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Tensor Tomography on Negatively Curved Manifolds of Low Regularity

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Abstract
We prove solenoidal injectivity for the geodesic X-ray transform of tensor fields on simple Riemannian manifolds with $C^{1,1}$ metrics and non-positive sectional curvature. The proof of the result rests on Pestov energy estimates for a transport equation on the non-smooth unit sphere bundle of the manifold. Our low regularity setting requires keeping track of regularity and making use of many functions on the sphere bundle having more vertical than horizontal regularity. Some of the methods, such as boundary determination up to gauge and regularity estimates for the integral function, have to be changed substantially from the smooth proof. The natural differential operators such as covariant derivatives are not smooth.

Keywords Geodesic X-ray tomography · Non-smooth geometry · Tensor tomography · Integral geometry · Inverse problems

Mathematics Subject Classification 44A12 · 53C22 · 53C65 · 58C99

1 Introduction

What are the minimal smoothness assumptions on a Riemannian metric under which the geodesic X-ray transform of tensor fields on the Riemannian manifold is solenoidally injective? Solenoidal injectivity on smooth simple manifolds with negative curvature was proved in [44]. Since [44], many solenoidal injectivity results have been shown under different variations of the geometric setup. Solenoidal injectivity is known for all real analytic simple Riemannian metrics [51] and for all smooth simple

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Riemannian metrics with certain bounds on their terminator values [43]. The study of the X-ray transform on manifolds with Riemannian metrics of low regularity was started recently [18], where the authors prove that the X-ray transform of scalar functions is injective on all simple manifolds with $C^{1,1}$ Riemannian metrics. We extend this result and prove that the X-ray transform of tensor fields of any order is solenoidally injective for all simple $C^{1,1}$ Riemannian metrics with almost everywhere non-positive sectional curvature.

X-ray tomography problems of 2-tensor fields naturally arise as linearized problems of travel time tomography or boundary rigidity [49]. The travel time problem arises in applications, such as seismological imaging, where one asks whether the sound speed in a medium can uniquely be determined from the knowledge of the arrival times of waves on the boundary. Because of the geophysical nature of such problems, it is relevant to ask how well the studied model corresponds to the real world. From this point of view, the smoothness assumption of the model manifold is merely a mathematical convenience, which is why we have set out to relax such assumptions.

Our main objective is to optimize the regularity assumptions imposed on the Riemannian metric g of the manifold. We focus on global and uniform non-smoothness (as opposed to, say, interfaces with jump discontinuities), and as in [18] the natural optimality to aim at remains $C^{1,1}$. If g is only assumed to be in the Hölder space $C^{1,\alpha}$ for $\alpha < 1$, the geodesic equation fails to have unique solutions [15, 47] and the X-ray transform itself becomes ill defined. In this sense,our result is optimal on the Hölder scale, as we provide a solenoidal injectivity result (theorem 1) for the class of simple $C^{1,1}$ Riemannian metrics with almost everywhere non-positive sectional curvature.

The non-positivity assumption on the curvature is likely unnecessary — milder assumptions on top of simplicity could suffice. Even in the smooth case relaxing the curvature assumption causes technical difficulties and solenoidal injectivity for all simple Riemannian metrics is not understood. Since our setting is complicated enough as it is, we decided not to include manifolds with possible positive curvature.

A popular method for proving injectivity results relies on interplay between the X-ray transform and a transport equation. In the smooth case, the transport equation is studied using the so-called Pestov identity and energy estimates derived from it (see e.g. [16, 36, 42] and references therein).

We employ a similar approach in our non-smooth setting. Our proof is structurally the same as those in smooth geometry, so the main content of this article is to ensure that everything is well defined and behaved in our non-smooth setting: the unit sphere bundle and operators on it, commutator formulas, function spaces, Santaló's formula, and others.

1.1 Main Results

We record as our main result the following kernel description for the geodesic X-ray transform of tensor fields. In the literature of the geodesic X-ray transform, similar results are often called solenoidal injectivity results. Throughout the article, M will be a compact and connected smooth manifold with a smooth boundary ∂M. The dimension of M will always be $n \geq 2$. The manifold M comes equipped with a $C^{1,1}$ regular Riemannian metric g. That is, the metric g is continuously differentiable and the derivative is Lipschitz.

We define what it means for (M, g) to be simple in Sect. 2.1. Simple $C^{1,1}$ manifolds have global coordinates by definition, but for smooth simple manifolds, this is a consequence of the definitions. When $g \in C^\infty$,the definition of $C^{1,1}$ simplicity is equivalent to the classical definition [18, Theorem 2] and thus assuming existence of global coordinates is not superfluous. We say that g has almost everywhere non-positive sectional curvature if for almost all $x \in M$ we have $\langle R(w, v)v, w\rangle_{g(x)} \leq 0$ where $v, w \in T_x M$ are orthogonal. The curvature tensor R is well defined by the familiar formula almost everywhere in M. The X-ray transform of tensor fields is defined in section 2.1.4.

Theorem 1 *Let (M, g) be a simple $C^{1,1}$ manifold (see Sect. 2.1) with almost everywhere non-positive sectional curvature. Let $m \geq 1$ be an integer.*

(1) *If $p \in C^{1,1}(M)$ is a symmetric $(m-1)$-tensor field vanishing on ∂M, then the X-ray transform $I(\sigma\nabla p)$ of its symmetrized covariant derivative vanishes.*
(2) *If the X-ray transform If of a symmetric m-tensor field $f \in C^{1,1}(M)$ vanishes, there is a symmetric $(m-1)$-tensor field $p \in \mathrm{Lip}(M)$ vanishing on ∂M so that $f = \sigma\nabla p$ almost everywhere on M.*

1.2 Regularity Discussion

Claims 1 and 2 in theorem 1 are not symmetric. The difference is in the regularity of the potential p and we believe this is only a consequence of our proof techniques.

There are two notions of smoothness of any given order of a tensor field: regularity with respect to the smooth structure and existence of high-order covariant derivatives. The covariant concept of smoothness is more natural on a Riemannian manifold. For a typical tensor field f that is C^∞ smooth in the sense of the smooth structure, the covariant derivative ∇f is typically only Lipschitz when $g \in C^{1,1}$. The metric tensor g and its tensor powers are examples of non-vanishing and non-smooth (in the sense of the smooth structure) tensor fields for which covariant derivatives of all orders are well defined. Thus neither of the two notions of smoothness implies the other in general. The two notions of $C^{1,1}$ and less regular Hölder spaces of tensor fields agree, but they disagree for higher regularity. Therefore there are, for example, two different spaces $C^{2,1}$ and we do not use such confusing spaces at all.

We focus on optimizing the regularity of the Riemannian metric g, but we did not pursue optimizing regularity of the tensor fields f or p, the boundary ∂M or the integral function u^f of f (see equation (3)).

It is important for our key regularity result (lemma 3 below) that the boundary values of the tensor field are determined by the data to the extent allowed by gauge freedom. A boundary determination result for 2-tensor fields in the smooth case, where g is C^∞, can be found in [51, Lemma 4.1]. Their result is based on clever analysis of equation $2f_{ij} = p_{i;j} + p_{j;i}$ in boundary normal coordinates. Although the argument in [51] works nicely in the smooth case, it does not give the desired result if g is only $C^{1,1}$ and f is $C^{1,1}$. The immediate conclusion of their argument in the non-smooth case would be that p has derivatives in some directions and is Lipschitz continuous, whereas in lemma 2,we find a p in the class $C^{1,1}$. The other difficulty in adapting similar arguments to the non-smooth case is the regularity of boundary normal coordinates.

To avoid these issues,we prove a boundary determination result (lemma 2) by a more explicit approach. Our construction gives a potential $p \in C^{1,1}(M)$ satisfying $\sigma\nabla p|_{\partial M} = f|_{\partial M}$ when $f \in C^{1,1}(M)$. The cost of our method compared to the method of [51] is losing control of the 1-jets in any neighbourhood of the boundary, but leading order boundary determination suffices for our needs.

We lose a derivative in the regularity of p twice in our argument:

(1) We lose a derivative of p in the boundary determination result. Even if the tensor field $f \in C^{l,1}(M)$ and the Riemannian metric $g \in C^{k,1}(M)$ are assumed to have any (finite) amounts of derivatives, we only get $p \in C^{\min(k,l),1}(M)$. Particularly, p is only $C^{1,1}$, when g and f are $C^{1,1}$. To our knowledge, our boundary determination result is optimal in the literature for differentiability of the potential p with properties $\sigma\nabla p = f$ and $p = 0$ on the boundary.
One might expect $f|_{\partial M} = \sigma\nabla p|_{\partial M}$, where $f \in C^{1,1}(M)$ and $p \in C^{2,1}(M)$. The space $C^{2,1}(M)$ is problematic as described above. In order to improve the regularity of p, one needs to make sense of higher regularity and prove a suitable ellipticity result, but we will not explore this avenue.

(2) Secondly, we lose a derivative of p in the transition of regularity from the spherical harmonic components of f to the spherical harmonic components of the integral function $u{:=}u^f$ of f (see Sect. 2.1). Consider the smooth case, where $g \in C^\infty$, and let $f = f_m + f_{m-2} + f_{m-4} + \cdots$ and $u = u_{m-1} + u_{m-3} + u_{m-5} + \cdots$ be the spherical harmonic decompositions of f and u. The geodesic vector field X on the unit sphere bundle of M splits into the two operators X_+ and X_- in each spherical harmonic degree (see Sect. 2.1). Projecting the transport equation $Xu = -f$ into each spherical harmonic degree gives $X_+u_{m-1} = -f_m$ and $X_+u_{k-1} = -f_k - X_-u_{k+1}$ for $k \leq m-2$ with $k \equiv m \pmod 2$. The operator X_+ is known to be an elliptic pseudodifferential operator of order one (see,e.g. [43]) and thus by elliptic regularity, we see that each u_k has one more derivative than the corresponding component f_{k+1}. This argument shows that u has one more derivative than f, proving that p is $C^{1,1}$ when f is Lipschitz.
However, when $g \in C^{1,1}(M)$,the phase space SM is not equipped with a smooth structure and the meaning of ellipticity and its implications,such as existence of a parametrix, become less clear. The exact formulation and application of ellipticity in the present low regularity setting would be a considerable task and would still not give fully matching regularities in the two parts of theorem 1. Therefore,we take a simpler route and do not pursue a fully symmetric version of our main theorem.

1.3 Related Results

The study of the X-ray transform via the transport equation and Pestov identity approach begun with the work of Mukhometov [30, 31, 33], where injectivity results for the transform of scalar functions were proved. Since Mukhometov's seminal articles, the Pestov identity method has been applied to the case of 1-forms in [2] and to higher-order tensors in [40, 43]. Besides manifolds with boundaries, Pestov identities are useful in the study of integral data of functions and tensor fields over closed curves on closed Anosov manifolds [7, 8, 41, 43, 48]. The method is even applicable in non-compact geometries. For results on Cartan–Hadamard manifolds, see [26, 27]. There are plenty of other geometrical variations of the problem, which have been studied employing a Pestov identity. These include reflecting obstacles inside the manifold [20, 21], attenuations and Higgs fields [13, 39, 46], manifolds with magnetic flows [1, 10, 22, 23, 28], and non-Abelian variations [12, 29, 35, 37]. The Pestov identity approach has been studied in more general geometries than Riemannian. For results in Finsler geometry, see [3, 19] and for pseudo-Riemannian geometry, [17].

Only few injectivity results exist outside smooth geometry, whether Riemannian or not. Injectivity of the scalar X-ray transform is known spherically symmetric $C^{1,1}$ regular manifolds satisfying the Herglotz condition, when the conformal factor of the metric is $C^{1,1}$ [11]. The scalar (and 1-form) X-ray transform is (solenoidally) injective on simple $C^{1,1}$ manifolds [18]. The proof of injectivity in [18] is based on a Pestov identity.

The boundary rigidity problem is a geometrization of the travel time tomography problem and its linearization is the X-ray tomography problem of 2-tensor fields. For results in boundary rigidity, see [4–6, 14, 24, 32, 34, 45, 50, 52]. For a comprehensive survey on results in travel time tomography and tensor tomography, see [16, 49].

2 Proof of the Main Theorem

2.1 Basic Definitions and Notation

In this subsection,we present enough terminology and notation to state and prove our main theorem. The preliminaries of the non-smooth setting are complemented in Sect. 3.

Throughout the article, M will be a compact and connected smooth manifold with a smooth boundary ∂M. The manifold M is equipped with a $C^{1,1}$ regular Riemannian metric g.

2.1.1 Bundles

The tangent bundle TM of M has a subbundle SM called the unit sphere bundle, which consists of the unit vectors in TM. As the level set $F^{-1}(1)$ of the $C^{1,1}$ map $F\colon TM \to \mathbb{R}$ defined by $F(x, v) = g_x(v, v)$, the unit sphere bundle is a $C^{1,1}$ submanifold[1] of TM. The boundary

$$\partial(SM) := \{ (x, v) \in SM : x \in \partial M \} \tag{1}$$

[1] It is easily verified by inspecting the vertical component that the differential $\mathrm{d}F$ is non-zero when $F = 1$. The smooth regular level set theorem [25] can easily be adapted to our case.

of SM is divided into inwards and outwards pointing parts $\partial_{\text{in}}(SM)$ and $\partial_{\text{out}}(SM)$ with respect to the inner product $\langle\cdot,\cdot\rangle_g$ and a unit normal vector field ν to the boundary ∂M. The subset of $\partial(SM)$ consisting of the vectors v such that $\langle v, \nu\rangle_g = 0$ is denoted by $\partial_0(SM)$ and it is disjoint from $\partial_{\text{in}}(SM)$ and $\partial_{\text{out}}(SM)$.

Let $\pi\colon SM \to M$ be the standard projection and let $\pi^*(TM)$ be the pullback of TM over SM. We denote by N the subbundle of $\pi^*(TM)$ with the fibre $N_{(x,v)}$ being the g-orthogonal complement of v in $T_x M$.

2.1.2 Horizontal–Vertical Decomposition

The tangent bundle $T(SM)$ of SM has an orthogonal splitting $T(SM) = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$ with respect to the so-called Sasaki metric, where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical subbundles,respectively, and X is the geodesic vector field on SM. We denote $\mathbb{R}X \oplus \mathcal{H}$ by $\overline{\mathcal{H}}$ and call it the total horizontal subbundle. Elements of $\mathcal{H}$ and $\mathcal{V}$ are, respectively, referred to as horizontal and vertical derivatives or vectors on SM. The summands $\mathcal{H}$ and $\mathcal{V}$ are each naturally identified with a copy of the bundle N. The horizontal–vertical geometry is essentially the same as the smooth one (see [38]) and works fine when $g \in C^{1,1}$.

