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Energy Stability for a Class of Semilinear Elliptic Problems

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Abstract

In this paper, we consider semilinear elliptic problems in a bounded domain Ω contained in a given unbounded Lipschitz domain $\mathcal{C} \subset \mathbb{R}^N$. Our aim is to study how the energy of a solution behaves with respect to volume-preserving variations of the domain Ω inside $\mathcal{C}$. Once a rigorous variational approach to this question is set, we focus on the cases when $\mathcal{C}$ is a cone or a cylinder and we consider spherical sectors and radial solutions or bounded cylinders and special one-dimensional solutions, respectively. In these cases, we show both stability and instability results, which have connections with related overdetermined problems.

Keywords Semilinear elliptic equations · Variational methods · Stability · Shape optimization in unbounded domains

Mathematics Subject Classification 35J61 · 35B35 · 35B38 · 49Q10

1 Introduction

Let $\mathcal{C} \subset \mathbb{R}^N$, $N \geq 2$, be an unbounded uniformly Lipschitz domain and let $\Omega \subset \mathcal{C}$ be a bounded Lipschitz domain with smooth relative boundary $\Gamma_\Omega := \partial\Omega \cap \mathcal{C}$. More

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precisely, we assume that Γ_Ω is a smooth manifold of dimension $N-1$ with smooth boundary $\partial\Gamma_\Omega$. We set $\Gamma_{1,\Omega} := \partial\Omega \setminus \overline{\Gamma}_\Omega$ and assume that $\mathcal{H}^{N-1}(\Gamma_{1,\Omega}) > 0$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. Hence $\partial\Omega = \Gamma_\Omega \cup \Gamma_{1,\Omega} \cup \partial\Gamma_\Omega$.

We consider the following semilinear elliptic problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\Omega} \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally $C^{1,\alpha}$ nonlinearity and ν denotes the exterior unit normal vector to $\partial\Omega$.

Let u_Ω be a positive weak solution of (1.1) in the Sobolev space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$, which is the space of functions in $H^1(\Omega)$ whose trace vanishes on Γ_Ω. By standard variational methods we have that under suitable hypotheses on f such a solution exists and is a critical point of the energy functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega F(v)\,dx, \quad v \in H_0^1(\Omega \cup \Gamma_{1,\Omega}), \tag{1.2}$$

where $F(s) = \int_0^s f(\tau)\,d\tau$.

A classical example of a nonlinearity for which a positive solution exists for any domain Ω in $\mathcal{C}$ is the Lane-Emden nonlinearity, namely

$$f(u) = u^p, \text{ with } \begin{cases} 1 < p < \frac{N+2}{N-2} & \text{if } N \geq 3, \\ 1 < p < +\infty & \text{if } N = 2. \end{cases} \tag{1.3}$$

In this case, u_Ω can be obtained, for instance, by minimizing the functional J on the Nehari manifold

$$\mathcal{N}(\Omega) = \{v \in H_0^1(\Omega \cup \Gamma_{1,\Omega}) \setminus \{0\} \;:\; J'(v)[v] = 0\}.$$

Given the unbounded region $\mathcal{C}$, an interesting question is to understand how the energy $J(u_\Omega)$ behaves with respect to variations of a domain Ω inside $\mathcal{C}$. In particular, one could ask whether the energy $J(u_\Omega)$ increases or decreases by deforming Ω into a domain $\widetilde{\Omega}$ sufficiently close to Ω and with the same measure.

Loosely speaking, one could consider the function $\Omega \mapsto T(\Omega) = J(u_\Omega)$ and study it in a suitable "neighborhood" of Ω. Under this aspect, domains Ω which are local minima of T could be particularly interesting. This question could be attacked by differentiating $T(\Omega)$ with respect to variations of Ω which leave the volume invariant and studying the stability or instability of its critical points. However, since (1.1) is a nonlinear problem and solutions of (1.1) are not unique in general, it is not clear a priori how to well define the functional $T(\Omega)$.

We will show in Sect. 2 that for nondegenerate solutions u_Ω of (1.1) the energy functional $T(\Omega)$ is well defined for domains obtained by small deformations of Ω induced by vector fields which leave $\mathcal{C}$ invariant.

We remark that the study of the stationary domains of the energy functional $T(\Omega)$ with a volume constraint is strictly related to the overdetermined problem obtained from (1.1) by adding the condition that the normal derivative $\frac{\partial u}{\partial \nu}$ is constant on Γ_Ω, see Proposition 2.6. This is well-known for a Dirichlet problem in $\mathbb{R}^N$ and when $T(\Omega)$ is globally defined for all domains $\Omega \subset \mathbb{R}^N$ (as in the case of the torsion problem, i.e. $f \equiv 1$). It has been observed in [21] and [17] in the relative setting of the cone.

The existence or not of domains that are local minimizers of the energy and their shapes obviously depend on the unbounded region $\mathcal{C}$ where the domains Ω are contained. In this paper, we consider unbounded cones and cylinders, in which there are some particular domains that, for symmetry or other geometric reasons, could be natural candidates for being local minimizers of the energy.

Let us first describe the case when $\mathcal{C}$ is a cone Σ_D defined as

$$\Sigma_D := \{x \in \mathbb{R}^N \ : \ x = tq, \ q \in D, \ t > 0\}, \tag{1.4}$$

where D is a smooth domain on the unit sphere $\mathbb{S}^{N-1}$.

In Σ_D we consider the spherical sector Ω_D obtained by intersecting the cone with the unit ball centered at the origin, i.e. $\Omega_D = \Sigma_D \cap B_1$. In Ω_D we can consider a radially symmetric solution u_D of problem (1.1), for the nonlinearities f for which they exist. Obviously, u_D is a radial solution of the analogous Dirichlet problem in the unit ball B_1.

In Sect. 3 we show that, whenever u_D is a nondegenerate solution of (1.1), then the pair (Ω_D, u_D) is energy-stationary in the sense of Definition 2.4 and investigate its "stability" as a critical point of the energy functional T, which is well defined for small perturbations of Ω_D. This means to investigate the sign of the quadratic form corresponding to the second domain derivative of T (see Sects. 2 and 3).

The main result we get is that the stability of (Ω_D, u_D) depends on the first nontrivial Neumann eigenvalue $\lambda_1(D)$ of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on the domain $D \subset \mathbb{S}^{N-1}$ which spans the cone. In particular, we obtain a precise threshold for stability/instability which is independent of the nonlinearity, and on the radial positive solution considered, whenever multiple radial positive solutions exist. Let us remark that for several nonlinearities the radial positive solution is unique (see [19]). For example, this is the case if $f(u) = u^p$, $p > 1$.

To state precisely our result we need to introduce the first eigenvalue $\widehat{\nu}_1$ of the following singular eigenvalue problem:

$$\begin{cases} -z'' - \frac{N-1}{r} z' - f'(u_D) z = \frac{\widehat{\nu}}{r^2} z & \text{in } (0,1) \\ z(1) = 0 \end{cases} \tag{1.5}$$

This problem arises naturally when studying the spectrum of the linearized operator $-\Delta - f'(u_D)$. We refer to Sect. 3 for more details.

Theorem 1.1 *Let Σ_D be the cone spanned by the smooth domain $D \subset \mathbb{S}^{N-1}$, $N \geq 3$, and let $\lambda_1(D)$ be the first nonzero Neumann eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on D. Let u_D be a radial positive solution of (1.1) in the spherical sector Ω_D. We have:*

(i) if $-\widehat{\nu}_1 < \lambda_1(D) < N-1$, then the pair (Ω_D, u_D) is an unstable energy-stationary pair;
(ii) if $\lambda_1(D) > N - 1$, then (Ω_D, u_D) is a stable energy-stationary pair.

Remark 1.2 The case $N = 2$ is special and in this case, the overdetermined torsion problem has been completely solved in [20] using that the boundary of any cone in dimension 2 is flat. In the nonlinear case, the condition $N \geq 3$ arises from the study of an auxiliary singular problem (see Proposition 3.12). It is important to observe that the singular eigenvalue $\widehat{\nu}_1$ which appears in *(i)* is larger than $-(N-1)$ for all autonomous nonlinearities $f(u)$ (see [7, Proposition 3.4]). Thus the condition $\lambda_1(D) \in (-\widehat{\nu}_1, N-1)$ is consistent.

Remark 1.3 It is known that if D is a convex domain in $\mathbb{S}^{N-1}$, then $\lambda_1(D) > N-1$ (see [12, Theorem 4.3] or [2, Theorem 4.1]) and the same holds if D is almost convex ([5]). On the other side, examples of domains D in the sphere for which *(i)* holds are provided in [17], Appendix A.

Let us comment on the meaning of Theorem 1.1. The statement *(ii)* will be proved by showing that the quadratic form corresponding to the second derivative of the energy functional, with a fixed volume constraint, is positive definite in all directions. This means that the spherical sector locally minimizes the energy among small volume preserving perturbations of Ω_D and of the corresponding radial solution u_D.

On the contrary, when $-\widehat{\nu}_1 < \lambda_1(D) < N-1$, by *(i)* we have that the pair (Ω_D, u_D) is unstable and therefore Ω_D is not a local minimizer of the energy. This means that there exist small volume preserving deformations of the spherical sector Ω_D which produce domains Ω_t and solutions u_t of (1.1) in Ω_t whose energy $J(u_t)$ is smaller than the energy $J(u_D)$ of the positive radial solution u_D in the spherical sector Ω_D.

Moreover, observe that the function $f = f(s)$ could satisfy suitable hypotheses such that problem (1.1) has a unique positive solution u_Ω in any domain $\Omega \subset \Sigma_D$ (or more generally in $\Omega \subset \mathcal{C}$). This is the case, for example, when $f \equiv 1$, i.e., (1.1) is a "relative" torsion problem. Then the energy functional $T(\Omega) = J(u_\Omega)$ is well defined for any domain $\Omega \subset \Sigma_D$. Hence we may ask whether a global minimum for T exists, once the volume of Ω is fixed, and is given by the spherical sector Ω_D. This question has been addressed in [20], [21] and [17] when $f \equiv 1$, showing that Ω_D is a global minimizer if Σ_D is a convex cone ([21]), as a consequence of an isoperimetric inequality introduced in [18], see also [6, 14, 22]. Instead, in [17] it is proved that Ω_D is not a local minimizer whenever $\lambda_1(D) < N-1$, which is the same threshold we get in Theorem 1.1 for general nonlinearities. It would be very interesting to find a domain Ω in Σ_D which is a local minimizer for T, but not a global minimizer, at least for some nonlinearity for which T is globally defined. This seems to be a challenging question.

The other example of an unbounded domain we consider in the present paper is a half-cylinder, defined as

$$\Sigma_\omega := \omega \times (0, +\infty) \subset \mathbb{R}^N, \tag{1.6}$$

where $\omega \subset \mathbb{R}^{N-1}$ is a smooth bounded domain. We denote the points in Σ_ω by $x = (x', x_N)$, $x' \in \omega$. In this case, a geometrically simple domain we consider is the bounded cylinder

$$\Omega_\omega := \{(x', x_N) \in \mathbb{R}^{N-1} \ : \ x' \in \omega, \ 0 < x_N < 1\}. \tag{1.7}$$

In Ω_ω we consider a positive solution

$$u_\omega(x) = u_\omega(x_N) \tag{1.8}$$

which is obtained by trivially extending to Ω_ω a positive one-dimensional solution of the problem

$$\begin{cases} -u'' = f(u) & \text{in } (0,1) \\ u'(0) = u(1) = 0 \end{cases} \tag{1.9}$$

for a nonlinearity f for which such a solution exists.

Before stating the results concerning the stability of the pair $(\Omega_\omega, u_\omega)$ we again consider an auxiliary eigenvalue problem (but not singular):

$$\begin{cases} -z'' - f'(u_\omega)z = \alpha z & \text{in } (0,1) \\ z'(0) = z(1) = 0 \end{cases} \tag{1.10}$$

The problem (1.10) is considered in Sect. 4 to study the spectrum of the linearized operator $-\Delta - f'(u_\omega)$. We denote by α_1 the first eigenvalue of (1.10).

We start by stating a sharp stability/instability result for the torsion problem, i.e., taking $f \equiv 1$ in (1.1).

