}} \circ T_g(\ell_{h^{-1}} \circ r_h \circ \ell_{g^{-1}})v_g\\
&= \mathrm{pr}_{\mathfrak{h}} \circ T_e(\ell_{h^{-1}} \circ r_h) \circ T_g\ell_{g^{-1}}v_g\\
&= \mathrm{pr}_{\mathfrak{h}} \circ T_e\mathrm{Conj}_{h^{-1}} \circ (T_e\ell_g)^{-1}v_g
\end{aligned}$$

$$= \mathrm{pr}_{\mathfrak{h}} \circ \mathrm{Ad}_{h^{-1}} \circ (T_e \ell_g)^{-1} v_g$$
$$= \mathrm{Ad}_{h^{-1}} \circ \mathrm{pr}_{\mathfrak{h}} \circ (T_e \ell_g)^{-1} v_g$$
$$= \mathrm{Ad}_{h^{-1}} \big(\omega\big|_g (v_g)\big).$$

Hence ω is the connection one-form corresponding to the principal $\mathcal{P}$. □

By [7, Prop. 23.23], adapted to the pseudo-Riemannian case, we obtain the following remark concerning pseudo-Riemannian reductive homogeneous spaces.

Remark 3.13 Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ endowed with an invariant pseudo-Riemannian metric corresponding to the $\mathrm{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$. By [7, Prop. 23.23], which clearly extends to the pseudo-Riemannian case, the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ can be extended to a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that $\mathfrak{m} = \mathfrak{h}^{\perp}$ is fulfilled. Then $\mathrm{pr} : G \to G/H$ becomes a pseudo-Riemannian submersion by [7, Prop. 23.23], where G is equipped with the left-invariant metric defined by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Obviously, the horizontal bundle defined by $\mathrm{Hor}(G) = \mathrm{Ver}(G)^{\perp}$ yields the connection on G which coincides with the principal connection from Proposition 3.12 defined by the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

4 Invariant Covariant Derivatives

In this section, we consider the invariant covariant derivatives on a reductive homogeneous space G/H that correspond to the invariant affine connections investigated in [1]. These invariant covariant derivatives are expressed in terms of horizontally lifted vector fields yielding another proof for their existence. In particular, this expression is used to characterize parallel vector fields along curves in terms of an ODE on $\mathfrak{m}$.

Throughout this subsection, we use the following notation.

Notation 4.1 *If not indicated otherwise, we denote by G/H a reductive homogeneous space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.*

4.1 Invariant Covariant Derivatives

We start with introducing the notion of an invariant covariant derivative on a reductive homogeneous space. In view of the one-to-one correspondence of covariant derivatives and affine connections, see Remark 4.3 below, the next definition can be seen as a reformulation of [1, Eq. (2.3) and Sec. 8, p. 43].

Definition 4.2 Let G/H be a homogeneous space. Then a covariant derivative

$$\nabla : \Gamma^{\infty}\big(T(G/H)\big) \times \Gamma^{\infty}\big(T(G/H)\big) \to \Gamma^{\infty}\big(T(G/H)\big) \tag{4.1}$$

on G/H is called G-invariant, or invariant for short, if

$$\nabla_X Y = (\tau_{g^{-1}})_* \big(\nabla_{(\tau_g)_* X} (\tau_g)_* Y\big) \tag{4.2}$$

holds for all $g \in G$ and $X, Y \in \Gamma^\infty\big(T(G/H)\big)$, where $(\tau_g)_* X$ denotes the well-known push-forward of X by the diffeomorphism $\tau_g \colon G/H \to G/H$ given by $(\tau_g)_* X = T\tau_g \circ X \circ \tau_{g^{-1}}$.

Obviously, for a fixed $g \in G$ the push-forward $(\tau_g)_* X = T\tau_g \circ X \circ \tau_{g^{-1}}$ of a vector field $X \in \Gamma^\infty\big(T(G/H)\big)$ by $\tau_g \colon G/H \to G/H$ is point-wise given by

$$\big((\tau_g)_* X\big)(\mathrm{pr}(k)) = T_{\tau_{g^{-1}}(\mathrm{pr}(k))}\tau_g X\big(\tau_{g^{-1}}(\mathrm{pr}(k))\big), \quad \mathrm{pr}(k) \in G/H. \tag{4.3}$$

In the next remark, we relate the notion of affine connections from [1] to covariant derivatives.

Remark 4.3 Let M be a manifold and denote by $\mathrm{End}_{\mathscr{C}^\infty(M)}\big(\Gamma^\infty(TM)\big)$ the endomorphisms of the $\mathscr{C}^\infty(M)$-module $\Gamma^\infty(TM)$. An affine connection is defined in [1] as a map

$$t \colon \Gamma^\infty(TM) \ni X \mapsto t(X) \in \mathrm{End}_{\mathscr{C}^\infty(M)}\big(\Gamma^\infty(TM)\big) \tag{4.4}$$

such that

$$t(X_1 + X_1) = t(X_1) + t(X_2) \quad \text{and} \quad t(fX)(Y) = f t(X)(Y) + (\mathscr{L}_X f) t(X)(Y) \tag{4.5}$$

holds for all $X_1, X_2, X, Y \in \Gamma^\infty(TM)$. As pointed out in [12, Sec. 4.5], an affine connection $t \colon \Gamma^\infty(TM) \to \mathrm{End}_{\mathscr{C}^\infty(M)}\big(\Gamma^\infty(TM)\big)$ defines a covariant derivative $\nabla \colon \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ on TM by

$$\nabla_X Y = t(Y)(X), \quad X, Y \in \Gamma^\infty(TM). \tag{4.6}$$

Obviously, the converse is also true. Given a covariant derivative ∇ on TM, Equation (4.6) yields an affine connection.

In the sequel, we discuss the invariant covariant derivatives on G/H corresponding to the invariant affine connections on G/H from [1, Thm. 8.1]. This correspondence is made precise in Proposition 4.18, below.

We first recall the notion of an $\mathrm{Ad}(H)$-invariant bilinear map from [1, Sec. 8].

Definition 4.4 Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the bilinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is called $\mathrm{Ad}(H)$-invariant if

$$\mathrm{Ad}_h\big(\alpha(X, Y)\big) = \alpha\big(\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)\big) \tag{4.7}$$

holds for all $X, Y \in \mathfrak{m}$ and $h \in H$. More generally, for $\ell \in \mathbb{N}$, we call a ℓ-linear map $\alpha \colon \mathfrak{m}^\ell \to \mathfrak{m}$ $\mathrm{Ad}(H)$-invariant if

$$\mathrm{Ad}_h\big(\alpha(X_1, \dots, X_\ell)\big) = \alpha\big(\mathrm{Ad}_h(X_1), \dots, \mathrm{Ad}_h(X_\ell)\big) \tag{4.8}$$

holds for all $X_1, \ldots, X_\ell \in \mathfrak{m}$ and $h \in H$.

Remark 4.5 As we have already pointed out in the introduction, the one-to-one correspondence between invariant affine connections and $\mathrm{Ad}(H)$-invariant bilinear maps $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is well-known by [1, Thm. 8.1]. Nevertheless, the discussion in this text differs from the discussion in [1]. Inspired by [7, Sec. 23.6], we consider invariant covariant derivatives evaluated at the fundamental vector fields of the action $\tau \colon G \times G/H \to G/H$ at the point $\mathrm{pr}(e)$ which already determines them uniquely. Moreover, we express invariant covariant derivatives on G/H in terms of horizontally lifted vector fields on G. Beside yielding another proof for the existence of an invariant covariant derivative associated with an $\mathrm{Ad}(H)$-invariant bilinear map $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, this point of view allows in particular for an easy characterization of parallel vector fields, see Sect. 4.4.

4.1.1 Invariant Covariant Derivatives Evaluated on Fundamental Vector Fields

Before we continue with considering invariant covariant derivatives, we take a closer look on the fundamental vector fields on G/H associated with the action $\tau \colon G \times G/H \to G/H$ from (3.9). Let $X \in \mathfrak{g}$. The fundamental vector field $X_{G/H} \in \Gamma^\infty\big(T(G/H)\big)$ associated with X is defined by

$$X_{G/H}(\mathrm{pr}(g)) = \tfrac{\mathrm{d}}{\mathrm{d}t} \tau_{\exp(tX)}(\mathrm{pr}(g))\big|_{t=0} \tag{4.9}$$

for $\mathrm{pr}(g) \in G/H$ with $g \in G$. In the next lemma, we state some properties of $X_{G/H}$. Note that its third claim is well-known.

Lemma 4.6 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Moreover, let $X \in \mathfrak{m}$, let $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ be a basis of $\mathfrak{m}$ and let $\{A^1, \ldots, A^N\} \subseteq \mathfrak{m}^*$ be its dual basis. Let $A_i^L \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ denote the left-invariant vector field on G defined by A_i for $i \in \{1, \ldots, N\}$. Then the following assertions are fulfilled:*

1. *The horizontal lift of $X_{G/H}$ is given by*

$$\overline{X_{G/H}}(g) = A^i\big(\mathrm{Ad}_{g^{-1}}(X)\big) A_i^L(g) \tag{4.10}$$

   for all $g \in G$.
2. *Let $Y \in \mathfrak{m}$ and define the smooth functions $y^j \colon G \ni g \mapsto y^j(g) = A^j\big(\mathrm{Ad}_{g^{-1}}(Y)\big) \in \mathbb{R}$, where $j \in \{1, \ldots, N\}$. Then one has*

$$\big(\mathscr{L}_{\overline{X_{G/H}}} y^j\big)(e) A_j^L(e) = -[X, Y]_{\mathfrak{m}}. \tag{4.11}$$

3. *One has*

$$(\tau_g)_* X_{G/H}(\mathrm{pr}(k)) = \big(\mathrm{Ad}_g(X)\big)_{G/H}(\mathrm{pr}(k)) \tag{4.12}$$

   for all $g \in G$ and $\mathrm{pr}(k) \in G/H$.

Proof We first show Claim 1. To this end, we compute for $g \in G$

$$\begin{aligned} X_{G/H}(\mathrm{pr}(g)) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(\tau_{\exp(tX)} \circ \mathrm{pr})(g)\big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{pr} \circ \ell_{\exp(tX)})(g)\big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{pr} \circ r_g(\exp(tX))\big|_{t=0} \\ &= T_g\mathrm{pr} \circ T_e r_g X \\ &= T_g\mathrm{pr} \circ X^R(g) \end{aligned} \tag{4.13}$$

showing that $X_{G/H}$ is pr-related to the right-invariant vector field X^R. Next we express X^R in terms of left-invariant vector fields. Let $g \in G$. We now compute

$$\begin{aligned} X^R(g) &= \big(T_e\ell_g \circ (T_e\ell_g)^{-1}\big)T_e r_g X \\ &= T_e\ell_g \circ T_g\ell_{g^{-1}} \circ T_e r_g X \\ &= T_e\ell_g \circ T_e(\ell_{g^{-1}} \circ r_g)X \\ &= T_e\ell_g \circ T_e\mathrm{Conj}_{g^{-1}}X \\ &= T_e\ell_g\mathrm{Ad}_{g^{-1}}(X) \\ &= \big(\mathrm{Ad}_{g^{-1}}(X)\big)^L(g). \end{aligned} \tag{4.14}$$

Let $\mathcal{P}$ be the principal connection from Proposition 3.12. Then the horizontal lift of $X_{G/H}$ is given by $\overline{X_{G/H}} = (\mathrm{id}_{TG} - \mathcal{P}) \circ X^R$ because of $X_{G/H} \circ \mathrm{pr} = T\mathrm{pr} \circ X^R$ according to (4.13). Using (4.14) and $\mathrm{pr}_{\mathfrak{m}}(Y) = A^i\big(\mathrm{pr}_{\mathfrak{m}}(Y)\big)A_i = A^i(Y)A_i$ for all $Y \in \mathfrak{g}$, we have for $g \in G$

$$\begin{aligned} \overline{X_{G/H}}(g) &= \big(\mathrm{id}_{TG} - \mathcal{P}\big) \circ X^R(g) \\ &= (\mathrm{id}_{TG} - \mathcal{P})\big(\mathrm{Ad}_{g^{-1}}(X)\big)^L(g) \\ &= \big(T_e\ell_g \circ \mathrm{pr}_{\mathfrak{m}} \circ (T_e\ell_g)^{-1}\big)T_e\ell_g\mathrm{Ad}_{g^{-1}}(X) \\ &= T_e\ell_g\big(A^i(\mathrm{Ad}_{g^{-1}}(X))A_i\big) \\ &= A^i\big(\mathrm{Ad}_{g^{-1}}(X)\big)A_i^L(g). \end{aligned}$$

Next we show Claim 2. The curve $\gamma\colon \mathbb{R} \ni t \mapsto \exp(tX) \in G$ fulfills $\gamma(0) = e$ and $\dot{\gamma}(0) = X$. Therefore we compute, again by $\mathrm{pr}_{\mathfrak{m}}(Y) = A^i(Y)A_i$ for all $Y \in \mathfrak{g}$

$$\begin{aligned} \big(\mathscr{L}_{\overline{X_{G/H}}}y^j\big)(e)A_j^L(e) &= \big(\tfrac{\mathrm{d}}{\mathrm{d}t}y^j(\gamma(t))\big|_{t=0}\big)A_j \\ &= \big(\tfrac{\mathrm{d}}{\mathrm{d}t}A^j\big(\mathrm{Ad}_{\exp(tX)^{-1}}(Y)\big)\big|_{t=0}\big)A_j \\ &= \big(A^j\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(-tX)}(Y)\big|_{t=0}\big)\big)A_j \\ &= A^j\big(-[X,Y]\big)A_j \\ &= -[X,Y]_{\mathfrak{m}} \end{aligned}$$

as desired.

Although a proof of Claim 3 can be found for example in [7, Prop. 23.20], following this reference, we repeat it here for the reader's convenience. We compute for $g, k \in G$

$$\begin{aligned}\big((\tau_g)_* X_{G/H}\big)(\mathrm{pr}(k)) &= \big(T_{\tau_{g^{-1}}(\mathrm{pr}(k))}\tau_g\big) X_{G/H}\big(\tau_g^{-1}(\mathrm{pr}(k))\big) \\ &= \big(T_{\tau_{g^{-1}}(\mathrm{pr}(k))}\tau_g\big) \tfrac{\mathrm{d}}{\mathrm{d}t} \tau_{\exp(tX)}\big(\tau_{g^{-1}}(\mathrm{pr}(k))\big)\big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t} \tau_{g\exp(tX)g^{-1}}(\mathrm{pr}(k))\big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t} \tau_{\exp(t\mathrm{Ad}_g(X))}(\mathrm{pr}(k))\big|_{t=0} \\ &= \big(\mathrm{Ad}_g(X)\big)_{G/H}(\mathrm{pr}(k)),\end{aligned}$$

where (4.3) is used in the first equality and we also exploited $\mathrm{Conj}_g \circ \exp = \exp \circ \mathrm{Ad}_g$. This yields the desired result. □

It is well-known that there is a one-to-one correspondence between $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$ and invariant tensor fields on G/H, see e.g. [6, Chap. 11, p. 312]. In the sequel, we need the following lemma which can be regarded as a special case of this assertion. In order to keep this text as self-contained as possible, we include proof which is inspired by the proof of [7, Prop. 23.23] and [6, Chap. 11, Prop. 22].

Lemma 4.7 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. There is a one-to-one-correspondence between $\mathrm{Ad}(H)$-invariant ℓ-linear maps $d\colon \mathfrak{m}^\ell \to \mathfrak{m}$ with $\ell \in \mathbb{N}$ and tensor fields $D \in \Gamma^\infty\big(T^*(G/H)^{\otimes \ell} \otimes T(G/H)\big)$ on G/H fulfilling*

$$(\tau_g)_*\big(D(X_1, \ldots, X_\ell)\big) = D\big((\tau_g)_* X_1, \ldots, (\tau_g)_* X_\ell\big) \tag{4.15}$$

for all $X_1, \ldots, X_\ell \in \Gamma^\infty(TM)$ and $g \in G$ by requiring

$$D\big|_{\mathrm{pr}(e)}\big(T_e\mathrm{pr} X_1, \ldots, T_e\mathrm{pr} X_\ell\big) = T_e\mathrm{pr}\big(d(X_1, \ldots, X_\ell)\big) \tag{4.16}$$

for all $X_1, \ldots, X_\ell \in \mathfrak{m}$.