2.1.3 Geodesic Flow

Since the Christoffel symbols of a $C^{1,1}$ metric are Lipschitz, there is a unique unit speed geodesic γ_z corresponding to a given initial condition $z \in SM$ by standard ODE theory. We define the geodesic flow on the unit sphere bundle to be the collection of (partially defined) maps $\phi_t\colon SM \to SM$, $\phi_t(z) = (\gamma_z(t), \dot{\gamma}_z(t))$, where t goes through all real numbers so that the right-hand side is defined. The infinitesimal generator X of the flow is called the geodesic vector field on SM. For any $z \in SM$, the geodesic γ_z is defined on a maximal interval of existence $[-\tau_-(z), \tau_+(z)]$, where $\tau_-(z)$ and $\tau_+(z)$ are positive. We call $\tau(z):=\tau_+(z)$ the travel time function on SM. The geodesic vector field X acts naturally on functions by differentiation and on sections W of the bundle N, it acts by

$$XW(z) = D_t W(\phi_t(z))|_{t=0}, \tag{2}$$

where D_t is the covariant derivative along the curve $t \mapsto \phi_t(z)$. The result XW of the action (2) is again a section of N.

2.1.4 The X-Ray Transform

Any symmetric m-tensor field f on M can be considered as a function on the unit sphere bundle. Given $(x, v) \in SM$,we let $f(x, v):=f_x(v, \dots, v)$. In lemma 7 and proposition 11 and their proofs, we denote the induced maps by $\lambda_x f\colon S_x M \to \mathbb{R}$ and $\lambda f\colon SM \to \mathbb{R}$ with $\lambda f(x, v) = \lambda_x f(v)$. Otherwise, we freely identify f with λf since there is no danger of confusion.

The integral function $u^f : SM \to \mathbb{R}$ of a continuous symmetric m-tensor field f is defined by

$$u^f(x, v) := \int_0^{\tau(x,v)} \lambda f(\phi_t(x, v))\, \mathrm{d}t \tag{3}$$

for all $(x, v) \in SM$. The X-ray transform of f is the restriction of the integral function to the inward pointing part of the boundary $\partial(SM)$, so we may declare $If := u^f|_{\partial_{\text{in}}(SM)}$.

2.1.5 Differentiability

We exclude the rank of the tensor field from our notations for function spaces. For tensor fields,the derivatives are covariant. We use the subscript 0 to indicate zero boundary values. Thus, for example, $f \in C_0^{1,\alpha}(M)$ for a tensor field f means that $f|_{\partial M} = 0$ and ∇f is α-Hölder. We use two kinds of functions on the sphere bundle SM, scalars (e.g. $C^1(SM)$) and sections of the bundle N (e.g. $C^1(N)$) defined in subsection 2.1.1.

We define $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$ as the subset of $C(SM)$ consisting of functions with k many α-Hölder horizontal derivatives and l many β-Hölder vertical derivatives as well as any combination of k horizontal and l vertical derivatives, which are assumed to be ω-Hölder for $\omega := \min(\alpha, \beta)$. We let

$$C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{\infty}(SM) := \bigcap_{l=0}^{\infty} C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,1}(SM). \tag{4}$$

According to the splitting $T(SM) = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}$, the gradient of a C^1 function u on SM can be written as

$$\nabla u = ((Xu)X, \overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u). \tag{5}$$

This gives rise to two new differential operators; the vertical gradient $\overset{\mathrm{h}}{\nabla}$ and the horizontal gradient $\overset{\mathrm{v}}{\nabla}$. Both $\overset{\mathrm{h}}{\nabla} u$ and $\overset{\mathrm{v}}{\nabla} u$ are naturally identified with sections of the bundle N. The horizontal and vertical divergences are the L^2 adjoints of the corresponding gradients. The L^2 adjoint of X is $-X$. The vertical Laplacian on the sphere bundle is $\overset{\mathrm{v}}{\Delta} := -\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{v}}{\nabla}$; see [43, Appendix A] for details on the differential operators.

2.1.6 Curvature

By Rademacher's theorem, a Lipschitz continuous scalar function on a Euclidean domain is differentiable almost everywhere and the derivative is in L^∞. Using local coordinates and studying the individual components show that the Riemann curvature tensor $R_{ijkl}(x)$ corresponding to a Riemannian metric $g \in C^{1,1}$ has all components well defined for almost all $x \in M$. Thus we may interpret the curvature tensor R as an L^∞ tensor field. The curvature tensor $R\colon L^\infty(N) \to L^\infty(N)$ acts on sections of the bundle N by $R(x, v)W(x, v) := R(W(x, v), v)v$ producing again L^∞ sections of the bundle N.

We say that the sectional curvature of the manifold M is almost everywhere non-positive, if for almost all $x \in M$,it holds that $\langle R(w, v)v, w\rangle_{g(x)} \leq 0$ for all linearly independent $v, w \in T_x M$.

2.1.7 Sobolev Spaces

There are natural L^2 spaces for functions on the sphere bundle as well as for sections of the bundle N, which we will denote by $L^2(SM)$ and $L^2(N)$. We define the Sobolev spaces $H^1(SM)$ and $H^1(N, X)$,respectively, defined as completions of $C^1(SM)$ and $C^1(N)$ with respect to the norms

$$\|u\|^2_{H^1(SM)} := \|u\|^2_{L^2(SM)} + \|Xu\|^2_{L^2(SM)} + \left\|\overset{\mathrm{h}}{\nabla}u\right\|^2_{L^2(SM)} + \left\|\overset{\mathrm{v}}{\nabla}u\right\|^2_{L^2(SM)}, \quad \text{and}$$
$$\|W\|^2_{H^1(N,X)} := \|W\|^2_{L^2(N)} + \|XW\|^2_{L^2(N)}. \tag{6}$$

We denote zero boundary values by a subindex 0. For example, $H^1_0(SM)$ is the subspace of $H^1(SM)$ with zero boundary values.

2.1.8 Spherical Harmonics

Given $x \in M$, the unit sphere $S_x M$ has the Laplace–Beltrami operator $\overset{\mathrm{v}}{\Delta}_x := -g^{ij}(x)\partial_{v^i}\partial_{v^j}$. Letting $x \in M$ vary we get a second-order operator $\overset{\mathrm{v}}{\Delta} = -\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{v}}{\nabla}$ on the unit sphere bundle called the vertical Laplacian, where $-\overset{\mathrm{v}}{\mathrm{div}}$ is the formal L^2-adjoint of $\overset{\mathrm{v}}{\nabla}$.

Let $S^{n-1} \subseteq \mathbb{R}^n$ be the Euclidean unit sphere. It is well known that any function $f \in L^2(S^{n-1})$ can be decomposed as an L^2-convergent series $f = \sum_{k=0}^\infty f_k$, where f_k are eigenfunctions of the spherical Laplacian on S^{n-1} corresponding to the eigenvalues $k(k+n-2)$. Similarly, any function $u \in L^2(SM)$ can be decomposed as an $L^2(SM)$-convergent series $u = \sum_{k=0}^\infty u_k$, where $\overset{\mathrm{v}}{\Delta}u_k = k(k+n-2)u_k$ for all $k \in \mathbb{N}$. We call u_k the kth spherical harmonic component of u. For $k \in \{0, 1\}$, $k, l \in \mathbb{N}$ and $\alpha, \beta \in [0, 1]$ we let

$$\Omega^{k,\alpha}_{\mathrm{h}}\Omega^{l,\beta}_{\mathrm{v}}(m) := \{\, u \in C^{k,\alpha}_{\mathrm{h}} C^{l,\beta}_{\mathrm{v}}(SM) : \overset{\mathrm{v}}{\Delta}u = m(m+n-2)u \,\} \tag{7}$$

and

$$\Omega^{k,\alpha}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(m) := \bigcap_{l\in\mathbb{N}} \Omega^{k,\alpha}_{\mathrm{h}}\Omega^{l,1}_{\mathrm{v}}(m). \tag{8}$$

Furthermore, we denote

$$\Lambda_{\mathrm{h}}^{k}\Lambda_{\mathrm{v}}^{l}(m) = \{\, u \in H_{\mathrm{h}}^{k}H_{\mathrm{v}}^{l}(SM) : \overset{\mathrm{v}}{\Delta} u = m(m+n-2)u \,\}. \tag{9}$$

For all $m \in \mathbb{N}$,there are operators $X_{\pm}\colon \Omega_{\mathrm{h}}^{1}\Omega_{\mathrm{v}}^{\infty}(m) \to \Omega_{\mathrm{h}}^{0}\Omega_{\mathrm{v}}^{\infty}(m \pm 1)$ with the convention that $\Omega_{\mathrm{h}}^{0}\Omega_{\mathrm{v}}^{\infty}(-1) = 0$ so that $X = X_{+} + X_{-}$. These mapping properties and validity of this decomposition in low regularity are addressed in proposition 12.

2.1.9 Simple $C^{1,1}$ Manifolds

The global index form Q of the manifold (M, g) (not of a single geodesic) is the quadratic form defined for $W \in H_0^1(N, X)$ by

$$Q(W) := \|XW\|_{L^2(N)}^2 - (RW, W)_{L^2(N)}\,. \tag{10}$$

It was proved in [18, Lemma 11] that there are no conjugate points on a Riemannian manifold (M, g), $g \in C^{\infty}$, if the global index form Q of (M, g) is positive definite.

We conclude this subsection by recalling a definition of a simple manifold in the case $g \in C^{1,1}$. Our definition is equivalent to the definition of traditional simple manifold when $g \in C^{\infty}$ [18]. Let $M \subseteq \mathbb{R}^n$ be the closed Euclidean unit ball and let g be a $C^{1,1}$ regular Riemannian metric on M. We say that (M, g) is *a simple $C^{1,1}$ Riemannian manifold* if the following hold:

- A1: There is $\varepsilon > 0$ so that $Q(W) \geq \varepsilon \|W\|_{L^2(N)}^2$ for all $W \in H_0^1(N, X)$.
- A2: Any two points of M can be joined by a unique geodesic in the interior of M, whose length depends continuously on its end points.
- A3: The squared travel time function τ^2 (see 2.1.3) is Lipschitz on SM.

2.2 Proof of the Theorem

In this subsection, we prove our main result, theorem 1. We state the lemmas required for the proof of 1, and the proofs of the lemmas are postponed to sections 4, 5, and 6 .

Lemma 2 (Boundary determination) *Let (M, g) be a simple $C^{1,1}$ manifold. If $f \in C^{1,1}(M)$ is a symmetric m-tensor field with $If = 0$, then there is a symmetric $(m-1)$-tensor field $p \in C^{1,1}(M)$ so that $f|_{\partial M} = \sigma\nabla p|_{\partial M}$ and $p|_{\partial M} = 0$.*

Lemma 3 (Regularity of spherical harmonic components) *Let (M, g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field on M with $If = 0$ and let $u := u^f$ be the integral function of f defined by (3). If the spherical harmonic decomposition of u is $u = \sum_{k=0}^{\infty} u_k$, then $u_k \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and $u_k|_{\partial(SM)} = 0$ for all $k \in \mathbb{N}$.*

Lemma 4 *Let (M, g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field on M with $If = 0$ and let $u := u^f$ be the integral function of f defined by (3). Then $X_{+}u \in L^2(SM)$.*

Lemma 4 follows immediately from lemmas 3 and 17 .

Recall that n is the dimension of M. For natural numbers k and l,we define the two constants

$$C(n,k):=\frac{2k+n-1}{2k+n-3} \quad \text{and} \quad B(n,l,k):=\prod_{p=1}^{l} C(n,k+2p). \tag{11}$$

Lemma 5 *Let (M,g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field with $If = 0$ and denote by $u:=u^f$ the integral function of f defined by (3). If the spherical harmonic decomposition of u is $u = \sum_{k=0}^{\infty} u_k$, then for all $k \geq m$ and $l \in \mathbb{N}$,we have*

$$\|X_+u_k\|^2_{L^2(SM)} \leq B(n,l,k)\,\|X_+u_{k+2l}\|^2_{L^2(SM)}\,. \tag{12}$$

Lemma 6 (Injectivity of X_+) *Let (M,g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Suppose that $u \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ and $u|_{\partial(SM)} =$ 0. Then $X_+u = 0$ implies that $u = 0$.*

Lemma 7 *Let (M,g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}(M)$ be a symmetric m-tensor field. Suppose that p is a symmetric $(m-1)$-tensor field and $u = -\lambda p$ is a Lipschitz function in SM so that $Xu = -\lambda f$ everywhere in SM. Then $\sigma\nabla p = f$ almost everywhere in M.*

Proof of theorem 1 Item 1: Suppose that $p \in C^{1,1}(M)$ is a symmetric $(m-1)$-tensor field vanishing on $\partial(M)$. Then using the fundamental theorem of calculus along each geodesic gives $If = I(\sigma\nabla p) = 0$ (see [36, Lemma 6.4.2]), which proves item 1.

Item 2: Suppose that the X-ray transform of a symmetric m-tensor field $f \in C^{1,1}(M)$ vanishes. We will prove that there is a symmetric $(m-1)$-tensor field p vanishing on ∂M so that $f = \sigma\nabla p$.

By boundary determination in lemma 2, there is a symmetric $(m-1)$-tensor field $p_0 \in C^{1,1}(M)$ so that $p_0|_{\partial M} = 0$ and $f|_{\partial M} = \sigma\nabla p_0|_{\partial M}$. Let $\hat{f}:=f - \sigma\nabla p_0$. Then $\hat{f} \in \mathrm{Lip}_0(M)$ is a symmetric m-tensor field on M and $I\hat{f} = If = 0$.

Let $u = \sum_{k=0}^{\infty} u_k$ be the spherical harmonic decomposition of $u:=u^{\hat{f}}$. Then $u_k \in \Omega^{0,1}_{\mathrm{h}}\Omega^{\infty}_{\mathrm{v}}(k)$ by lemma 3. First, we prove that $u_k = 0$ for all k for which $k \equiv m \pmod 2$.

Since for all $(x,v) \in SM$ it holds that $\hat{f}(x,-v) = (-1)^m\hat{f}(x,v)$, we have

$$\begin{aligned} u(x,-v) &= \int_0^{\tau_+(x,-v)} \hat{f}(\gamma_{x,-v}(t), \dot{\gamma}_{x,-v}(t))\,\mathrm{d}t \\ &= (-1)^m \int_{-\tau_-(x,v)}^{0} \hat{f}(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))\,\mathrm{d}t. \end{aligned} \tag{13}$$

Therefore,

$$u(x,-v) + (-1)^m u(x,v) = (-1)^m I\hat{f}(\phi_{-\tau_-(x,v)}(x,v)) = 0. \tag{14}$$

This shows that $u(x, -v) = (-1)^{m+1} u(x, v)$ for all $(x, v) \in SM$ and thus $u_k = 0$ whenever $k \equiv m \pmod 2$. Next, we will show that $u_k = 0$ for all $k \geq m$.