Theorem 1.4 *Let $\Sigma_\omega \subset \mathbb{R}^N$, $N \geq 2$, and Ω_ω be respectively, as in (1.6) and (1.7), and let u_ω be the one-dimensional positive solution of (1.1) in Ω_ω obtained by (1.9) for $f \equiv 1$. Let $\lambda_1(\omega)$ be the first nontrivial Neumann eigenvalue of the Laplace operator $-\Delta_{\mathbb{R}^{N-1}}$ in the domain $\omega \subset \mathbb{R}^{N-1}$. Then there exists a number $\beta \approx 1,439$ such that*

(i) if $\lambda_1(\omega) < \beta$, then the pair $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair;
(ii) if $\lambda_1(\omega) > \beta$, then the pair $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair.

Note that the number β that gives the threshold for the stability is independent of the dimension N. Its value is obtained by solving numerically the equation $\sqrt{\lambda_1}\tanh(\sqrt{\lambda_1}) - 1 = 0$ (see (4.44) in the proof of Theorem 1.4).

It is interesting to observe that the instability result of Theorem 1.4 is related to a bifurcation theorem obtained in [13]. Indeed, if we consider the cylinder Σ_ω in $\mathbb{R}^2$, in which case ω is simply an interval in $\mathbb{R}$ and Ω_ω is a rectangle, a byproduct of Theorem 1.1 of [13] is the existence of a domain $\widetilde{\Omega}_\omega$ in Σ_ω that is a small deformation of the rectangle Ω_ω and in which the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in } \widetilde{\Omega}_\omega \\ u = 0 & \text{on } \Gamma_{\widetilde{\Omega}_\omega} \\ \frac{\partial u}{\partial \nu} = c < 0 & \text{on } \Gamma_{\widetilde{\Omega}_\omega} \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\widetilde{\Omega}_\omega} \end{cases}$$

has a solution.

By looking at the proof of [13] and relating it to our instability result it is clear that the bifurcation should occur when the eigenvalue $\lambda_1(\omega)$ crosses the value β provided by Theorem 1.4.

The proof of Theorem 1.4 can be derived from a general condition for the stability of the pair $(\Omega_\omega, u_\omega)$ in the nonlinear case, which is obtained in Theorem 4.11. The proof of Theorem 4.11 involves auxiliary functions that appear naturally in the study of derivatives of the energy functional T, see Sect. 4.

Let us remark that in the case when $f \equiv 1$ we succeed in obtaining the sharp bound of Theorem 1.4 because the solution given by (1.8) and (1.9) is explicit:

$$u_\omega(x) = u_\omega(x_N) = \frac{1 - x_N^2}{2},$$

and so are the auxiliary functions which are solutions of simple linear ODEs. This allows us to use the condition of Theorem 4.11 to obtain Theorem 1.4.

The result of Theorem 1.4 gives a striking difference between the torsional energy problem and the isoperimetric problem in cylinders. Indeed, Proposition 2.1 of [1] shows that the only stationary cartesian graphs for the perimeter functional are the flat ones. Instead, Theorem 1.4 (as well as the result of [13]) indicate that there are domains for which the overdetermined problem relative to (1.1), with $f \equiv 1$, has a solution and whose relative boundary is a non-flat cartesian graph.

For the semilinear problem, we obtain a stability result for a large class of nonlinearities as soon as the eigenvalue $\lambda_1(\omega)$ is sufficiently large. Indeed, we have

Theorem 1.5 *Let Σ_ω and Ω_ω be as in (1.6) and (1.7), and let u_ω be a positive one-dimensional solution of (1.1) in Ω_ω. Let α_1 be the first eigenvalue of (1.10) and let $\lambda_1(\omega)$ be as in Theorem 1.4. If the nonlinearity f satisfies $f(0) = 0$ and*

$$\lambda_1(\omega) > \max\{-\alpha_1, \|f'(u_\omega)\|_\infty\}, \tag{1.11}$$

then the pair $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair.

The condition (1.11) shows that the stability depends on an interplay between the geometry of the cylinder Σ_ω (through the eigenvalue $\lambda_1(\omega)$) and the nonlinearity f.

On the contrary, numerical evidence shows, for the Lane-Emden nonlinearity (1.3), that, if λ_1 is sufficiently close to $-\alpha_1$, instability occurs, see Remark 4.13.

Concerning the eigenvalue α_1 in the bound (1.11), as well as the analogous one, $\lambda_1(D) > -\widehat{\nu}_1$, of Theorem 1.1, we point out that they are used in the proofs of both theorems to deduce the positivity of some auxiliary functions. It is an open problem to understand if they really play a role in the stability/instability result.

We delay further comments on the results and their proofs to the respective sections.

The paper is organized as follows. In Sect. 2 we study problem (1.1) in domains Ω contained in a general unbounded set $\mathcal{C}$. We define the energy functional and its derivative with respect to variations of Ω which leave $\mathcal{C}$ invariant and preserve the measure of Ω. This is done by considering nondegenerate solutions of (1.1) in Ω.

In Sect. 3 we consider the case when $\mathcal{C}$ is a cone Σ_D. In this setting we take domains which are defined by smooth radial graphs over D, in particular we consider the spherical sector Ω_D and a corresponding radial solution u_D for which we prove the stability/instability result.

Finally in Sect. 4 we study the case of the cylinder Σ_ω and prove the corresponding stability/instability result for the pair $(\Omega_\omega, u_\omega)$ when Ω_ω is a bounded cylinder and u_ω is as in (1.8) and (1.9).

2 Semilinear Elliptic Problems in Unbounded Sets

In this section we consider problem (1.1) in a bounded Lipschitz domain Ω contained in an unbounded open set $\mathcal{C}$ which we assume to be (uniformly) Lipschitz regular.

Starting from a positive nondegenerate solution of (1.1) in Ω we show how to define an energy functional for small variations of Ω which preserve the volume.

2.1 Nondegenerate Solutions

Let $\Omega \subset \mathcal{C}$ be a bounded domain whose relative boundary $\Gamma_\Omega = \partial\Omega \cap \mathcal{C}$ is a smooth manifold (with boundary). As in Sect. 1 we set $\Gamma_{1,\Omega} = \partial\Omega \setminus \overline{\Gamma}_\Omega$.

We consider a positive weak solution u_Ω of (1.1) in the Sobolev space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$, which is the subspace of $H^1(\Omega)$ of functions whose trace vanishes on Γ_Ω. By standard variational methods, such as constrained minimization, Mountain-Pass Theorem etc, it is easy to exhibit many nonlinearities $f = f(s)$ for which such a solution exists. Moreover, with suitable assumptions on the growth of f we also have, by regularity results, that u_Ω is a classical solution of (1.1) inside Ω and at any regular point of $\partial\Omega$, and that u_Ω is bounded (see also [7, Proposition 3.1]).

We assume that u_Ω is nondegenerate, i.e., the linearized operator

$$L_{u_\Omega} = -\Delta - f'(u_\Omega) \tag{2.1}$$

does not have zero as an eigenvalue in $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ or, in other words, L_{u_Ω} defines an isomorphism between $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ and its dual space. We consider small deformations of Ω which leave $\mathcal{C}$ invariant and would like to show that the nondegeneracy of u_Ω induces a local uniqueness result for solutions of (1.1) in the deformed domains. Thus we take a one-parameter family of diffeomorphisms ξ_t, for $t \in (-\eta, \eta)$, $\eta > 0$, associated to a smooth vector field V such that $V(x) \in T_x\partial\mathcal{C}$ for every $x \in \partial\mathcal{C}^{\mathrm{reg}}$, $V(x) = 0$ for $x \in \partial\mathcal{C} \setminus \partial\mathcal{C}^{\mathrm{reg}}$, and set $\Omega_t := \xi_t(\Omega)$, where $T_x\partial\mathcal{C}$ denotes the tangent space to $\partial\mathcal{C}$ at the point x, and $\partial\mathcal{C}^{\mathrm{reg}}$ denotes the regular part of $\partial\mathcal{C}$. In particular $\Omega_0 = \Omega$ and in order to simplify the notations we set

$$\Gamma_t := \Gamma_{\Omega_t}, \quad \Gamma_{1,t} := \Gamma_{1,\Omega_t}. \tag{2.2}$$

Proposition 2.1 *Let u_Ω be a positive nondegenerate solution of (1.1) which belongs to $W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Let V be a smooth vector field and let ξ_t be the associated family of diffeomorphisms. Then there exists $\delta > 0$ such that for any $t \in (-\delta, \delta)$ there is a unique solution u_t of the problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_t \\ u = 0 & \text{on } \Gamma_t \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,t} \end{cases} \tag{2.3}$$

in a neighborhood of the function $u_\Omega \circ \xi_t^{-1}$ in the space $H_0^1(\Omega_t \cup \Gamma_{1,t})$. Moreover, the map $t \mapsto u_t$ is differentiable.

Proof By using the diffeomorphism ξ_t we can pass from the space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ to the space $H_0^1(\Omega_t \cup \Gamma_{1,t})$. Indeed,

$$H_0^1(\Omega \cup \Gamma_{1,\Omega}) = \{v \circ \xi_t \;:\; v \in H_0^1(\Omega_t \cup \Gamma_{1,t})\}. \tag{2.4}$$

Moreover, u_t is a weak solution of (2.3), i.e.,

$$\int_{\Omega_t} \nabla u_t \cdot \nabla v \, dx - \int_{\Omega_t} f(u_t) v \, dx = 0 \quad \forall v \in H_0^1(\Omega_t \cup \Gamma_{1,t})$$

if and only if the function $\widehat{u}_t = u_t \circ \xi_t \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ satisfies

$$\int_{\Omega} (M_t \nabla \widehat{u}_t) \cdot \nabla w \, J_t \, dx - \int_{\Omega} f(\widehat{u}_t) w \, J_t \, dx = 0 \quad \forall w \in H_0^1(\Omega \cup \Gamma_{1,\Omega}) \tag{2.5}$$

where

$$J_t(x) = |\det(\mathrm{Jac}\, \xi_t(x))|$$

and

$$M_t = [\mathrm{Jac}\, \xi_t^{-1}(\xi_t(x))][\mathrm{Jac}\, \xi_t^{-1}(\xi_t(x))]^T. \tag{2.6}$$

In other words, setting $\widehat{M}_t := M_t J_t$, we have that $\widehat{u}_t$ is a solution of

$$-\operatorname{div}(\widehat{M}_t \nabla \widehat{u}_t) - f(\widehat{u}_t) J_t = 0$$

in the space $H_0^1(\Omega \cup \Gamma_{1,\Omega})$. Now we consider the map

$$\mathcal{F} : (-\eta, \eta) \times H_0^1(\Omega \cup \Gamma_{1,\Omega}) \to H_0^1(\Omega \cup \Gamma_{1,\Omega})^*$$

defined as

$$\mathcal{F}(t, v) = -\operatorname{div}(\widehat{M}_t \nabla v) - f(v) J_t. \tag{2.7}$$

Since u_Ω is a solution in Ω and ξ_0 is the identity map we have

$$\mathcal{F}(0, u_\Omega) = 0.$$

Notice that $\mathcal{F}$ is differentiable with respect to to v, and

$$\partial_v \mathcal{F}(0, u_\Omega) = -\Delta - f'(u_\Omega). \tag{2.8}$$

Indeed, for any $h \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ we have

$$\begin{aligned}
\frac{\mathcal{F}(t, v + \varepsilon h) - \mathcal{F}(t, v)}{\varepsilon} &= \frac{-\operatorname{div}(\widehat{M}_t(\nabla v + \varepsilon \nabla h)) - f(v + \varepsilon h) J_t - (-\operatorname{div}(\widehat{M}_t \nabla v) - f(v) J_t)}{\varepsilon} \\
&= -\frac{\operatorname{div}(\varepsilon \widehat{M}_t \nabla h)}{\varepsilon} - \frac{(f(v + \varepsilon h) - f(v)) J_t}{\varepsilon} \\
&\to -\operatorname{div}(\widehat{M}_t \nabla h) - f'(v) J_t
\end{aligned}$$

as $\varepsilon \to 0$. Hence $\mathcal{F}$ is differentiable and evaluating $\partial_v \mathcal{F}$ at $(0, u_\Omega)$ we obtain (2.8).