Proof We use ideas that can be found in [7, Prop. 23.23], see also [6, Chap. 11, Prop. 22]. In this proof, we write $o = \mathrm{pr}(e) = e \cdot H \in G/H$ for the coset defined by $e \in G$. Let $D \in \Gamma^\infty\big(T^*(G/H)^{\otimes \ell} \otimes T(G/H)\big)$ be a tensor field satisfying (4.15). Using Lemma 3.2, i.e. $T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr}\big|_{\mathfrak{m}} = T_o\tau_h \circ T_e\mathrm{pr}\big|_{\mathfrak{m}} = T_e\mathrm{pr} \circ \mathrm{Ad}_h\big|_{\mathfrak{m}}$ for all $h \in H$, and Lemma 4.6, Claim 3, i.e. $(\tau_h)_* X_{G/H} = \big(\mathrm{Ad}_h(X)\big)_{G/H}$ for all $X \in \mathfrak{m}$ as well as $T_e\mathrm{pr} X = X_{G/H}(o)$ we compute for $X_1, \ldots, X_\ell \in \mathfrak{m}$ and $h \in H$

$$\begin{aligned}&(T_e\mathrm{pr} \circ \mathrm{Ad}_h) d(X_1, \ldots, X_\ell)\big) \\ &\quad= (T_o\tau_h \circ T_e\mathrm{pr}) d(X_1, \ldots, X_\ell) \\ &\quad= (T_o\tau_h)\big(D\big|_o\big(T_e\mathrm{pr} X_1, \ldots T_e\mathrm{pr} X_\ell\big)\big) \\ &\quad= (T_o\tau_h) D\big|_o\big((X_1)_{G/H}(o), \ldots, (X_\ell)_{G/H}(o)\big) \\ &\quad= (T_{\tau_{h^{-1}}(o)}\tau_h)\big(D\big((X_1)_{G/H}, \ldots, (X_\ell)_{G/H}\big)(\tau_{h^{-1}}(o)\big)\end{aligned}$$

$$
\begin{aligned}
&= (\tau_h)_*\big(D\big((X_1)_{G/H}, \dots, (X_\ell)_{G/H}\big)\big)\big|_o \\
&= D\big((\tau_h)_*(X_1)_{G/H}, \dots, (\tau_h)_*(X_\ell)_{G/H}\big)\big|_o \\
&= D\big|_o\big((\mathrm{Ad}_h(X_1))_{G/H}(o), \dots, (\mathrm{Ad}_h(X_\ell))_{G/H}(o)\big) \\
&= T_e\mathrm{pr}\big(d\big(\mathrm{Ad}_h(X_1), \dots, \mathrm{Ad}_h(X_\ell)\big)\big),
\end{aligned} \tag{4.17}
$$

where we exploited $\tau_h(o) = o$ for all $h \in H$. Thus the tensor field D fulfilling (4.15) yields an $\mathrm{Ad}(H)$-invariant ℓ-linear map d via (4.16) since $T_e\mathrm{pr}\big|_{\mathfrak{m}} : \mathfrak{m} \to T_{\mathrm{pr}(e)}(G/H)$ is a linear isomorphism. Conversely, assume that $d : \mathfrak{m}^\ell \to \mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant ℓ-linear map. Then (4.16) defines a unique invariant tensor field on G/H fulfilling (4.15) by setting for $v_{1,o}, \dots v_{\ell,o} \in T_o(G/H)$

$$
D\big|_o(v_{1,o}, \dots, v_{\ell,o}) = T_e\mathrm{pr}\Big(d\Big(\big(T_e\mathrm{pr}\big|_{\mathfrak{m}}\big)^{-1}v_{1,o}, \dots, \big(T_e\mathrm{pr}\big|_{\mathfrak{m}}\big)^{-1}v_{\ell,o}\Big)\Big) \tag{4.18}
$$

and defining for $g \in G$ with $p = \mathrm{pr}(g) = \tau_g(o) \in G/H$ for $v_{1p}, \dots, v_{\ell p} \in T_p(G/H)$

$$
D\big|_p(v_{1p}, \dots, v_{\ell p}) = T_o\tau_g\Big(D\big|_o\big(T_p\tau_{g^{-1}}v_{1p}, \dots, (T_p\tau_{g^{-1}}v_{\ell p})\big)\Big). \tag{4.19}
$$

We first show that this yields a well-defined expression. Let $k \in G$ be another element with $\mathrm{pr}(g) = p = \mathrm{pr}(k)$, i.e. there exists a $h \in H$ with $g = kh$ and therefore $k^{-1} = hg^{-1}$ as well as $k = gh^{-1}$ is fulfilled. Using the definition of D in (4.18) and (4.19), we compute

$$
\begin{aligned}
D\big|_p\big(v_{1p}, \dots, v_{\ell p}\big) &= D\big|_{\mathrm{pr}(k)}\big(v_{1p}, \dots, v_{\ell p}\big) \\
&= T_o\tau_k\big(D\big|_o\big(T_p\tau_{k^{-1}}v_{1p}, \dots, T_p\tau_{k^{-1}}v_{\ell p}\big)\big) \\
&= (T_o\tau_g \circ T_o\tau_{h^{-1}})\big(D\big|_o\big((T_o\tau_h \circ T_p\tau_{g^{-1}})v_{1p}, \dots, \\
&\qquad (T_o\tau_h \circ T_p\tau_{g^{-1}})v_{\ell p}\big)\big) \\
&= (T_o\tau_g \circ T_o\tau_{h^{-1}})\big(T_o\tau_h\big(D\big|_o\big(T_p\tau_{g^{-1}}v_{1p}, \dots, (T_p\tau_{g^{-1}}v_{\ell p})\big)\big)\big) \\
&= D\big|_{\mathrm{pr}(g)}\big(v_{1p}, \dots, v_{\ell p}\big),
\end{aligned}
$$

where the fourth equality follows by a calculation similar to (4.17) exploiting the $\mathrm{Ad}(H)$-invariance of $d : \mathfrak{m}^\ell \to \mathfrak{m}$. It remains to proof that D has the desired invariance property. To this end, let $X_1, \dots, X_\ell \in \Gamma^\infty\big(T(G/H)\big)$ be vector fields and let $g \in G$. Suppressing the "foot points" of the tangent maps, we compute by the definition of D for $q = \mathrm{pr}(k) \in G/H$ represented by some $k \in G$

$$
\begin{aligned}
&\big((\tau_g)_*D(X_1, \dots, X_\ell)\big)(q) \\
&\quad = T\tau_g \circ D(X_1, \dots, X_\ell) \circ \tau_{g^{-1}}(q) \\
&\quad = T\tau_g\big(D\big|_{\tau_{g^{-1}}(q)}\big(X(\tau_{g^{-1}}(q)), \dots, X_\ell(\tau_{g^{-1}}(q))\big)\big) \\
&\quad = T\tau_g\big(T\tau_{g^{-1}k}D\big|_o\big(T\tau_{(g^{-1}k)^{-1}}X_1(\tau_{g^{-1}}(q)), \dots, T\tau_{(g^{-1}k)^{-1}}X_\ell(\tau_{g^{-1}}(q))\big)\big) \\
&\quad = T\tau_k D\big|_o\big(T\tau_{k^{-1}} \circ T\tau_g \circ X_1 \circ \tau_{g^{-1}}(q), \dots, T\tau_{k^{-1}} \circ T\tau_g \circ X_\ell \circ \tau_{g^{-1}}(q)\big)
\end{aligned}
$$

$$= T\tau_k D\big|_o\big(T\tau_{k^{-1}}\big((\tau_g)_* X_1\big)(q), \dots, T\tau_{k^{-1}}\big((\tau_g)_* X_\ell\big)(q)\big)$$
$$= D\big|_q\big((\tau_g)_* X_1(q), \dots, (\tau_g)_* X_\ell(q)\big)$$
$$= \big(D\big((\tau_g)_* X_1, \dots, (\tau_g)_* X_\ell\big)\big)(q).$$

Thus $(\tau_g)_* D(X_1, \dots, X_\ell) = D\big((\tau_g)_* X_1, \dots, (\tau_g)_* X_\ell\big)$ is shown for all $g \in G$ and $X_1, \dots, X_\ell \in \Gamma^\infty\big(T(G/H)\big)$ as desired. □

In the remainder part of this subsection, we investigate invariant covariant derivatives and their relation to $\mathrm{Ad}(H)$-invariant bilinear maps $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. We first show that an invariant covariant derivative on G/H yields an $\mathrm{Ad}(H)$-invariant bilinear map by evaluating it on fundamental vector fields and considering its value at $\mathrm{pr}(e) \in G/H$. This is motivated by the discussion in [7, Sec. 23.6].

Before we proceed, we point out that the right-hand side of (4.20) in the next lemma is chosen such that it coincides with the expression from Definition 4.16, below.

Lemma 4.8 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Moreover, let $\nabla\colon \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ be an invariant covariant derivative. Then the following assertions are fulfilled:*

1. *Let $X, Y \in \mathfrak{m}$. Then*
$$\nabla_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(-[X, Y]_{\mathfrak{m}} + \alpha(X, Y)\big) \tag{4.20}$$
defines an $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$.
2. *Let ∇^1 and ∇^2 be both invariant covariant derivatives on G/H. Then $\nabla^1_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = \nabla^2_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)}$ for all $X, Y \in \mathfrak{m}$ implies $\nabla^1 = \nabla^2$.*

Proof We first show Claim 1. Obviously, the map $\mathfrak{m} \times \mathfrak{m} \ni (X, Y) \mapsto [X, Y]_{\mathfrak{m}} \in \mathfrak{m}$ is bilinear and $\mathrm{Ad}(H)$-invariant. Moreover, by exploiting that $T_e\mathrm{pr}\big|_{\mathfrak{m}}\colon \mathfrak{m} \to T_{\mathrm{pr}(e)}(G/H)$ is a linear isomorphism, Claim 1 is equivalent to the assertion that

$$\beta\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}, \quad (X, Y) \mapsto \big(T_e\mathrm{pr}\big|_{\mathfrak{m}}\big)^{-1}\Big(\nabla_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)}\Big)$$

is an $\mathrm{Ad}(H)$-invariant bilinear map. The map β is bilinear since the covariant derivative $\nabla\colon \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ is $\mathbb{R}$-bilinear. Next, let $h \in H$. Using ideas of [1, Sec. 8], we obtain by Lemma 4.6, Claim 3 for $X, Y \in \mathfrak{m}$

$$\begin{aligned}\beta\big(\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)\big) &= \big(T_e\mathrm{pr}\big|_{\mathfrak{m}}\big)^{-1}\Big(\nabla_{\left(\mathrm{Ad}_h(X)\right)_{G/H}}\big(\mathrm{Ad}_h(Y)\big)_{G/H}\big|_{\mathrm{pr}(e)}\Big) \\ &= \big(T_e\mathrm{pr}\big|_{\mathfrak{m}}\big)^{-1}\Big(\nabla_{(\tau_h)_* X_{G/H}} (\tau_h)_* Y_{G/H}\big|_{\mathrm{pr}(e)}\Big).\end{aligned} \tag{4.21}$$

Moreover, using Lemma 3.2, i.e. $T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr}\big|_{\mathfrak{m}} = T_e\mathrm{pr} \circ \mathrm{Ad}_h\big|_{\mathfrak{m}}$, and exploiting that $T_e\mathrm{pr}\big|_{\mathfrak{m}}\colon \mathfrak{m} \to T_{\mathrm{pr}(e)}(G/H)$ is a linear isomorphism, we obtain by the invariance of ∇

$$
\begin{aligned}
T_e\mathrm{pr}\big(\mathrm{Ad}_h\big(\beta(X,Y)\big)\big) &= T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr}\beta(X,Y) \\
&= T_{\mathrm{pr}(e)}\tau_h\big(\nabla_{X_{G/H}}Y_{G/H}\big|_{\mathrm{pr}(e)}\big) \\
&= T_{\tau_h^{-1}(\mathrm{pr}(e))}\tau_h\big(\big(\nabla_{X_{G/H}}Y_{G/H}\big)(\tau_{h^{-1}}(\mathrm{pr}(e)))\big) \\
&= \big((\tau_h)_*\nabla_{X_{G/H}}Y_{G/H}\big)(\mathrm{pr}(e)) \\
&= \nabla_{(\tau_h)_*X_{G/H}}((\tau_h)_*Y_{G/H})\big|_{\mathrm{pr}(e)} \\
&= T_e\mathrm{pr}\big(\beta(\mathrm{Ad}_h(X),\mathrm{Ad}_h(Y))\big),
\end{aligned}
\tag{4.22}
$$

where the last equality holds by (4.21). Obviously, (4.22) is equivalent to

$$
\beta\big(\mathrm{Ad}_h(X),\mathrm{Ad}_h(Y)\big) = \mathrm{Ad}_h\big(\beta(X,Y)\big)
$$

for all $h \in H$ and $X, Y \in \mathfrak{m}$. Hence Claim 1 is proven.

We now show Claim 2. Let $\nabla^1, \nabla^2\colon \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ be two invariant covariant derivatives. Then their difference

$$
D(X,Y) = \nabla^1_X Y - \nabla^2_X Y, \quad X, Y \in \Gamma^\infty(T(G/H))
$$

defines a tensor field $D \in \Gamma^\infty\big(T^*(G/H)^{\otimes 2} \otimes T(G/H)\big)$ on G/H according to [13, Prop. 4.13]. Moreover, this tensor field corresponds to an $\mathrm{Ad}(H)$-invariant bilinear map $d\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ via $D\big|_{\mathrm{pr}(e)}\big(T_e\mathrm{pr}X, T_e\mathrm{pr}Y\big) = T_e\mathrm{pr}\big(d(X,Y)\big)$ for all $X, Y \in \mathfrak{m}$ by Lemma 4.7 because of

$$
\begin{aligned}
(\tau_g)_*\big(D(X,Y)\big) &= (\tau_g)_*\big(\nabla^1_X Y - \nabla^2_X Y\big) \\
&= \nabla^1_{(\tau_g)_*X}(\tau_g)_*Y - \nabla^2_{(\tau_g)_*X}(\tau_g)_*Y \\
&= D\big((\tau_g)_*X, (\tau_g)_*Y\big)
\end{aligned}
$$

for all $X, Y \in \Gamma^\infty\big(T(G/H)\big)$. By $\nabla^1_{X_{G/H}}Y_{G/H}\big|_{\mathrm{pr}(e)} = \nabla^2_{X_{G/H}}Y_{G/H}\big|_{\mathrm{pr}(e)}$ for all $X, Y \in \mathfrak{m}$, we obtain

$$
0 = \nabla^1_{X_{G/H}}Y_{G/H}\big|_{\mathrm{pr}(e)} - \nabla^2_{X_{G/H}}Y_{G/H}\big|_{\mathrm{pr}(e)} = D(X_{G/H}(e), Y_{G/H}(e)) = T_e\mathrm{pr}\big(d(X,Y)\big).
$$

Hence $d(X,Y) = 0$ is fulfilled for all $X, Y \in \mathfrak{m}$. This implies $D = 0$ as desired. □

4.1.2 Invariant Covariant Derivatives in Terms Horizontal Lifts

Lemma 4.8, Claim 1 shows that an invariant covariant derivative ∇ on G/H defines an $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by (4.20). Moreover, it shows that ∇ is uniquely determined by an $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by (4.20).

However, it does not show that such an invariant covariant derivative ∇ on G/H exists. In the sequel, we obtain another proof for the existence of an invariant covariant derivative ∇^α on G/H for a given $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ by expressing ∇^α in terms of horizontally lifted vector fields on G. To this end, we state some lemmas as preparation.