Let $m_0 \geq m$ and suppose that $A_1 := \left\| X_+ u_{m_0} \right\|^2_{L^2(SM)} > 0$. For all $l \in \mathbb{N}$, lemma 5 yields the estimate

$$B(n, l, m_0)^{-1} \left\| X_+ u_{m_0} \right\|^2_{L^2(SM)} \leq \left\| X_+ u_{m_0+2l} \right\|^2_{L^2(SM)} . \tag{15}$$

By an elementary estimate (see [20, Lemma 13]), there is a constant $A_2 > 0$ only depending on m_0 and n so that

$$B(n, l, m_0)^{-1} \geq \left(1 + \frac{4l}{2m_0 + n - 3}\right)^{-1/2} \geq A_2 l^{-1/2}. \tag{16}$$

Thus the estimate (15) gives

$$\sum_{l=1}^{\infty} \left\| X_+ u_{m_0+2l} \right\|^2_{L^2(SM)} \geq A_1 A_2 \sum_{l=1}^{\infty} l^{-1/2} = \infty. \tag{17}$$

On the other hand, $X_+ u \in L^2(SM)$ by lemma 4. Hence orthogonality implies that

$$\sum_{l=1}^{\infty} \left\| X_+ u_{m_0+2l} \right\|^2_{L^2(SM)} \leq \sum_{k=0}^{\infty} \| X_+ u_k \|^2_{L^2(SM)} \leq \| X_+ u \|^2_{L^2(SM)} < \infty. \tag{18}$$

This contradiction proves that $\| X_+ u_k \|^2_{L^2(SM)} = 0$ for all $k \geq m$. Since additionally $u_k|_{\partial(SM)} = 0$ for $k \geq m$, lemma 6 says $u_k = 0$ for all $k \geq m$.

We have shown $u_k = 0$ for $k \geq m$ and $u_k = 0$ for $k \equiv m \pmod 2$. Thus $-u \in \mathrm{Lip}_0(SM)$ is identified with a symmetric $(m-1)$-tensor field $p_1 \in \mathrm{Lip}_0(M)$. As u solves the transport equation $Xu = -\hat{f}$ everywhere on SM we have $\sigma \nabla p_1 = \hat{f}$ almost everywhere on M by lemma 7. Thus we conclude that $f = \sigma \nabla p$ almost everywhere in M, where $p := p_0 + p_1 \in \mathrm{Lip}(M)$ is a symmetric $(m-1)$-tensor field with $p|_{\partial M} = 0$. □

3 Preliminaries

In this article, we consider compact and connected smooth manifolds with smooth boundaries. We assume that such a manifold M comes equipped with a symmetric and positive definite 2-tensor field g so that its component functions g_{jk} are $C^{1,1}$-functions on M. In this case, we refer to g as a $C^{1,1}$ *Riemannian metric* and to (M, g) as a *(non-smooth) Riemannian manifold*.

3.1 Spaces of Tensor Fields

Since g is a $C^{1,1}$ Riemannian metric, componentwise differentiability and existence of covariant derivatives are not the same. Even if the components of a tensor field f in any local coordinates are C^k functions for $k \geq 2$ (which is possible since M is assumed to have a smooth structure), the covariant derivative ∇f falls into $\mathrm{Lip}(M)$. Since most of our considerations are related to the metric structure and componentwise differentiability is not compatible with the covariant derivative, the correct definition of a $C^{1,1}$ tensor field is by covariant differentiability. However, with covariant differentiability, we are restricted to $C^{1,1}(M)$ and higher regularity does not exist on the Hölder scale.

The space $L^2(M)$ of L^2-tensor fields of order m on M is defined to be the completion of the space of continuous m-tensor fields with respect to the norm induced by the inner product

$$(f, h)_{L^2(M)} := \int_M g^{j_1k_1} \cdots g^{j_mk_m} f_{j_1\cdots j_m} h_{k_1\cdots k_m} \,\mathrm{d}V_g. \tag{19}$$

Here $\mathrm{d}V_g$ is the Riemannian volume form of M. The space $H^1(M)$ of H^1-tensor fields of order m on M is defined to be the closure of the space of continuously differentiable m-tensor fields with respect to the norm

$$\|f\|^2_{H^1(M)} := \|f\|^2_{L^2(M)} + \|\nabla f\|^2_{L^2(M)}. \tag{20}$$

Let $p \in [1, \infty)$. The spaces $L^p(M)$ and $W^{1,p}(M)$ of L^p- and $W^{1,p}$-tensor fields of order m are defined analogously to the spaces $L^2(M)$ and $H^1(M)$.

We could give definitions of the spaces $H^2(M)$ and $W^{2,p}(M)$ for tensor fields of any order similar to the definitions of spaces $H^1(M)$ and $W^{1,p}(M)$. Again, since g is only a $C^{1,1}$ regular Riemannian metric, there are no spaces $H^3(M)$ and $W^{3,p}(M)$ compatible with the geometry. A compatible space should be defined using covariant derivatives in the norms, which would force the spaces $W^{k,p}(M)$ trivial, when $k \geq 3$.

If $f \in C^1(M)$ is a symmetric m-tensor field on M, its symmetrized covariant derivative is $\sigma\nabla f$. The symmetrization σ is defined for all m-tensor fields h on M by

$$(\sigma h)_{j_1\cdots j_m} := \frac{1}{m!} \sum_\pi h_{j_{\pi(1)}\cdots j_{\pi(m)}} \tag{21}$$

where the summation is over all permutations π of $\{1, \ldots, m\}$. Note that since $\|\sigma\nabla f\|_{L^2} \leq \|\nabla f\|_{L^2}$, the symmetrized covariant derivative is bounded between Sobolev spaces.

The trace of a symmetric m-tensor field f on M is denoted by $\mathrm{tr}_g(f)$. In local coordinates, $\mathrm{tr}_g(f)_{i_1\cdots i_{m-2}} = g^{jk} f_{jki_1\cdots i_{m-2}}$. A symmetric m-tensor field is called trace-free, if its trace is zero.

3.2 Vertical and Horizontal Differentiability

Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. Let $k \in \mathbb{N}$ and $\alpha \in [0, 1]$ be so that $k + \alpha \leq 2$. For $l \in \mathbb{N}$ and $\beta \in [0, 1]$, the set $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$ consists of all functions $u \in C(SM)$ with

$$H_1 \cdots H_k u \in C^{0,\alpha}(SM) \quad \text{and} \quad V_1 \cdots V_l u \in C^{0,\beta}(SM) \tag{22}$$

for any k vector fields $H_1, \ldots, H_k \in \overline{\mathcal{H}}$ and any l vector fields $V_1, \ldots, V_l \in \mathcal{V}$. Additionally, we require that for any $k + l$ vector fields $Z_1, \ldots, Z_{k+l} \in T(SM)$ out of which exactly k are in $\overline{\mathcal{H}}$ and exactly l are in $\mathcal{V}$,we have

$$Z_1 \cdots Z_{k+l} u \in C^{0,\omega}(SM), \quad \text{where} \quad \omega := \min(\alpha, \beta). \tag{23}$$

We let

$$C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{\infty}(SM) := \bigcap_{l \in \mathbb{N}} C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,1}(SM). \tag{24}$$

Remark 8 In the definition of $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$,the vertical differentiability indices l and β can surpass the smoothness of charts of SM. It is not necessary for SM to have C^{∞} smooth charts, since vertical vector fields operate on a fixed fibre and for a fixed point x in M, the scaling $s(x, v) = (x, v\,|v|_g^{-1})$ is smooth on $T_x M \backslash 0$. The slit tangent space $T_x M \setminus 0$ has a smooth structure even if M does not.

Remark 9 Any commutator $[H, V] = HV - VH$, where $H \in \overline{\mathcal{H}}$ and $V \in \mathcal{V}$, can be defined classically on the space $C_{\mathrm{h}}^{1} C_{\mathrm{v}}^{1}(SM)$, since for any $u \in C_{\mathrm{h}}^{1} C_{\mathrm{v}}^{1}(SM)$, the derivatives HVu and VHu are in $C(SM)$.

The set $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N)$ consists of all continuous sections W of the bundle N with W^j in $C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM)$ when $W = W^j \partial_{x^j}$. A section W of the bundle N is continuous, if it is continuous as a map $SM \to TM$.

As one might expect, vertical operators preserve horizontal differentiability and horizontal operators preserve vertical differentiability. That is

$$X : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(SM), \tag{25}$$

$$X : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(N), \tag{26}$$

$$\overset{\mathrm{v}}{\nabla} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM) \to C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l-1,\beta}(N), \tag{27}$$

$$\overset{\mathrm{v}}{\mathrm{div}} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N) \to C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l-1,\beta}(SM), \tag{28}$$

$$\overset{\mathrm{h}}{\nabla} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(SM) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(N), \quad \text{and} \tag{29}$$

$$\overset{\mathrm{h}}{\mathrm{div}} : C_{\mathrm{h}}^{k,\alpha} C_{\mathrm{v}}^{l,\beta}(N) \to C_{\mathrm{h}}^{k-1,\alpha} C_{\mathrm{v}}^{l,\beta}(SM). \tag{30}$$

3.3 Sobolev Spaces of Different Vertical and Horizontal Indices

Standard Sobolev spaces on SM are defined in Sect. 2.1.7. Here, we define Sobolev spaces for scalar functions on SM of different vertical and horizontal indices. If $k, l \in \{0, 1\}$ and u is a scalar function in $C^k_{\mathrm{h}} C^l_{\mathrm{v}}(SM)$, we define the $H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)$-norm of u to be

$$\|u\|^2_{H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)} := \|u\|^2_{L^2(SM)} + k\,\|Xu\|^2_{L^2(SM)} + k\left\|\overset{\mathrm{h}}{\nabla} u\right\|^2_{L^2(N)} + l\left\|\overset{\mathrm{v}}{\nabla} u\right\|^2_{L^2(N)}. \tag{31}$$

The Sobolev space $H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)$ for $k, l \in \{0, 1\}$ is defined to be the completion of $C^k_{\mathrm{h}} C^l_{\mathrm{v}}(SM)$ with respect to the norm $\|\cdot\|_{H^k_{\mathrm{h}} H^l_{\mathrm{v}}(SM)}$.

Similarly, we define spaces $H^0_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ and $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ to be the completions of $C^0_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ and of $C^1_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ with respect to the norms

$$\|u\|^2_{H^0_{\mathrm{h}} H^2_{\mathrm{v}}(SM)} := \|u\|^2_{L^2(SM)} + \left\|\overset{\mathrm{v}}{\Delta} u\right\|^2_{L^2(SM)}, \quad \text{and} \tag{32}$$

$$\|u\|^2_{H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)} := \|u\|^2_{H^1_{\mathrm{h}} H^1_{\mathrm{v}}(SM)} + \|u\|^2_{H^0_{\mathrm{h}} H^2_{\mathrm{v}}(SM)} \tag{33}$$

$$+ \left\|X\overset{\mathrm{v}}{\Delta} u\right\|^2_{L^2(SM)} + \left\|\overset{\mathrm{v}}{\Delta} Xu\right\|^2_{L^2(SM)}. \tag{34}$$

Note that the norm on $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ does not cover all possible combinations of a horizontal derivative and two vertical derivatives (e.g. $\overset{\mathrm{v}}{\mathrm{div}} X \overset{\mathrm{v}}{\nabla}$). This is intentional, since the missing combinations will not be needed.

Proposition 10 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. The following commutator formulas hold on $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$:*

$$[X, \overset{\mathrm{v}}{\nabla}] = -\overset{\mathrm{h}}{\nabla}, \tag{35}$$

$$\overset{\mathrm{h}}{\mathrm{div}} \overset{\mathrm{v}}{\nabla} - \overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla} = (n-1)X, \tag{36}$$

$$[X, \overset{\mathrm{v}}{\Delta}] = 2\overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla} + (n-1)X. \tag{37}$$

The following commutator formula holds on $H^1_{\mathrm{h}} H^1_{\mathrm{v}}(N)$:

$$[X, \overset{\mathrm{v}}{\mathrm{div}}] = -\overset{\mathrm{h}}{\mathrm{div}}. \tag{38}$$

Proof Formulas (35), (36) and (37) on $C^1_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ and (38) on $C^1_{\mathrm{h}} C^1_{\mathrm{v}}(N)$ can be proved by a computation similar to [43, Appendix], since the computations use one horizontal derivative and two vertical for (35), (36) and (37) and one horizontal and one vertical derivative for (38). The same formulas hold on $H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ and $H^1_{\mathrm{h}} H^1_{\mathrm{v}}(N)$ by approximation. □

3.4 Vertical Fourier Analysis

In this subsection, we recall the identification of trace-free symmetric tensor fields and spherical harmonics (the vertical Fourier modes). We state and prove proposition 11 in order to emphasize what changes in these well known results when applied to a case of non-smooth Riemannian metrics. More details in the case of C^∞-smooth Riemannian metrics can be found for example in [36] and [9].

Proposition 11 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. Let $k \in \{0, 1\}$ and $\alpha \in [0, 1]$. The map $\lambda\colon f \mapsto \lambda f$ is defines a linear isomorphism from the space of symmetric trace-free m-tensor fields in $C^{k,\alpha}(M)$ to the space $\Omega^{k,\alpha}_{\mathrm{h}} \Omega^{\infty}_{\mathrm{v}}(m)$. There is a constant $C_{m,n} > 0$ so that for all symmetric trace-free m-tensor fields $f \in C^0(M)$, we have*

$$\|\lambda f\|_{L^2(SM)} = C_{m,n} \|f\|_{L^2(M)} . \tag{39}$$

Furthermore, there are positive constants $c, C > 0$ so that for any two m-tensor fields f and h in $C^0(M)$, we have

$$c\,(\lambda f, \lambda h)_{L^2(SM)} \leq (f, h)_{L^2(M)} \leq C\,(\lambda f, \lambda h)_{L^2(SM)} . \tag{40}$$

Proof As in the smooth case [9, Lemma 2.5.],the map λ_x isomorphically maps trace-free m-tensors to spherical harmonics $S_x M$ of degree m. Since the dependence on x is of the form $\lambda f(x, v) = f_{j_1 \ldots j_m}(x) v^{j_1} \cdots v^{j_m}$, the map λ maps on trace-free m-tensor fields in $C^{k,\alpha}(M)$ into $\Omega^{k,\alpha}_{\mathrm{h}} \Omega^{\infty}_{\mathrm{v}}(m)$.

For any symmetric and trace-free m-tensor fields $f, h \in C^0(M)$, a fibrewise calculation [9, Lemma 2.4.] shows that for all $x \in M$,we have

$$\int_{S_x M} (\lambda_x f)(\lambda_x h)\, \mathrm{d}S_x = C_{m,n} \langle f, h \rangle_{g(x)} \tag{41}$$

for some $C_{m,n} > 0$. Since the computation is fibrewise, it remains valid when $g \in C^{1,1}$. Integrating equation (41) over M gives

$$(\lambda f, \lambda h)_{L^2(SM)} = C_{m,n} (f, h)_{L^2(M)} , \tag{42}$$

which proves (39). Furthermore, the last claim (40) follows from (41), since any symmetric m-tensor field can be decomposed into a sum of symmetric trace-free tensor fields of orders less than or equal to m [36]. □

3.5 Decomposition of the Geodesic Vector Field

In this subsection,we recall the fact that the geodesic vector field maps from spherical harmonic degree m to spherical harmonic degrees $m-1$ and $m+1$. This mapping property induces a decomposition of X into operators X_+ and X_-. See [36, Section 6.6.] for details of the decomposition when $g \in C^\infty$. We record in proposition 12 what changes in the decomposition, when the Riemannian metric g is only $C^{1,1}$-smooth.