By the nondegeneracy assumption on the solution u_Ω, we infer that (2.8) gives an isomorphism between $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ and $H_0^1(\Omega \cup \Gamma_{1,\Omega})^*$. Then, by the Implicit Function Theorem, there exists an interval $(-\delta, \delta)$ and a neighborhood $\mathcal{B}$ of u_Ω in $H_0^1(\Omega \cup \Gamma_{1,\Omega})$ such that for every $t \in (-\delta, \delta)$ there exists a unique function $\widehat{u}_t \in H_0^1(\Omega \cup \Gamma_{1,\Omega})$ in $\mathcal{B}$ such that $\mathcal{F}(t, \widehat{u}_t) = 0$, that is, $\widehat{u}_t$ is the unique solution (in $\mathcal{B}$) of (2.5). It follows that $u_t = \widehat{u}_t \circ \xi_t^{-1}$ is the unique solution of (2.3) in a neighborhood of $u_\Omega \circ \xi_t^{-1}$ in $H_0^1(\Omega_t \cup \Gamma_{1,t})$.

Finally, since the map $t \mapsto \widehat{u}_t$ is smooth, so is the map $t \mapsto u_t$. In addition

$$\widetilde{u} := \left.\frac{d}{dt}\right|_{t=0} u_t = \left(\left.\frac{d}{dt}\right|_{t=0} \widehat{u}_t\right) - \langle \nabla u_\Omega, V \rangle. \tag{2.9}$$

The proof is complete. □

Note that, as for u_Ω, u_t is a classical solution of (2.3) in Ω_t and on the regular part of $\partial \Omega_t$. By Proposition 2.1 we have that the energy functional

$$T(\Omega_t) = J(u_t) = \frac{1}{2} \int_{\Omega_t} |\nabla u_t|^2 \, dx - \int_{\Omega_t} F(u_t) \, dx, \tag{2.10}$$

where $F(s) = \int_0^s f(\tau)\,d\tau$, is well defined for all sufficiently small t. Observe that, since u_t is a solution to (2.3), we have

$$\int_{\Omega_t} |\nabla u_t|^2\,dx = \int_{\Omega_t} f(u_t)u_t\,dx,$$

so we can also write

$$T(\Omega_t) = \frac{1}{2}\int_{\Omega_t} f(u_t)u_t\,dx - \int_{\Omega_t} F(u_t)\,dx. \tag{2.11}$$

In the next result we show that T is differentiable with respect to t and compute its derivative at $t = 0$, that is, at the initial domain Ω.

Proposition 2.2 *Assume that u_Ω is a positive nondegenerate solution of (1.1) which belongs to $W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} T(\Omega_t) = -\frac{1}{2}\int_{\Gamma_\Omega} |\nabla u_\Omega|^2 \langle V, \nu\rangle\,d\sigma. \tag{2.12}$$

Proof Recall from Proposition 2.1 that $t \mapsto u_t$ is smooth and (2.9) holds. Differentiating the equation $-\Delta u_t = f(u_t)$ with respect to t we obtain

$$-\Delta\widetilde{u} = f'(u_\Omega)\widetilde{u} \quad \text{in } \Omega. \tag{2.13}$$

Now observe that by the hypotheses on u_Ω we have that

$$\widetilde{u} + \langle \nabla u_\Omega, V\rangle = \left(\left.\frac{d}{dt}\right|_{t=0} \widehat{u}_t\right) \in H_0^1(\Omega \cup \Gamma_{1,\Omega}), \tag{2.14}$$

thus

$$\widetilde{u} = -\frac{\partial u_\Omega}{\partial \nu}\langle V, \nu\rangle \quad \text{on } \Gamma_\Omega. \tag{2.15}$$

Finally, since ξ_t maps $\partial\mathcal{C}$ into itself we have that, for all small t and $x \in (\partial\mathcal{C} \cap \partial\Omega)^{\text{reg}}$

$$\langle \nabla u_t(\xi_t(x)), \nu(\xi_t(x))\rangle = 0.$$

Differentiating this relation with respect to t and evaluating at $t = 0$ we obtain

$$0 = \langle \nabla\widetilde{u}(x), \nu(x)\rangle + d_x(\langle \nabla u_\Omega, \nu\rangle)[V(x)],$$

where $d_x(\langle \nabla u_\Omega, \nu\rangle)[V(x)]$ is the differential of the function $\langle \nabla u_\Omega, \nu\rangle\big|_{(\partial\mathcal{C}\cap\partial\Omega)^{\text{reg}}}$ computed at x, along $V(x)$. Then, since $\langle \nabla u_\Omega, \nu\rangle = 0$ on $(\partial\mathcal{C} \cap \partial\Omega)^{\text{reg}}$, and in view of

(2.13), (2.15), we infer that $\widetilde{u}$ satisfies

$$\begin{cases} -\Delta\widetilde{u} = f'(u_\Omega)\widetilde{u} & \text{in } \Omega \\ \widetilde{u} = -\dfrac{\partial u_\Omega}{\partial \nu}\langle V, \nu\rangle & \text{on } \Gamma_\Omega \\ \dfrac{\partial \widetilde{u}}{\partial \nu} = 0 & \text{on } \Gamma_{1,\Omega} \end{cases} \tag{2.16}$$

in the classical sense in the interior of Ω and on the regular part of $\partial\Omega$.

Recalling (2.11) we can write

$$T(\Omega_t) = \int_{\Omega_t} \frac{1}{2}\left(f(u_t)u_t - F(u_t)\right)\,dx.$$

Since $t \mapsto f(u_t)u_t - F(u_t)$ is differentiable at $t = 0$, $\partial\Omega$ is Lipschitz and taking into account that $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$, then, applying [16, Theorem 5.2.2], we can compute the derivative with respect to t of the functional T obtaining that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} T(\Omega_t) &= \frac{1}{2}\int_\Omega (f'(u_\Omega)\widetilde{u}u_\Omega + f(u_\Omega)\widetilde{u})\,dx - \int_\Omega f(u_\Omega)\widetilde{u}\,dx \\ &\quad + \int_{\partial\Omega}\left(\frac{1}{2}f(u_\Omega)u_\Omega - F(u_\Omega)\right)\langle V, \nu\rangle\,d\sigma \\ &= \frac{1}{2}\int_\Omega \left(f'(u_\Omega)\widetilde{u}u_\Omega - f(u_\Omega)\widetilde{u}\right)\,dx \\ &= \frac{1}{2}\int_\Omega \left((-\Delta\widetilde{u})u_\Omega + \Delta u_\Omega \widetilde{u}\right)\,dx \\ &= \frac{1}{2}\int_{\partial\Omega}\left(\widetilde{u}\frac{\partial u_\Omega}{\partial \nu} - u_\Omega\frac{\partial \widetilde{u}}{\partial \nu}\right)\,d\sigma \\ &= -\frac{1}{2}\int_{\Gamma_\Omega} |\nabla u_\Omega|^2\langle V, \nu\rangle\,d\sigma. \end{aligned} \tag{2.17}$$

The previous applications of the Divergence Theorem are justified by arguing as in [20, Lemma 2.1], where the regularity hypothesis on u_Ω comes into play. □

Remark 2.3 It is not difficult to see that $\widetilde{u}$ is also a weak solution of (2.16). Indeed, let $\varphi \in C_c^\infty(\Omega \cup \Gamma_{1,\Omega})$. Then, for all sufficiently small t, we also have $\varphi \in C_c^\infty(\Omega_t \cup \Gamma_{1,t})$. Hence, since u_t is a weak solution to (2.3), we have

$$0 = \int_{\Omega_t} \nabla u_t \nabla\varphi\,dx - \int_{\Omega_t} f(u_t)\varphi\,dx = \int_\Omega \nabla u_t \nabla\varphi\,dx - \int_\Omega f(u_t)\varphi\,dx. \tag{2.18}$$

Now, as proved in [17, Claim (3.17)], it holds that

$$\left.\frac{d}{dt}\right|_{t=0} \nabla u_t = \nabla\widetilde{u}.$$

Then, taking the derivative with respect to t in (2.18), evaluating at $t = 0$, and since φ is arbitrary, we easily conclude.

Let us now consider domains $\Omega \subset \mathcal{C}$ of fixed measure $c > 0$ and define

$$\mathcal{A} := \{\Omega \subset \mathcal{C} \ : \ \Omega \text{ is admissible and } |\Omega| = c\}, \tag{2.19}$$

where admissible means that $\Omega \subset \mathcal{C}$ is a bounded domain with smooth relative boundary $\Gamma_\Omega := \partial\Omega \cap \mathcal{C}$, $\partial\Gamma_\Omega$ is a smooth $(N-2)$-dimensional manifold and $\Gamma_{1,\Omega} := \partial\Omega \setminus \overline{\Gamma}_\Omega$ is such that $\mathcal{H}^{N-1}(\Gamma_{1,\Omega}) > 0$. We consider vector fields that induce deformations that preserve the volume. More precisely we take a one-parameter family of diffeomorphisms $\xi_t, t \in (-\eta, \eta)$, associated to a smooth vector field V such that $V(x) \in T_x\partial\mathcal{C}^{\text{reg}}$ for all $x \in \partial\mathcal{C}^{\text{reg}}$, and satisfying the condition $|\Omega_t| = |\Omega|$, for all $t \in (-\eta, \eta)$, where $\Omega_t = \xi_t(\Omega)$.

Definition 2.4 We say that the pair (Ω, u_Ω) is energy-stationary under a volume constraint if

$$\left.\frac{d}{dt}\right|_{t=0} T(\Omega_t) = 0 \tag{2.20}$$

for any vector field tangent to $\partial\mathcal{C}$ such that the associated one-parameter family of diffeomorphisms preserves the volume.

Definition 2.5 Let (Ω, u_Ω) be an energy-stationary pair under a volume constraint. We say that it is stable if, for any volume-preserving vector field V, the second derivative

$$\left.\frac{d^2}{dt^2}\right|_{t=0} T(\Omega_t)$$

is positive.

Since the computation of second domain derivatives is quite involved, we do not present a general formula. Explicit expressions are given in Sects. 3 and 4, in the special cases of cones and cylinders.

A characterization of energy-stationary pairs in $\mathcal{C}$ is the following:

Proposition 2.6 *Let $\Omega \in \mathcal{A}$ and assume that $u_\Omega \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ is a nondegenerate positive solution of (1.1). Then (Ω, u_Ω) is energy-stationary under a volume constraint if and only if u_Ω satisfies the overdetermined condition $|\nabla u_\Omega| = constant$ on Γ_Ω.*

Proof Let ξ_t be an arbitrary admissible one-parameter family of diffeomorphisms and let V be the associated vector field. Since the volume is preserved and $V(x) \in T_x\partial\mathcal{C}$ on $\partial\mathcal{C}$,

$$0 = \left.\frac{d}{dt}\right|_{t=0} |\Omega_t| = \int_{\partial\Omega} \langle V, \nu \rangle \, d\sigma = \int_{\Gamma_\Omega} \langle V, \nu \rangle \, d\sigma. \tag{2.21}$$

If $|\nabla u_\Omega|$ is constant on Γ_Ω, then (Ω, u_Ω) is energy-stationary, in view of (2.12) and (2.21). On the other hand, if (Ω, u_Ω) is energy stationary, then

$$\int_{\Gamma_\Omega} (|\nabla u_\Omega|^2 - a)\langle V, \nu\rangle \, d\sigma = 0 \tag{2.22}$$

for every constant a and every admissible vector field V. Assume by contradiction that $|\nabla u_\Omega|$ is not constant on Γ_Ω. Then there exists a compact set $K \subset \Gamma_\Omega$, with nonempty interior part, such that $|\nabla u_\Omega|$ is not constant on K. Take a nonnegative cutoff function Θ such that $\Theta \equiv 1$ in K, and choose

$$a = \frac{\int_{\Gamma_\Omega} \Theta |\nabla u_\Omega|^2 \, d\sigma}{\int_{\Gamma_\Omega} \Theta \, d\sigma}. \tag{2.23}$$

Then we can build a deformation from the vector field $V = (|\nabla u_\Omega|^2 - a)\Theta\nu$, and in this case, since (Ω, u_Ω) is energy stationary, we would have

$$\int_K (|\nabla u_\Omega|^2 - a)^2 \, d\sigma = 0, \tag{2.24}$$

which contradicts the fact that $|\nabla u_\Omega|$ is not constant on K. The proof is complete. □

Remark 2.7 It is relevant to observe that all concepts introduced in this section apply to the case when $\Gamma_{1,\Omega}$ is empty, or, equivalently, when $\mathcal{C} = \mathbb{R}^N$. Thus all the above results hold for Dirichlet problems in domains in the whole space. In this case it is known, by Serrin's Theorem (see [23]) that if a positive solution for the overdetermined problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases} \tag{2.25}$$

exists, then Ω is a ball. Therefore, in view of Proposition 2.6, it follows that the only energy-stationary pairs in $\mathbb{R}^N$ are (B, u_B), where B is a ball and u_B is a nondegenerate positive solution.