Lemma 4.9 *Let $g\in G$. Then the following assertions are fulfilled:*

1. *Let $\overline{X}\in\Gamma^\infty\big(\mathrm{Hor}(G)\big)$ and $g\in G$. Then the push-forward of $\overline{X}$ by $\ell_g\colon G\to G$ is a horizontal vector field on G, i.e. $(\ell_g)_*\overline{X}=T\ell_g\circ\overline{X}\circ\ell_{g^{-1}}\in\Gamma^\infty\big(\mathrm{Hor}(G)\big)$ holds.*
2. *Let $f\colon G\to\mathbb{R}$ be smooth and let $X\in\mathfrak{m}$. Moreover, let $g\in G$. Denoting by $(\ell_g)_*X^L=T\ell_g\circ X^L\circ\ell_{g^{-1}}$ the push-forward of $X^L\in\Gamma^\infty\big(\mathrm{Hor}(G)\big)$ by $\ell_g\colon G\to G$, one has*

$$(\ell_g)_*(fX^L)=\big((\ell_{g^{-1}})^*f\big)X^L. \tag{4.23}$$

Proof The first claim is obvious. It remains to prove the second claim. To this end, we compute for $g,k\in G$

$$\begin{aligned}\big((\ell_g)_*(fX^L)\big)(k)&=T_{g^{-1}k}\ell_g\circ\big(fX^L\big)\circ\ell_{g^{-1}}(k)\\&=T_{g^{-1}k}\ell_g\big(f(g^{-1}k)X^L(g^{-1}k)\big)\\&=f\big(\ell_{g^{-1}}(k)\big)\big(T_{g^{-1}k}\ell_g\circ X^L\circ\ell_{g^{-1}}(k)\big)\\&=\big((\ell_{g^{-1}})^*f\big)(k)X^L(k),\end{aligned}$$

where exploited that $X^L\in\Gamma^\infty\big(\mathrm{Hor}(G)\big)$ is a left-invariant vector field. This yields the desired result. □

Lemma 4.10 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ and let $\mathrm{Hor}(G)\subseteq TG$ be the horizontal bundle from Proposition 3.12. Moreover, let $\alpha\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant billinear map. Let $\{A_1,\dots,A_N\}\subseteq\mathfrak{m}$ be a basis of $\mathfrak{m}$ and denote by $A_1^L,\dots,A_N^L\in\Gamma^\infty\big(\mathrm{Hor}(G)\big)$ the corresponding left-invariant frame. Let $\overline{X},\overline{Y}\in\Gamma^\infty(\mathrm{Hor}(G))$ be horizontal vector fields on G and expand them in the frame $A_1^L,\dots A_N^L$, i.e. $\overline{X}=x^iA_i^L$ and $\overline{Y}=y^jA_j^L$, with some uniquely determined smooth functions $x^i,y^j\colon G\to\mathbb{R}$, where $i,j\in\{1,\dots,N\}$. Using this notation and Einstein summation convention, as usual, we set*

$$\nabla^{\mathrm{Hor},\alpha}_{\overline{X}}\overline{Y}=\big(\mathscr{L}_{\overline{X}}y^j\big)A_j^L+x^iy^j\big(\alpha(A_i,A_j)\big)^L. \tag{4.24}$$

Then (4.24) defines a map $\nabla^{\mathrm{Hor},\alpha}\colon\Gamma^\infty\big(\mathrm{Hor}(G)\big)\times\Gamma^\infty\big(\mathrm{Hor}(G)\big)\to\Gamma^\infty\big(\mathrm{Hor}(G)\big)$ fulfilling

$$\nabla^{\mathrm{Hor},\alpha}_{f\overline{X}}\overline{Y}=f\nabla^{\mathrm{Hor},\alpha}_{\overline{X}}\overline{Y}\quad\text{and}\quad\nabla^{\mathrm{Hor},\alpha}_{\overline{X}}(f\overline{Y})=\big(\mathscr{L}_{\overline{X}}f\big)\overline{Y}+f\nabla^{\mathrm{Hor},\alpha}_{\overline{X}}\overline{Y} \tag{4.25}$$

for all $f\in\mathscr{C}^\infty(G)$ and $\overline{X},\overline{Y}\in\Gamma^\infty\big(\mathrm{Hor}(G)\big)$. Moreover, $\nabla^{\mathrm{Hor},\alpha}$ has the following properties:

1. *For each $g \in G$, the map $\nabla^{\mathrm{Hor},\alpha}$ is invariant under $\ell_g \colon G \to G$ in the sense that*

$$\nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y} = \big(\ell_{g^{-1}}\big)_* \big(\nabla^{\mathrm{Hor},\alpha}_{(\ell_g)_* \overline{X}} \big((\ell_g)_* \overline{Y}\big)\big), \quad \overline{X}, \overline{Y} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big). \tag{4.26}$$

holds.

2. *The map $\nabla^{\mathrm{Hor},\alpha} \colon \Gamma^\infty\big(\mathrm{Hor}(G)\big) \times \Gamma^\infty\big(\mathrm{Hor}(G)\big) \to \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ fulfills*

$$\nabla^{\mathrm{Hor},\alpha}_{X^L} Y^L\big|_e = \big(\alpha(X,Y)\big)^L(e) = \alpha(X,Y) \quad X, Y \in \mathfrak{m}. \tag{4.27}$$

Proof We first show that $\nabla^{\mathrm{Hor},\alpha}$ is well-defined. Let $\{B_1, \ldots, B_N\} \subseteq \mathfrak{m}$ be another basis of $\mathfrak{m}$. Then one has $A_i = a_i^k B_k$ and therefore $A_i^L = a_i^k (B_k^L)$, where $\big(a_i^k\big) \in \mathbb{R}^{N\times N}$ is some invertible matrix. Writing $\overline{X} = x^i A_i^L$ and $\overline{Y} = y^j A_j^L$ yields $\overline{X} = (x^i a_i^k) B_k^L$ as well as $\overline{Y} = (y^j a_j^\ell) B_\ell^L$. Using the $\mathbb{R}$-linearity of Lie derivatives and the bilinearity of $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ we compute

$$\begin{aligned} \nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y} &= \big(\mathscr{L}_{\overline{X}}(y^j a_j^\ell)\big) B_\ell^L + (x^i a_i^k)(y^j a_j^\ell)\big(\alpha(B_k, B_\ell)\big)^L \\ &= \big(\mathscr{L}_{\overline{X}} y^j\big) a_j^\ell B_\ell^L + x^i y^j \big(\alpha(a_i^k B_k, a_j^\ell B_\ell)\big)^L \\ &= \big(\mathscr{L}_{\overline{X}} y^j\big) A_j^L + x^i y^j \big(\alpha(A_i, A_j)\big)^L. \end{aligned}$$

Thus $\nabla^{\mathrm{Hor},\alpha} \colon \Gamma^\infty\big(\mathrm{Hor}(G)\big) \times \Gamma^\infty\big(\mathrm{Hor}(G)\big) \to \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ is well-defined. By a straightforward computation, one verifies that $\nabla^{\mathrm{Hor},\alpha}$ fulfills (4.25).

Next we show Claim 1. By Lemma 4.9, we obtain

$$(\ell_g)_* \overline{X} = (\ell_g)_* \big(x^i A_i^L\big) = \big((\ell_{g^{-1}})^* x^i\big) A_i^L \tag{4.28}$$

for all $g \in G$ and analogously $(\ell_g)_* \overline{Y} = \big((\ell_{g^{-1}})^* y^j\big) A_j^L$. By (4.28) and using $\ell_g^* A_i^L = A_i^L$ due the left-invariance of A_i^L, we compute

$$\begin{aligned} &\nabla^{\mathrm{Hor},\alpha}_{((\ell_g)_* \overline{X})} ((\ell_g)_* \overline{Y}) \\ &= \Big(\mathscr{L}_{((\ell_{g^{-1}})^* x^i) A_i^L} \big((\ell_{g^{-1}})^* y^j\big)\Big) A_j^L + \big((\ell_{g^{-1}})^* x^i\big)\big((\ell_{g^{-1}})^* y^j\big)\big(\alpha(A_i, A_j)\big)^L \\ &= \Big(\big((\ell_{g^{-1}})^* x^i\big) \mathscr{L}_{(\ell_{g^{-1}})^* A_i^L} \big((\ell_{g^{-1}})^* y^j\big)\Big) A_j^L + \big((\ell_{g^{-1}})^* (x^i y^j)\big)\big(\alpha(A_i, A_j)\big)^L \\ &= \big((\ell_{g^{-1}})^* x^i\big)\Big((\ell_{g^{-1}})^* \big(\mathscr{L}_{A_i^L} y^j\big)\Big) A_j^L + \big((\ell_{g^{-1}})^* (x^i y^j)\big)\big(\alpha(A_i, A_j)\big)^L \\ &= \big((\ell_{g^{-1}})^* \big(\mathscr{L}_{x^i A_i^L} y^j\big)\big) A_j^L + \big((\ell_{g^{-1}})^* (x^i y^j)\big)\big(\alpha(A_i, A_j)\big)^L \\ &= (\ell_g)_* \big(\nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\big), \end{aligned}$$

where we used $\mathscr{L}_{(\ell_{g^{-1}})^* A_i^L} \big((\ell_{g^{-1}})^* y^j\big) = (\ell_{g^{-1}})^* (\mathscr{L}_{A_i^L} y^j)$, see e.g. [13, Prop. 8.16] and the last equality follows by Lemma 4.9, Claim 2. Thus Claim 1 is shown.

It remains to prove Claim 2. To this end, let $X, Y \in \mathfrak{m}$. Then we can write

$$X^L = x^i A_i^L \quad \text{and} \quad Y^L = y^j A_j^L,$$

where the functions $x^i, y^j : G \to \mathbb{R}$ are clearly constant. By this notation, we compute by exploiting that $\mathscr{L}_{X^L} y^j = 0$ since $y^j : G \to \mathbb{R}$ is constant for all $j \in \{1, \dots, N\}$

$$\nabla^{\mathrm{Hor},\alpha}_{X^L} Y^L = \big(\mathscr{L}_{X^L} y^j\big) A_j^L + x^i y^j \big(\alpha(A_i, A_j)\big)^L = x^i y^j \big(\alpha(A_i, A_j)\big)^L = \big(\alpha(X, Y)\big)^L.$$

For $g = e$, the equation above yields

$$\nabla^{\mathrm{Hor},\alpha}_{X^L} Y^L\big|_e = \big(\alpha(X, Y)\big)^L(e) = \alpha(X, Y) \tag{4.29}$$

as desired. □

Remark 4.11 The map $\nabla^{\mathrm{Hor},\alpha} : \Gamma^\infty\big(\mathrm{Hor}(G)\big) \times \Gamma^\infty\big(\mathrm{Hor}(G)\big) \to \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ in Lemma 4.10 has properties that are similar to those of a covariant derivative on $\mathrm{Hor}(G) \to G$ although its first argument is only defined on $\Gamma^\infty\big(\mathrm{Hor}(G)\big) \subsetneq \Gamma^\infty(TG)$.

Horizontal lifts are compatible with push-forwards in the following sense.

Lemma 4.12 *Let $X \in \Gamma^\infty\big(T(G/H)\big)$ and let $\overline{X} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ be its horizontal lift. Then*

$$\overline{(\tau_g)_* X} = (\ell_g)_* \overline{X} \tag{4.30}$$

holds for $g \in G$, where $\overline{(\tau_g)_* X}$ denotes the horizontal lift of $(\tau_g)_* X \in \Gamma^\infty\big(T(G/H)\big)$.

Proof Let $g \in G$. We have $\mathrm{pr} \circ \ell_g = \tau_g \circ \mathrm{pr}$ implying $T\mathrm{pr} \circ T\ell_g = T\tau_g \circ T\mathrm{pr}$. Using this equality as well as $T\mathrm{pr} \circ \overline{X} = X \circ \mathrm{pr}$ we compute

$$\begin{aligned} T\mathrm{pr} \circ \big((\ell_g)_* \overline{X}\big) &= T\mathrm{pr} \circ \big(T\ell_g \circ \overline{X} \circ \ell_{g^{-1}}\big) \\ &= T\tau_g \circ \big(T\mathrm{pr} \circ \overline{X}\big) \circ \ell_{g^{-1}} \\ &= T\tau_g \circ \big(X \circ \mathrm{pr}\big) \circ \ell_{g^{-1}} \\ &= T\tau_g \circ X \circ \tau_{g^{-1}} \circ \mathrm{pr} \\ &= \big((\tau_g)_* X\big) \circ \mathrm{pr}. \end{aligned} \tag{4.31}$$

Since $(\ell_g)_* \overline{X} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ is horizontal and $T\mathrm{pr} \circ \big((\ell_g)_* \overline{X}\big) = \big((\tau_g)_* X\big) \circ \mathrm{pr}$ holds by (4.31), we obtain $\overline{(\tau_g)_* X} = (\ell_g)_* \overline{X}$ as desired. □

Lemma 4.13 *Let G/H be a reductive homogeneous space with reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Moreover, let $\{A_1, \dots, A_M\} \subseteq \mathfrak{m}$ be some vectors, not necessarily forming a basis of $\mathfrak{m}$, and let $x^i : G \to \mathbb{R}$ for $i \in \{1, \dots, M\}$ be smooth. Define the horizontal vector field $\overline{X} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ by*

$$\overline{X}(g) = x^i(g) A_i^L(g), \quad g \in G. \tag{4.32}$$

Then $\overline{X}$ is the horizontal lift of $X \in \Gamma^\infty\big(T(G/H)\big)$ given by

$$T\mathrm{pr} \circ \overline{X} = X \circ \mathrm{pr} \tag{4.33}$$

iff

$$x^i(g)A^L(g) = x^i(gh)\big(\mathrm{Ad}_h(A_i)\big)^L(g) \quad \Longleftrightarrow \quad x^i(g)A_i = x^i(gh)\mathrm{Ad}_h(A_i) \tag{4.34}$$

holds for all $g \in G$ and $h \in H$.

Proof We first assume that $\overline{X} = x^i A_i^L$ is the horizontal lift of the vector field $X \in \Gamma^\infty\big(T(G/H)\big)$. Then (4.33) holds. Using $\mathrm{pr}(gh) = \mathrm{pr}(g)$ for all $g \in G$ and $h \in H$, we can rewrite (4.33) equivalently as

$$\begin{aligned}
T_g\mathrm{pr}\big(x^i(g)A_i^L(g)\big) &= (T_g\mathrm{pr})\overline{X}(g) \\
&= X \circ \mathrm{pr}(g) \\
&= X \circ \mathrm{pr}(gh) \\
&= \big(T\mathrm{pr} \circ \overline{X}\big)(gh) \\
&= T_{gh}\mathrm{pr}\big(x^i(gh)A_i^L(gh)\big) \\
&= x^i(gh)\big((T_{gh}\mathrm{pr} \circ T_e\ell_{gh})A_i\big) \\
&= x^i(gh)\big(T_e(\mathrm{pr} \circ \ell_{gh})A_i\big) \\
&= x^i(gh)\big((T_{\mathrm{pr}(e)}\tau_{gh} \circ T_e\mathrm{pr})A_i\big) \\
&= x^i(gh)\big(T_{\mathrm{pr}(e)}(\tau_g \circ \tau_h) \circ T_e\mathrm{pr}A_i\big) \\
&= x^i(gh)\big(T_{\mathrm{pr}(e)}\tau_g \circ (T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr})A_i\big) \\
&= x^i(gh)\big(T_{\mathrm{pr}(e)}\tau_g \circ (T_e\mathrm{pr} \circ \mathrm{Ad}_h)A_i\big) \\
&= x^i(gh)\big(T_e(\tau_g \circ \mathrm{pr})(\mathrm{Ad}_h(A_i))\big) \\
&= x^i(gh)\big(T_e(\mathrm{pr} \circ \ell_g)(\mathrm{Ad}_h(A_i))\big) \\
&= x^i(gh)\big((T_g\mathrm{pr} \circ T_e\ell_g)\mathrm{Ad}_h(A_i)\big) \\
&= x^i(gh)\big(T_g\mathrm{pr}\big(\mathrm{Ad}_h(A_i)\big)^L(g)\big),
\end{aligned} \tag{4.35}$$

where we exploited Lemma 3.2, i.e. $T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr}\big|_{\mathfrak{m}} = T_e\mathrm{pr} \circ \mathrm{Ad}_h\big|_{\mathfrak{m}}$ for all $h \in H$. Since $T_g\mathrm{pr}\colon \mathrm{Hor}(G)_g \to T_{\mathrm{pr}(g)}(G/H)$ is a linear isomorphism for each $g \in G$, Equation (4.35) is equivalent to the left-hand side of (4.34). Applying the linear isomorphism $(T_e\ell_g)^{-1}\colon \mathrm{Hor}(G)_g \to \mathfrak{m}$ to both sides of this equality shows the equivalence to right-hand side of (4.34).