Proposition 12 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. The geodesic vector field maps*

$$X\colon \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m) \to \Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m-1) \oplus \Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m+1). \tag{43}$$

Therefore X decomposes into operators X_+ and X_- in each spherical harmonic degree so that

$$X_\pm\colon \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m) \to \Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m \pm 1). \tag{44}$$

Proof Let $u \in \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m)$ and pick a point $x \in M$. Then $Xu(x, v) = v^j \delta_j u(x, v)$ for all $v \in S_x M$, where v^j is a spherical harmonic of degree 1 on $S_x M$ and $\delta_j u(x, \cdot)$ is a spherical harmonic of degree m on $S_x M$. Since any product of spherical harmonics of degrees 1 and m is a sum of spherical harmonics of degrees $m-1$ and $m+1$ we see that

$$X\colon \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m) \to \Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m-1) \oplus \Omega^0_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m+1). \tag{45}$$

Here the spherical harmonic components of Xu have one horizontal derivative less than u since $X \in \overline{\mathcal{H}}$. □

Remark 13 Since X maps continuously with respect to the H^1- and L^2-norms the mapping properties from proposition 12 carry over to the Sobolev space. In other words

$$\begin{aligned} X\colon \Lambda^1_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m) &\to \Lambda^0_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m-1) \oplus \Lambda^0_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m+1), \quad \text{and} \\ X_\pm\colon \Lambda^1_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m) &\to \Lambda^0_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m \pm 1). \end{aligned} \tag{46}$$

As stated above, proposition 12 gives degreewise defined operators X_- and X_+ acting on $\Lambda^1_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(SM)$. If $u \in H^1_{\mathrm{h}}H^2_{\mathrm{v}}(SM)$ and $u = \sum_{k=0}^\infty u_k$ is the spherical harmonic decomposition of u, we define

$$X_\pm u = \sum_{k=0}^\infty X_\pm u_k. \tag{47}$$

We prove in lemma 17 that the series in (47) converges (absolutely) in $L^2(SM)$.

The following lemma 14 is a low regularity version of [43, Lemma 3.3.], the only difference being the regularity of u.

Lemma 14 *Let M be a compact smooth manifold with a smooth boundary and let g be a $C^{1,1}$ Riemannian metric on M. If $u \in \Lambda^1_{\mathrm{h}}\Lambda^2_{\mathrm{v}}(m)$ then*

$$[X_+, \overset{\mathrm{v}}{\Delta}]u = -(2m+n-1)X_+u, \quad \textit{and} \tag{48}$$

$$[X_-, \overset{\mathrm{v}}{\Delta}]u = (2m+n-3)X_-u. \tag{49}$$

Proof By density, it is enough to prove the claimed formulas for $u \in \Omega^1_{\mathrm{h}}\Omega^\infty_{\mathrm{v}}(m)$. By eigenvalue property of u and by the mapping property of X_+, we have

$$X_+\overset{\mathrm{v}}{\Delta}u = m(m+n-2)X_+u. \tag{50}$$

Similarly, by the eigenvalue property of X_+u, we have

$$\overset{\mathrm{v}}{\Delta}X_+u = (m+1)((m+1)+n-2)X_+u. \tag{51}$$

Subtracting (50) from (51) shows that

$$[X_+, \overset{\mathrm{v}}{\Delta}]u = -(2m+n-1)X_+u. \tag{52}$$

The identity (49) can be proved similarly. □

4 Boundary Determination and Regularity Lemmas

This section is devoted to the study of the integral function u^f of a tensor field f with vanishing X-ray transform. We prove a vital boundary determination result (lemma 2) that allows us to prove that u^f is a Lipschitz function on SM in subsection 4.2. In subsection 4.3, we exploit the particular form of the identification of trace-free tensor fields and spherical harmonics to prove our main regularity lemma 3.

4.1 Boundary Determination

The boundary determination lemma 2 is proved in two parts. In lemma 15,we give an explicit local construction. In more detail, we prove that if If vanishes for some tensor field f, then in local coordinates near any boundary point, we construct a tensor field p so that the symmetrized covariant derivative of p equals f when restricted to the boundary. We prove that lemma 2 follows from the local construction by a partition of unity argument.

Lemma 15 *Let (M, g) be a simple $C^{1,1}$ manifold and suppose that $f \in C^{1,1}(M)$ is a symmetric m-tensor field on M so that in $If = 0$. For each $x \in \partial M$, there is a neighbourhood $W \subseteq M$ of x and a symmetric $(m-1)$-tensor field $p \in C^{1,1}(W)$ so that $p|_{W\cap\partial M} = 0$ and $\sigma\nabla p|_{W\cap\partial M} = f|_{W\cap\partial M}$.*

Proof Let $x_0 \in \partial M$ be a boundary point. Choose a neighbourhood $W \subseteq M$ of x_0, where we have C^∞ coordinates $\phi\colon W \to \mathbb{R}^n$ so that

$$\phi(W \cap \partial M) = \{x^n = 0\} \quad \text{and} \quad \phi(W \cap M^{\text{int}}) = \{x^n > 0\}. \tag{53}$$

The smooth coordinate function ϕ exists, since M is a smooth manifold with a smooth boundary. Denote $\hat{x}:=(x^1, \dots, x^{n-1})$ so that $x = (\hat{x}, x^n)$.

In these coordinates,the required tensor field p can be defined in the following way. Given $l \in \{0, \dots, m-1\}$ and $j_1, \dots, j_l \in \{1, \dots, n-1\}$,we let the component of p corresponding to the indices $j_1 \cdots j_l n \cdots n$ be

$$p_{j_1 \cdots j_l n \cdots n}(\hat{x}, x^n):=\frac{m}{m-l} x^n f_{j_1 \cdots j_l n \cdots n}(\hat{x}, 0). \tag{54}$$

Here the index n appears $m-1-l$ times in $p_{j_1 \cdots j_l n \cdots n}$ and $m-l$ times in $f_{j_1 \cdots j_l n \cdots n}$. We can insist that p is symmetric by requiring

$$p_{j_1 \cdots j_{m-1}}(\hat{x}, x^n) = p_{j_{\pi(1)} \cdots j_{\pi(m-1)}}(\hat{x}, x^n), \tag{55}$$

where π is any permutation of $\{1, \dots, m-1\}$ so that $j_{\pi(1)} \leq \cdots \leq j_{\pi(m-1)}$. This causes no contradictions, since f is symmetric. Clearly, it holds that $p|_{x^n=0} = 0$ and $p \in C^{1,1}(M)$ since $f \in C^{1,1}(M)$.

It remains to show that $\sigma \nabla p|_{x^n=0} = f|_{x^n=0}$, which follows from two claims:

(1) We prove $f_{j_1 \cdots j_m}(\hat{x}, 0) = 0$ in the coordinates in W when $j_1, \dots, j_m \in \{1, \dots, n-1\}$.
(2) We verify that $(\sigma \nabla p)_{j_1 \dots j_m}|_{x^n=0} = f_{j_1 \dots j_m}|_{x^n=0}$ in the coordinates in W.

Both claims are proved in appendix A. The idea is that item 1 follows from the fact $If = 0$, and item 2 can then be verified by a straightforward computation in the coordinates in W. □

Proof of lemma 2 Let $f \in C^{1,1}(M)$ be a symmetric m-tensor field with $If = 0$. We construct a symmetric $(m-1)$-tensor field $p \in C^{1,1}(M)$ so that $p|_{\partial M} = 0$ and $\sigma \nabla p|_{\partial M} = f|_{\partial M}$.

For each $x \in \partial M$ pick a neighbourhood $W_x \subseteq M$ of x and a symmetric $(m-1)$-tensor field $p_x \in C^{1,1}(W_x)$. Such neighbourhoods W_x and tensor fields p_x exist by lemma 15. Since ∂M is compact, there is a finite subcover $\{W_{x_i}\}_{i=1}^k$ of the open cover $\{W_x\}_{x \in \partial M}$ of ∂M. Denote $W_i:=W_{x_i}$ and $p_i:=p_{x_i}$. We add $W_0:=M \setminus \partial M$ to get a finite open cover of M. Choose a partition of unity $\{\psi_i\}_{i=1}^n \cup \{\psi_0\}$ subordinate to $\{W_i\}_{i=1}^n \cup \{W_0\}$. We let the tensor field p_0 corresponding to W_0 to be identically zero. The products $\psi_i p_i$ are $C^{1,1}$ tensor fields in neighbourhoods W_i and we can extend them by zero outside W_i to get $C^{1,1}$ tensor fields on M since each $W_i \setminus \operatorname{supp} \psi_i$ is open. We define an $(m-1)$-tensor field p by

$$p(x) = \sum_{i=0}^{n} \psi_i(x) p_i(x). \tag{56}$$

Since $\psi_i p_i$ are zero outside $\operatorname{supp}\psi_i$ and $p_i|_{\partial M\cap \operatorname{supp}\psi_i} = 0$ by construction, we see that $p|_{\partial M} = 0$. The final step is to check that $\sigma\nabla p = f$ on the boundary ∂M. By the product rule,we have $\nabla(\psi_i p_i) = \nabla\psi_i \otimes p_i + \psi_i(\nabla p_i)$ for all i. Since symmetrization commutes with multiplication by a scalar function and ψ_i is a scalar, we have

$$\sigma\nabla p = \sum_{i=0}^{n}[\sigma((\nabla\psi_i)\otimes p_i) + \psi_i\sigma(\nabla p_i)]. \tag{57}$$

Since symmetrization and tensor product commute with pointwise evaluations,we have $\sigma((\nabla\psi_i)\otimes p_i)|_{\partial M} = 0$. Since $\psi_i = 0$ in $M\setminus \operatorname{supp}\psi_i$ we have $\sigma\nabla\psi_i = 0$ in the same open set $M\setminus \operatorname{supp}\psi_i$. Together with $p_i = 0$ on $\partial M \cap \operatorname{supp}\psi_i \subseteq \partial M \cap W_i$, vanishing of the covariant derivative $\sigma\nabla\psi_i$ in $M\setminus \operatorname{supp}\psi_i$ implies

$$\begin{aligned}\sigma\nabla p|_{\partial M} &= \sum_{i=0}^{n}(\psi_i(\sigma\nabla p_i))|_{\partial M} = \sum_{i=0}^{n}\psi_i(\sigma\nabla p_i|_{\partial M\cap W_i})\\ &= \sum_{i=0}^{n}\psi_i(f|_{\partial M\cap W_i}) = \sum_{i=0}^{n}(\psi_i f)|_{\partial M} = f|_{\partial M}.\end{aligned} \tag{58}$$

Thus p has the desired properties. □

4.2 Regularity of the Integral Function

Let (M, g) be a simple $C^{1,1}$ manifold and let $f \in C^{1,1}(M)$ be a symmetric m-tensor field with $If = 0$. Since the main objective is to prove that there is a symmetric $(m-1)$-tensor field p on M so that $\sigma\nabla p = f$ and by lemma 2, we can find a tensor field $p \in C^{1,1}(M)$ with this property on the boundary ∂M, we can move to studying tensor fields $f \in \mathrm{Lip}_0(M)$ vanishing on the boundary. The following lemma is a special case of [18, Lemma 21]. We record it for the convenience of the reader.

Lemma 16 *Let (M, g) be a simple $C^{1,1}$ manifold. Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field on M and let $u{:=}u^f$ be the integral function of f defined by (3). Then $u \in \mathrm{Lip}(SM)$.*

Proof Since f is in $\mathrm{Lip}_0(M)$ the corresponding function on the sphere bundle is in $\mathrm{Lip}_0(SM)$. It was shown in [18, Lemma 21] that the integral function of a function in $\mathrm{Lip}_0(SM)$ is again a Lipschitz function on SM. □

Next we prove lemma 7 which states that if a Lipschitz function u on SM arising from of tensor field $-p$ satisfies the transport equation $Xu = -f$, then $\sigma\nabla p = f$ holds pointwise almost everywhere.

Proof of lemma 7 Let $f \in \mathrm{Lip}(M)$ is a symmetric m-tensor field. Suppose that $p \in \mathrm{Lip}(M)$ is a symmetric m-tensor field so that the Lipschitz function $u{:=}-\lambda p$ solves the transport equation $Xu = -f$ everywhere in SM. We prove that $\sigma\nabla p = f$ almost everywhere on SM by proving that

$$(\sigma\nabla p - f, \eta)_{L^2(M)} = 0 \tag{59}$$

for all symmetric m-tensor fields $\eta \in C_0^1(M)$. Since by proposition 11, there are positive constants $c, C > 0$ so that

$$c\,(\lambda h_1, \lambda h_2)_{L^2(SM)} \leq (h_1, h_2)_{L^2(M)} \leq C\,(\lambda h_1, \lambda h_2)_{L^2(SM)} \tag{60}$$

for all symmetric m-tensor fields $h_1, h_2 \in \mathrm{Lip}(M)$ it is enough to prove that

$$(\lambda\sigma\nabla p - \lambda f, \lambda\eta)_{L^2(SM)} = 0. \tag{61}$$

Consider a maximal geodesic γ of M so that $\gamma(0) = x \in \partial M$ and $\dot\gamma(0) = v \in \partial_{\text{in}}(SM)$. We denote $z{:=}(x, v)$ and write $\eta{:=}\lambda\eta$ and $f{:=}\lambda f$. Furthermore, we denote $\theta(t){:=}\phi_t(z)$ and $\eta(t){:=}\eta(\theta(t))$. Then we have

$$\int_0^{\tau(z)} (\lambda\sigma\nabla p)(\theta(t))\eta(t)\,\mathrm{d}t = \int_0^{\tau(z)} (\nabla p)_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\eta(t)\,\mathrm{d}t. \tag{62}$$

Since γ is a geodesic, it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$. Therefore,the Leibniz rule implies

$$\begin{aligned}\int_0^{\tau(z)} (\nabla p)_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\eta(t)\,\mathrm{d}t &= \int_0^{\tau(z)} \partial_t(p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t)))\eta(t)\,\mathrm{d}t \\ &= -\int_0^{\tau(z)} p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\partial_t\eta(t)\,\mathrm{d}t.\end{aligned} \tag{63}$$

By assumption $u(\theta(t)) = -p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))$ for all $t \in [0, \tau(z)]$ and thus

$$\begin{aligned}-\int_0^{\tau(z)} p_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\partial_t\eta(t)\,\mathrm{d}t &= \int_0^{\tau(z)} u(\theta(t))\partial_t\eta(t)\,\mathrm{d}t \\ &= -\int_0^{\tau(z)} \partial_t u(\theta(t))\eta(t)\,\mathrm{d}t \\ &= \int_0^{\tau(z)} f(\theta(t))\eta(t)\,\mathrm{d}t,\end{aligned} \tag{64}$$

where the last equality holds since $Xu = -f$ and X is the infinitesimal generator of the geodesic flow ϕ_t. Together, equations (62), (63) and (64) show that

$$\int_0^{\tau(z)} (\lambda\sigma\nabla p)(\theta(t))\eta(t)\,\mathrm{d}t = \int_0^{\tau(z)} f(\theta(t))\eta(t)\,\mathrm{d}t. \tag{65}$$

We integrate (65) over $\partial_{\text{in}}(SM)$ and use Santaló's formula (lemma 24) to see that

$$\begin{aligned}\int_{SM}(\lambda\sigma\nabla p)\eta\,\mathrm{d}\Sigma_g &= \int_{\partial_{\text{in}}(SM)}\int_0^{\tau(z)}(\lambda\sigma\nabla p)(\theta(t))\eta(t)\,\mathrm{d}t\,\mu\mathrm{d}\Sigma_{j^*g}\\ &= \int_{\partial_{\text{in}}(SM)}\int_0^{\tau(z)} f(\theta(t))\eta(t)\,\mathrm{d}t\,\mu\mathrm{d}\Sigma_{j^*g}\\ &= \int_{SM} f\eta\,\mathrm{d}\Sigma_g.\end{aligned} \tag{66}$$

Equation (61) follows immediately from (66), which finishes the proof. □

4.3 Regularity of the Spherical Harmonic Components

In this subsection,we use the special form of spherical harmonics and the identification of trace-free tensor fields and spherical harmonics to prove lemma 3. Also, we prove that the degreewise definition of operators $X_\pm$ acting on functions on SM is reasonable by proving that series in (47) converge absolutely in $L^2(SM)$.