Remark 2.8 We observe that all the results in this section hold true also for non-degenerate sign-changing solutions u_Ω to (1.1). However, since in the sequel we study the stability in the case of positive solutions, we have considered only this case

3 The Case of the Cone

Let $D \subset \mathbb{S}^{N-1}$ be a smooth domain on the unit sphere and let Σ_D be the cone spanned by D, which is defined as

$$\Sigma_D := \{x \in \mathbb{R}^N \ : \ x = tq, \ q \in D, \ t > 0\}. \tag{3.1}$$

In Σ_D we consider admissible domains Ω, in the sense of (2.19), that are strictly star-shaped with respect to the vertex of the cone, which we choose to be the origin 0 in $\mathbb{R}^N$. In other words, we consider domains whose relative boundary is the radial graph in Σ_D of a function in $C^2(\overline{D}, \mathbb{R})$. Hence for $\varphi \in C^2(\overline{D}, \mathbb{R})$ we set

$$\Gamma_\varphi := \{x \in \mathbb{R}^N \ : \ x = e^{\varphi(q)} q, \ q \in D\} \tag{3.2}$$

and consider the strictly star-shaped domain Ω_φ defined as

$$\Omega_\varphi := \{x \in \mathbb{R}^N \ : \ x = tq, \ 0 < t < e^{\varphi(q)}, \ q \in D\}. \tag{3.3}$$

To simplify the notation we set

$$\Gamma_{1,\varphi} := \Gamma_{1,\Omega_\varphi} = \partial\Omega_\varphi \setminus \overline{\Gamma}_\varphi.$$

3.1 Energy Functional for Star-Shaped Domains

In Ω_φ we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_\varphi \\ u = 0 & \text{on } \Gamma_\varphi \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\varphi} \setminus \{0\} \end{cases} \tag{3.4}$$

and assume throughout this section that a bounded positive nondegenerate solution u_{Ω_φ} exists and belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$. Then we can apply the results of Sect. 2 and define the energy functional T as in (2.10) for small variations of Ω_φ. Since Ω_φ is strictly star-shaped, this property also holds for the domains obtained by small regular deformations. Thus it is convenient to parametrize the domains and their variations by C^2 functions defined on $\overline{D}$. Hence, for $v \in C^2(\overline{D}, \mathbb{R})$ and $t \in (-\eta, \eta)$, where $\eta > 0$ is a fixed number sufficiently small, we consider the domain variations $\Omega_{\varphi+tv} \subset \Sigma_D$.

Let $\xi : (-\eta, \eta) \times \overline{\Sigma}_D \setminus \{0\} \to \overline{\Sigma}_D \setminus \{0\}$ be the map defined by

$$\xi(t, x) = e^{tv\left(\frac{x}{|x|}\right)} x. \tag{3.5}$$

Then $\xi|_{\Omega_\varphi}(t, \cdot) : \Omega_\varphi \to \Omega_{\varphi+tv}$ is a diffeomorphism, whose inverse is

$$(\xi|_{\Omega_\varphi})^{-1}(t, x) = e^{-tv\left(\frac{x}{|x|}\right)} x = \xi(-t, x). \tag{3.6}$$

By definition, $\xi(t, x) \in \partial\Sigma_D \setminus \{0\}$ for all $(t, x) \in (-\eta, \eta) \times (\partial\Sigma_D \setminus \{0\})$ and ξ is the flow associated to the vector field

$$V(x) = v\left(\frac{x}{|x|}\right) x, \tag{3.7}$$

since $\xi(0, x) = x$ and

$$\frac{d}{dt}\xi(t, x) = e^{tv\left(\frac{x}{|x|}\right)} v\left(\frac{x}{|x|}\right) x = V(\xi(t, x)).$$

The energy functional T in (2.10) becomes a functional defined on functions in $C^2(\overline{D}, \mathbb{R})$. More precisely, we define, for every $v \in C^2(\overline{D}, \mathbb{R})$,

$$T(\varphi + tv) := T(\Omega_{\varphi+tv}) = J(u_{\varphi+tv}), \tag{3.8}$$

for $t \in (-\delta, \delta)$ with $\delta > 0$ small, where

$$u_{\varphi+tv} := u_{\Omega_{\varphi+tv}}$$

is the unique positive solution of (3.4) in the domain $\Omega_{\varphi+tv}$, in a neighborhood of $u_\varphi \circ \xi(t, \cdot)^{-1}$.

We now compute the first derivative of the functional T at φ along a direction $v \in C^2(\overline{D}, \mathbb{R})$, i.e. the derivative with respect to t of (3.8) computed at $t = 0$.

Lemma 3.1 *Let $\varphi \in C^2(\overline{D}, \mathbb{R})$ and assume that u_φ is a bounded positive nondegenerate solution to (3.4) and that u_φ belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$. Then for any $v \in C^2(\overline{D}, \mathbb{R})$ it holds that*

$$T'(\varphi)[v] = -\frac{1}{2}\int_D \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 e^{N\varphi} v \, d\sigma \tag{3.9}$$

Proof The result follows from Proposition 2.2. Indeed, since the exterior unit normal to Γ_φ is given by

$$\nu(x) = \frac{\frac{x}{|x|} - \nabla_{\mathbb{S}^{N-1}}\varphi\left(\frac{x}{|x|}\right)}{\sqrt{1 + \left|\nabla_{\mathbb{S}^{N-1}}\varphi\left(\frac{x}{|x|}\right)\right|^2}}, \quad x \in \Gamma_\varphi,$$

where $\nabla_{\mathbb{S}^{N-1}}$ is the gradient in $\mathbb{S}^{N-1}$ (see [17, Sect. 2]), then, from (3.7), it follows that

$$\langle V, \nu\rangle = \frac{|x|}{\sqrt{1 + \left|\nabla_{\mathbb{S}^{N-1}}\varphi\left(\frac{x}{|x|}\right)\right|^2}} v\left(\frac{x}{|x|}\right) \quad \text{on} \Gamma_\varphi.$$

Hence, using the parametrization $x = e^{\varphi(q)}q$, for $q \in D$, taking into account that the induced $(N-1)$-dimensional area element on Γ_φ is given by

$$d\sigma_{\Gamma_\varphi} = e^{(N-1)\varphi}\sqrt{1 + |\nabla_{\mathbb{S}^{N-1}}\varphi|^2}\, d\sigma,$$

and since $u_\varphi = 0$ on Γ_φ, then, from (2.12), we readily obtain (3.9). □

The next step is to compute the second derivative of T at Ω_φ with respect to directions $v, w \in C^2(\overline{D}, \mathbb{R})$

Lemma 3.2 *Let φ and u_φ be as in Lemma 3.1. Then for any $v, w \in C^2(\overline{D}, \mathbb{R})$ it holds*

$$\begin{aligned} T''(\varphi)[v, w] = &-\frac{N}{2}\int_D e^{N\varphi} vw \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 d\sigma \\ &- \int_D e^{N\varphi} v \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q) \frac{\partial \widetilde{u}_w}{\partial \nu}(e^\varphi q)\, d\sigma \\ &- \int_D e^{N\varphi} vw \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)(D^2 u_\varphi(e^\varphi q) e^\varphi q) \cdot \nu \, d\sigma \\ &+ \int_D e^{N\varphi} v \frac{\partial u_\varphi}{\partial \nu}(e^\varphi q) \frac{\nabla u_\varphi(e^\varphi q) \cdot \nabla_{\mathbb{S}^{N-1}} w}{\sqrt{1 + |\nabla_{\mathbb{S}^{N-1}}\varphi|^2}}\, d\sigma \\ &+ \int_D e^{N\varphi} \left(\frac{\partial u_\varphi}{\partial \nu}(e^\varphi q)\right)^2 \frac{\nabla_{\mathbb{S}^{N-1}}\varphi \cdot \nabla_{\mathbb{S}^{N-1}} w}{1 + |\nabla_{\mathbb{S}^{N-1}}\varphi|^2}\, d\sigma, \end{aligned} \tag{3.10}$$

where $\widetilde{u}_w = \frac{d}{ds}\big|_{s=0} u_{\varphi+sw}$ satisfies (2.16) with $V(x) = w\left(\frac{x}{|x|}\right)x$.

Proof The proof is the same as that of [17, Lemma 3.2] and therefore we omit it. □

In view of Definition 2.4, we are interested in studying pairs $(\Omega_\varphi, u_\varphi)$ which are energy-stationary under a volume constraint. Thus we need to consider domains Ω_φ with a fixed volume. We recall that the volume of the domain defined by the radial graph of a function $\varphi \in C^2(\overline{D}, \mathbb{R})$ is given by

$$\mathcal{V}(\varphi) := \mathcal{V}(\Omega_\varphi) = |\Omega_\varphi| = \frac{1}{N}\int_D e^{N\varphi}\, d\sigma.$$

Simple computations yield, for $v, w \in C^2(\overline{D}, \mathbb{R})$:

$$\mathcal{V}'(\varphi)[v] = \int_D e^{N\varphi} v \, d\sigma, \qquad \mathcal{V}''(\varphi)[v, w] = N\int_D e^{N\varphi} vw\, d\sigma. \tag{3.11}$$

Then, for $c > 0$ we define the manifold

$$M := \{\varphi \in C^2(\overline{D}, \mathbb{R}) \; : \; \mathcal{V}(\varphi) = c\}, \tag{3.12}$$

whose tangent space at any point $\varphi \in M$ is given by

$$T_\varphi M = \left\{ v \in C^2(\overline{D}, \mathbb{R}) \; : \; \int_D e^{N\varphi} v \, d\sigma = 0 \right\}.$$

We restrict the energy functional to the manifold M and denote it by

$$I(\varphi) := T\big|_M(\varphi).$$

Clearly, if the pair $(\Omega_\varphi, u_\varphi)$ is energy-stationary under a volume constraint, in the sense of Definition 2.4, then $\varphi \in M$ is a critical point of I. Hence, by the Theorem of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that

$$T'(\varphi) = \mu \mathcal{V}'(\varphi). \tag{3.13}$$

Moreover, the following result holds true:

Proposition 3.3 *Let $\varphi \in M$ such that $(\Omega_\varphi, u_\varphi)$ is energy-stationary under the volume constraint. Then the Lagrange multiplier μ is negative and*

$$\frac{\partial u_\varphi}{\partial \nu} = -\sqrt{-2\mu} \quad on \quad \Gamma_\varphi. \tag{3.14}$$

Proof The proof is the same as in [17, Lemma 4.1] □

For the second derivative of I we have

Lemma 3.4 *Let $\varphi \in M$ and let $v, w \in T_\varphi M$. If $(\Omega_\varphi, u_\varphi)$ is energy-stationary under the volume constraint, then*

$$I''(\varphi)[v, w] = T''(\varphi)[v, w] - \mu \mathcal{V}''(\varphi)[v, w]. \tag{3.15}$$

Proof The proof is the same as in [17, Lemma 4.3]. □

3.2 Spherical Sectors and Radial Solutions

Given a cone Σ_D we consider the spherical sector Ω_D obtained by intersecting Σ_D with the unit ball B_1. Obviously its relative boundary Γ_{Ω_D} is the radial graph obtained by taking $\varphi \equiv 0$ in (3.2), which is in fact the domain D which spans the cone, that is $\Gamma_{\Omega_D} = D$.