Conversely, assuming that the functions $x^i\colon G \to \mathbb{R}$ in the definition of $\overline{X} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ in (4.32) fulfill (4.34) for all $i \in \{1, \ldots, M\}$, we define the map

$$X\colon G/H \to T(G/H), \quad \mathrm{pr}(g) = g \cdot H \mapsto (T_g\mathrm{pr}) \circ \overline{X}(g),$$

where the coset $\mathrm{pr}(g) = g \cdot H \in G/H$ is represented by $g \in G$. Then the computation in (4.35) shows that $X\colon G/H \to T(G/H)$ is well-defined, i.e. we have for all $g \in G$ and $h \in H$

$$X(\mathrm{pr}(g)) = T_g\mathrm{pr} \circ \overline{X}(g) = T_{gh}\mathrm{pr} \circ \overline{X}(gh) = X(\mathrm{pr}(gh)). \tag{4.36}$$

Then $X \circ \mathrm{pr} = T\mathrm{pr} \circ \overline{X}$ holds by construction. Since $\mathrm{pr}\colon G \to G/H$ is a surjective submersion and $T\mathrm{pr} \circ \overline{X}\colon G \to T(G/H)$ is smooth, the map $X\colon G/H \to T(G/H)$ is smooth by [13, Thm. 4.29]. Clearly, for $\mathrm{pr}(g) \in G/H$, one has $X(\mathrm{pr}(g)) \in T_{\mathrm{pr}(g)}(G/H)$. Hence $X \in \Gamma^\infty\big(T(G/H)\big)$ is a smooth vector field on G/H. Obviously, its horizontal lift is given by $\overline{X}$. □

Lemma 4.14 *Let $\overline{X}, \overline{Y} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ be the horizontal lifts of $X, Y \in \Gamma^\infty\big(T(G/H)\big)$, respectively, and let $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ be a basis of $\mathfrak{m}$. Denote by $A_1^L, \ldots, A_N^L$ the corresponding left-invariant frame of $\Gamma^\infty(\mathrm{Hor}(G))$. Moreover, expand $\overline{X} = x^i A_i^L$ and $\overline{Y} = y^j A_j^L$, where $x^i, y^j\colon G \to \mathbb{R}$ are smooth. Then*

$$(\mathscr{L}_{\overline{X}} y^j)(g) A_j^L(g) = (\mathscr{L}_{\overline{X}} y^j)(gh)\big(\mathrm{Ad}_h(A_j)\big)^L(g) \tag{4.37}$$

holds. In particular, $\big(\mathscr{L}_{\overline{X}} y^j\big) A_j^L \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ is the horizontal lift of the vector field $X \in \Gamma^\infty\big(T(G/H)\big)$ given by $X \circ \mathrm{pr} = T\mathrm{pr} \circ \big((\mathscr{L}_{\overline{X}} y^j) A_j^L\big)$.

Proof Let $\{A^1, \ldots A^N\} \subseteq \mathfrak{m}^*$ be the dual basis of $\{A_1, \ldots, A_N\}$, i.e. $A^i(A_j) = \delta^i_j$ for all $i, j \in \{1, \ldots, N\}$ with δ^i_j denoting Kronecker deltas. Since $\overline{Y} = y^j A_j^L$ is the horizontal lift of Y, one has

$$y^j(g) A_j = y^j(gh)\mathrm{Ad}_h(A_j) \tag{4.38}$$

for all $g \in G$ and $h \in H$ by Lemma 4.13. Let $j \in \{1, \ldots, N\}$. Applying $A^j \in \mathfrak{m}^*$ to (4.38) yields by $A^j(A_k) = \delta^j_k$

$$y^j(g) = A^j\big(y^k(g)A_k\big) = A^j\big(y^k(gh)\mathrm{Ad}_h(A_k)\big) = y^k(gh)A^j\big(\mathrm{Ad}_h(A_k)\big) \tag{4.39}$$

for all $g \in G$ and $h \in H$. Next we define the curves $c_1\colon \mathbb{R} \ni t \mapsto g\exp\big(tx^i(g)A_i\big) \in G$ and $c_2\colon \mathbb{R} \ni t \mapsto gh\exp\big(tx^i(gh)A_i\big) \in G$. Then

$$\dot{c_1}(0) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big(g\exp\big(tx^i(g)A_i\big)\big)\big|_{t=0} = T_e\ell_g\big(x^i(g)A_i\big) = x^i(g)A_i^L(g) = \overline{X}(g)$$

holds and analogously one obtains

$$\dot{c_2}(0) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big(gh\exp\big(tx^i(gh)A_i\big)\big)\big|_{t=0} = T_e\ell_{gh}\big(x^i(gh)A_i\big) = x^i(gh)A_i^L(gh) = \overline{X}(gh).$$

Expressing $y^j : G \to \mathbb{R}$ by (4.39) and using the definition of c_1 and c_2, we compute for $g \in G$ and $h \in H$ by $\mathrm{Conj}_h \circ \exp = \exp \circ \mathrm{Ad}_h$

$$
\begin{aligned}
\big(\mathscr{L}_{\overline{X}} y^j\big) A_j^L(g) &= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(y^j\big(c_1(t)\big)\big|_{t=0}\big)\Big) A_j^L(g) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(y^k\big(c_1(t)h\big)A^j\big(\mathrm{Ad}_h(A_k)\big)\Big)\Big|_{t=0}\Big) A_j^L(g) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^k\Big(g \exp(tx^i(g)A_i)h\Big)\Big|_{t=0}\Big)\Big(A^j\big(\mathrm{Ad}_h(A_k)\big)A_j^L(g)\Big) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^k\Big(g \exp\Big(tx^i(gh)\mathrm{Ad}_h(A_i)\Big)h\Big)\Big|_{t=0}\Big)\big(\mathrm{Ad}_h(A_k)\big)^L(g) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^k\Big(g \exp\Big(t\mathrm{Ad}_h\big(x^i(gh)A_i\big)\Big)h\Big)\Big|_{t=0}\Big)\big(\mathrm{Ad}_h(A_j)\big)^L(g) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^k\Big(g\mathrm{Conj}_h\Big(\exp\big(tx^i(gh)A_k\big)\Big)h\Big)\Big|_{t=0}\Big)\big(\mathrm{Ad}_h(A_j)\big)^L(g) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^k\Big(gh \exp\big(tx^i(gh)A_k\big)\Big)\Big|_{t=0}\Big)\big(\mathrm{Ad}_h(A_j)\big)^L(g) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^k\big(c_2(t)\big)\big|_{t=0}\Big)\big(\mathrm{Ad}_h(A_j)\big)^L(g) \\
&= \big(\mathscr{L}_{\overline{X}} y^k\big)(gh)\big(\mathrm{Ad}_h(A_j)\big)^L(g)
\end{aligned}
$$

showing (4.37). Thus $\big(\mathscr{L}_{\overline{X}} y^j\big) A_j^L$ is the horizontal lift of the vector field X on G/H given by $X \circ \mathrm{pr} = T\mathrm{pr} \circ \big(\mathscr{L}_{\overline{X}} y^j\big) A_j^L$ according to Lemma 4.13. □

After this preparation, we are in the position to prove the existence of an invariant covariant derivative on G/H associated with an $\mathrm{Ad}(H)$-invariant bilinear map by expressing it in terms of horizontally lifted vector fields.

Theorem 4.15 *Let $X, Y \in \Gamma^\infty\big(T(G/H)\big)$ and let $\overline{X}, \overline{Y} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big)$ denote their horizontal lifts. Let $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ be a basis and let $A_1^L, \ldots A_N^L$ denote the corresponding left-invariant vector fields. Moreover, expand $\overline{X} = x^i A_i^L$ and $\overline{Y} = y^j A_j^L$ with smooth functions $x^i, y^j : G \to \mathbb{R}$ for $i, j \in \{1, \ldots, N\}$. Let $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map. Then*

$$
(\nabla^\alpha_X Y) \circ \mathrm{pr} = T\mathrm{pr}\big((\mathscr{L}_{\overline{X}} y^j) A_j^L + x^i y^j \big(\alpha(A_i, A_j)\big)^L\big) \tag{4.40}
$$

defines an invariant covariant derivative $\nabla^\alpha : \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ and

$$
\overline{\nabla^\alpha_X Y} = \big(\mathscr{L}_{\overline{X}} y^j\big) A_j^L + x^i y^j \big(\alpha(A_i, A_j)\big)^L \tag{4.41}
$$

holds, where $\overline{\nabla^\alpha_X Y}$ denotes the horizontal lift of $\nabla^\alpha_X Y$. Moreover, for all $X, Y \in \mathfrak{m}$

$$
\nabla^\alpha_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(-[X, Y]_{\mathfrak{m}} + \alpha(X, Y)\big) \tag{4.42}
$$

is fulfilled. In addition, ∇^α is the unique invariant covariant derivative on G/H satisfying (4.42).

Proof We define the covariant derivative ∇^α on G/H by

$$\left(\nabla^\alpha_X Y\right) \circ \mathrm{pr} = T\mathrm{pr} \circ \left(\nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\right), \tag{4.43}$$

where $\nabla^{\mathrm{Hor},\alpha}$ is given by Lemma 4.10. We first show that this definition yields a well-defined expression, i.e. $\left(\nabla^\alpha_X Y\right) \circ \mathrm{pr}(g) = \left(\nabla^\alpha_X Y\right) \circ \mathrm{pr}(gh)$ holds for $g \in G$ and $h \in H$. To this end, we calculate by exploiting Lemmas 4.13 and 4.14 as well as the $\mathrm{Ad}(H)$-invariance of $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$

$$\begin{aligned}
\nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\big|_g &= \left(\mathscr{L}_{\overline{X}} y^j\right)(g) A_j^L(g) + x^i(g) y^j(g) \big(\alpha(A_i, A_j)\big)^L(g) \\
&= \left(\mathscr{L}_{\overline{X}} y^j\right)(g) A_j^L(g) + \big(\alpha\big(x^i(g) A_i, y^j(g) A_j\big)\big)^L(g) \\
&= \left(\mathscr{L}_{\overline{X}} y^j\right)(gh) \big(\mathrm{Ad}_h(A_j)\big)^L(g) + \big(\alpha\big(x^i(gh)\mathrm{Ad}_h(A_i), y^j(gh)\mathrm{Ad}_h(A_j)\big)\big)^L(g) \\
&= \left(\mathscr{L}_{\overline{X}} y^j\right)(gh) \big(\mathrm{Ad}_h(A_j)\big)^L(g) + x^i(gh) y^j(gh) \big(\alpha\big(\mathrm{Ad}_h(A_i), \mathrm{Ad}_h(A_j)\big)\big)^L(g) \\
&= \left(\mathscr{L}_{\overline{X}} y^j\right)(gh) \big(\mathrm{Ad}_h(A_j)\big)^L(g) + x^i(gh) y^j(gh) \big(\mathrm{Ad}_h\big(\alpha(A_i, A_j)\big)\big)^L(g).
\end{aligned}$$

Hence (4.43) yields a well-defined vector field on G/H by Lemma 4.13.

Next we show that ∇^α yields a covariant derivative on G/H. Let $f : G/H \to \mathbb{R}$ be smooth. By $\overline{fX} = \mathrm{pr}^*(f)\overline{X}$ and the properties of $\nabla^{\mathrm{Hor},\alpha}$ from (4.25) in Lemma 4.10, we obtain

$$\begin{aligned}
\nabla^\alpha_X (fY) \circ \mathrm{pr} &= T\mathrm{pr} \circ \left(\nabla^{\mathrm{Hor}}_{\overline{X}} (\mathrm{pr}^*(f)\overline{Y})\right) \\
&= T\mathrm{pr} \circ \left((\mathscr{L}_{\overline{X}}(\mathrm{pr}^* f))\overline{Y} + \mathrm{pr}^*(f) \nabla^{\mathrm{Hor}}_{\overline{X}} \overline{Y}\right) \\
&= T\mathrm{pr} \circ \left(\mathrm{pr}^*(\mathscr{L}_X f)\overline{Y}\right) + T\mathrm{pr} \circ \left(\mathrm{pr}^*(f) \nabla^{\mathrm{Hor}}_{\overline{X}} \overline{Y}\right) \\
&= \left((\mathscr{L}_X f) Y + f \nabla^\alpha_X Y\right) \circ \mathrm{pr}
\end{aligned}$$

due to $\mathscr{L}_{\overline{X}}(\mathrm{pr}^* f) = \mathrm{pr}^*(\mathscr{L}_X f)$ by [13, Prop. 8.16] since X and $\overline{X}$ are pr-related. Moreover, we have

$$\begin{aligned}
\nabla^\alpha_{fX} Y \circ \mathrm{pr} &= T\mathrm{pr} \circ \left(\nabla^{\mathrm{Hor},\alpha}_{\mathrm{pr}^*(f)\overline{X}} \overline{Y}\right) \\
&= T\mathrm{pr} \circ \left(\mathrm{pr}^*(f) \nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\right) \\
&= (\mathrm{pr}^* f)\left(T\mathrm{pr}\left(\nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\right)\right) \\
&= \left(f \nabla^\alpha_X Y\right) \circ \mathrm{pr}
\end{aligned}$$

by Lemma 4.10. Hence ∇^α is indeed a covariant derivative. In addition, ∇^α is invariant. Indeed, by Lemma 4.10, Claim 1 and Lemma 4.12, one has

$$\begin{aligned}
\left(\nabla^\alpha_{(\tau_g)_* X} (\tau_g)_* Y\right) \circ \mathrm{pr} &= T\mathrm{pr} \circ \left(\nabla^{\mathrm{Hor},\alpha}_{(\ell_g)_* \overline{X}} (\ell_g)_* \overline{Y}\right) \\
&= T\mathrm{pr} \circ \left((\ell_g)_* \nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\right) \\
&= \left((\tau_g)_* \nabla^\alpha_X Y\right) \circ \mathrm{pr}.
\end{aligned}$$

Next let $X, Y \in \mathfrak{m}$ and let $\{A^1, \ldots, A^N\} \subseteq \mathfrak{m}^*$ be the dual basis of $\{A_1, \ldots, A_N\}$. By Lemma 4.6, Claim 1, we have $\overline{Y_{G/H}} = y^j A_j^L$ with $y^j : G \ni g \mapsto y^j(g) = A^j(\mathrm{Ad}_{g^{-1}}(Y)) \in \mathbb{R}$ for $j \in \{1, \ldots, N\}$. Thus we obtain by Lemma 4.6, Claim 2

$$\begin{aligned}
&\nabla^{\mathrm{Hor},\alpha}_{\overline{X_{G/H}}} \overline{Y_{G/H}}\Big|_e \\
&\quad = -[X, Y]_{\mathfrak{m}} + A^i(\mathrm{Ad}_e(X)) A^j(\mathrm{Ad}_e(Y)) \alpha(A_i, A_j)^L(e) \\
&\quad = -[X, Y]_{\mathfrak{m}} + \alpha(X, Y),
\end{aligned} \tag{4.44}$$

where we used that $\overline{X_{G/H}}(e) = A^i\big(\mathrm{Ad}_{e^{-1}}(X)\big) A_i^L(e) = X$ is fulfilled for all $X \in \mathfrak{m}$ by Lemma 4.6, Claim 1. Equation (4.44) is equivalent to (4.42) because of

$$\nabla^{\alpha}_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\Big(\overline{\nabla^{\alpha}_{X_{G/H}} Y_{G/H}}(e)\Big) = T_e\mathrm{pr}\Big(\nabla^{\mathrm{Hor},\alpha}_{\overline{X_{G/H}}} \overline{Y_{G/H}}\Big|_e\Big). \tag{4.45}$$

Moreover, ∇^α is uniquely determined by (4.42) according to Lemma 4.8, Claim 2. This yields the desired result. □

The next definition makes sense due to Theorem 4.15.

Definition 4.16 Let $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map. Then the invariant covariant derivative $\nabla^\alpha : \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ which is uniquely determined by

$$\nabla^{\alpha}_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(-[X, Y]_{\mathfrak{m}} + \alpha(X, Y)\big), \quad X, Y \in \mathfrak{m}, \tag{4.46}$$

is called the invariant covariant derivative associated with α or corresponding to α.

Remark 4.17 The right hand side of (4.46) in Definition 4.16 is chosen such that the invariant covariant derivative ∇^α corresponds to the invariant affine connection from [1, Thm. 8.1] associated with the $\mathrm{Ad}(H)$-invariant bilinear map α, see Proposition 4.18 below.