Proof of lemma 3 Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field with vanishing X-ray transform and let $u:=u^f$ be the integral function of f defined by (3). The integral function u is in $\mathrm{Lip}(SM)$ by lemma 16. We prove that the spherical harmonic components u_k of u are in $\Omega_{\text{h}}^{0,1}\Omega_{\text{v}}^{\infty}(k)$ and that $u_k|_{\partial(SM)} = 0$.

For a fixed $x \in M$,the fibre S_xM is isometric to the Euclidean unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ via the map

$$s_x\colon S_xM \to S^{n-1}, \quad s_x(v) = g(x)^{1/2}v, \tag{67}$$

where $g(x)^{1/2}$ is the unique square root of a positive definite matrix $g(x)$. Since u is in $\mathrm{Lip}(SM)$, its restriction $u_x:=u(x,\,\cdot\,)$ to S_xM is in $\mathrm{Lip}(S_xM)$. Thus the functions $\tilde{u}_x$ on S^{n-1} corresponding to u_x via s_x has a decomposition

$$\tilde{u}_x = \sum_{k=0}^{\infty}(\tilde{u}_x, \phi_k)_{L^2(S^{n-1})}\,\phi_k, \tag{68}$$

where ϕ_k is the eigenfunction of the Laplacian on S^{n-1} corresponding to the eigenvalue $k(k+n-2)$. Tracing back through s_x,we find a $L^2(S_xM)$ convergent decomposition

$$u_x = \sum_{k=0}^{\infty}(u_x, \psi_k)_{L^2(S_xM)}\,\psi_k, \tag{69}$$

where $\psi_k(v) = \phi_k(s_x^{-1}(v))$. On the level of the bundle SM, we denote $\psi_k(x,v):=\phi_k(s_x^{-1}(v))$, and thus get the formula $u_k = (u,\psi_k)_{L^2(S_xM)}\,\psi_k$. Here ψ_k is in $C^{1,1}(SM)$, since ϕ_k is in $C^\infty(S^{n-1})$ and the map $(x,v)\mapsto s_x(v)$ is in $C^{1,1}(SM)$. This proves that $u_k \in \mathrm{Lip}(SM)$. We note that by lemma 11 for all k,there is a symmetric and trace-free k-tensor field $h_k \in \mathrm{Lip}(M)$ so that $u_k(x,v) = (h_k)_{j_1\cdots j_k}(x)v^{j_1}\cdots v^{j_k}$. This proves that $u_k \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ for all k, since u_k is polynomial in v.

Finally, we prove that $u_k|_{\partial(SM)} = 0$. Since the X-ray transform of f is zero, the restriction of u on the boundary $\partial(SM)$ is zero. Thus for any $x \in \partial M$ we have

$$0 = \|u(x,\cdot)\|^2_{L^2(S_xM)} = \sum_{k=0}^{\infty} \|u_k(x,\cdot)\|^2_{L^2(S_xM)}. \tag{70}$$

Therefore, since $u_k(x,\cdot) \in C^{\infty}(S_xM)$, we have $u_k(x,\cdot) = 0$ pointwise on S_xM for all k, which implies that $u_k|_{\partial(SM)} = 0$ for all k. □

Lemma 17 *Let (M,g) be a simple $C^{1,1}$ manifold. Given $u \in H_{\mathrm{h}}^1H_{\mathrm{v}}^2(SM)$, if $u = \sum_{k=0}^{\infty} u_k$ is the spherical harmonic decomposition of u, then the series $\sum_{k=0}^{\infty} X_{\pm}u_k$ converge absolutely in $L^2(SM)$. Here we use the convention that $X_-u_0 = 0$.*

Proof We prove convergence of both of series $\sum_{k=0}^{\infty} X_{\pm}u_k$ at once by proving that

$$\sum_{k=0}^{\infty} \|X_+u_k\|^2_{L^2(SM)} + \sum_{k=1}^{\infty} \|X_-u_k\|^2_{L^2(SM)} \le \|u\|^2_{H_{\mathrm{h}}^1H_{\mathrm{v}}^0(SM)}. \tag{71}$$

The proof of (71) is identical to the proofs of [43, Lemma 4.4] and [26, Lemma 5.1], where the authors proved that

$$\|X_+u\|^2_{L^2(SM)} + \|X_-u\|^2_{L^2(SM)} \le \|Xu\|^2_{L^2(SM)} + \left\|\overset{\mathrm{h}}{\nabla}u\right\|_{L^2(SM)}. \tag{72}$$

The major difference to the results in [43] and [26] is that we work in non-smooth geometry instead of a smooth geometry, so the tools in the proof have changed. For completeness, we repeat the arguments in appendix B to document the fact that all steps go through in lower regularity with suitably chosen function spaces. □

Remark 18 For $u \in H_{\mathrm{h}}^1H_{\mathrm{v}}^2(SM)$,we defined $X_{\pm}u$ to be the series $\sum_{k=0}^{\infty} X_{\pm}u_k$, when $u = \sum_{k=0}^{\infty} u_k$ is the spherical harmonic decomposition of u. By lemma 17 both X_+u and X_-u are well- defined functions in $L^2(SM)$ and by orthogonality

$$\|X_{\pm}u\|^2_{L^2(SM)} = \sum_{k=0}^{\infty} \|X_{\pm}u_k\|^2_{L^2(SM)}. \tag{73}$$

5 Energy Estimates and a Santaló Formula

In this section, we show that the L^2-estimate in lemma 5 follows from the Pestov identity, and we establish the Santaló's formula in low regularity in lemma 24.

5.1 Pestov Energy Identity

Let (M, g) be a simple $C^{1,1}$ manifold. Recall that the global index form Q of (M, g) is defined by

$$Q(W) := \|XW\|_{L^2(N)} - (RW, W)_{L^2(N)} \tag{74}$$

for $W \in H_0^1(N, X)$.

Lemma 19 (Pestov identity) *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. If $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and $u|_{\partial(SM)} = 0$, then*

$$\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|_{L^2(N)}^2 = Q\left(\overset{\mathrm{v}}{\nabla} u\right) + (n-1)\,\|Xu\|_{L^2(SM)}^2 . \tag{75}$$

Proof Since $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$, we have $u \in \mathrm{Lip}_0(SM)$, $\overset{\mathrm{v}}{\nabla} Xu \in L^2(N)$ and $X\overset{\mathrm{v}}{\nabla} u \in L^2(N)$. It was proved in [18, Lemma 9] that the Pestov identity (75) holds for this class of functions on simple $C^{1,1}$ manifolds. □

When $g \in C^{\infty}$, the estimate in Lemma 20 was derived in [20, Section 6]. We present a proof compatible with low regularity employing the Pestov identity in Lemma 19.

Lemma 20 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. If $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and $u|_{\partial(SM)} = 0$, then*

$$\left(Xu, [X, \overset{\mathrm{v}}{\Delta}]u\right)_{L^2(SM)} \leq 0. \tag{76}$$

Proof Since the sectional curvature of (M, g) is almost everywhere non-positive, $Q(W) \geq \|XW\|^2$ for all $W \in H_0^1(N, X)$ and we have

$$\left\| \overset{\mathrm{v}}{\nabla} Xu \right\|_{L^2(N)}^2 \geq \left\| X\overset{\mathrm{v}}{\nabla} u \right\|_{L^2(N)}^2 + (n-1)\,\|Xu\|_{L^2(SM)}^2 \tag{77}$$

by the Pestov identity (lemma 19). On the other hand, using commutator formulas from proposition 10,we see that

$$\begin{aligned} \left\| X\overset{\mathrm{v}}{\nabla} u \right\|^2 &= \left\| \overset{\mathrm{v}}{\nabla} Xu - \overset{\mathrm{h}}{\nabla} u \right\|^2 \\ &= \left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2 - 2\left(\overset{\mathrm{v}}{\nabla} Xu, \overset{\mathrm{h}}{\nabla} u\right) + \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 \\ &= \left\| \overset{\mathrm{v}}{\nabla} Xu \right\|^2 + \left(Xu, 2\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{h}}{\nabla} u\right) + \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 . \end{aligned} \tag{78}$$

Combining estimate (77) and equation (78) and applying the commutator formula (37),we get

$$
\begin{aligned}
0 &\geq \left(Xu, 2\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{h}}{\nabla}u\right) + \left\|\overset{\mathrm{h}}{\nabla}u\right\|^2 + (n-1)\,\|Xu\|^2 \\
&\geq \left(Xu, 2\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{h}}{\nabla}u + (n-1)Xu\right) \\
&= \left(Xu, [X, \overset{\mathrm{v}}{\Delta}]u\right)
\end{aligned}
\tag{79}
$$

as claimed. □

Lemma 21 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Suppose that $f \in \mathrm{Lip}_0(M)$ is a symmetric m-tensor field on M with vanishing X-ray transform If. Let $u:=u^f$ be the integral function of f defined by (3). If $k \geq m$ or $k \equiv m \pmod 2$, we have*

$$
\|X_+u_k\|^2_{L^2(SM)} = \|X_-u_{k+2}\|^2_{L^2(SM)}\,. \tag{80}
$$

Proof Since $f \in \mathrm{Lip}_0(M)$ and the X-ray transform of f vanishes, we have $u \in \mathrm{Lip}_0(SM)$ by lemma 16. By the fundamental theorem of calculus u solves $Xu = -f$. Projecting this transport equation onto spherical harmonic degree $k+1$ gives

$$
-f_{k+1} = X_+u_k + X_-u_{k+2}. \tag{81}
$$

If $k \geq m$ or $k \equiv m \pmod 2$, then $f_{k+1} = 0$ and the claim (80) follows by taking L^2-norms. □

Recall that the constants $C(n, k)$ and $B(n, l, k)$ in lemma 5 are

$$
C(n, k):=\frac{2k+n-1}{2k+n-3} \quad \text{and} \quad B(n, l, k):=\prod_{p=1}^{l} C(n, k+2p). \tag{82}
$$

Lemma 22 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Suppose that $f \in \mathrm{Lip}_0(M)$ is a symmetric m-tensor field with $If = 0$. Let $u:=u^f$ be integral function of f defined by (3). If $2k+n-3 > 0$, we have*

$$
\|X_-u_k\|^2_{L^2(SM)} \leq C(n, k)\,\|X_+u_k\|^2_{L^2(SM)}\,, \tag{83}
$$

where u_k are the spherical harmonic components of u.

Proof Let $2k+n-3>0$. Since $u_k \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ by lemma 3, we can use lemma 20, which together with commutator formulas in 14 gives

$$\begin{aligned}(2k+n-1)\,\|X_+u_k\|^2 &\geq (2k+n-1)\,\|X_+u_k\|^2 + \left(Xu_k, [X, \overset{\mathrm{v}}{\Delta}]u_k\right)\\ &= (2k+n-1)\,\|X_+u_k\|^2 + \left(X_+u_k, [X_+, \overset{\mathrm{v}}{\Delta}]u_k\right)\\ &\quad + \left(X_-u_k, [X_-, \overset{\mathrm{v}}{\Delta}]u_k\right)\\ &= (2k+n-3)\,\|X_-u_k\|^2 .\end{aligned} \tag{84}$$

Dividing by $2k+n-3>0$ proves the claimed estimate (83). □

Proof of lemma 5 Let $f \in \mathrm{Lip}_0(M)$ be a symmetric m-tensor field so that $If=0$ and denote by $u:=u^f$ its integral function defined by (3). Let $k \geq m$. By lemma 3,we have $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ and thus lemmas 21 and 22,we get

$$\|X_+u_k\|^2_{L^2(SM)} = \|X_-u_{k+2}\|^2_{L^2(SM)} \leq C(n,k+2)\,\|X_+u_{k+2}\|^2_{L^2(SM)} . \tag{85}$$

Iterating lemmas 21 and 22 a total of $l \in \mathbb{N}$ times yields

$$\|X_+u_k\|^2 \leq \|X_+u_{k+2l}\|^2 \prod_{p=1}^{l} C(n,k+2p) = B(n,l,k)\,\|X_+u_{k+2l}\|^2 \tag{86}$$

as claimed. □

5.2 Santaló's Formula

The proof of Santaló's formula on a smooth simple manifolds (M,g) is based on the so-called Liouville's theorem and can be found e.g. in [36]. We give a similar proof of the formula on a simple $C^{1,1}$ manifold based on the following formulation of Liouville's theorem.

Lemma 23 *Let (M,g) be a simple $C^{1,1}$ manifold. Denote by L_X the Lie derivative into the direction of the geodesic vector field X on SM. Then for any $u \in \mathrm{Lip}(SM)$ it holds that*

$$\int_{SM} uL_X(\mathrm{d}\Sigma_g) = 0. \tag{87}$$

The proof of lemma 23 is based on smooth approximation of the Riemannian metric g and can be found in Appendix C.

If ν is the inner unit normal vector field to ∂M, let $\mu(x,v):=\langle \nu(x), v\rangle_{g(x)}$ for all $(x,v)\in SM$. If ω is a differential k-form on SM, then denote by $i_X\omega$ the contraction of ω with the geodesic vector field X. That is, for any vector fields $Y_1, \dots, Y_{k-1}$ on SM, we define $i_X\omega$ by letting $i_X\omega(Y_1, \dots, Y_{k-1}) = \omega(X, Y_1, \dots, Y_{k-1})$.

Lemma 24 (Santaló's formula) *Let (M, g) be a simple $C^{1,1}$ manifold. For any function $f \in \mathrm{Lip}_0(SM)$ the integral of f over SM with respect to $\mathrm{d}\Sigma_g$ can be written as*

$$\int_{SM} f\, d\Sigma_g = \int_{\partial_{\mathrm{in}} SM} \int_0^{\tau(z)} f(\phi_t(z))\,\mathrm{d}t\, \mu(z)\mathrm{d}\Sigma_{j^*g}. \tag{88}$$

Here $j\colon \partial(SM) \to SM$ is the inclusion map and j^*g is the Riemannian metric of ∂M induced by the inclusion j.