In the spherical sector Ω_D we would like to consider a nondegenerate positive radial solution $u_D := u_{\Omega_D}$ of (3.4), hence we first recall conditions on the nonlinearity f which ensure that a positive radial solution of (3.4) in Ω_D exists. Observe that such u_D is just the restriction to Ω_D of a positive radial solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases} \tag{3.16}$$

Proposition 3.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Assume that f satisfies one of the following:*

(i) $|f(s)| \le a|s| + b$ *for all* $s > 0$*, where* $b > 0$ *and* $a \in (0, \mu_1(B_1))$*, where* $\mu_1(B_1)$ *is the first eigenvalue of the operator* $-\Delta$ *in* $H_0^1(B_1)$.

(ii) $f : [0, +\infty) \to [0, +\infty)$ *is non-increasing.*

(iii) • $|f(s)| < c|s|^p + d$*, where* $c, d > 0$ *and* $p \in \left(1, \frac{N+2}{N-2}\right)$ *if* $N \ge 3$*,* $p > 1$ *if* $N = 2$*;*

- *$f(s) = o(s)$ as $s \to 0$;*
- *There exist $\gamma > 2$, $\kappa > 0$ such that for $|s| > \kappa$ it holds*

$$0 < \gamma F(s) < sf(s);$$

- *$f'(s) > \dfrac{f(s)}{s}$ for all $s > 0$.*

Then a radial positive solution of (3.16) in B_1, and hence of (3.4) in Ω_D, exists.

Proof In cases *(i)* and *(ii)*, the corresponding functional

$$J(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2\, dx - \int_{B_1} F(u)\, dx$$

is coercive and weakly lower semicontinuous in the space $H^1_{0,rad}(B_1)$, which is the subspace of $H^1_0(B_1)$ of radial functions, and so it has a minimum which is a solution of (3.16). In the case *(iii)* standard variational methods, such as minimization on the Nehari manifold or Mountain Pass type theorems give a positive solution of (3.16), which is then radial by the Gidas-Ni-Nirenberg Theorem (see [15]). We refer to [4] and [9] for the details. □

We point out that a radial solution u_D is always a classical solution of (3.16) in B_1, and hence in Ω_D. In particular, u_D is bounded and $u_D \in C^2(\overline{B}_1)$

Now we would like to study the nondegeneracy of a radial solution u_D of (3.4) in Ω_D.

As recalled in Sect. 2.1, we need conditions that ensure that zero is not an eigenvalue of the linearized operator

$$L_{u_D} = -\Delta - f'(u_D) \tag{3.17}$$

in the space $H^1_0(\Omega_D \cup \Gamma_{1,0})$, where $\Gamma_{1,0} = \partial\Omega_D \setminus \overline{\Gamma}_{\Omega_D}$. Obviously, if the linearized operator L_{u_D} admits only positive eigenvalues, then u_D is nondegenerate. This is the case of stable solutions of (3.4), which occur when f satisfies conditions *(i)* or *(ii)* in Proposition 3.5, in particular, if f is a constant.

In general, L_{u_D} could have negative eigenvalues, so to detect the nondegeneracy of u_D we have to analyze the spectrum of the linear operator (3.17) in $H^1_0(\Omega_D \cup \Gamma_{1,0})$. As we will see, the fact that Ω_D is a spherical sector in the cone Σ_D (and not the ball B_1) plays a role.

The first remark is that zero is an eigenvalue for L_{u_D} if and only if it is an eigenvalue for the following singular problem:

$$\begin{cases} -\Delta\psi - f'(u_D)\psi = \dfrac{\widehat{\Lambda}}{|x|^2}\psi & \text{in } \Omega_D \\ \psi = 0 & \text{on } D \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \Gamma_{1,0} \setminus \{0\}. \end{cases} \tag{3.18}$$

Indeed, since $N \geq 3$, problem (3.18) is well-defined in the space $H_0^1(\Omega_D \cup \Gamma_{1,0})$ due to Hardy's inequality (see [7, Proposition 2.1], for (3.18), and also [3] for the analogous Dirichlet problem).

Therefore we investigate the eigenvalues of (3.18). The advantage of considering this singular eigenvalue problem is that, since u_D is radial, its eigenfunctions can be obtained by separation of variables, using polar coordinates in $\mathbb{R}^N$. To this aim we denote by $\{\lambda_j(D)\}_{j\in\mathbb{N}}$, the eigenvalues of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on the domain D with Neumann boundary conditions. It is well-known that

$$0 = \lambda_0(D) < \lambda_1(D) \leq \lambda_2(D) \leq \dots, \tag{3.19}$$

and the only accumulation point is $+\infty$. Then we consider the following singular eigenvalue problem in the interval (0, 1):

$$\begin{cases} -z'' - \frac{N-1}{r}z' - f'(u_D)z = \dfrac{\widehat{\nu}}{r^2}z & \text{in } (0,1) \\ z(1) = 0 \end{cases} \tag{3.20}$$

It is shown in [3] (see also [7]) that nonpositive eigenvalues for (3.20) can be defined. They are a finite number and we denote them by $\widehat{\nu}_i$, $i = 1, \dots, k$. It is immediate to check that the eigenvalues $\widehat{\nu}_i$ are the eigenvalues of (3.18) which correspond to radial eigenfunctions. In particular, we consider the first eigenvalue $\widehat{\nu}_1$ of (3.20), referring to [3] for a variational definition and a study of its main properties.

By using (3.18)-(3.20) we obtain the following result:

Proposition 3.6 *The problem (3.18) admits zero as an eigenvalue if and only if there exist $i \in \mathbb{N}^+$ and $j \in \mathbb{N}$ such that*

$$\widehat{\nu}_i + \lambda_j(D) = 0. \tag{3.21}$$

Proof The proof follows by [7, Proposition 2.6], where it is proved that the nonpositive eigenvalues of (3.18) are obtained by summing the eigenvalues of the one-dimensional problem (3.20) and the Neumann eigenvalues of $-\Delta_{\mathbb{S}^{N-1}}$ on D. We refer also to [11] for another approach, which consists in approximating the ball by annuli in order to avoid the singularity at 0. □

From Proposition 3.6 we get the following sufficient condition for a radial solution u_D to be nondegenerate.

Corollary 3.7 *A radial solution u_D of (3.4) in Ω_D (i.e. for $\varphi = 0$) is nondegenerate if both the following conditions are satisfied:*

(I) the eigenvalue problem (3.20) does not admit zero as an eigenvalue;
(II) $\lambda_1(D) > -\widehat{\nu}_1$.

Proof From Condition *(I)* we have

$$\widehat{\nu}_i \neq 0 \quad \forall i \in \mathbb{N}^+, \tag{3.22}$$

which means that zero is not an eigenvalue of (3.18) with a corresponding radial eigenfunction. This, in turn, is equivalent to saying that zero is not a "radial" eigenvalue of the linearized operator (3.17), i.e., u_D is a radial solution of (3.4) in Ω_D (or of (3.16) in B_1) which is nondegenerate in the subspace $H^1_{0,rad}(\Omega_D \cup \Gamma_{1,0})$, which is the subspace of $H^1_0(\Omega_D \cup \Gamma_{1,0})$ given by radial functions.

Now, since $\lambda_0(D) = 0$, $\lambda_1(D) > 0$ and since $\widehat{\nu}_1$ is the smallest eigenvalue of (3.20), then, from Condition *(II)* and (3.22) we infer that the sum (3.21) can never be zero. Hence, thanks to Proposition 3.6, we have that zero is not an eigenvalue of (3.18) and so cannot be an eigenvalue for the linearized operator (3.17) in the whole $H^1_0(\Omega_D \cup \Gamma_{1,0})$, i.e. u_D is a non-degenerate solution to (3.4) in Ω_D. □

Remark 3.8 Condition *(I)* in Corollary 3.7, i.e., the nondegeneracy of u_D in the space $H^1_{0,rad}(\Omega_D \cup \Gamma_{1,0})$, is satisfied by positive radial solutions of (3.4) corresponding to many kinds of nonlinearities.

It holds if f satisfies conditions *(i)* or *(ii)* of Proposition 3.5, because in this case all the eigenvalues of (3.17) and of (3.18) are positive. It then follows that *(II)* holds as well. More precisely, in the case *(i)*, since $0 < a < \mu_1(B_1)$, the first eigenvalue of L_{u_D} is positive, so

$$\underbrace{\lambda_0(D)}_{=0} + \widehat{\nu}_1 > 0. \tag{3.23}$$

In the case *(ii)*, since $f'(u_D) \leq 0$, it follows that $\widehat{\nu}_1 > 0$.

Among the nonlinearities satisfying condition *(iii)* of Proposition 3.5 we could consider $f(u) = u^p$, $1 < p < \frac{N+2}{N-2}$, $N \geq 3$. Then it is known that the positive radial solution of (3.16) is unique and nondegenerate (see [8, 15]), so *(I)* holds. It is also well-known that for this nonlinearity it holds $\widehat{\nu}_1 < 0$ and $\widehat{\nu}_1$ is the only negative eigenvalue of (3.20), because u_D can be obtained by the Mountain Pass Theorem or by minimization on the Nehari manifold and thus it has Morse index one. Then the validity of *(II)* depends on the cone, since it depends on $\lambda_1(D)$. However, once p is fixed, since $\widehat{\nu}_1$ does not depend on the cone, it is obvious that, by varying D, there are many cones for which *(II)* holds. Moreover, it has been proved in [7] that $\widehat{\nu}_1 > -(N-1)$ for every autonomous nonlinearity, so that whenever $\lambda_1(D) > N-1$ all radial solutions of (3.4) are nondegenerate.

3.3 Stability of (Ω_D, u_D)

Let us first observe that if u_D is a positive nondegenerate radial solution of (3.4) for $\varphi = 0$, belonging to $W^{1,\infty}(\Omega_D) \cap W^{2,2}(\Omega_D)$, then (Ω_D, u_D) is energy-stationary in the sense of Definition 2.4. Indeed, since u_D is radial, we have that $\frac{\partial u_D}{\partial \nu} =$ constant on $\Gamma_0 = D$ and thanks to Proposition 2.6 we easily conclude.

To investigate the stability of (Ω_D, u_D) we analyze the quadratic form corresponding to the second derivative $I''(\varphi)$ at $\varphi = 0$. Fixing the constant c in the definition of M (see (3.12)) as $c = |\Omega_D|$, we have that the tangent space to M at $\varphi = 0$ is given by

$$T_0M = \left\{ v \in C^2(\overline{D}, \mathbb{R}) \ : \ \int_D v \, d\sigma = 0 \right\}. \tag{3.24}$$

Writing $u_D(r) = u_D(|x|)$, we denote by u'_D and u''_D the derivatives of u_D with respect to r, so that

$$u'_D(1) = \left.\frac{\partial u_D}{\partial \nu}\right|_D, \qquad u''_D(1) = [(D^2 u_D \nu) \cdot \nu]|_D. \tag{3.25}$$

By Hopf's Lemma we know that $u'_D(1) < 0$ and actually

$$u'_D(1) = -\sqrt{-2\mu_D}, \tag{3.26}$$

where μ_D denotes the Lagrange multiplier in the case $\varphi = 0$, see (3.13).

For $v \in T_0M$, we will denote by $\widetilde{u}_v$ the solution of

$$\begin{cases} -\Delta \widetilde{u}_v - f'(u_D)\widetilde{u}_v = 0 & \text{in } \Omega_D \\ \widetilde{u}_v = -u'_D(1)v & \text{on } D \\ \dfrac{\partial \widetilde{u}_v}{\partial \nu} = 0 & \text{on } \Gamma_{1,0} \setminus \{0\} \end{cases}. \tag{3.27}$$

Let us remark that for every $q \in D$ the outer unit normal vector $\nu(q)$ is precisely q, hence (3.27) corresponds to (2.16) in Ω_D.

Note that, since u_D is a nondegenerate radial solution, then the weak solution $\widetilde{u}_v$ of (3.27) is unique for every v.

Our next result shows that the quadratic form corresponding to the second derivative of I at $\varphi = 0$ has a simple expression.