As already mentioned above, the one-to-one correspondence between invariant affine connections on G/H and $\mathrm{Ad}(H)$-invariant bilinear maps $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is proven in [1, Thm. 8.1]. Clearly, an invariant covariant derivative on G/H yields an invariant affine connection on G/H and vice versa by Remark 4.3. In addition, Theorem 4.15 provides another proof for the existence and uniqueness of an invariant covariant derivative on G/H corresponding to an $\mathrm{Ad}(H)$-invariant bilinear map $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ via (4.46) from Definition 4.2. The next proposition shows that ∇^α associated to the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ corresponds indeed the invariant affine connection associated with α from [1, Thm. 8.1].

Proposition 4.18 *Let G/H be a reductive homogeneous space with fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and let $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map. Moreover, let $t^\alpha : \Gamma^\infty\big(T(G/H)\big) \to \mathrm{End}_{\mathscr{C}^\infty(G/H)}\big(\Gamma^\infty\big(T(G/H)\big)\big)$ denote the invariant affine connection corresponding to α from* [1, Thm. 8.1]. *Then $\nabla^\alpha_X Y = t^\alpha(Y)(X)$ holds for all $X, Y \in \Gamma^\infty\big(T(G/H)\big)$, i.e. t^α is the affine connection corresponding to ∇^α by Remark 4.3.*

Proof Obviously, an invariant affine connection corresponds to an invariant covariant derivative and vice versa by Remark 4.3. We now briefly recall some parts of the construction of the invariant affine connections from [1, Sec. 7–8], where we adapt some notations. Let $N = \dim(\mathfrak{m})$ and $n = \dim(\mathfrak{g})$. Let (V, x) be a chart of G, where $V \subseteq G$ is an open neighborhood of $e \in G$ such that V is diffeomorphic to $M \times K$, where M and K are the submanifolds of V defined by

$$M = \big\{g \in V \mid x^{N+1}(g) = \cdots = x^n(g) = 0\big\},$$
$$K = \big\{g \in V \mid x^1(g) = \cdots = x^N(g) = 0\big\},$$

where M is denoted by N in [1, Sec. 7]. Moreover, it is assumed that V is chosen such that the restriction of the canonical projection $\mathrm{pr}\big|_M : M \to G/H$ is a diffeomorphism onto its image denoted by $M^* = \mathrm{pr}(M)$. It is pointed out in [1, Sec. 7] that the existence of such a chart is well-known referring to [14, Chap. IV, §V]. In addition to the assumptions from [1, Sec. 7], we assume that $T_e M = \mathfrak{m}$ holds. Clearly, a chart (V, x) of G centered at $e \in G$ with the properties listed above can be constructed by exploiting that the map

$$\mathfrak{g} \to G, \quad X \mapsto \exp(X_{\mathfrak{m}}) \exp(X_{\mathfrak{h}}) \tag{4.47}$$

restricted to a suitable open neighborhood of $0 \in \mathfrak{g}$ is a diffeomorphism onto its image which is an open neighborhood of $e \in G$, see e.g. [15, p. 76]. Obviously, M^* is an open submanifold of G/H. Following [1, Eq. (7.1)], we now define for $X \in \mathfrak{m}$ the vector field $X^* \in \Gamma^\infty\big(TM^*\big)$ by

$$X^*(\mathrm{pr}(g)) = X^*(\tau_g(\mathrm{pr}(e))) = \big(T_{\mathrm{pr}(e)}\tau_g\big)(T_e\mathrm{pr}X), \quad \mathrm{pr}(g) \in M^*, \quad g \in M, \tag{4.48}$$

where we exploit that $\mathrm{pr}\big|_M : M \to M^*$ is a diffeomorphism. We now relate ∇^α to t^α which is uniquely determined by

$$t^\alpha(Y^*)(X^*)\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(\alpha(X, Y)\big), \quad X, Y \in \mathfrak{m} \tag{4.49}$$

according to [1, Thm. 8.1], see in particular [1, Eq. (8.1)]. To this end, we rewrite (4.48) as

$$\begin{aligned} X^*(\mathrm{pr}(g)) &= \big(T_{\mathrm{pr}(e)}\tau_g\big) \circ (T_e\mathrm{pr}X) \\ &= T_e(\tau_g \circ \mathrm{pr})X \\ &= T_e(\mathrm{pr} \circ \ell_g)X \\ &= T_g\mathrm{pr} \circ T_e\ell_g X \\ &= T_g\mathrm{pr} \circ X^L(g) \end{aligned} \tag{4.50}$$

for all $g \in M$, where we used $\tau_g \circ \mathrm{pr} = \mathrm{pr} \circ \ell_g$ and $\tau_g(\mathrm{pr}(e)) = \mathrm{pr}(g)$. Thus the horizontal lift $\overline{X^*} \in \Gamma^\infty\big(T\mathrm{pr}^{-1}(M^*)\big)$ of X^* restricted to $M \subseteq \mathrm{pr}^{-1}(M^*) \subseteq G$ fulfills $\overline{X^*}\big|_M = X^L\big|_M$ due to (4.50) since X^L is horizontal. Next let $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ be a basis of $\mathfrak{m}$ and expand $\overline{X^*} = x^i A_i^L$ with uniquely determined smooth functions $x^i: \mathrm{pr}^{-1}(M^*) \to \mathbb{R}$. Analogously, one defines for $Y \in \mathfrak{m}$ the vector field Y^* on M^* whose horizontal lift $\overline{Y^*} \in \Gamma^\infty\big(T\mathrm{pr}^{-1}(M^*)\big)$ is expanded as $\overline{Y^*} = y^j A_j^L$. Clearly, the unique smooth functions $y^j: \mathrm{pr}^{-1}(M^*) \to \mathbb{R}$ restricted to M, i.e. $y^j\big|_M: M \to \mathbb{R}$, are constant for all $j \in \{1, \ldots, N\}$. We now compute $\nabla^\alpha_{X^*} Y^*\big|_{\mathrm{pr}(e)}$ which makes sense since Y^* is defined on the open neighborhood M^* of $\mathrm{pr}(e) \in G/H$. Moreover, by the assumption $T_e M = \mathfrak{m}$, there exists a smooth curve $c: (-\epsilon, \epsilon) \to M$ for some $\epsilon > 0$ with $c(0) = e$ and $\dot{c}(0) = X \in \mathfrak{m}$. Since the functions $y^j\big|_M: M \to \mathbb{R}$ are constant for all $j \in \{1, \ldots, N\}$, Theorem 4.15 yields for $X, Y \in \mathfrak{m}$

$$
\begin{aligned}
\nabla^\alpha_{X^*} Y^*\big|_{\mathrm{pr}(e)} &= T_e\mathrm{pr}\Big(\big(\mathscr{L}_{\overline{X^*}} y^j\big) A_j^L(e) + x^i(e) y^j(e) \alpha(A_i, A_j)^L(e)\Big) \\
&= T_e\mathrm{pr}\Big(\big(\tfrac{\mathrm{d}}{\mathrm{d}t} y^j(c(t))\big|_{t=0}\big) A_i + \alpha(X, Y)\Big) \\
&= T_e\mathrm{pr}\big(\alpha(X, Y)\big) \\
&= t^\alpha(Y^*)(X^*)\big|_{\mathrm{pr}(e)},
\end{aligned}
\tag{4.51}
$$

where we used (4.49) in the last equality. Moreover, ∇^α is the unique invariant covariant derivative on G/H satisfying (4.51). Indeed, let ∇^β be the invariant covariant derivative associated with the $\mathrm{Ad}(H)$-invariant bilinear map $\beta: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ fulfilling $\nabla^\beta_{X^*} Y^*\big|_{\mathrm{pr}(e)} = t^\alpha(Y^*)(X^*)\big|_{\mathrm{pr}(e)}$ for all $X, Y \in \mathfrak{m}$. Then

$$
\nabla^\beta_{X^*} Y^*\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(\beta(X, Y)\big) = t^\alpha(Y^*)(X^*)\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(\alpha(X, Y)\big) = \nabla^\alpha_{X^*} Y^*\big|_{\mathrm{pr}(e)}
$$

yields $\beta = \alpha$ implying $\nabla^\alpha = \nabla^\beta$. In addition, t^α is uniquely determined by (4.49). Hence ∇^α and t^α are both uniquely determined by (4.51). Thus (4.51) implies $\nabla^\alpha_X Y = t^\alpha(Y)(X)$ for all $X, Y \in \Gamma^\infty\big(T(G/H)\big)$ as desired. □

4.2 Torsion and Curvature

Next we consider the torsion of an invariant covariant derivative. This is the next lemma whose result coincides with [1, Eq. (9.2)].

Lemma 4.19 *Let ∇^α be the invariant covariant derivative on G/H associated to the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. The torsion of ∇^α is the G-invariant tensor field $\mathrm{Tor}^\alpha \in \Gamma^\infty\big(\Lambda^2(T^*(G/H)) \otimes T(G/H)\big)$ defined by*

$$
\mathrm{Tor}^\alpha\big(X_{G/H}, Y_{G/H}\big)\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(\alpha(X, Y) - \alpha(Y, X) - [X, Y]_{\mathfrak{m}}\big) \tag{4.52}
$$

for all $X, Y \in \mathfrak{m}$

Proof We first note that $(\tau_g)_*[X_{G/H}, Y_{G/H}] = [(\tau_g)_* X_{G/H}, (\tau_g)_* Y_{G/H}]$ holds all for $g \in G$, see e.g. [13, Cor. 8.31]. This identity and the invariance of ∇^α yields that Tor^α is G-invariant. Thus Tor^α corresponds to an $\mathrm{Ad}(H)$-invariant bilinear map

$\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ by Lemma 4.7. In order to determine this bilinear map, writing $\mathrm{pr}(e)=o$, we compute

$$\begin{aligned}\mathrm{Tor}^\alpha\big(X_{G/H},Y_{G/H}\big)\big|_o &= \nabla^\alpha_{X_{G/H}}Y_{G/H}\big|_o-\nabla^\alpha_{Y_{G/H}}X_{G/H}\big|_o-[X_{G/H},Y_{G/H}]\big|_o\\ &= \nabla^\alpha_{X_{G/H}}Y_{G/H}\big|_o-\nabla^\alpha_{Y_{G/H}}X_{G/H}\big|_o+[X,Y]_{G/H}(o)\\ &= T_e\mathrm{pr}\big(-[X,Y]_\mathfrak{m}+\alpha(X,Y)-\big(\alpha(Y,X)-[Y,X]_\mathfrak{m}\big)+[X,Y]_\mathfrak{m}\big)\\ &= T_e\mathrm{pr}\big(\alpha(X,Y)-\alpha(Y,X)-[X,Y]_\mathfrak{m}\big)\end{aligned}$$

for all $X,Y\in\mathfrak{m}$, where we exploited that $\mathfrak{g}\ni X\mapsto X_{G/H}\in\Gamma^\infty\big(T(G/H)\big)$ is an anti-morphism of Lie algebras, see e.g. [9, Sec. 6.2]. □

Moreover, one can compute the curvature of ∇^α given by

$$R^\alpha(X,Y)Z=\nabla^\alpha_X\nabla^\alpha_Y Z-\nabla^\alpha_Y\nabla^\alpha_X Z-\nabla^\alpha_{[X,Y]}Z,\quad X,Y,Z\in\Gamma^\infty\big(T(G/H)\big)\tag{4.53}$$

by using the expression for ∇^α from Theorem 4.15. This is the next proposition which yields an alternative derivation for the curvature obtained in [1, Eq. (9.6)].

Proposition 4.20 *Let ∇^α be the invariant covariant derivative on G/H associated to the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$. The curvature of ∇^α is the G-invariant tensor field $R^\alpha\in\Gamma^\infty\big((\Lambda^2(T^*(G/H)))\otimes T^*(G/H)\otimes T(G/H)\big)$ given by*

$$\begin{aligned}&R^\alpha(X_{G/H},Y_{G/H})Z_{G/H}\big|_{\mathrm{pr}(e)}\\ &\quad=T_e\mathrm{pr}\Big(\alpha\big(X,\alpha(Y,Z)\big)-[[X,Y]_\mathfrak{h},Z]-\alpha\big([X,Y]_\mathfrak{m},Z\big)-\alpha\big(Y,\alpha(X,Z)\big)\Big)\end{aligned}\tag{4.54}$$

for all $X,Y,Z\in\mathfrak{m}$.

Proof Obviously, the curvature R^α fulfills

$$(\tau_g)_*\big(R^\alpha(X,Y)Z\big)=R^\alpha\big((\tau_g)_*X,(\tau_g)_*Y\big)(\tau_g)_*Z$$

for all vector fields $X,Y,Z\in\Gamma^\infty\big(T(G/H)\big)$ by the invariance of ∇^α. Hence R^α is uniquely determined by an $\mathrm{Ad}(H)$-invariant 3-linear map $\mathfrak{m}^3\to\mathfrak{m}$ according to Lemma 4.7. We now determine this 3-linear map. To this end, let $X,Y,Z\in\mathfrak{m}$ and let $X_{G/H},Y_{G/H},Z_{G/H}\in\Gamma^\infty\big(T(G/H)\big)$ be the associated fundamental vector fields. In order to compute the curvature $R^\alpha(X_{G/H},Y_{G/H})Z_{G/H}$ defined by (4.53) evaluated at the point $\mathrm{pr}(e)=e\cdot H\in G/H$, we need some computations as preparation. Let $\{A_1,\dots,A_N\}\subseteq\mathfrak{m}$ be a basis of $\mathfrak{m}$ and denote by $\{A^1,\dots,A^N\}\subseteq\mathfrak{m}^*$ its dual basis. By Lemma 4.6, Claim 1, we have $\overline{X_{G/H}}=x^iA_i^L$ and $\overline{Y_{G/H}}=y^jA_j^L$ as well as $\overline{Z_{G/H}}=z^kA_k^L$, where the functions $x^i,y^j,z^k\colon G\to\mathbb{R}$ are defined by

$$x^i(g)=A^i\big(\mathrm{Ad}_{g^{-1}}(X)\big),\quad y^j(g)=A^j\big(\mathrm{Ad}_{g^{-1}}(Y)\big),\quad z^k(g)=A^k\big(\mathrm{Ad}_{g^{-1}}(Z)\big)$$

for all $g \in G$ and $i, j, k \in \{1, \ldots, N\}$. Using this notation, we obtain by Theorem 4.15

$$\overline{\nabla^{\alpha}_{X_{G/H}} Z_{G/H}} = \big(\mathscr{L}_{\overline{Y_{G/H}}} z^k\big) A_k^L + y^j z^k \big(\alpha(A_j, A_k)\big)^L = a^\ell A_\ell^L,$$

where the functions $a^\ell \colon G \to \mathbb{R}$ for $\ell \in \{1, \ldots, N\}$ are given by

$$a^\ell = A^\ell\big(\big(\mathscr{L}_{\overline{Y_{G/H}}} z^k\big) A_k + y^j z^k \big(\alpha(A_j, A_k)\big)\big) = \mathscr{L}_{\overline{Y_{G/H}}} z^\ell + y^j z^k A^\ell\big(\alpha(A_j, A_k)\big). \tag{4.55}$$

In particular, evaluating $a^\ell \colon G \to \mathbb{R}$ at $g = e$ yields by Lemma 4.6, Claim 2

$$\begin{aligned} a^\ell(e) A_\ell &= \big(\mathscr{L}_{Y_{G/H}} z^\ell\big)(e) A_\ell^L(e) + y^j(e) z^k(e) A^\ell\big(\alpha(A_j, A_k)\big) A_\ell \\ &= -[Y, Z]_{\mathfrak{m}} + \alpha(Y, Z). \end{aligned} \tag{4.56}$$

Moreover, we obtain by Theorem 4.15 and (4.56)

$$\begin{aligned} \overline{\nabla^{\alpha}_{X_{G/H}} \nabla^{\alpha}_{Y_{G/H}} Z_{G/H}}\big|_e &= \big(\mathscr{L}_{\overline{X_{G/H}}} a^\ell\big)(e) A_\ell + x^i(e) a^\ell(e) \alpha(A_i, A_\ell) \\ &= \big(\mathscr{L}_{\overline{X_{G/H}}} a^\ell\big)(e) A_\ell + \alpha\big(X, \alpha(Y, Z)\big) - \alpha\big(X, [Y, Z]_{\mathfrak{m}}\big). \end{aligned} \tag{4.57}$$