Proof Let $f \in \mathrm{Lip}_0(SM)$ and consider its integral function $u{:=}u^f$. The integral function satisfies $Xu = -f$ and $u \in \mathrm{Lip}(SM)$ by lemma 16. By Cartan's formula, we have

$$\int_{SM} L_X(u\,\mathrm{d}\Sigma) = \int_{SM} i_X d(u\,\mathrm{d}\Sigma) + \int_{SM} d(i_X u\,\mathrm{d}\Sigma), \tag{89}$$

where d is the exterior derivative. Since $u\,\mathrm{d}\Sigma$ is a volume form, the first term on the right in (89) vanishes. By Stoke's theorem

$$\int_{SM} d(i_X u\,\mathrm{d}\Sigma_g) = \int_{\partial(SM)} j^*(u i_X \mathrm{d}\Sigma_g). \tag{90}$$

As in the smooth case ([36, Proposition 3.6.6.]), we compute that

$$\begin{aligned}\int_{\partial(SM)} j^*(u i_X \mathrm{d}\Sigma_g) &= \int_{SM} (j^*u)(j^* i_X \mathrm{d}\Sigma_g)\\ &= \int_{SM} (j^*u)\,\langle X, \nu\rangle\,\mathrm{d}\Sigma_{j^*g}\\ &= \int_{SM} (j^*u)\mu\,\mathrm{d}\Sigma_{j^*g}.\end{aligned} \tag{91}$$

Finally, since j^*u is merely a restriction to the boundary, we invoke the definition of u and lemma 23 to see that

$$\begin{aligned}\int_{SM} f\,\mathrm{d}\Sigma_g &= \int_{SM} L_X(u)\,\mathrm{d}\Sigma\\ &= \int_{SM} L_X(u\,\mathrm{d}\Sigma) - \int_{SM} u L_X(\mathrm{d}\Sigma)\\ &= \int_{SM} L_X(u\,\mathrm{d}\Sigma) = \int_{\partial(SM)} (j^*u)\mu\,\mathrm{d}\Sigma_{j^*g}\\ &= \int_{\partial(SM)} \int_0^{\tau(z)} f(\phi_t(z))\,\mathrm{d}t \mu\,\mathrm{d}\Sigma_{j^*g}.\end{aligned} \tag{92}$$

Since $\tau(z) = 0$ for $z \notin \partial_{\text{in}}(SM)$, the claim (88) follows at once from (92). □

6 Friedrich's Inequalities

In this section, we prove that L^2-norms of scalar functions on SM and sections of the bundle N are bounded above by constant multiples of L^2-norms of their derivatives along the geodesic flow. We call these estimates Friedrich's inequalities on SM. We apply the inequalities to prove lemma 6.

Lemma 25 *Let (M, g) be a simple $C^{1,1}$ manifold with almost everywhere non-positive sectional curvature. Let d be the diameter of M. Then*

$$d^2 \|Xu\|^2_{L^2(SM)} \geq \|u\|^2_{L^2(SM)} \quad \textit{and} \quad d^2 \|XW\|^2_{L^2(N)} \geq \|W\|^2_{L^2(N)} \tag{93}$$

for any $u \in H^1_0(SM)$ and $W \in H^1_0(N, X)$.

Proof First, we prove the inequality for functions. By density it is enough to consider the case $u \in C^1_0(SM)$. By Santaló's formula (lemma 24), we can write

$$\|Xu\|^2_{L^2(SM)} = \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} |Xu(\phi_t(z)|^2 \, \mathrm{d}t \, \mu \mathrm{d}\Sigma_{j^*g}, \tag{94}$$

where $j\colon \partial(SM) \to SM$ is the inclusion. Let us denote $u_z(t) := u(\phi_t(z))$. Then $u_z \in H^1_0([0, \tau(z)])$ and we have

$$Xu(\phi_t(z)) = \frac{\mathrm{d}}{\mathrm{d}s} u(\phi_{t+s}(z)) \Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} u_z(t+s) \Big|_{s=0} = \dot{u}_z(t). \tag{95}$$

By the usual Friedrich's inequality of $H^1_0([0, \tau(z)])$, we see that

$$d^2 \int_0^{\tau(z)} |\dot{u}_z(t)|^2 \, \mathrm{d}t \geq \tau(z)^2 \int_0^{\tau(z)} |\dot{u}_z(t)|^2 \, \mathrm{d}t \geq \int_0^{\tau(z)} |u_z(t)|^2 \, \mathrm{d}t. \tag{96}$$

Combining equation (95) with inequality (96), we get

$$\begin{aligned} d^2 \|Xu\|^2_{L^2(SM)} &\geq d^2 \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} |\dot{u}_z(t)|^2 \, \mathrm{d}t \, \mu \mathrm{d}\Sigma_{j^*g} \\ &\geq \int_{\partial_{\text{in}}(SM)} \int_0^{\tau(z)} |u_z(t)|^2 \, \mathrm{d}t \, \mu \mathrm{d}\Sigma_{j^*g} \\ &= \|u\|^2_{L^2(SM)}, \end{aligned} \tag{97}$$

which is the claimed inequality for functions.

Next, we prove the inequality for sections of the bundle N. Let $W \in H_0^1(N, X)$. In this case, Santaló's formulas (lemma 24) gives

$$\|XW\|^2_{L^2(SM)} = \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} |XW(\phi_t(z))|_g^2 \, \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)}. \tag{98}$$

We let $W_z(t) := W(\phi_t(z))$. Then $W_z(t)$ is a H_0^1 vector field along γ_z and it holds that $XW(\phi_t(z)) = D_t W_z(t)$. Choose a parallel frame $(E_1, \dots, E_n)$ along γ_z. Then we have $D_t W_z = \dot{W}_z^i E_i$, when $W_z = W_z^i E_i$. Since W_z is a H_0^1 vector field along γ_z we have $W_z^i \in H_0^1([0, \tau(z)])$ for all i. Thus we read from equation (96) that

$$d^2 \int_0^{\tau(z)} \left|\dot{W}_z^i\right|^2 \mathrm{d}t \geq \int_0^{\tau(z)} \left|W_z^i\right|^2 \mathrm{d}t. \tag{99}$$

From equations (98) and (99) we see that

$$\begin{aligned} d^2 \|XW\|^2_{L^2(N)} &= d^2 \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} |D_t W_z(t)|_g^2 \, \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)} \\ &= d^2 \sum_{i=1}^n \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} \left|\dot{W}_z^i(t)\right|^2 \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)} \\ &\geq \sum_{i=1}^n \int_{\partial_{\mathrm{in}}(SM)} \int_0^{\tau(z)} \left|W_z^i(t)\right|^2 \mathrm{d}t \, \mu(z) \, \mathrm{d}\Sigma_{\partial(SM)} \\ &= \|W\|^2_{L^2(N)}, \end{aligned} \tag{100}$$

which is the second claimed inequality. □

Proof of lemma 6 Let $u \in \Omega_{\mathrm{h}}^{0,1}\Omega_{\mathrm{v}}^{\infty}(k)$ be so that $u|_{\partial(SM)} = 0$ and $X_+ u = 0$. By lemma 14,we have

$$\begin{aligned} (2k + n - 3)\|X_- u\|^2 &= -(2k + n - 1)\|X_+ u\|^2 + (2k + n - 3)\|X_- u\|^2 \\ &= \left([X_+, \overset{\mathrm{v}}{\Delta}]u, X_+ u\right) + \left([X_-, \overset{\mathrm{v}}{\Delta}]u, X_- u\right) \\ &= \left([X_+, \overset{\mathrm{v}}{\Delta}]u, Xu\right) + \left([X_-, \overset{\mathrm{v}}{\Delta}]u, Xu\right) \\ &= \left([X, \overset{\mathrm{v}}{\Delta}]u, Xu\right). \end{aligned} \tag{101}$$

The last inner product in (101) is non-positive by lemma 20. Thus $X_- u = 0$ almost everywhere on SM. Let d be the diameter of M. Lemma 25 then provides

$$\|u\|^2_{L^2(SM)} \leq d^2 \|Xu\|^2_{L^2(SM)} = d^2(\|X_+ u\|^2_{L^2(SM)} + \|X_- u\|^2_{L^2(SM)}) = 0. \tag{102}$$

Thus $u = 0$ almost everywhere on SM, but since u is continuous, we have shown that $u = 0$ everywhere on SM. □

Even though we do not need the result, we next show for completeness that there are no conjugate points in the sense of the global index form Q when the sectional curvature is non-positive.

Proposition 26 *Let M be the closed Euclidean unit ball in $\mathbb{R}^n$. Suppose that M comes equipped with a $C^{1,1}$ Riemannian metric g so that the sectional curvature of (M, g) is almost everywhere non-positive. Then there is $\varepsilon > 0$ so that $Q(W) \geq \varepsilon \|W\|^2_{L^2(N)}$ for all $W \in H^1_0(N, X)$.*

Proof Since the sectional curvature is almost everywhere non-positive,

$$(RW, W)_{L^2(N)} = \int_{(x,v)\in SM} \langle R(W(x, v), v)v, W(x, v)\rangle_g \, \mathrm{d}\Sigma_g \leq 0 \tag{103}$$

for all $W \in H^1_0(N, X)$, since $W(x, v)$ and v are always orthogonal. Thus $Q(W) \geq \|XW\|^2_{L^2(N)}$ for all $W \in H^1_0(N, X)$. Then it follows from lemma 25 that for all $W \in H^1_0(N, X)$,we have

$$Q(W) \geq \|XW\|^2_{L^2(N)} \geq \frac{1}{d^2} \|W\|^2_{L^2(N)} . \tag{104}$$

We take $\varepsilon = 1/d^2$ which finishes the proof. □

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Appendix A: Completion of the Proof of Boundary Determination

We complete the details in the proof of lemma 15 by proving items 1 and 2. Recall that we work in local coordinates $\phi \colon W \to \mathbb{R}^n$ so that

$$\phi(W \cap \partial M) = \{x^n = 0\}, \quad \text{and} \quad \phi(W \cap M^{\text{int}}) = \{x^n > 0\}. \tag{105}$$

We denote $\hat{x} = (x^1, \dots, x^{n-1})$. The local tensor field p is defined in these coordinates by

$$p_{j_1 \cdots j_l n \cdots n}(\hat{x}, x^n) = \frac{m}{m-l} x^n f_{j_1 \cdots j_l n \cdots n}(\hat{x}, 0), \tag{106}$$

where n appears $m-1-l$ times in $p_{j_1 \cdots j_l n \cdots n}$ and $m-l$ times in $f_{j_1 \cdots j_l n \cdots n}$.

First we prove item 1. We begin by proving that $f_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$ and $x \in W \cap \partial M$. Given $v \in S_x(W \cap \partial M)$,we choose a sequence (v_k) of vectors $v_k \in S_x(W \cap \partial M)$ so that $\tau(x, v_k) > 0$, and $\tau(x, v_k) \to 0$ and $v_k \to v$ when $k \to \infty$. Such a sequence of vectors exists by $C^{1,1}$ simplicity as proved in [18, Lemma 23]. Since the lengths of the geodesics corresponding to (x, v_k) become arbitrarily short and $If = 0$, we find that

$$\begin{aligned} f_x(v, \dots, v) &= \lim_{k \to \infty} \frac{1}{\tau(x, v_k)} \int_0^{\tau(x, v_k)} f(\phi_t(x, v_k))\, \mathrm{d}t \\ &= \lim_{k \to \infty} \frac{If(x, v_k)}{\tau(x, v_k)} \\ &= 0. \end{aligned} \tag{107}$$

We have shown that $f_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$. Next, we prove that $f_{j_1 \cdots j_m}(\hat{x}, 0) = 0$ in $W \cap \partial M$ for all $j_1, \dots, j_m \in \{1, \dots, n-1\}$.

Let $\iota \colon \partial M \to M$ be the inclusion map. The pullback $\iota^* f$ is an m-tensor field on ∂M. Since $f_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$ we have $(\iota^* f)_x(v, \dots, v) = 0$ for all $v \in S_x(W \cap \partial M)$. Then a fibrewise computation [9, Lemma 2.4] shows that

$$0 = \int_{W \cap \partial M} (\iota^* f)_x(v, \dots, v)^2 \, \mathrm{d}S_x = C_{m,n-1} \left| \iota^* f \right|^2_{g(x)} \tag{108}$$

for all $x \in W \cap \partial M$. We have shown that $\iota^* f|_{W \cap \partial M} = 0$ which written in the coordinates in W gives $f_{j_1 \cdots j_m}(\hat{x}, 0) = 0$ for all $j_1, \dots, j_m \in \{1, \dots, n-1\}$. We have proved item 1.

We proceed to proving item 2. Let $l \in \{0, \dots, m-1\}$ and $j_1, \dots, j_l \in \{1, \dots, n-1\}$. To compute the restriction to boundary of the component functions of $\sigma \nabla p$, we first compute $\nabla_n p_{j_1 \cdots j_l n \cdots n}(\hat{x}, 0)$ and $\nabla_{j_s} p_{j_1 \cdots \widehat{j_s} \cdots j_l n \cdots n}(\hat{x}, 0)$. We have

$$\begin{aligned} \nabla_n p_{j_1 \cdots j_l n \cdots n} &= \partial_n p_{j_1 \cdots j_l n \cdots n} \\ &\quad - \sum_{s=1}^{l} \Gamma^k_{n j_s} p_{j_1 \cdots k \cdots j_l n \cdots n} - \sum_{s=l+1}^{m-1} \Gamma^k_{nn} p_{j_1 \cdots j_l n \cdots k \cdots n}. \end{aligned} \tag{109}$$

Thus by the construction of p,we find that

$$\nabla_n p_{j_1\cdots j_l n\cdots n}(\hat{x}, x^n) = \frac{m}{m-l} f_{j_1\cdots j_l n\cdots nn}(\hat{x}, 0)$$
$$-\frac{m}{m-l} x^n \sum_{s=1}^{l} \Gamma^k_{nj_s} f_{j_1\cdots k\cdots j_l n\cdots nn}(\hat{x}, 0) - \frac{m}{m-l} x^n \sum_{s=l+1}^{m-1} \Gamma^k_{nn} f_{j_1\cdots j_l n\cdots k\cdots nn}(\hat{x}, 0). \quad (110)$$

On the boundary $\{x^n = 0\}$, equation (110) reduces to

$$\nabla_n p_{j_1\cdots j_l n\cdots n}(\hat{x}, 0) = \frac{m}{m-l} f_{j_1\cdots j_l n\cdots nn}(\hat{x}, 0). \quad (111)$$

As in equation (109), we have

$$\begin{aligned}\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n} &= \partial_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n} \\ &- \sum_{r=1}^{l-1} \Gamma^k_{j_s j_r} p_{j_1\cdots k\cdots j_l n\cdots n} \\ &- \sum_{r=l}^{m-1} \Gamma^k_{j_s j_r} p_{j_1\cdots j_l n\cdots k\cdots n}.\end{aligned} \quad (112)$$

By the construction of p, equation (112) gives

$$\begin{aligned}\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n}(\hat{x}, x^n) &= \frac{m}{m-l} x^n \partial_{j_s} f_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots nn}(\hat{x}, 0) \\ &- \frac{m}{m-l} x^n \sum_{r=1}^{l-1} \Gamma^k_{j_s j_r} f_{j_1\cdots k\cdots j_l n\cdots nn}(\hat{x}, 0) \\ &- \frac{m}{m-l} x^n \sum_{r=l}^{m-1} \Gamma^k_{j_s n} f_{j_1\cdots j_l n\cdots k\cdots nn}(\hat{x}, 0).\end{aligned} \quad (113)$$