Lemma 3.9 *For any $v \in T_0M$ it holds*

$$I''(0)[v, v] = -u'_D(1) \left(\int_D v \frac{\partial \widetilde{u}_v}{\partial \nu} \, d\sigma + u''_D(1) \int_D v^2 \, d\sigma \right), \tag{3.28}$$

where $\widetilde{u}_v$ is the solution of (3.27).

Proof From Lemma 3.2, (3.11) and Lemma 3.4, with $w = v$, by simple substitutions and elementary computations we obtain:

$$\begin{aligned} I''(0)[v, v] = &-\frac{N}{2} \int_D (u'_D(1))^2 v^2 \, d\sigma - \int_D u'_D(1) v \frac{\partial \widetilde{u}_v}{\partial \nu} \, d\sigma \\ &- \int_D u'_D(1) v^2 (D^2 u_D \nu) \cdot \nu \, d\sigma - N\mu_D \int_D v^2 \, d\sigma. \end{aligned} \tag{3.29}$$

Since $\widetilde{u}_v = -u'_D(1)v$ on D, by (3.25) and (3.26), we deduce that

$$-\frac{N}{2} \int_D (u'_D(1))^2 v^2 \, d\sigma = -\frac{N}{2} \int_D \widetilde{u}_v^2 \, d\sigma; \tag{3.30}$$

$$- N\mu_D \int_D v^2 \, d\sigma = \frac{N}{2} \int_D \widetilde{u}_v^2 \, d\sigma. \tag{3.31}$$

Then (3.28) follows by substituting (3.30)-(3.31) into (3.29). □

To investigate the stability of (Ω_D, u_D) as an energy stationary pair for I we need to study the solution $\widetilde{u}_v$ of (3.27), for any $v \in T_0 M$ (that is, for functions with mean value zero on D). As we will see, it will be enough to consider only functions v which are eigenfunctions of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ with Neumann boundary conditions on D. Hence we consider the eigenvalue problem

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{on } D \\ \dfrac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial D \end{cases} \tag{3.32}$$

and denote its eigenvalues as in (3.19), counted with multiplicity: $0 = \lambda_0(D) < \lambda_1(D) \le \lambda_2(D) \le \ldots$. The corresponding L^2-normalized eigenfunctions are denoted by $\{\psi_j\}_{j\in\mathbb{N}}$, with $\int_D \psi_j^2 \, d\sigma = 1$, $\psi_0 =$ constant and $\int_D \psi_j \, d\sigma = 0$ for $j \ge 1$.

Theorem 3.10 *Let $j \ge 1$ and $\widetilde{u}_j$ be the unique solution of (3.27) for $v = \psi_j$. Then, writing $\widetilde{u}_j = \widetilde{u}_j(r, q)$, the function*

$$h_j(r) = \int_D \widetilde{u}_j(r, q)\psi_j(q) \, d\sigma, \quad r \in (0, 1) \tag{3.33}$$

satisfies

$$\begin{cases} -h_j'' - \dfrac{N-1}{r} h_j' - f'(u_D) h_j = -\dfrac{\lambda_j(D)}{r^2} h_j & \text{in } (0, 1) \\ h_j(1) = -u_D'(1) \end{cases} \tag{3.34}$$

Proof Since the proof is the same for all j, we drop the index and the dependence on D and write simply h, ψ and λ.

It is immediate to check that $h(1) = -u_D'(1)$. Moreover, since we can bring the radial derivative inside the integral on D, for every $r \in (0, 1]$ we have:

$$\begin{aligned} -h''(r) - \frac{N-1}{r} h'(r) &= \int_D \left(-\widetilde{u}_{rr}(r, q) - \frac{N-1}{r} \widetilde{u}_r(r, q) \right) \psi(q) \, d\sigma \\ &= \int_D \left(-\Delta \widetilde{u} + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} \widetilde{u} \right) \psi \, d\sigma \\ &= \int_D f'(u_D(r)) \widetilde{u} \psi \, d\sigma + \frac{1}{r^2} \int_D (\Delta_{\mathbb{S}^{N-1}} \widetilde{u}) \psi \, d\sigma. \end{aligned} \tag{3.35}$$

Now, on the one hand,

$$\int_D f'(u_D(r)) \widetilde{u} \psi \, d\sigma = f'(u_D(r)) h(r). \tag{3.36}$$

On the other hand, applying Green's formula, taking into account the Neumann conditions on ψ and $\widetilde{u}$, we infer that

$$\begin{aligned}\frac{1}{r^2}\int_D (\Delta_{\mathbb{S}^{N-1}}\widetilde{u})\psi\, d\sigma &= \frac{1}{r^2}\int_D \widetilde{u}\Delta_{\mathbb{S}^{N-1}}\psi\, d\sigma \\ &= -\frac{\lambda}{r^2}\int_D \widetilde{u}\psi\, d\sigma \\ &= -\frac{\lambda}{r^2}h(r).\end{aligned} \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.35) we conclude the proof. □

Remark 3.11 Note that with $\widetilde{u}_j$ and h_j as in Theorem 3.10 we have that

$$\widetilde{u}_j(r,q) = h_j(r)\psi_j(q).$$

Indeed, the boundary conditions are clearly satisfied by this function, and it holds

$$\begin{aligned}-\Delta(h_j\psi_j) &= -h_j''\psi_j - \frac{(N-1)}{r}h_j'\psi_j - \frac{h_j}{r^2}\Delta_{\mathbb{S}^{N-1}}\psi_j \\ &= f'(u_D)h_j\psi_j - \frac{\lambda_j(D)}{r^2}h_j\psi_j + \frac{\lambda_j(D)}{r^2}h_j\psi_j \\ &= f'(u_D)h_j\psi_j.\end{aligned}$$

Proposition 3.12 *Let $N \geq 3$. For any $j \geq 1$ we have*

$$\int_0^1 r^{N-3}h_j^2\, dr < +\infty \tag{3.38}$$

and

$$\int_0^1 r^{N-1}(h_j')^2\, dr < +\infty. \tag{3.39}$$

Moreover, $h_j \in L^\infty(0,\infty)$ and $h_j(0) = 0$.

Proof Again, for simplicity, we drop the index j. Since $\widetilde{u} \in H^1(\Omega_D)$ (see Sect. 2), writing $\widetilde{u} = \widetilde{u}(r,q)$ and recalling that ψ is a $L^2(D)$-normalized solution to (3.32), we get that

$$\begin{aligned}+\infty &> \int_{\Omega_D}|\nabla\widetilde{u}|^2\, dx \\ &= \int_0^1 r^{N-1}(h')^2\int_D \psi^2\, d\sigma\, dr + \int_0^1 r^{N-3}h^2\int_D |\nabla_{\mathbb{S}^{N-1}}\psi|^2\, d\sigma\, dr \\ &= \int_0^1 r^{N-1}(h')^2\, dr + \lambda\int_0^1 r^{N-3}h^2\, dr,\end{aligned}$$

which proves (3.38) and (3.39). Once we have these estimates, we can proceed as in [10, Lemma A.9] to get the boundness of h and $h(0) = 0$. □

Proposition 3.13 *Let $\lambda_j(D)$, $j \geq 1$ be a nontrivial Neumann eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}$ on D. Assume that*

$$-\widehat{\nu}_1 < \lambda_j(D),$$

where $\widehat{\nu}_1$ is the smallest eigenvalue of (3.20). Then for the solution h_j of (3.34) it holds that

$$h_j > 0 \quad in \quad (0, 1).$$

Proof Let z_1 be an L^2-normalized first eigenfunction of (3.20). From [3, Sect. 3.1] we know that z_1 does not change sign.

Writing the equations satisfied by h_j and z_1 in Sturm-Liouville form we have:

$$\begin{aligned}
&(r^{N-1}h_j')' + r^{N-1}(f'(u_D) - r^{-2}\lambda_j(D))h_j = 0,\\
&(r^{N-1}z_1')' + r^{N-1}(f'(u_D) + r^{-2}\widehat{\nu}_1)z_1 = 0.
\end{aligned}$$

By Proposition 3.12 we know that $h_j(0) = 0$ and $h_j(1) = -u_D'(1) > 0$.

Now, assume by contradiction that h_j changes sign in $(0, 1)$. Then there would exist $r_0 \in (0, 1)$ such that $h_j(0) = 0$. Since $-\widehat{\nu}_1 < \lambda_j(D)$, then, by the Sturm-Picone Comparison Theorem it would follow that z_1 has a zero in $(0, r_0)$. This is a contradiction, because z_1 does not change sign. Hence the only possibility is that h_j is strictly positive in $(0, 1)$. □

We are ready to prove our main result for problem (1.1) in the case of the cone, i.e., Theorem 1.1, which is a sharp instability/stability result for the pair (Ω_D, u_D).

Proof of Theorem 1.1

Let us fix the domain D which spans the cone, so that we denote $\lambda_1(D)$ simply by λ_1.

For *(i)*, let $\widetilde{u}_1 = h_1\psi_1$ be the solution of (3.27) with $v = \psi_1$. Then

$$I''(0)[\psi_1, \psi_1] = -u_D'(1)(h_1'(1) + u_D''(1)). \tag{3.40}$$

Putting (3.34) in Sturm-Liouville form we get

$$-(r^{N-1}h_1')' - r^{N-1}f'(u_D)h_1 = -r^{N-3}\lambda_1 h_1. \tag{3.41}$$

On the other hand, writing $-\Delta u_D = f(u_D)$ in polar coordinates and differentiating with respect to $r = |x|$ we get

$$-(u_D')'' - \frac{N-1}{r}(u_D')' - f'(u_D)u_D' = -\frac{N-1}{r^2}u_D',$$

which in Sturm-Liouville form is

$$-(r^{N-1}u_D'')' - r^{N-1}f'(u_D)u_D' = -r^{N-3}(N-1)u_D'. \tag{3.42}$$

Multiplying (3.41) by u_D' and integrating by parts in $(\bar{r}, 1)$ we get that

$$\begin{aligned}&\int_{\bar{r}}^1 r^{N-1}h_1'u_D''\,dr - (r^{N-1}h_1'u_D')\big|_{\bar{r}}^1 - \int_{\bar{r}}^1 r^{N-1}f'(u_D)h_1u_D'\,dr\\ &= -\lambda_1\int_{\bar{r}}^1 r^{N-3}h_1u_D'\,dr.\end{aligned} \tag{3.43}$$

Similarly, multiplying (3.42) by h_1 and integrating by parts we deduce that

$$\begin{aligned}&\int_{\bar{r}}^1 r^{N-1}h_1'u_D''\,dr - (r^{N-1}h_1u_D'')\big|_{\bar{r}}^1 - \int_{\bar{r}}^1 r^{N-1}f'(u_D)u_D'h_1\,dr\\ &= -(N-1)\int_{\bar{r}}^1 r^{N-3}u_D'h_1\,dr.\end{aligned} \tag{3.44}$$

Notice that, in view of Proposition 3.12, the right-hand sides of (3.43), (3.44) remain finite when taking the limit as $\bar{r} \to 0^+$. In addition, we claim that

$$\lim_{\bar{r}\to 0^+} r^{N-1}h_1'(\bar{r})u_D'(\bar{r}) = 0. \tag{3.45}$$

Indeed, integrating (3.41) and taking the absolute value we obtain

$$\begin{aligned}\left|\int_{\bar{r}}^1 -(r^{N-1}h_1')'\,dr\right| &= \left|\bar{r}^{N-1}h_1'(\bar{r}) - h_1'(1)\right|\\ &\le \int_{\bar{r}} r^{N-1}|f'(u_D)|h_1\,dr + \int_0^1 r^{N-3}\lambda_1h_1\,dr\\ &\le C_1\end{aligned}$$

for some $C_1 > 0$. Hence

$$\limsup_{\bar{r}\to 0^+} \bar{r}^{N-1}|h_1'(\bar{r})| \le C_2 \tag{3.46}$$

for some $C_2 > 0$, and thus, since $\lim_{\bar{r}\to 0^+} u_D'(\bar{r}) = 0$, (3.45) follows.