In order to obtain a more explicit expression for (4.57), we consider the first summand on the right-hand side. Recalling that a^ℓ is given by (4.55) one obtains by the Leibniz rule

$$\begin{aligned} &\big(\mathscr{L}_{\overline{X_{G/H}}} a^\ell\big) A_\ell \\ &= \mathscr{L}_{\overline{X_{G/H}}}\big(\mathscr{L}_{\overline{Y_{G/H}}} z^\ell\big) A_\ell + \big((\mathscr{L}_{\overline{X_{G/H}}} y^j) z^k + y^j (\mathscr{L}_{\overline{X_{G/H}}} z^k)\big) A^\ell\big(\alpha(A_j, A_k)\big) A_\ell. \end{aligned} \tag{4.58}$$

We now take a closer look at (4.58) evaluated at $g = e$. We obtain for second summand of its right-hand side by Lemma 4.6, Claim 2 and $A^\ell\big(\alpha(A_j, A_k)\big) A_\ell = \alpha(A_j, A_k)$

$$\begin{aligned} &\big((\mathscr{L}_{\overline{X_{G/H}}} y^j) z^k + y^j (\mathscr{L}_{\overline{X_{G/H}}} z^k)\big)(e) \alpha(A_j, A_k) \\ &= \alpha\big(((\mathscr{L}_{\overline{X_{G/H}}} y^j)(e) A_j, Z\big) + \alpha\big(Y, (\mathscr{L}_{\overline{X_{G/H}}} z^k)(e) A_k\big) \\ &= -\alpha\big([X, Y]_{\mathfrak{m}}, Z\big) - \alpha\big(Y, [X, Z]_{\mathfrak{m}}\big). \end{aligned} \tag{4.59}$$

Next we consider the first summand of the right-hand side of (4.58). As preparation, we note that for fixed $g \in G$, the curve $\gamma_Y \colon \mathbb{R} \to G$ defined by

$$\gamma_Y(t) = g \exp\big(t A^j\big(\mathrm{Ad}_{g^{-1}}(X)\big) A_j\big), \quad t \in \mathbb{R}$$

fulfills $\gamma_Y(0) = g$ and $\dot{\gamma}_Y(0) = T_e\ell_g A^j(\mathrm{Ad}_{g^{-1}}(Y))A_j = \overline{Y_{G/H}}(g)$, where the last equality follows by Lemma 4.6, Claim 1. Thus we obtain

$$\begin{aligned}\mathscr{L}_{\overline{Y_{G/H}}} z^\ell(g) &= \tfrac{\mathrm{d}}{\mathrm{d}t} z^\ell\big(\gamma_Y(t)\big)\big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t} A^\ell\Big(\mathrm{Ad}_{\left(g\exp\left(tA^j(\mathrm{Ad}_{g^{-1}}(Y)A_j)\right)\right)^{-1}}(Z)\Big)\Big|_{t=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t} A^\ell\Big(\mathrm{Ad}_{\exp\left(-A^j(\mathrm{Ad}_{g^{-1}}(Y)A_j)\right)}\big(\mathrm{Ad}_{g^{-1}}(Z)\big)\Big)\Big|_{t=0} \\ &= -A^\ell\big(\big[A^j\big(\mathrm{Ad}_{g^{-1}}(Y)\big)A_j, \mathrm{Ad}_{g^{-1}}(Z)\big]\big).\end{aligned} \tag{4.60}$$

Since the curve $\gamma_X\colon \mathbb{R} \ni t \mapsto \exp(tX) \in G$ fulfills $\gamma_X(0) = e$ and $\dot{\gamma}(0) = X = \overline{X_{G/H}}(e)$, Equation (4.60) yields

$$\begin{aligned}\big(\mathscr{L}_{\overline{X_{G/H}}}\big(\mathscr{L}_{\overline{Y_{G/H}}} z^\ell\big)\big)(e)A_\ell &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathscr{L}_{\overline{Y_{G/H}}} z^\ell\big(\exp(tX)\big)\big)\big|_{t=0} \\ &= -\tfrac{\mathrm{d}}{\mathrm{d}t} A^\ell\big(\big[A^j(\mathrm{Ad}_{\exp(-tX)}(Y))A_j, \mathrm{Ad}_{\exp(-tX)}(Z)\big]\big)A_\ell\big|_{t=0} \\ &= [[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} + [Y,[X,Z]]_{\mathfrak{m}}.\end{aligned} \tag{4.61}$$

Plugging (4.59) and (4.61) into (4.58) yields by (4.57)

$$\begin{aligned}&\overline{\nabla^\alpha_{X_{G/H}} \nabla^\alpha_{Y_{G/H}} Z_{G/H}}\big|_e \\ &\quad= [[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} + [Y,[X,Z]]_{\mathfrak{m}} - \alpha\big([X,Y]_{\mathfrak{m}}, Z\big) - \alpha\big(Y,[X,Z]_{\mathfrak{m}}\big) \\ &\qquad+ \alpha\big(X, \alpha(Y,Z)\big) - \alpha\big(X,[Y,Z]_{\mathfrak{m}}\big).\end{aligned} \tag{4.62}$$

By exchanging X with Y in (4.62), one obtains

$$\begin{aligned}&\overline{\nabla^\alpha_{Y_{G/H}} \nabla^\alpha_{X_{G/H}} Z_{G/H}}\big|_e \\ &\quad= [[Y,X]_{\mathfrak{m}}, Z]_{\mathfrak{m}} + [X,[Y,Z]]_{\mathfrak{m}} - \alpha\big([Y,X]_{\mathfrak{m}}, Z\big) - \alpha\big(X,[Y,Z]_{\mathfrak{m}}\big) \\ &\qquad+ \alpha\big(Y, \alpha(X,Z)\big) - \alpha\big(Y,[X,Z]_{\mathfrak{m}}\big).\end{aligned} \tag{4.63}$$

Moreover, we obtain by Theorem 4.15

$$\begin{aligned}\overline{\nabla^\alpha_{[X_{G/H},Y_{G/H}]} Z_{G/H}}\big|_e &= \overline{\nabla^\alpha_{-[X,Y]_{G/H}} Z_{G/H}} \\ &= -\big(\big(\mathscr{L}_{\overline{[X,Y]_{G/H}}} z^k\big)(e)A_k + A^i([X,Y])z^k\alpha(A_i,A_k)\big) \\ &= [[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} - \alpha([X,Y]_{\mathfrak{m}}, Z),\end{aligned} \tag{4.64}$$

where we exploited that $\mathfrak{g} \ni X \mapsto X_{G/H} \in \Gamma^\infty(T(G/H))$ is an anti-morphism of Lie algebras, see e.g. [9, Sec. 6.2]. Combining (4.62) with (4.63) and (4.64) yields the following expression for the curvature

$$
\begin{aligned}
&\overline{R^\alpha(X_{G/H}, Y_{G/H})Z_{G/H}}(e) \\
&\quad= \Big([[X,Y]_\mathfrak{m}, Z]_\mathfrak{m} + [Y,[X,Z]]_\mathfrak{m} - \alpha\big([X,Y]_\mathfrak{m}, Z\big) - \alpha\big(Y, [X,Z]_\mathfrak{m}\big) \\
&\qquad + \alpha\big(X, \alpha(Y,Z)\big) - \alpha\big(X, [Y,Z]_\mathfrak{m}\big)\Big) \\
&\qquad - \Big([[Y,X]_\mathfrak{m}, Z]_\mathfrak{m} + [X,[Y,Z]]_\mathfrak{m} - \alpha\big([Y,X]_\mathfrak{m}, Z\big) - \alpha\big(X, [Y,Z]_\mathfrak{m}\big) \\
&\qquad + \alpha\big(Y, \alpha(X,Z)\big) - \alpha\big(Y, [X,Z]_\mathfrak{m}\big)\Big) \\
&\qquad - \Big([[X,Y]_\mathfrak{m}, Z]_\mathfrak{m} - \alpha([X,Y]_\mathfrak{m}, Z)\Big) \\
&\quad= [Y,[X,Z]]_\mathfrak{m} - \alpha\big([X,Y]_\mathfrak{m}, Z\big) + \alpha\big(X, \alpha(Y,Z)\big) \\
&\qquad - [[Y,X]_\mathfrak{m}, Z]_\mathfrak{m} - [X,[Y,Z]]_\mathfrak{m} - \alpha\big(Y, \alpha(X,Z)\big) \\
&\quad= -[[X,Y]_\mathfrak{h}, Z] - \alpha\big([X,Y]_\mathfrak{m}, Z\big) + \alpha\big(X, \alpha(Y,Z)\big) - \alpha\big(Y, \alpha(X,Z)\big),
\end{aligned}
$$

where the last holds due to

$$
\begin{aligned}
[Y,[X,Z]]_\mathfrak{m} - [[Y,X]_\mathfrak{m}, Z]_\mathfrak{m} - [X,[Y,Z]]_\mathfrak{m} &= \big([Y,[X,Z]] \\
&\quad - [X,[Y,Z]]\big)_\mathfrak{m} - [[Y,X]_\mathfrak{m}, Z]_\mathfrak{m} \\
&= -[[X,Y]_\mathfrak{h}, Z]
\end{aligned}
$$

by the Jacobi identity and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. This yields the desired result. □

4.3 Invariant Metric Covariant Derivatives

In this short subsection, we assume that G/H carries an invariant pseudo-Riemannian metric defined by an $\mathrm{Ad}(H)$-invariant scalar product $\langle\cdot,\cdot\rangle\colon \mathfrak{m}\times\mathfrak{m}\to\mathbb{R}$. We characterize all $\mathrm{Ad}(H)$-invariant bilinear maps $\alpha\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ such that ∇^α is an invariant metric covariant derivative with respect to the invariant pseudo-Riemannian metric corresponding to $\langle\cdot,\cdot\rangle$. To this end, we first recall that a covariant derivative ∇ on a manifold M is called compatible with the pseudo-Riemannian metric $g \in \Gamma^\infty\big(\mathrm{S}^2(T^*M)\big)$, or metric for short, if

$$
\mathscr{L}_Z\big(g(X,Y)\big) = g\big(\nabla_Z X, Y\big) + g\big(X, \nabla_Z Y\big), \quad X, Y, Z \in \Gamma^\infty(TM) \qquad (4.65)
$$

holds, see e.g. [9, Sec. 22.5].

Notation 4.21 *In this subsection, we denote by g and $\overline{g}$ a pseudo-Riemannian metric on G and a fiber metric on* $\mathrm{Hor}(G)$*, respectively, while in the previous sections as well as in the sequel, we usually denote by g an element in a Lie group G.*

Proposition 4.22 *Let $\alpha\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ be an* $\mathrm{Ad}(H)$*-invariant bilinear map defining the invariant covariant derivative ∇^α on G/H. Then ∇^α is metric with respect to the* *invariant pseudo-Riemannian metric on G/H defined by the* $\mathrm{Ad}(H)$*-invariant scalar product* $\langle\cdot,\cdot\rangle\colon \mathfrak{m}\times\mathfrak{m}\to\mathbb{R}$ *iff for each* $X\in\mathfrak{m}$ *the linear map*

$$\alpha(X,\cdot)\colon \mathfrak{m}\to\mathfrak{m},\quad Y\mapsto\alpha(X,Y) \tag{4.66}$$

is skew-adjoint with respect to $\langle\cdot,\cdot\rangle$*, i.e.*

$$\big\langle\alpha(X,Y),Z\big\rangle = -\big\langle Y,\alpha(X,Z)\big\rangle \tag{4.67}$$

holds for all $X, Y, Z\in\mathfrak{m}$.

Proof We denote the invariant pseudo-Riemannian metric on G/H corresponding to $\langle\cdot,\cdot\rangle$ by $g\in\Gamma^\infty\big(\mathrm{S}^2(T^*(G/H))\big)$. Let $X, Y, Z\in\Gamma^\infty(T(G/H))$ be vector fields with horizontal lifts $\overline{X},\overline{Y},\overline{Z}\in\Gamma^\infty(\mathrm{Hor}(G))$. We expand these vector fields in a left-invariant frame, i.e.

$$\overline{X}=x^iA_i^L,\quad \overline{Y}=y^jA_j^L\quad\text{and}\quad \overline{Z}=z^kA_k^L,$$

where $\{A_1,\ldots,A_N\}\subseteq\mathfrak{m}$ is a basis of $\mathfrak{m}$ and $x^i, y^j, z^k\colon G\to\mathbb{R}$ are uniquely determined smooth functions for $i, j, k\in\{1,\ldots,N\}$. Moreover, we endow $\mathrm{Hor}(G)\to G$ with the fiber metric $\overline{g}\in\Gamma^\infty\big(\mathrm{S}^2\mathrm{Hor}(G)^*\big)$ defined by left translating the scalar product $\langle\cdot,\cdot\rangle$. Then

$$g(X,Y)\circ\mathrm{pr}=\mathrm{pr}^*\big(g(X,Y)\big)=\overline{g}(\overline{X},\overline{Y})\colon G\to\mathbb{R} \tag{4.68}$$

holds by the definition of $g\in\Gamma^\infty\big(\mathrm{S}^2T^*(G/H)\big)$. Since Z and $\overline{Z}$ are pr-related, we obtain by [13, Prop. 8.16] and (4.68)

$$\begin{aligned}\mathrm{pr}^*\big(\mathscr{L}_Z\big(g(X,Y)\big)\big)&=\mathscr{L}_{\overline{Z}}\big(\mathrm{pr}^*\big(g(X,Y)\big)\big)\\&=\mathscr{L}_{\overline{Z}}\big(\overline{g}(\overline{X},\overline{Y})\big)\\&=\mathscr{L}_{\overline{Z}}\big(\overline{g}\big(x^iA_i^L,y^jA_j^L\big)\big)\\&=\mathscr{L}_{\overline{Z}}\big(x^iy^j\langle A_i,A_j\rangle\big)\\&=\big(\mathscr{L}_{\overline{Z}}x^i\big)y^j\langle A_i,A_j\rangle+x^i\big(\mathscr{L}_{\overline{Z}}y^j\big)\langle A_i,A_j\rangle,\end{aligned} \tag{4.69}$$

where we exploited that $\overline{g}\big(A_i^L,A_j^L\big)=\langle A_i,A_j\rangle$ holds by the definition of $\overline{g}\in\Gamma^\infty\big(\mathrm{S}^2\mathrm{Hor}(G)^*\big)$. Moreover, with $\nabla^{\mathrm{Hor},\alpha}$ from Lemma 4.10, we compute

$$\begin{aligned}&\overline{g}\big(\nabla^{\mathrm{Hor},\alpha}_{\overline{Z}}\overline{X},\overline{Y}\big)+\overline{g}\big(\overline{X},\nabla^{\mathrm{Hor},\alpha}_{\overline{Z}}\overline{Y}\big)\\&\quad=\overline{g}\big(\big(\mathscr{L}_{\overline{Z}}x^i\big)A_i^L+z^kx^i(\alpha(A_k,A_i))^L,y^jA_j^L\big)\\&\qquad+\overline{g}\big(x^iA_i^L,\big(\mathscr{L}_{\overline{Z}}y^j\big)A_j^L+z^ky^j(\alpha(A_k,A_j))^L\big)\\&\quad=\big(\mathscr{L}_{\overline{Z}}x^i\big)y^j\langle A_i,A_j\rangle+x^i\big(\mathscr{L}_{\overline{Z}}y^j\big)\langle A_i,A_j\rangle\\&\qquad+z^kx^iy^j\big(\big\langle\alpha(A_k,A_i),A_j\big\rangle+\big\langle A_i,\alpha(A_k,A_j)\big\rangle\big).\end{aligned} \tag{4.70}$$

By comparing (4.69) with (4.70), we obtain by $(\nabla^{\alpha}_X Y) \circ \mathrm{pr} = T\mathrm{pr} \circ \big(\nabla^{\mathrm{Hor},\alpha}_{\overline{X}} \overline{Y}\big)$

$$\begin{aligned}
\mathrm{pr}^*\big(\mathscr{L}_Z\big(g(X,Y)\big)\big) &= \big(\mathscr{L}_{\overline{Z}} x^i\big) y^j \langle A_i, A_j\rangle + x^i \big(\mathscr{L}_{\overline{Z}} y^j\big)\langle A_i, A_j\rangle \\
&= \overline{g}\big(\nabla^{\mathrm{Hor},\alpha}_{\overline{Z}} \overline{X}, \overline{Y}\big) + \overline{g}\big(\overline{X}, \nabla^{\mathrm{Hor},\alpha}_{\overline{Z}} \overline{Y}\big) \\
&= \overline{g}\big(\overline{\nabla^{\alpha}_Z X}, \overline{Y}\big) + \overline{g}\big(\overline{X}, \overline{\nabla^{\alpha}_Z Y}\big) \\
&= \mathrm{pr}^*\big(g\big(\nabla^{\alpha}_Z X, Y\big) + g\big(X, \nabla^{\alpha}_Z, Y\big)\big),
\end{aligned}$$

where the second equality holds iff

$$\big\langle \alpha(A_k, A_i), A_j \big\rangle + \big\langle A_i, \alpha(A_k, A_j)\big\rangle = 0 \tag{4.71}$$

is satisfied for all $i, j, k \in \{1, \ldots, N\}$. Since $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ is a basis of $\mathfrak{m}$, Equation (4.71) is equivalent to (4.67). This yields the desired result since the pull-back by the surjective map $\mathrm{pr}\colon G \to G/H$ yields clearly an injective map $\mathrm{pr}^*\colon \mathscr{C}^\infty(G/H) \to \mathscr{C}^\infty(G)$. □

We now recall an expression for the Levi-Civita covariant derivative on a reductive homogeneous space G/H equipped with an invariant pseudo-Riemannian metric corresponding to the $\mathrm{Ad}(H)$-invariant scalar product $\langle\cdot,\cdot\rangle\colon \mathfrak{m}\times\mathfrak{m} \to \mathbb{R}$. This is the next proposition which is taken from [7, Sec. 23.6], where it is stated for the Riemannian case. However, since its proof only relies on the non-degeneracy of the invariant pseudo-Riemannian metric and its associated $\mathrm{Ad}(H)$-invariant scalar product, it can be generalized to the pseudo-Riemannian setting.