Therefore, on the boundary $\{x^n = 0\}$,we get

$$\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n}(\hat{x}, 0) = 0. \quad (114)$$

Now we are ready to compute $(\sigma\nabla p)_{j_1\ldots j_l n\cdots n}$, when $l \in \{0, \ldots, m-1\}$. Denote $j_{l+1} = \cdots = j_m = n$. There are $(m-l)(m-1)!$ permutations π of $\{1, \ldots, m\}$ so that $j_{\pi(1)} = n$, when no restrictions are set on the remaining indices $j_{\pi(2)}, \ldots, j_{\pi(m)}$. Thus using symmetry of p we find that

$$(\sigma\nabla p)_{j_1\cdots j_l n\cdots n} = \frac{(m-l)(m-1)!}{m!}\nabla_n p_{j_1\cdots j_l n\cdots n} + \frac{(m-1)!}{m!}\sum_{s=1}^{l}\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n}$$
$$= \frac{m-l}{m}\nabla_n p_{j_1\cdots j_l n\cdots n} + \frac{1}{m}\sum_{s=1}^{l}\nabla_{j_s} p_{j_1\cdots\widehat{j_s}\cdots j_l n\cdots n}. \tag{115}$$

Evaluating (115) on the boundary $\{x^n = 0\}$ and substituting (111) and (114) results in

$$(\sigma\nabla p)_{j_1\ldots j_l n\cdots n}(\hat{x}, 0) = f_{j_1\cdots j_l n\cdots n}(\hat{x}, 0). \tag{116}$$

The last step is to prove that

$$(\sigma\nabla p)_{j_1\cdots j_m}(\hat{x}, 0) = f_{j_1\cdots j_m}(\hat{x}, 0) \tag{117}$$

when $j_1, \ldots, j_m \in \{1, \ldots, n-1\}$. By the definition of the symmetrized covariant derivative,

$$(\sigma\nabla p)_{j_1\cdots j_m} = \frac{1}{m!}\sum_{\pi}\nabla_{j_{\pi(1)}} p_{j_{\pi(2)}\cdots j_{\pi(m)}} \tag{118}$$

where the summation is over all permutations π of $\{1, \ldots, m\}$. Since $j_{\pi(k)} < n$ for all $k \in \{1, \ldots, m\}$, we can compute as in (113) to see that

$$\nabla_{j_{\pi(1)}} p_{j_{\pi(2)}\cdots j_{\pi(m)}}|_{x^n=0} = 0 \tag{119}$$

for all permutations π of $\{1, \ldots, m\}$. Thus

$$(\sigma\nabla p)_{j_1\cdots j_m}|_{x^n=0} = 0 = f_{j_1\cdots j_m}|_{x^n=0}. \tag{120}$$

We have finally used item 1 of the proof, where we proved that $f_{j_1\cdots j_m}(\hat{x}, 0) = 0$ for all $j_1, \ldots, j_m \in \{1, \ldots, n-1\}$. This concludes the proof item 2 and thus the proof of lemma 15 is completed.

Appendix B: A Regularity Computation

The following calculation completes the proof of lemma 3. It is based on the proofs of [43, Lemma 4.4] and [26, Lemma 5.1].

Let $u \in H^1_{\mathrm{h}} H^2_{\mathrm{v}}(SM)$ and let $w_k \in \Omega^1_{\mathrm{h}} \Omega^\infty_{\mathrm{v}}(k)$ be so that $w_k|_{\partial(SM)} = 0$. Then $\overset{\mathrm{h}}{\nabla} u \in H^1_{\mathrm{h}} H^1_{\mathrm{v}}(SM)$ and thus

$$\left(\overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w_k\right)_{L^2(N)} = -\left(\overset{\mathrm{v}}{\operatorname{div}} \overset{\mathrm{h}}{\nabla} u, w_k\right)_{L^2(N)}. \tag{121}$$

Using proposition 10,the right side can rewritten as

$$-\left(\overset{\mathrm{v}}{\operatorname{div}} \overset{\mathrm{h}}{\nabla} u, w_k\right) = -\frac{1}{2}\left([X, \overset{\mathrm{v}}{\Delta}]u, w_k\right) + \frac{n-1}{2}(Xu, w_k). \tag{122}$$

If $u_k \in \Lambda^1_{\mathrm{h}} \Lambda^2_{\mathrm{v}}(k)$ are the spherical harmonic components of u, then by orthogonality and lemma 14 we have

$$\begin{aligned} \left([X, \overset{\mathrm{v}}{\Delta}]u, w_k\right) &= \left([X_+, \overset{\mathrm{v}}{\Delta}]u_{k-1} + [X_-, \overset{\mathrm{v}}{\Delta}]u_{k+1}, w_k\right) \\ &= \left(-\frac{2k+n-3}{2} X_+ u_{k-1} + \frac{2k+n-1}{2} X_- u_{k+1}, w_k\right). \end{aligned} \tag{123}$$

Together, equations (121), (122) and (123) show that

$$\left(\overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w_k\right) = ((k+n-2)X_+ u_{k-1} - kX_- u_{k+1}, w_k). \tag{124}$$

Then we let $w \in C^1_{\mathrm{h}} C^2_{\mathrm{v}}(SM)$ so that $w|_{\partial(SM)} = 0$. If we decompose w into spherical harmonics w_k, then $w_k \in \Omega^1_{\mathrm{h}} \Omega^\infty_{\mathrm{v}}(k)$. We sum equation (124) over $k \in \mathbb{N}$ and use $k(k+n-2)w_k = \overset{\mathrm{v}}{\Delta} w_k$ to get

$$\begin{aligned} \left(\overset{\mathrm{h}}{\nabla} u, \overset{\mathrm{v}}{\nabla} w\right) &= \sum_{k=0}^{\infty} ((k+n-2)X_+ u_{k-1} + kX_- u_{k+1}, w_k) \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1}, \overset{\mathrm{v}}{\Delta} w_k\right) \\ &= \left(\sum_{k=0}^{\infty} \overset{\mathrm{v}}{\nabla}\left[\frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1}\right], \overset{\mathrm{v}}{\nabla} w_k\right). \end{aligned} \tag{125}$$

Thus there is $W(u) \in H^0_{\mathrm{h}} H^1_{\mathrm{v}}(N)$ so that $\overset{\mathrm{v}}{\operatorname{div}}(W(u)) = 0$ and

$$\overset{\mathrm{h}}{\nabla} u = \sum_{k=0}^{\infty} \overset{\mathrm{v}}{\nabla}\left[\frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1}\right] + W(u). \tag{126}$$

It follows from the eigenvalue property that

$$\left\| \overset{\mathrm{v}}{\nabla} u_k \right\|_{L^2(N)}^2 = k(k+n-2) \left\| u_k \right\|_{L^2(SM)}^2 . \tag{127}$$

Thus equation (126) yields

$$\begin{aligned} \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 &= \sum_{k=0}^{\infty} k(k+n-2) \left\| \frac{1}{k} X_+ u_{k-1} + \frac{1}{k+n-2} X_- u_{k+1} \right\|^2 + \left\| W(u) \right\|^2 \\ &= \sum_{k=0}^{\infty} \left(\frac{k+n-2}{k} \left\| X_+ u_{k-1} \right\|^2 - 2 \left(X_+ u_{k-1}, X_- u_{k+1} \right) \right. \\ &\quad \left. + \frac{k}{k+n-2} \left\| X_- u_{k+1} \right\|^2 \right) + \left\| W(u) \right\|^2 . \end{aligned} \tag{128}$$

Again, by orthogonality, we have

$$\begin{aligned} \left\| Xu \right\|^2 &= \sum_{k=0}^{\infty} \left\| X_+ u_{k-1} + X_- u_{k+1} \right\|^2 \\ &= \sum_{k=0}^{\infty} \left(\left\| X_+ u_{k-1} \right\|^2 + 2 \left(X_+ u_{k-1}, X_- u_{k+1} \right) + \left\| X_- u_{k+1} \right\|^2 \right) \end{aligned} \tag{129}$$

We add equations (128) and (129) to get

$$\begin{aligned} \left\| u \right\|_{H^1_{\mathrm{h}} H^0_{\mathrm{v}}(SM)}^2 &= \left\| Xu \right\|^2 + \left\| \overset{\mathrm{h}}{\nabla} u \right\|^2 \\ &= \sum_{k=0}^{\infty} \left(1 + \frac{k+n-2}{k} \right) \left\| X_+ u_{k-1} \right\|^2 \\ &\quad + \sum_{k=0}^{\infty} \left(1 + \frac{k}{k+n-2} \right) \left\| X_- u_{k+1} \right\|^2 + \left\| W(u) \right\|^2 \\ &\geq \sum_{k=0}^{\infty} \left\| X_+ u_{k-1} \right\|^2 + \sum_{k=0}^{\infty} \left\| X_- u_{k+1} \right\|^2 , \end{aligned} \tag{130}$$

This is estimate (71).

Appendix C: Proof of Liouville's Theorem

This appendix is devoted to the proof of lemma 23. We let M be a compact smooth manifold with a smooth boundary. Suppose that we are given two $C^{1,1}$ Riemannian metrics g and h on M. Let the corresponding unit sphere bundles be $S_g M$ and $S_h M$.

There is a natural radial $C^{1,1}$-diffeomorphism $(x, v) \mapsto (x, v\,|v|_h^{-1})$ from $S_g M$ to $S_h M$, the inverse map from $S_h M$ to $S_g M$ being $(x, w) \mapsto (x, w\,|w|_g^{-1})$.

In the proof of lemma 23, we use three types of Riemannian metrics on M. We will have a $C^{1,1}$ Riemannian metric g and two types of smooth Riemannian metrics h and $\overset{\alpha}{g}$. We denote the corresponding radial diffeomorphisms by

$$\overset{\alpha}{s}\colon S_h M \to \overset{\alpha}{S}M, \quad s\colon S_h M \to S_g M, \quad \text{and} \quad \overset{\alpha}{r}\colon \overset{\alpha}{S}M \to S_g M. \tag{131}$$

In the proof of lemma 23, we will use the convention that the unit sphere bundle related $\overset{\alpha}{g}$ is denoted $\overset{\alpha}{S}M{:=}S_{\overset{\alpha}{g}}M$, the operators and differential forms related to $\overset{\alpha}{g}$ are decorated with α on top or as a subscript, the sphere bundle, operators and differential forms related to h are decorated with subscripts h and the bundles and the operators related to the metric g are written without decorations.

Proof of lemma 23 The proof is based on smooth approximations of the Riemannian metric g. Let h be a smooth fixed reference Riemannian metric on M. Let $\big(\overset{\alpha}{g}\big)$ be a sequence of smooth Riemannian metrics on M so that

$$\overset{\alpha}{g}_{jk} \to g_{jk} \quad \text{in } W_h^{1,\infty}(M) \quad \text{and} \quad \overset{\alpha}{\Gamma}{}^i{}_{jk} \to \Gamma^i{}_{jk} \quad \text{in } L_h^\infty(M). \tag{132}$$

Existence of such sequence was proved in [18, Lemma 18]. Let $u \in \mathrm{Lip}(SM)$ and denote $\overset{\alpha}{u}{:=}\overset{\alpha}{r}{}^* u$ and $\tilde{u}{:=}s^* u$. We note that $\tilde{u} = \overset{\alpha}{s}{}^*\overset{\alpha}{u}$. We will prove that

$$\lim_{\alpha\to\infty} \int_{\overset{\alpha}{S}M} \overset{\alpha}{u} L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) = \int_{SM} u L_X(\mathrm{d}\Sigma). \tag{133}$$

Establishing equation (133) proves the claim, since by Liouville's theorem [36, Lemma 3.6.4.], we have

$$L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) = 0 \tag{134}$$

for all $\alpha \in \mathbb{N}$ and thus the limit integral in equation (133) is zero.

Recall that $\tilde{u} = s^* u = \overset{\alpha}{s}{}^*\overset{\alpha}{u}$. Thus by basic properties of pullback,it is enough prove that

$$\lim_{\alpha\to\infty} \int_{S_h M} \tilde{u}\overset{\alpha}{s}{}^*(L_{\overset{\alpha}{X}}\mathrm{d}\overset{\alpha}{\Sigma}) = \int_{S_h M} \tilde{u} s^*(L_X \mathrm{d}\Sigma) \tag{135}$$

The manifold M is the Euclidean unit ball in $\mathbb{R}^n$ and we let $(x^1, \ldots, x^n)$ be usual Cartesian coordinates on M. We consider coordinates $(x^1, \ldots, x^n, w^1, \ldots, w^n)$ on $S_h M$ and corresponding coordinates

$$(x^1, \ldots, x^n, \overset{\alpha}{v}{}^1, \ldots, \overset{\alpha}{v}{}^n) \quad \text{on } \overset{\alpha}{S}M \quad \text{and} \quad (x^1, \ldots, x^n, v^1, \ldots, v^n) \quad \text{on } SM$$

so that $\overset{\alpha}{s}(x, w) = (x, \overset{\alpha}{v})$ and $s(x, w) = (x, v)$. We associate to (x, w) the coordinate vector fields $\partial_{x^1}, \ldots, \partial_{x^n}, \partial_{w^1}, \ldots, \partial_{w^n}$ and similarly to $(x, \overset{\alpha}{v})$ we associate

$\partial_{x^1}, \dots, \partial_{x^n}, \partial_{\overset{\alpha}{v}^1}, \dots, \partial_{\overset{\alpha}{v}^n}$ and to (x, v) we associate $\partial_{x^1}, \dots, \partial_{x^n}, \partial_{v^1}, \dots, \partial_{v^n}$. We let

$$\begin{aligned} &\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}w^1, \dots, \mathrm{d}w^n, \\ &\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}\overset{\alpha}{v}^1, \dots, \mathrm{d}\overset{\alpha}{v}^n, \quad \text{and} \\ &\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}v^1, \dots, \mathrm{d}v^n \end{aligned} \tag{136}$$

be the dual basis one-forms characterized by

$$\begin{aligned} &\mathrm{d}x^j(\partial_{x^k}) = \delta^j_k, \quad \mathrm{d}x^j(\partial_{w^k}) = 0, \quad \mathrm{d}w^j(\partial_{x^k}) = 0, \quad \mathrm{d}w^j(\partial_{w^k}) = \delta^j_k, \\ &\mathrm{d}x^j(\partial_{x^k}) = \delta^j_k, \quad \mathrm{d}x^j(\partial_{\overset{\alpha}{v}^k}) = 0, \quad \mathrm{d}\overset{\alpha}{v}^j(\partial_{x^k}) = 0, \quad \mathrm{d}\overset{\alpha}{v}^j(\partial_{\overset{\alpha}{v}^k}) = \delta^j_k, \\ &\mathrm{d}x^j(\partial_{x^k}) = \delta^j_k, \quad \mathrm{d}x^j(\partial_{v^k}) = 0, \quad \mathrm{d}v^j(\partial_{x^k}) = 0, \quad \mathrm{d}v^j(\partial_{v^k}) = \delta^j_k. \end{aligned} \tag{137}$$

Next, we will write the integrals in equation (135) in coordinates on $S_h M$ and we will argue that equation (135) follows from (132). We will derive a local coordinate formula for $L_X(\mathrm{d}\Sigma)$. A similar formula for $L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma})$ can be derived analogously. Then we will compute how the coordinate presentations transform under the pullbacks s^* and $\overset{\alpha}{s}{}^*$.