Now, subtracting (3.44) from (3.43) and taking the limit as $\bar{r} \to 0^+$, then, thanks to (3.45) and since $h_1(0) = 0$, $h_1(1) = -u_D'(1)$, we obtain

$$-u_D'(1)(h_1'(1) + u_D''(1)) = (N-1-\lambda_1)\int_0^1 r^{N-3}h_1u_D'\,dr. \tag{3.47}$$

Since $\lambda_1 > -\widehat{\nu}_1$, then, by Proposition 3.13, we have that $h_1 > 0$ in $(0, 1)$. On the other hand $u_D' < 0$ in $(0, 1)$ and $\lambda_1 < N-1$ by assumption. Hence by (3.40) and

(3.47) we obtain

$$I''(0)[\psi_1, \psi_1] < 0,$$

which proves *(i)*.

For *(ii)*, we choose an orthonormal basis $(\psi_j)_j$ of $L^2(D)$ made of normalized eigenfunctions of (3.32). Then any $v \in T_0 M$ can be written as

$$v = \sum_{j=1}^{\infty} (v, \psi_j)\psi_j,$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(D)$. We assume without loss of generality that $\int_D v^2 \, d\sigma = 1$. Let $\widetilde{u}_j$ be the solution of (3.27) with $v = \psi_j$, then we can check that

$$\widetilde{v} = \sum_{j=1}^{\infty} (v, \psi_j)\widetilde{u}_j$$

is the solution of (3.27). As observed in Remark 3.11, $\widetilde{u}_j(r, q) = h_j(r)\psi_j(q)$ for every $j \in \mathbb{N}$, so

$$\frac{\partial \widetilde{u}_j}{\partial \nu}(1, q) = h_j'(1)\psi_j(q) \quad \text{on } D.$$

By an argument analogous to the one presented in the proof of *(i)*, we have that if $k > j$, then $h_k'(1) \geq h_j'(1)$ and in fact $h_k'(1) > h_j'(1)$ if $k > j$ are such that $\lambda_k > \lambda_j$.

Indeed, writing the equations for h_j, h_k, multiplying the first one by h_k and the second one by h_j, integrating by parts and subtracting we get

$$-u_D'(1)(h_k'(1) - h_j'(1)) = (-\lambda_j + \lambda_k) \int_0^1 r^{N-3} h_i h_j \geq 0.$$

Exploiting the orthogonality of the basis $(\psi_j)_j$ and exploiting (3.47) we obtain

$$\begin{aligned}
I''(0)[v, v] &= -u_D'(1) \left(\int_D \left(\sum_{j=1}^{\infty} (v, \psi_j)\psi_j \right) \left(\sum_{k=1}^{\infty} (v, \psi_j) h_k'(1)\psi_k \right) d\sigma + u_D''(1) \int_D v^2 \, d\sigma \right) \\
&= -u_D'(1) \left(\left(\sum_{j=1}^{\infty} (v, \psi_j)^2 h_j'(1) \right) + u_D''(1) \right) \\
&\geq -u_D'(1) \left(h_1'(1) \left(\sum_{j=1}^{\infty} (v, \psi_j)^2 \right) + u_D''(1) \right) \\
&= -u_D'(1)(h_1'(1) + u_D''(1)) \\
&= (N - 1 - \lambda_1) \int_0^1 r^{N-3} h_1 u_D' \, dr > 0,
\end{aligned}$$

because $h_1 > 0$ in $(0, 1)$, $u'_D < 0$ in $(0, 1)$ and $\lambda_1 > N - 1$ by assumption. The proof is complete. □

Remark 3.14 As already pointed out in Remark 2.7, in the case when $\mathcal{C} = \mathbb{R}^N$, the couples (B, u_B), where B is a ball and u_B is a positive nondegenerate radial solution, are the only energy-stationary pairs. Thus it remains to study the stability of (B, u_B) as critical point of the energy functional T. This can be done by looking at the problem as the case of a cone spanned by the domain $D = \mathbb{S}^{N-1}$.

As observed in Remark 1.2, the first eigenvalue $\widehat{\nu}_1$ of the singular eigenvalue problem (3.20) is always larger than $-(N-1)$. On the other hand, it is known that the first nontrivial eigenvalue of the Laplace-Beltrami operator on the whole $\mathbb{S}^{N-1}$ is precisely $N-1$. Then any radial solution u_B is nondegenerate and we obtain that the pair (B, u_B) is a semistable stationary-point.

4 The Case of the Cylinder

Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain and let Σ_ω be the half-cylinder spanned by ω, namely

$$\Sigma_\omega := \omega \times (0, +\infty).$$

We denote by $x = (x', x_N)$ the points in $\overline{\Sigma}_\omega$, where $x' = (x_1, \ldots, x_{N-1}) \in \overline{\omega}$ and $x_N \geq 0$.

In analogy with the case of the cone, we consider domains whose relative boundaries are the cartesian graphs of functions in $C^2(\overline{\omega})$. More precisely, for $\varphi \in C^2(\overline{\omega})$ we set

$$\Gamma_\varphi := \{(x', x_N) \in \Sigma_\omega \; : \; x_N = e^{\varphi(x')}\}$$

and consider domains of the type

$$\Omega_\varphi = \{(x', x_N) \in \Sigma_\omega \; : \; x_N < e^{\varphi(x')}\}.$$

Finally, let

$$\Gamma_{1,\varphi} := (\partial\Omega_\varphi \setminus \overline{\Gamma}_\varphi).$$

Observe that the outer unit normal vector on Γ_φ at a point $(x', e^{\varphi(x')})$ is given by

$$\nu = \nu_\varphi(x') = \frac{(-e^{\varphi(x')}\nabla_{\mathbb{R}^{N-1}}\varphi(x'), 1)}{\sqrt{1 + |e^{\varphi(x')}\nabla_{\mathbb{R}^{N-1}}\varphi(x')|^2}}, \tag{4.1}$$

where $\nabla_{\mathbb{R}^{N-1}}$ denotes the gradient with respect to the variables $x_1, \ldots, x_{N-1}$.

4.1 Energy Functional in Cylindrical Domains

We study the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_\varphi \\ u = 0 & \text{on } \Gamma_\varphi \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,\varphi} \end{cases} \tag{4.2}$$

and consider bounded positive weak solutions of (4.2) in the Sobolev space $H_0^1(\Omega_\varphi \cup \Gamma_{1,\varphi})$, which is the space of functions in $H^1(\Omega_\varphi)$ whose trace vanishes on Γ_φ.

As before, we assume that a bounded nondegenerate positive solution u_φ of (4.2) exists and belongs to $W^{1,\infty}(\Omega_\varphi) \cap W^{2,2}(\Omega_\varphi)$, so that we can apply the results of Sect. 2.

We consider variations of the domain Ω_φ in the class of cartesian graphs of the type $\Omega_{\varphi+tv}$, for $v \in C^2(\overline{\omega})$, which amounts to consider a one-parameter family of diffeomorphisms $\xi : (-\eta, \eta) \times \overline{\Sigma}_\omega \to \overline{\Sigma}_\omega$ of the type

$$\xi(t, x) = (x', e^{tv(x')} x_N),$$

whose inverse, for any fixed $t \in (-\eta, \eta)$, is given by

$$\xi(t, x)^{-1} = (x', e^{-tv(x')} x_N) = \xi(-t, x).$$

This one-parameter family of diffeomorphisms is generated by the vector field

$$V(x) = (0', v(x') x_N), \tag{4.3}$$

where $0' := (0, \dots, 0) \in \mathbb{R}^{N-1}$. Indeed, $\xi(0, x) = x$ for every $x \in \overline{\Sigma}_\omega$,

$$\frac{d\xi}{dt}(t, x) = (0', e^{tv(x')} v(x') x_N) = V(\xi(t, x)) \quad \forall (t, x) \in (-\eta, \eta) \times \Sigma_\omega$$

and $\xi(t, x) \in \partial\Sigma_\omega$, for all $(t, x) \in (-\eta, \eta) \times \partial\Sigma_\omega$. We also observe that, in view of (4.1), it holds

$$\langle V, \nu \rangle = \left\langle (0', ve^\varphi), \frac{(-e^\varphi \nabla_{\mathbb{R}^{N-1}} \varphi, 1)}{\sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}} \varphi|^2}} \right\rangle = \frac{ve^\varphi}{\sqrt{1 + |e^\varphi \nabla_{\mathbb{R}^{N-1}} \varphi|^2}} \quad \text{on} \Gamma_\varphi. \tag{4.4}$$

The energy functional T defined in (2.10) becomes a functional depending only on functions in $C^2(\overline{\omega})$. More precisely, for every $v \in C^2(\overline{\omega})$, in view of Proposition 2.1, there exists $\delta > 0$ sufficiently small such that for all $t \in (-\delta, \delta)$

$$T(\varphi + tv) = T(\Omega_{\varphi+tv}) = J(u_{\varphi+tv}),$$

is well defined, where $u_{\varphi+tv} := u_{\Omega_{\varphi+tv}}$ is the unique positive solution of (4.2) in the domain $\Omega_{\varphi+tv}$, in a neighborhood of $u_\varphi \circ \xi_t^{-1}$.

By the results of Sect. 2 we know that the map $t \mapsto u_{\varphi+tv}$ is differentiable at $t = 0$, and the derivative $\widetilde{u}$ is a weak solution of

$$\begin{cases} -\Delta\widetilde{u} = f'(u_\varphi)\widetilde{u} & \text{in } \Omega_\varphi \\ \widetilde{u} = -\dfrac{\partial u_\varphi}{\partial \nu} \dfrac{ve^\varphi}{\sqrt{1+|e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}} & \text{on } \Gamma_\varphi \\ \dfrac{\partial \widetilde{u}}{\partial \nu} = 0 & \text{on } \Gamma_{1,\varphi} \end{cases} \tag{4.5}$$

We now compute the first derivative of T at Ω_φ, i.e., for $t = 0$, with respect to variations $v \in C^2(\overline{\omega})$.

Lemma 4.1 *Let $\varphi \in C^2(\overline{\omega})$ and assume that u_φ is a positive nondegenerate solution of (4.2) which belongs to $W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then, for any $v \in C^2(\overline{\omega})$ we have*

$$T'(\varphi)[v] = -\frac{1}{2}\int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)\right)^2 ve^\varphi \, dx'. \tag{4.6}$$

Proof The proof is similar to that of Lemma 3.1. It suffices to observe that for the parametrization of Γ_φ given by $x = (x', e^{\varphi(x')})$, for $x' \in \omega$, the induced $(N-1)$-dimensional area element on Γ_φ is expressed by

$$d\sigma_{\Gamma_\varphi} = \sqrt{1+|e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}\, dx'.$$

Then the result follows immediately from Proposition 2.2, taking into account (4.4). □

Lemma 4.2 *Let φ and u_φ be as in Lemma 4.1. Then for any $v, w \in C^2(\overline{\omega})$ it holds*

$$\begin{aligned} T''(\varphi)[v,w] = &-\frac{1}{2}\int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)\right)^2 e^\varphi vw\, dx' \\ &- \int_\omega \frac{\partial \widetilde{u}_w}{\partial \nu}(x', e^\varphi)\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)e^\varphi v\, dx' \\ &- \int_\omega \frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)[(D^2 u_\varphi(x', e^\varphi)(0', e^\varphi)) \cdot \nu]vw\, dx' \\ &+ \int_\omega \frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)e^{2\varphi} v \frac{\nabla u_\varphi(x', e^\varphi)\cdot(w\nabla_{\mathbb{R}^{N-1}}\varphi + \nabla_{\mathbb{R}^{N-1}}w, 0)}{\sqrt{1+|e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}}\, dx' \\ &+ \int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^\varphi)\right)^2 e^{3\varphi} v \frac{\nabla_{\mathbb{R}^{N-1}}\varphi \cdot (w\nabla_{\mathbb{R}^{N-1}}\varphi + \nabla_{\mathbb{R}^{N-1}}w)}{1+|e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}\, dx', \end{aligned} \tag{4.7}$$

where $\widetilde{u}_w$ is the solution of (4.5), with w in the place of v.