Proposition 4.23 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Moreover, let $\langle\cdot,\cdot\rangle\colon \mathfrak{m}\times\mathfrak{m} \to \mathbb{R}$ be an $\mathrm{Ad}(H)$-invariant scalar product corresponding to an invariant pseudo-Riemannian metric on G/H. Then the Levi-Civita covariant derivative defined by this metric fulfills for all $X, Y \in \mathfrak{m}$*

$$\nabla^{\mathrm{LC}}_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(-\tfrac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y)\big), \tag{4.72}$$

where $U\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ is uniquely determined by

$$2\big\langle U(X,Y), Z\big\rangle = \big\langle [Z,X]_{\mathfrak{m}}, Y\big\rangle + \big\langle X, [Z,Y]_{\mathfrak{m}}\big\rangle \tag{4.73}$$

for all $Z \in \mathfrak{m}$.

Proposition 4.23 can be simplified for naturally reductive homogeneous spaces. This is the next corollary which can be seen as a reformulation of [7, Prop. 23.25] adapted to the pseudo-Riemannian setting.

Corollary 4.24 *Let G/H be a naturally reductive homogeneous space. Then*

$$\nabla^{\mathrm{LC}}_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(-\tfrac{1}{2}[X,Y]_{\mathfrak{m}}\big) \tag{4.74}$$

holds for all $X, Y \in \mathfrak{m}$.

Proof This is proven in [7, Prop. 23.25]. Nevertheless, we include the proof here, as well. Since G/H is naturally reductive, we have $\langle [X, Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y, Z]_{\mathfrak{m}}\rangle$ for all $X, Y, Z \in \mathfrak{m}$ implying $\langle [Y, X]_{\mathfrak{m}}, Z\rangle + \langle X, [Y, Z]_{\mathfrak{m}}\rangle = 0$. Using the definition of $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ in (4.73) of Proposition 4.23 yields for $X, Y, Z \in \mathfrak{m}$

$$\langle U(X, Z), Y\rangle = \langle [Y, X]_{\mathfrak{m}}, Z\rangle + \langle X, [Y, Z]_{\mathfrak{m}}\rangle = 0$$

implying $U(X, Y) = 0$ for all $X, Y \in \mathfrak{m}$. Thus Proposition 4.23 yields the desired result. □

Next we relate the Levi-Civita covariant derivative on G/H, equipped with an invariant pseudo-Riemannian metric, to an invariant covariant derivative on G/H. This is the next remark which coincides with [1, Sec. 13].

Remark 4.25 Let G/H be a reductive homogeneous space equipped with an invariant pseudo-Riemannian metric corresponding the $\mathrm{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot\rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$. Then the action $\tau : G \times G/H \to G/H$ is isometric, see e.g. [6, Chap. 11, Prop. 22]. Therefore ∇^{LC} is an invariant covariant derivative on G/H by [16, Prop. 5.13]. Thus Lemma 4.8, Claim 2 implies by Proposition 4.23 that $\nabla^{\mathrm{LC}} = \nabla^{\alpha}$ holds, where $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined by

$$\alpha(X, Y) = \tfrac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y) \tag{4.75}$$

for all $X, Y \in \mathfrak{m}$ in accordance with [1, Thm. 13.1]. In particular, one has $\nabla^{\mathrm{LC}} = \nabla^{\alpha}$ for $\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$ by Corollary 4.24 if G/H is naturally reductive. This coincides with [1, Eq. (13.1)].

4.4 Parallel Vector Fields Along Curves

Having an expression for ∇^{α} on a reductive homogeneous space G/H in terms of horizontally lifted vector fields on G allows for determining the associated covariant derivative of vector fields along a given curve on G/H in terms of horizontal lifts, as well. In this subsection, an ODE for a specific curve in $\mathfrak{m}$ is determined which is fulfilled iff the corresponding vector field along the given curve is parallel. Let

$$\gamma : I \to G/H \tag{4.76}$$

be a curve and let

$$\widehat{Z} : I \to T(G/H) \tag{4.77}$$

be a vector field along γ, i.e

$$\widehat{Z}(t) \in T_{\gamma(t)}(G/H), \quad t \in I. \tag{4.78}$$

Moreover, let

$$g\colon I \to G \tag{4.79}$$

denote a horizontal lift of γ with respect to the principal connection $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$ from Proposition 3.12. It is well-known that g is unique up to the initial condition $g(t_0) = g_0 \in G_{\gamma(0)}$. Furthermore, the curve g is defined on the whole interval I since principal connections are complete, see e.g. [9, Thm. 19.6]. Let $\overline{Z}\colon I \to \mathrm{Hor}(G)$ be the horizontal lift of $\widehat{Z}$ along g, i.e.

$$\overline{Z}(t) = \big(T_{g(t)}\mathrm{pr}\big|_{\mathrm{Hor}(G)_{g(t)}}\big)^{-1}\widehat{Z}(t), \quad t \in I. \tag{4.80}$$

Next we define the curves in $\mathfrak{m}$ associated with g and $\overline{Z}$, namely

$$x\colon I \to \mathfrak{m}, \quad t \mapsto x(t) = \big(T_e\ell_{g(t)}\big)^{-1}\dot{g}(t) \tag{4.81}$$

and

$$z\colon I \to \mathfrak{m}, \quad t \mapsto z(t) = \big(T_e\ell_{g(t)}\big)^{-1}\overline{Z}(t). \tag{4.82}$$

We now consider the covariant derivative of $\widehat{Z}$ along γ. This is next proposition which can be seen as a generalization of [2, Lem. 1], where we use the notation which has been introduced above.

Proposition 4.26 *Let G/H be a reductive homogeneous space and let $\gamma\colon I \to G/H$ be smooth. Let $g\colon I \to G$ be a horizontal lift of γ. Moreover, let $\widehat{Z}\colon I \to T(G/H)$ be a vector field along γ with horizontal lift $\overline{Z}\colon I \to \mathrm{Hor}(G)$ along $g\colon I \to G$. Let $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ be a basis and write*

$$\dot{g}(t) = x^i(t)A_i^L(g(t)) \quad \textit{and} \quad \overline{Z}(t) = z^j(t)A_j^L(g(t)) \tag{4.83}$$

for some uniquely determined smooth functions $x^i, z^j\colon I \to \mathbb{R}$. Let $\alpha\colon \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map and let ∇^α be the corresponding invariant covariant derivative on G/H. Then the associated covariant derivative of $\widehat{Z}$ along γ lifted to a horizontal vector field along $g\colon I \to G$ is given by

$$\begin{aligned} \overline{\nabla^\alpha_{\dot{\gamma}(t)}\widehat{Z}}\Big|_t &= \big(\tfrac{\mathrm{d}}{\mathrm{d}t}z^j(t)\big)A_j^L(g(t)) + x^i(t)z^j(t)\big(\alpha(A_i, A_j)\big)^L(g(t)) \\ &= \big(\dot{z}(t)\big)^L(g(t)) + \big(\alpha(x(t), z(t)\big)^L(g(t)) \end{aligned} \tag{4.84}$$

for all $t \in I$, where $z\colon I \ni t \mapsto z^i(t)A_i = (T_e\ell_{g(t)})^{-1}\overline{Z}(t) \in \mathfrak{m}$ and $x\colon I \ni t \mapsto x^i(t)A_i = (T_e\ell_{g(t)})^{-1}\dot{g}(t) \in \mathfrak{m}$.

Proof The proof is essentially given by applying Theorem 4.15. To this end, we define the vector field $X\colon I \to T(G/H)$ along $\gamma\colon I \to G/H$ by

$$X(t) = \dot{\gamma}(t), \quad t \in I$$

and we denote by $\overline{X}\colon I \to TG$ the horizontal lift of X along $g\colon I \to G$. Moreover, for fixed $t_0 \in I$, we extend X and $\widehat{Z}$ to vector fields defined on an open neighborhood $O \subseteq G/H$ of $\gamma(t_0)$. These vector fields are denoted by

$$\widetilde{X} \in \Gamma^\infty\big(T(G/H)\big|_O\big) \quad \text{and} \quad \widetilde{Z} \in \Gamma^\infty\big(T(G/H)\big|_O\big),$$

respectively. In particular,

$$\widetilde{X}(\gamma(t)) = X(t) = \dot{\gamma}(t) \quad \text{and} \quad \widehat{Z}(t) = \widetilde{Z}(\gamma(t))$$

is fulfilled for all t in a suitable open neighborhood of t_0 in I. Moreover, their horizontal lifts $\overline{\widetilde{X}}, \overline{\widetilde{Y}} \in \Gamma^\infty\big(\mathrm{Hor}(G)\big|_{\mathrm{pr}^{-1}(O)}\big)$ fulfill

$$\overline{X}(t) = \overline{\widetilde{X}}(g(t)) = \dot{g}(t) \quad \text{and} \quad \overline{Z}(t) = \overline{\widetilde{Z}}(g(t)).$$

These horizontal lifts can be expanded in the global frame $A_1^L \dots, A_N^L$ of $\mathrm{Hor}(G)$. We write for $t \in I$ in a suitable open neighborhood of t_0

$$\overline{X}(t) = x^i(t) A_i^L(g(t)) = \big(x(t)\big)^L(g(t)) \quad \text{and} \quad \overline{Z}(t) = z^j(t) A_j^L(g(t)) = \big(z(t)\big)^L(g(t)).$$

Similarly, we expand

$$\overline{\widetilde{X}} = \widetilde{x}^i A_i^L\big|_{\mathrm{pr}^{-1}(O)} \quad \text{and} \quad \overline{\widetilde{Z}}(t) = \widetilde{z}^j A_j\big|_{\mathrm{pr}^{-1}(O)},$$

where $\widetilde{x}^i, \widetilde{z}^j\colon \mathrm{pr}^{-1}(O) \subseteq G \to \mathbb{R}$ are uniquely determined smooth functions for $i, j \in \{1, \dots, N\}$. By construction

$$x^i(t) = \widetilde{x}^i(g(t)) \quad \text{and} \quad z^j(t) = \widetilde{z}^j(g(t))$$

holds for all t in a suitable open neighborhood of t_0 in I. We now use [16, Thm. 4.24] as well as Theorem 4.15 to compute the horizontal lift of the covariant derivative of

$\widehat{Z}$ along γ. We obtain for $t \in I$ in a suitable neighborhood of t_0

$$\begin{aligned}
\overline{\nabla^{\alpha}_{\dot{\gamma}(t)} \widehat{Z}}\Big|_t &= \overline{\nabla^{\alpha}_{\widetilde{X}} \widetilde{Z}}\Big|_{g(t)} \\
&= \nabla^{\mathrm{Hor},\alpha}_{\widetilde{\widetilde{X}}} \widetilde{\widetilde{Z}}\Big|_{g(t)} \\
&= \big(\mathscr{L}_{\widetilde{\widetilde{X}}} \widetilde{z}^j\big)(g(t)) A_j^L(g(t)) + \widetilde{x}^i(g(t)) \widetilde{z}^j(g(t)) \big(\alpha(A_i, A_j)\big)^L(g(t)) \\
&= \big(\tfrac{\mathrm{d}}{\mathrm{d}t} \widetilde{z}^j(g(t))\big) A_j^L(g(t)) + \widetilde{x}^i(g(t)) \widetilde{z}^j(g(t)) \big(\alpha(A_i, A_j)\big)^L(g(t)) \\
&= \Big(\big(\tfrac{\mathrm{d}}{\mathrm{d}t} z^j(t)\big) A_i\Big)^L(g(t)) + \Big(\alpha\big(x^i(t) A_i, z^i(t) A_i\big)\Big)^L(g(t)) \\
&= \big(\dot{z}(t)\big)^L(g(t)) + \big(\alpha(x(t), z(t))\big)^L(g(t)).
\end{aligned}$$

Applying this argument for each $t_0 \in I$ yields the desired result. □

Proposition 4.26 allows for characterizing parallel vector fields along curves.

Corollary 4.27 *Let G/H be a reductive homogeneous space equipped with the invariant covariant derivative ∇^α. Let $\widehat{Z} \colon I \to T(G/H)$ be a vector field along $\gamma \colon I \to G/H$. Then $\widehat{Z}$ is parallel along γ iff the ODE*

$$\dot{z}(t) = -\alpha\big(x(t), z(t)\big) \tag{4.85}$$

is fulfilled, where $x, z \colon I \to \mathfrak{m}$ are defined as in Proposition 4.26.

Proof Let $g \in G$. The map $(T_e \ell_g)^{-1} \colon \mathrm{Hor}(G) \to \mathfrak{m}$ is a linear isomorphism which fulfills $(T_e \ell_g)^{-1} \xi^L(g) = \xi$ for all $\xi \in \mathfrak{m}$ by the definition of left-invariant vector fields. Hence Proposition 4.26 yields the desired result due to

$$0 = \overline{\nabla^{\alpha}_{\dot{\gamma}(t)} \widehat{Z}\big|_t} \quad \Longleftrightarrow \quad 0 = \big(T_e \ell_{g(t)}\big)^{-1} \overline{\nabla^{\alpha}_{\dot{\gamma}(t)} \widehat{Z}}\Big|_t = \dot{z}(t) + \alpha\big(x(t), z(t)\big)$$

for $t \in I$. □

4.5 Geodesics

In this short section, we consider geodesics on the reductive homogeneous space G/H with respect to an invariant covariant derivative ∇^α. Recall that a curve $\gamma \colon I \to G/H$ is a geodesic if the vector field $\dot{\gamma} \colon I \to T(G/H)$ along γ is parallel. Thus Corollary 4.27 can be used to obtain the following characterization of the geodesics on G/H with respect to ∇^α.

Lemma 4.28 *Let G/H be a reductive homogeneous space endowed with the invariant covariant derivative ∇^α corresponding to the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. Let $\gamma \colon I \to G/H$ be a curve in G/H and $g \colon I \to G$ be a horizontal lift of*

γ. Define $x\colon I \ni t \mapsto x(t) = (T_e \ell_{g(t)})^{-1} \dot{g}(t) \in \mathfrak{m}$. Then $\gamma\colon I \to G/H$ is a geodesic with respect to ∇^α iff the ODE

$$\dot{x}(t) = -\alpha\big(x(t), x(t)\big) \tag{4.86}$$

is satisfied for all $t \in I$.