We denote by $|g|$ the determinant of g. Since $\mathrm{d}\Sigma$ is a volume form (differential form of the highest order), Cartan's formula implies that

$$L_X(\mathrm{d}\Sigma) = d(i_X \mathrm{d}\Sigma). \tag{138}$$

Since

$$i_X \mathrm{d}x^i = \mathrm{d}x^i(X) = \mathrm{d}x^i(v^j \partial_{x^j} - \Gamma^l{}_{jk} v^j v^k \partial_{v^l}) = v^i \tag{139}$$

and

$$i_X \mathrm{d}v^i = \mathrm{d}v^i(X) = \mathrm{d}v^i(v^j \partial_{x^j} - \Gamma^l{}_{jk} v^j v^k \partial_{v^l}) = -\Gamma^i{}_{jk} v^j v^k \tag{140}$$

we see that

$$\begin{aligned} i_X \mathrm{d}\Sigma = &\sum_{i=1}^n v^i\, |g|\, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n \\ &+ \sum_{i=1}^n (-\Gamma^i{}_{jk} v^j v^k\, |g|)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \cdots \wedge \widehat{\mathrm{d}v^i} \wedge \cdots \wedge \mathrm{d}v^n, \end{aligned} \tag{141}$$

where $\widehat{\mathrm{d}x^i}$ and $\widehat{\mathrm{d}v^i}$ indicate that one-forms $\mathrm{d}x^i$ and $\mathrm{d}v^i$ are omitted from the wedge product. From (141),it follows that

$$\begin{aligned} d(i_X \mathrm{d}\Sigma) &= \sum_{i=1}^{n}(-1)^{i-1}\partial_{x^i}(v^i\,|g|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}v^1\wedge\cdots\wedge\mathrm{d}v^n \\ &\quad+\sum_{i=1}^{n}(-1)^{n+i-1}\partial_{v^i}(-\Gamma^i{}_{jk}v^jv^k\,|g|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}v^1\wedge\cdots\wedge\mathrm{d}v^n \\ &=\sum_{i=1}^{n}(-1)^{i-1}(\partial_{x^i}(v^i\,|g|)+(-1)^{n+1}\partial_{v^i}(\Gamma^i{}_{jk}v^jv^k\,|g|)) \\ &\qquad\times\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}v^1\wedge\cdots\wedge\mathrm{d}v^n. \end{aligned} \tag{142}$$

Similarly, we see that

$$\begin{aligned} L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) &= \sum_{i=1}^{n}(-1)^{i-1}\partial_{x^i}(\overset{\alpha}{v}{}^i\,|\overset{\alpha}{g}|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}\overset{\alpha}{v}{}^1\wedge\cdots\wedge\mathrm{d}\overset{\alpha}{v}{}^n \\ &\quad+\sum_{i=1}^{n}(-1)^{n+i-1}\partial_{\overset{\alpha}{v}{}^i}(-\overset{\alpha}{\Gamma}{}^i{}_{jk}\overset{\alpha}{v}{}^j\overset{\alpha}{v}{}^k\,|\overset{\alpha}{g}|)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}\overset{\alpha}{v}{}^1\wedge\cdots\wedge\mathrm{d}\overset{\alpha}{v}{}^n \\ &=\sum_{i=1}^{n}(-1)^{i-1}(\partial_{x^i}(\overset{\alpha}{v}{}^i\,|\overset{\alpha}{g}|)+(-1)^{n+1}\partial_{\overset{\alpha}{v}{}^i}(\overset{\alpha}{\Gamma}{}^i{}_{jk}\overset{\alpha}{v}{}^j\overset{\alpha}{v}{}^k\,|\overset{\alpha}{g}|)) \\ &\qquad\times\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}\overset{\alpha}{v}{}^1\wedge\cdots\wedge\mathrm{d}\overset{\alpha}{v}{}^n. \end{aligned} \tag{143}$$

Next, we pullback formulas (142) and (143) onto S_hM. We can compute

$$s^*\mathrm{d}v^j = \mathrm{d}(s^*v^j) = \mathrm{d}(w^j\,|w|_g^{-1}) = |w|_g^{-1}\,\mathrm{d}w^j + w^j\mathrm{d}(|w|_g^{-1}). \tag{144}$$

If we write

$$\mathrm{d}(|w|_g^{-1}) = \mu_i\mathrm{d}x^i + \lambda_i\mathrm{d}w^i, \tag{145}$$

then

$$\mu_k = \mu_i\mathrm{d}x^i(\partial_{x^k}) = \mathrm{d}(|w|_g^{-1})(\partial_{x^k}) = \partial_{x^k}\,|w|_g^{-1} \quad \text{and} \quad \lambda_k = \partial_{w^k}\,|w|_g^{-1}. \tag{146}$$

Thus

$$s^*\mathrm{d}v^j = w^j(\partial_{x^k}\,|w|_g^{-1})\mathrm{d}x^k + (|w|_g^{-1}\,\delta_k^j + w^j\,\partial_{w^k}\,|w|_g^{-1})\mathrm{d}w^k. \tag{147}$$

Similarly, we get

$$s^*\mathrm{d}\overset{\alpha}{v}{}^j = w^j(\partial_{x^k}\,|w|_\alpha^{-1})\mathrm{d}x^k + (|w|_\alpha^{-1}\,\delta_k^j + w^j\,\partial_{w^k}\,|w|_\alpha^{-1})\mathrm{d}w^k. \tag{148}$$

Since s and $\overset{\alpha}{s}$ act identically on the base point x, we have

$$s^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \quad \text{and} \quad \overset{\alpha}{s}{}^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{149}$$

Using the fact that a wedge product vanishes whenever repetition appears,we get

$$\begin{aligned} &s^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n) \\ &\quad = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge (|w|_g^{-1}\, \delta_k^j + w^1(\partial_{w^k}\, |w|_g^{-1}))\mathrm{d}w^k \wedge \cdots \\ &\qquad \cdots \wedge (|w|_g^{-1}\, \delta_k^n + w^n(\partial_{w^k}\, |w|_g^{-1}))\mathrm{d}w^k \\ &\quad = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \bigwedge_{j=1}^{n} (|w|_g^{-1}\, \delta_k^j + w^j(\partial_{w^k}\, |w|_g^{-1}))\mathrm{d}w^k. \end{aligned} \tag{150}$$

By a similar computation

$$\begin{aligned} &s^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n) \\ &\quad = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \bigwedge_{j=1}^{n} (|w|_g^{-1}\, \delta_k^j + w^j(\partial_{w^k}\, |w|_\alpha^{-1}))\mathrm{d}w^k. \end{aligned} \tag{151}$$

To complete formulas for the pullback of (142) and (143) we use the facts that $s^* = s_*^{-1}$ and $\overset{\alpha}{s}{}^* = \overset{\alpha}{s}{}_*^{-1}$ to compute

$$s^*\partial_{x^i} = \partial_{x^i} + (\partial_{x^i} w^j)\partial_{w^j} \quad \text{and} \quad \overset{\alpha}{s}{}^*\partial_{x^i} = \partial_{x^i} + (\partial_{x^i} w^j)\partial_{w^j} \tag{152}$$

as well as

$$s^*\partial_{v^i} = (\partial_{v^j} w^j)\partial_{w^j} \quad \text{and} \quad \overset{\alpha}{s}{}^*\partial_{\overset{\alpha}{v}{}^i} = (\partial_{\overset{\alpha}{v}{}^j} w^j)\partial_{w^j}. \tag{153}$$

Thus we get

$$s^*(\partial_{x^i} v^i\, |g|) = \partial_{x^i}(w^i\, |w|_g^{-1}\, |g|) + (\partial_{x^i} w^j)(\partial_{w^j}(w^i\, |w|_g^{-1}\, |g|)), \tag{154}$$

$$\overset{\alpha}{s}{}^*(\partial_{x^i} v^i\, \left|\overset{\alpha}{g}\right|) = \partial_{x^i}(w^i\, |w|_\alpha^{-1}\, \left|\overset{\alpha}{g}\right|) + (\partial_{x^i} w^j)(\partial_{w^j}(w^i\, |w|_\alpha^{-1}\, \left|\overset{\alpha}{g}\right|)), \tag{155}$$

and

$$s^*\partial_{v^i}(\Gamma^i{}_{jk} v^j v^k\, |g|) = \Gamma^i{}_{jk}\, |g|\, (\partial_{v^i} w^l)\partial_{w^l}(w^i\, |w|_g^{-1}\, w^k\, |w|_g^{-1}), \tag{156}$$

$$\overset{\alpha}{s}{}^*\partial_{\overset{\alpha}{v}{}^i}(\overset{\alpha}{\Gamma}{}^i{}_{jk} \overset{\alpha}{v}{}^j \overset{\alpha}{v}{}^k\, \left|\overset{\alpha}{g}\right|) = \overset{\alpha}{\Gamma}{}^i{}_{jk}\, \left|\overset{\alpha}{g}\right|\, (\partial_{\overset{\alpha}{v}{}^i} w^l)\partial_{w^l}(w^i\, |w|_\alpha^{-1}\, w^k\, |w|_\alpha^{-1}). \tag{157}$$

The formulas we get for the pullbacks of $L_X(\mathrm{d}\Sigma)$ along s and of $L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma})$ along $\overset{\alpha}{s}$ are

$$s^*L_X(\mathrm{d}\Sigma) = \sum_{i=1}^{n}(-1)^{i-1}\Big(\partial_{x^i}(w^i\,|w|_g^{-1}\,|g|) + (\partial_{x^k}w^j)(\partial_{w^j}(w^k\,|w|_g^{-1}\,|g|)) + (-1)^{n+1}\Gamma^i{}_{jk}\,|g|\,(\partial_{v^m}w^l)\partial_{w^l}(w^m\,|w|_g^{-1}\,w^k\,|w|_g^{-1})\Big) \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\bigwedge_{j=1}^{n}(|w|_g^{-1}\,\delta_k^j + w^j(\partial_{w^k}|w|_g^{-1}))\mathrm{d}w^k \tag{158}$$

and

$$\overset{\alpha}{s}{}^*L_{\overset{\alpha}{X}}(\mathrm{d}\overset{\alpha}{\Sigma}) = \sum_{i=1}^{n}(-1)^{i-1}\Big(\partial_{x^i}(w^i\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|) + (\partial_{x^k}w^j)(\partial_{w^j}(w^k\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|)) + (-1)^{n+1}\overset{\alpha}{\Gamma}{}^i{}_{jk}\,|\overset{\alpha}{g}|\,(\partial_{\overset{\alpha}{v}{}^m}w^l)\partial_{w^l}(w^m\,|w|_\alpha^{-1}\,w^k\,|w|_\alpha^{-1})\Big) \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\wedge\bigwedge_{j=1}^{n}(|w|_\alpha^{-1}\,\delta_k^j + w^j(\partial_{w^k}|w|_\alpha^{-1}))\mathrm{d}w^k. \tag{159}$$

From formulas (158) and (159) we see that can conclude the equation (135) if the following holds:

$$\partial_{x^i}(w^i\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|)\prod_{j\in S}(|w|_\alpha^{-1}\,\delta_k^j)\prod_{j\in S'}(w^j(\partial_{w^k}|w|_\alpha^{-1})) \to \partial_{x^i}(w^i\,|w|_\alpha^{-1}\,|g|)\prod_{j\in S}(|w|_g^{-1}\,\delta_k^j)\prod_{j\in S'}(w^j(\partial_{w^k}|w|_g^{-1})), \tag{160}$$

$$(\partial_{x^i}w^j)(\partial_{w^j}(w^k\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|))\prod_{j\in S}(|w|_\alpha^{-1}\,\delta_k^j)\prod_{j\in S'}(w^j(\partial_{w^k}|w|_\alpha^{-1})) \to (\partial_{x^i}w^j)(\partial_{w^j}(w^k\,|w|_g^{-1}\,|g|))\prod_{j\in S}(|w|_g^{-1}\,\delta_k^j)\prod_{j\in S'}(w^j(\partial_{w^k}|w|_g^{-1})), \tag{161}$$

$$\overset{\alpha}{\Gamma}{}^i{}_{jk}\,|\overset{\alpha}{g}|\,(\partial_{\overset{\alpha}{v}{}^m}w^l)(\partial_{w^l}(w^m\,|w|_\alpha^{-1}\,w^l\,|w|_\alpha^{-1}))\prod_{j\in S}(|w|_\alpha^{-1}\,\delta_k^j)\prod_{j\in S'}(w^j(\partial_{w^k}|w|_\alpha^{-1})) \to \Gamma^i{}_{jk}\,|g|\,(\partial_{v^m}w^l)(\partial_{w^l}(w^m\,|w|_g^{-1}\,w^l\,|w|_g^{-1}))\prod_{j\in S}(|w|_g^{-1}\,\delta_k^j)\prod_{j\in S'}(w^j(\partial_{w^k}|w|_g^{-1})) \tag{162}$$

in $L^1(S_h M)$, where S and S' are any subsets of $\{1, \ldots, n\}$. We chose the approximating sequence $\left(\overset{\alpha}{g}\right)$ so that

$$\overset{\alpha}{g}_{jk} \to g_{jk} \quad \text{in } W_h^{1,\infty}(M) \quad \text{and} \quad \overset{\alpha}{\Gamma}{}^i{}_{jk} \to \Gamma^i{}_{jk} \quad \text{in } L_h^\infty(M). \tag{163}$$

From (163),we see that

$$\begin{aligned}
\partial_{x^i}(w^i\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}|) &\to \partial_{x^i}(w^i\,|w|_g^{-1}\,|g|), \\
|w|_\alpha^{-1}\,\delta_k^j &\to |w|_g^{-1}\,\delta_k^j, \\
w^j(\partial_{w^k}|w|_\alpha^{-1})) &\to w^j(\partial_{w^k}|w|_g^{-1})), \\
\partial_{w^j}(w^k\,|w|_\alpha^{-1}\,|\overset{\alpha}{g}| &\to \partial_{w^j}(w^k\,|w|_g^{-1}\,|g|, \\
\overset{\alpha}{\Gamma}{}^i{}_{jk}\,|\overset{\alpha}{g}| &\to \Gamma^i{}_{jk}\,|g|, \\
\partial_{\overset{\alpha}{v}{}^m} w^l &\to \partial_{v^m} w^l, \\
\partial_{w^l}(w^m\,|w|_\alpha^{-1}\,w^l\,|w|_\alpha^{-1}) &\to \partial_{w^l}(w^m\,|w|_g^{-1}\,w^l\,|w|_g^{-1})
\end{aligned} \tag{164}$$

in $L^\infty(S_h M)$. Thus we can take products and we conclude that (160), (161) and (162) hold, which finishes the proof. □

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