Proof The curve $\gamma\colon I \to G/H$ is a geodesic with respect to ∇^α iff the vector field $\dot{\gamma}\colon I \to T(G/H)$ is parallel along $\gamma\colon I \to G/H$. Thus the desired result follows by Corollary 4.27. □

We now apply Lemma 4.28 to a reductive homogeneous space equipped with the Levi-Civita covariant derivative defined by some invariant pseudo-Riemannian metric. Inspired by the well-known characterization of geodesics on a Lie group equipped with a left-invariant metric given in [5, Ap. B], see also [17, Sec. 4] for a discussion in the complex setting, we obtain the next corollary which generalizes the description of geodesics on Lie groups equipped with left-invariant metrics.

Corollary 4.29 *Let G/H be a reductive homogeneous space and let $\langle \cdot, \cdot \rangle\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ be an $\mathrm{Ad}(H)$-invariant scalar product. Moreover, let ∇^{LC} denote the Levi-Civita covariant derivative defined by the invariant metric on G/H corresponding to $\langle \cdot, \cdot \rangle$. Let $\gamma\colon I \to G/H$ be a curve in G/H and $g\colon I \to G$ be a horizontal lift of γ. Define $x\colon I \ni t \mapsto x(t) = (T_e \ell_{g(t)})^{-1} \dot{g}(t) \in \mathfrak{m}$. Then $\gamma\colon I \to G/H$ is a geodesic with respect to ∇^{LC} iff the ODE*

$$\dot{x}(t) = (\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{x(t)})^*(x(t)) \tag{4.87}$$

is satisfied for all $t \in I$. Here $(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X)^*\colon \mathfrak{m} \to \mathfrak{m}$ denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$ of the linear map defined for fixed $X \in \mathfrak{m}$ by $\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X\colon \mathfrak{m} \to \mathfrak{m}$.

Proof We first recall Proposition 4.23. The Levi-Civita covariant derivative on G/H with respect to the invariant metric fulfills $\nabla^{\mathrm{LC}} = \nabla^\alpha$, where $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is given by $\alpha(X, Y) = -\frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y)$ for all $X, Y \in \mathfrak{m}$ with

$$2\langle U(X, Y), Z \rangle = \langle [Z, X]_{\mathfrak{m}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{m}} \rangle = -\big(\langle [X, Z]_{\mathfrak{m}}, Y \rangle + \langle X, [Y, Z]_{\mathfrak{m}} \rangle\big) \tag{4.88}$$

Obviously, (4.88) is equivalently to

$$U(X, Y) = -\tfrac{1}{2}\big(\langle (\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X)^*(Y), Z \rangle + \langle (\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_Y)^*(X), Z \rangle\big) \tag{4.89}$$

for all $X, Y, Z \in \mathfrak{m}$, where $(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X)^*$ and $(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_Y)^*$ denote the adjoints of the linear maps $(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_Y)$ and $(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_Y)$ with respect to $\langle \cdot, \cdot \rangle$, respectively. Since $\langle \cdot, \cdot \rangle$ is non-degenerated, we can rewrite (4.89) equivalently as

$$U(X, Y) = -\tfrac{1}{2}\big((\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X)^*(Y) + (\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_Y)^*(X)\big).$$

Thus we obtain

$$\alpha(X, X) = -\tfrac{1}{2}[X, X]_{\mathfrak{m}} + U(X, X) = -(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X)^*(X)$$

for all $X \in \mathfrak{m}$. Now Lemma 4.28 yields the desired result. □

As indicated above, by applying Corollary 4.29 to a Lie group equipped with a left-invariant pseudo-Riemannian metric considered as the reductive homogeneous space $G \cong G/\{e\}$, one obtains the following corollary concerning geodesics on G. Its statement is well-known and can be found in [5, Ap. 2]. We also refer to [17, Sec. 4] for a discussion of this characterization of geodesics in the complex setting, where it is named Euler-Arnold Formalism.

Corollary 4.30 *Let G be a Lie group equipped with a left-invariant metric defined by the scalar product $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Then $g \colon I \to G$ is a geodesic iff the curve $x \colon I \ni t \mapsto x(t) = (T_e \ell_{g(t)})^{-1} \dot{g}(t) \in \mathfrak{g}$ satisfies*

$$\dot{x}(t) = (\mathrm{ad}_{x(t)})^*(x(t)) \tag{4.90}$$

for all $t \in I$. Here $(\mathrm{ad}_X)^* \colon \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint of $\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$, where $X \in \mathfrak{g}$ is fixed.

Proof Clearly, the Lie group G equipped with the left-invariant metric defined by the scalar product $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ can be viewed as the reductive homogeneous space G/H for $H = \{e\}$ with reductive decomposition $\mathfrak{g} = \{0\} \oplus \mathfrak{g}$ equipped with the pseudo-Riemannian metric defined by the $\mathrm{Ad}(\{e\})$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Thus the assertion follows by Corollary 4.29 due to $\mathrm{pr}_{\mathfrak{m}} = \mathrm{id}_{\mathfrak{g}}$. □

4.6 Canonical Invariant Covariant Derivatives

We now relate two particular invariant covariant derivatives on G/H to the canonical affine connections of first and second kind from [1, Sec. 10]. To this end, we list the two properties concerning invariant covariant derivatives which correspond to the properties of invariant affine connections from [1, Sec. 10, (A1) and (A2)]. This is the next definition.

Definition 4.31 Let ∇^α be an invariant covariant derivative on G/H corresponding to the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. The following properties of ∇^α are of particular interest:

1. The curves $\gamma_X \colon \mathbb{R} \ni t \mapsto \mathrm{pr}(\exp(tX)) \in G/H$ are geodesics with respect to ∇^α for all $X \in \mathfrak{m}$.
2. The curves $\gamma_X \colon \mathbb{R} \ni t \mapsto \mathrm{pr}(\exp(tX)) \in G/H$ are geodesics with respect to ∇^α for all $X \in \mathfrak{m}$ and the parallel transport of $T_e \mathrm{pr} Z \in T_{\mathrm{pr}(e)}(G/H)$ along γ_X with respect to ∇^α is given by $\widehat{Z} \colon \mathbb{R} \ni t \mapsto \big(T_{\exp(tX)} \mathrm{pr} \circ T_e \ell_{\exp(tX)}\big) Z \in T(G/H)$ for all $Z \in \mathfrak{m}$.

The next lemma is very similar some parts of [1, Sec. 10].

Lemma 4.32 *Let G/H be a reductive homogeneous space equipped with an invariant covariant derivative ∇^α corresponding to the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$. Then the following assertions are fulfilled:*

1. *∇^α fulfills the property from Definition 4.31, Claim 1 iff $\alpha(X,X)=0$ holds for all $X\in\mathfrak{m}$.*
2. *∇^α fulfills the property from Definition 4.31, Claim 2 iff $\alpha(X,Y)=0$ is fulfilled for all $X,Y\in\mathfrak{m}$.*

Proof Let $X\in\mathfrak{m}$ be arbitrary. We define the curve $\gamma_X\colon\mathbb{R}\ni t\mapsto \mathrm{pr}(\exp(tX))\in G/H$. Obviously, the curve $\mathbb{R}\ni t\mapsto\exp(tX)\in G$ is a horizontal lift of γ. Define $x\colon\mathbb{R}\to\mathfrak{m}$ by $x(t)=(T_e\ell_{\exp(tX)})^{-1}\circ(T_e\mathrm{pr}\big|_{\mathfrak{m}})^{-1}\dot\gamma(t)$. Clearly, $x(t)=X$ holds for all $t\in\mathbb{R}$. By Lemma 4.28, the curve $\gamma\colon I\to G/H$ is a geodesic with respect to ∇^α iff $\alpha(X,X)=0$ holds, i.e. Claim 1 is shown.

It remains to prove Claim 2. To this end, let $Z\in\mathfrak{m}$ be arbitrary. We now define the vector field $\widehat{Z}\colon\mathbb{R}\ni t\mapsto\big(T_{\exp(tX)}\mathrm{pr}\circ T_e\ell_{\exp(tX)}\big)Z\in T(G/H)$ along the curve $\gamma_X\colon\mathbb{R}\ni t\mapsto\mathrm{pr}(\exp(tX))\in G/H$. Next we consider the curve $z\colon\mathbb{R}\to\mathfrak{m}$ given by $z(t)=\big(T_e\ell_{\exp(tX)}\big)^{-1}\circ\big(T_e\mathrm{pr}\big|_{\mathfrak{m}}\big)^{-1}\widehat{Z}(t)=Z$. According to Corollary 4.27, the vector field $\widehat{Z}\colon I\to T(G/H)$ is parallel along γ iff $\alpha(x(t),z(t))=\alpha(X,Z)=0$ holds for all $t\in\mathbb{R}$. This yields the desired result. □

The next proposition can be viewed as a reformulation of [1, Thm. 10.1] and [1, Thm. 10.2].

Proposition 4.33 *Let G/H be a reductive homogeneous space.*

1. *Define the $\mathrm{Ad}(H)$-invariant bilinear map*

$$\alpha\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m},\quad (X,Y)\mapsto\alpha(X,Y)=\tfrac{1}{2}[X,Y]_{\mathfrak{m}}. \tag{4.91}$$

The corresponding invariant covariant derivative ∇^α is the unique invariant covariant derivative on G/H which is torsion free and satisfies Definition 4.31, Claim 1.

2. *Define the $\mathrm{Ad}(H)$-invariant bilinear map*

$$\alpha\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m},\quad (X,Y)\mapsto\alpha(X,Y)=0. \tag{4.92}$$

The corresponding invariant covariant derivative ∇^α is the unique invariant covariant derivative which satisfies Definition 4.31, Claim 1 and Claim 2.

Proof Claim 2 is an immediate consequence of Lemma 4.32, Claim 2.

It remains to proof Claim 2. Obviously, ∇^α is torsion free for $\alpha(X,Y)=\frac{1}{2}[X,Y]_{\mathfrak{m}}$ by Lemma 4.19. Moreover, ∇^α fulfills Definition 4.31, Claim 1 by Lemma 4.32, Claim 1 because of $\alpha(X,X)=0$ for all $X\in\mathfrak{m}$. It remains to prove the uniqueness of α. To this end, let $\beta\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map and assume that the $\mathrm{Ad}(H)$-invariant bilinear map

$$\gamma=\alpha+\beta\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m},\quad (X,Y)\mapsto\tfrac{1}{2}[X,Y]_{\mathfrak{m}}+\beta(X,Y)$$

fulfills $\gamma(X, X) = 0$ for all $X \in \mathfrak{m}$ such that γ defines the torsion-free invariant covariant derivative ∇^{γ} on G/H, i.e. $\gamma(X, Y) - \gamma(Y, X) = [X, Y]_{\mathfrak{m}}$ holds for all $X, Y \in \mathfrak{m}$ by Lemma 4.19. This yields

$$\begin{aligned}\gamma(X, Y) - \gamma(Y, X) &= \tfrac{1}{2}[X, Y]_{\mathfrak{m}} + \beta(X, Y) - \big(\tfrac{1}{2}[Y, X]_{\mathfrak{m}} + \beta(Y, X)\big) \\ &= [X, Y]_{\mathfrak{m}} + \beta(X, Y) - \beta(Y, X) \\ &= [X, Y]_{\mathfrak{m}}.\end{aligned} \tag{4.93}$$

By (4.93), one obtains $\beta(X, Y) - \beta(Y, X) = 0$ for all $X, Y \in \mathfrak{m}$, i.e. $\beta(X, Y) = \beta(Y, X)$ is symmetric. Moreover, we have $\beta(X, X) = 0$ for all $X \in \mathfrak{m}$ due to

$$0 = \gamma(X, X) = \tfrac{1}{2}[X, X]_{\mathfrak{m}} + \beta(X, X) = \beta(X, X).$$

Thus $\beta\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is a symmetric bilinear map that fulfills $\beta(X, X) = 0$ for all $X \in \mathfrak{m}$. By polarization, we obtain $\beta(X, Y) = 0$ for all $X, Y \in \mathfrak{m}$. Hence $\gamma = \alpha + \beta = \alpha$ holds, i.e. $\alpha\colon \mathfrak{m} \times \mathfrak{m} \ni (X, Y) \mapsto \frac{1}{2}[X, Y]_{\mathfrak{m}} \in \mathfrak{m}$ is the unique $\mathrm{Ad}(H)$-invariant bilinear map that satisfies $\alpha(X, Y) - \alpha(Y, X) = [X, Y]_{\mathfrak{m}}$ and $\alpha(X, X) = 0$ for all $X, Y \in \mathfrak{m}$. This yields the desired result. □

Definition 4.34 Let G/H be a reductive homogeneous space.

1. The invariant covariant derivative defined by $\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$ for all $X, Y \in \mathfrak{m}$ is called the canonical invariant covariant derivative of first kind. It is denoted by ∇^{can1}.
2. The invariant covariant derivative defined by $\alpha(X, Y) = 0$ for all $X, Y \in \mathfrak{m}$ is called the canonical invariant covariant derivative of second kind. It is denoted by ∇^{can2}.

Remark 4.35 By Proposition 4.18, the canonical covariant derivatives of first kind ∇^{can1} and of second kind ∇^{can2} from Definition 4.34 correspond to the canonical affine connections of first and second kind form [1, Sec. 10], respectively.

Remark 4.36 Assume that G/H is a naturally reductive homogeneous space. Then the Levi-Civita covariant derivative coincides with the canonical covariant derivative of first kind by Remark 4.25, i.e $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can1}}$ holds. This has already been proven in [1, Thm. 13.1 and Eq. (13.2)].

Remark 4.37 Let G/H be equipped with an invariant pseudo-Riemannian metric. Then ∇^{can2} is an invariant metric covariant derivative on G/H by Proposition 4.22.

We briefly comment on the canonical covariant derivatives on symmetric homogeneous spaces in the next remark following [1, Thm. 15.1].

Remark 4.38 Let (G, H, σ) be a symmetric pair and let G/H be the corresponding symmetric homogeneous space. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ denote the canonical reductive decomposition. Then $[X, Y] \in \mathfrak{h}$ holds for all $X, Y \in \mathfrak{m}$ by Lemma 3.9. Therefore $\frac{1}{2}[X, Y]_{\mathfrak{m}} = 0$ is fulfilled for all $X, Y \in \mathfrak{m}$. Hence $\nabla^{\mathrm{can1}} = \nabla^{\mathrm{can2}}$ holds by Proposition 4.33.

Moreover, for pseudo-Riemannian symmetric spaces, we obtain the following remark whose statement can be found in [1, Thm. 15.6].

Remark 4.39 Let G/H be a pseudo-Riemannian symmetric homogeneous space. Then one has $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can1}} = \nabla^{\mathrm{can2}}$ by Remark 4.36 combined with Remark 3.11.

We end this section by specializing Corollary 4.27 on parallel vector fields along curves to the canonical covariant derivatives ∇^{can1} and ∇^{can2}.

Corollary 4.40 *Let $\widehat{Z}\colon I \to T(G/H)$ be a vector field along the curve $\gamma\colon I \to G/H$. Using the notation of Corollary 4.27, the following assertions are fulfilled:*

1. *$\widehat{Z}$ is parallel along γ with respect to ∇^{can1} iff*
$$\dot{z}(t) = -\tfrac{1}{2}[x(t), z(t)]_{\mathfrak{m}} \tag{4.94}$$
holds for all $t \in I$.
2. *$\widehat{Z}$ is parallel along γ with respect to ∇^{can2} iff*
$$\dot{z}(t) = 0 \tag{4.95}$$
is fulfilled for all $t \in I$.

Remark 4.41 A similar description of parallel vector fields as in Corollary 4.40, Claim 1 has already appeared in [18, Prop. 2.12] for the special case, where G/H is a normal naturally reductive space, see Remark 3.7 for this notion, and $\gamma\colon I \to G/H$ is a geodesic, i.e. for $x\colon I \to \mathfrak{m}$ being constant.

5 Conclusion

We considered invariant covariant derivatives on a reductive homogeneous space in detail. We proved that they are uniquely determined by evaluating them on fundamental vector fields. Moreover, we provided a new proof for their existence by expressing them in terms of horizontally lifted vector fields. By this result, a characterization of parallel vector fields along curves in a reductive homogeneous space equipped with an invariant covariant derivative is obtained. In addition, the so-called canonical covariant derivatives of first and second kind corresponding the canonical affine connections of first and second kind from [1] are considered.

Acknowledgements This work has been supported by the German Federal Ministry of Education and Research (BMBF-Projekt 05M20WWA: Verbundprojekt 05M2020 - DyCA).

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability Not applicable.

Declarations

Conflict of interest There is no conflict of interest.

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