Proof Let $v, w \in C^2(\overline{\omega})$. By definition, Lemma 4.1 and using the Leibniz rule, we have:

$$\begin{aligned} T''(\varphi)[v,w] &= \frac{d}{ds}\bigg|_{s=0}\left(-\frac{1}{2}\int_\omega \left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(x', e^{\varphi+sw})\right)^2 e^{\varphi+sw} v\, dx'\right) \\ &= -\int_\omega e^{\varphi} v \frac{\partial u_\varphi}{\partial \nu}\, \frac{d}{ds}\bigg|_{s=0}\left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(x', e^{\varphi+sw})\right) dx' \\ &\quad -\frac{1}{2}\int_\omega \left(\frac{\partial u_\varphi}{\partial \nu}(x', e^{\varphi})\right)^2 e^{\varphi} vw\, dx'. \end{aligned} \tag{4.8}$$

To conclude it suffices to compute the derivative in the first integral of the right-hand side of (4.8). To this end we observe that

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0}\left(\frac{\partial u_{\varphi+sw}}{\partial \nu}(x', e^{\varphi+sw})\right) &= \frac{d}{ds}\bigg|_{s=0}\left(\nabla u_{\varphi+sw}(x', e^{\varphi+sw}) \cdot \nu_{\varphi+sw}\right) \\ &= \frac{d}{ds}\bigg|_{s=0}\left(\nabla u_{\varphi+sw}(x', e^{\varphi+sw})\right) \cdot \nu_\varphi \\ &\quad + \nabla u_\varphi(x', e^\varphi) \cdot \frac{d}{ds}\bigg|_{s=0} \nu_{\varphi+sw} \end{aligned} \tag{4.9}$$

where ν_φ is given by (4.1) and

$$\nu_{\varphi+sw} = \frac{(-e^{\varphi+sw}\nabla_{\mathbb{R}^{N-1}}(\varphi+sw), 1)}{\sqrt{1+|e^{\varphi+sw}\nabla_{\mathbb{R}^{N-1}}(\varphi+sw)|^2}}.$$

Now, for the first term in the right-hand side of (4.9), thanks to the argument presented in [17, Lemma 3.2], we have

$$\frac{d}{ds}(\nabla u_{\varphi+sw}) = \nabla\left(\frac{d}{ds}u_{\varphi+sw}\right),$$

and thus we obtain

$$\frac{d}{ds}\bigg|_{s=0}\left(\nabla u_{\varphi+sw}(x', e^{\varphi+sw})\right) = \nabla \widetilde{u}_w(x', e^\varphi) + D^2 u_\varphi(x', e^\varphi)(0', we^\varphi). \tag{4.10}$$

On the other hand, for the last term in (4.9), we check that

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \nu_{\varphi+sw} &= -\frac{e^\varphi}{\sqrt{1+|e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}}(\nabla_{\mathbb{R}^{N-1}} w + w\nabla_{\mathbb{R}^{N-1}}\varphi, 0) \\ &\quad - \frac{(e^\varphi)^2(w|\nabla_{\mathbb{R}^{N-1}}\varphi|^2 + \nabla_{\mathbb{R}^{N-1}}\varphi \cdot \nabla_{\mathbb{R}^{N-1}} w)}{1+|e^\varphi \nabla_{\mathbb{R}^{N-1}}\varphi|^2}\nu_\varphi \end{aligned} \tag{4.11}$$

Finally, substituting (4.9)–(4.11) into (4.8) we obtain (4.7). □

As in Sect. 3, in view of Definition 2.4, we consider a volume constraint. In the case of cartesian graphs, the volume of the domain Ω_φ associated to $\varphi \in C^2(\overline{\omega})$ is expressed by

$$\mathcal{V}(\varphi) = |\Omega_\varphi| = \int_\omega e^\varphi \, dx'. \tag{4.12}$$

The functional $\mathcal{V}$ is of class C^2 and for every $v, w \in C^2(\overline{\omega})$ it holds

$$\mathcal{V}'(\varphi)[v] = \int_\omega e^\varphi v \, dx', \qquad \mathcal{V}''(\varphi)[v, w] = \int_\omega e^\varphi v w \, dx'. \tag{4.13}$$

For $c > 0$ we define the manifold

$$M := \left\{ \varphi \in C^2(\overline{\omega}) \ : \ \int_\omega e^\varphi \, dx' = c \right\},$$

whose tangent space at any point $\varphi \in M$ is given by

$$T_\varphi M = \left\{ v \in C^2(\overline{\omega}) \ : \ \int_\omega e^\varphi v \, dx' = 0 \right\}. \tag{4.14}$$

We consider the restricted functional

$$I(\varphi) = T|_M(\varphi), \quad \varphi \in M.$$

As before, if $\varphi \in M$ is a critical point for I, then there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$T'(\varphi) = \mu I'(\varphi).$$

Results analogous to Proposition 3.3 and Lemma 3.4 hold with the same proofs. In particular, we point out that for an energy stationary pair $(\Omega_\varphi, u_\varphi)$ under a volume constraint the function u_φ has constant normal derivative on Γ_φ. For the reader's convenience, we restate here these results.

Proposition 4.3 *Let $\varphi \in M$ and let $(\Omega_\varphi, u_\varphi)$ be energy-stationary under a volume constraint. Then the Lagrange multiplier μ is negative and*

$$\frac{\partial u_\varphi}{\partial \nu} = -\sqrt{-2\mu} \quad on \quad \Gamma_\varphi.$$

Proof The same as in [17, Lemma 4.1] □

For the second derivative of I we have

Lemma 4.4 *Let $\varphi \in M$ and let $v, w \in T_\varphi M$. If $(\Omega_\varphi, u_\varphi)$ is energy-stationary under a volume constraint, then*

$$I''(\varphi)[v, w] = T''(\varphi)[v, w] - \mu \mathcal{V}''(\varphi)[v, w]. \tag{4.15}$$

Proof The same as in [17, Lemma 4.3] □

4.2 The Case $\varphi \equiv 0$ and One-Dimensional Solutions

When $\varphi \equiv 0$ (that is, $\Gamma_\varphi = \Gamma_0$ is the intersection of the cylinder with the plane $x_N = 1$), the domain Ω_0 is just the finite cylinder

$$\Omega_\omega := \omega \times (0, 1).$$

Then, if f is a locally Lipschitz continuous function, any weak solution of (4.2) is also a classical solution up to the boundary, i.e., it belongs to $C^2(\overline{\Omega}_\omega)$. This follows by standard regularity theory by considering the boundary conditions and that $\partial\Omega_\omega$ is made by the union of three $(N-1)$-dimensional manifolds (with boundary) intersecting orthogonally (see also [20, Proposition 6.1]).

In Ω_ω, for suitable nonlinearities, we can find a solution of (4.2) in Ω_ω which depends only on x_N in the following way: first, we can apply some variational method to find a solution u of the ordinary differential equation

$$\begin{cases} -u'' = f(u) & \text{in } (0, 1) \\ u'(0) = u(1) = 0 \end{cases} \tag{4.16}$$

and then set

$$u_\omega(x', x_N) := u(x_N), \qquad (x', x_N) \in \Omega_\omega.$$

Recall that, in one dimension, there is no critical Sobolev exponent for the embedding into L^p. So one example of a suitable nonlinearity is $f(u) = u^p$ with $1 < p < \infty$, or those of Proposition 3.5 with the only caution that in (iii), for $N \geq 2$ we can take $1 < p < \infty$.

For our purposes we need to consider one-dimensional solutions u_ω of (4.2) in Ω_ω that are nondegenerate, which means that the linearized operator

$$L_{u_\omega} = -\Delta - f'(u_\omega)$$

does not admit zero as an eigenvalue. In other words, u_ω is nondegenerate if there are no nontrivial weak solutions $\phi \in H_0^1(\Omega_\omega \cup \Gamma_{1,0})$ of the problem

$$\begin{cases} -\Delta\phi - f'(u_\omega)\phi = 0 & \text{in } \Omega_\omega \\ \phi = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Gamma_{1,0} \end{cases} \tag{4.17}$$

To analyze the spectrum of L_{u_ω} it is convenient to consider the following auxiliary one-dimensional eigenvalue problem:

$$\begin{cases} -z'' - f'(u_\omega)z = \alpha z & \text{in } (0,1) \\ z'(0) = z(1) = 0 \end{cases} \tag{4.18}$$

We denote the eigenvalues of (4.18) by α_i, for $i \in \mathbb{N}$. Clearly, they correspond to the eigenvalues of the linear operator

$$\widehat{L}_{u_\omega}(z) = -z'' - f'(u_\omega)z \tag{4.19}$$

with the boundary conditions of (4.18).

We also consider the following Neumann eigenvalue problem in the domain $\omega \subset \mathbb{R}^{N-1}$:

$$\begin{cases} -\Delta_{\mathbb{R}^{N-1}}\psi = \lambda\psi & \text{in } \omega \\ \dfrac{\partial\psi}{\partial\nu_{\partial\omega}} = 0 & \text{on } \partial\omega \end{cases} \tag{4.20}$$

where $-\Delta_{\mathbb{R}^{N-1}} = -\sum_{i=1}^{N-1} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^{N-1}$, i.e. with respect to the variables $x_1, \ldots, x_{N-1}$. We denote its eigenvalues by

$$0 = \lambda_0(\omega) < \lambda_1(\omega) \le \lambda_2(\omega) \le \ldots. \tag{4.21}$$

It is well-known that $\lambda_j(\omega) \nearrow +\infty$ as $j \to \infty$ and that the normalized eigenfunctions form a basis $(\psi_j)_j$ of the tangent space T_0M defined in (4.14) when $\varphi \equiv 0$.

Lemma 4.5 *The spectra of L_{u_ω}, $\widehat{L}_{u_\omega}$ and $-\Delta_{\mathbb{R}^{N-1}}$ with respect to the above boundary conditions are related by*

$$\sigma(L_{u_\omega}) = \sigma(\widehat{L}_{u_\omega}) + \sigma(-\Delta_{\mathbb{R}^{N-1}}). \tag{4.22}$$

Proof We begin by showing that $\sigma(L_{u_\omega}) \subset \sigma(\widehat{L}_{u_\omega}) + \sigma(-\Delta_{\mathbb{R}^{N-1}})$. Let $\tau \in \sigma(L_{u_\omega})$ and let $\phi \in H_0^1(\Omega_\omega \cup \Gamma_{1,0})$ be an associated eigenfunction, that is, ϕ is a weak solution of

$$\begin{cases} -\Delta\phi - f'(u_\omega)\phi = \tau\phi & \text{in } \Omega_\omega \\ \phi = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Gamma_{1,0} \end{cases} \tag{4.23}$$

As observed at the beginning of this subsection for the the nonlinear problem (4.2), by the shape of Ω_ω and the boundary conditions, since $f \in C^{1,\alpha}(\mathbb{R})$, by standard elliptic regularity, we have that ϕ is a classical solution of (4.23) in $\overline{\Omega}_\omega$.

Let λ be an eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ with homogeneous Neumann boundary condition on ω and let ψ be an associated eigenfunction. Define

$$z(x_N) := \int_\omega \phi(x', x_N)\psi(x')\,dx'. \tag{4.24}$$

Then, differentiating with respect to x_N, using Green's formulas and the boundary conditions we have

$$\begin{aligned} -z'' &= \int_\omega -\frac{\partial^2\phi}{\partial x_N^2}\psi\,dx' \\ &= \int_\omega (-\Delta\phi + \Delta_{\mathbb{R}^{N-1}}\phi)\psi\,dx' \\ &= \int_\omega f'(u_\omega)\phi\psi\,dx' + \int_\omega \tau\phi\psi\,dx' + \int_\omega \Delta_{\mathbb{R}^{N-1}}\psi\phi\,dx' \\ &= f'(u_\omega)z + \tau z - \lambda z. \end{aligned}$$

Thus $(\tau - \lambda) \in \sigma(\widehat{L}_{u_\omega})$ and hence $\tau = (\tau - \lambda) + \lambda \in \sigma(\widehat{L}_{u_\omega}) + \sigma(-\Delta_{\mathbb{R}^{N-1}})$.

To show the reverse inclusion, let $\alpha \in \sigma(\widehat{L}_{u_\omega})$, $\lambda \in \sigma(-\Delta_{\mathbb{R}^{N-1}})$ and let z, ψ be, respectively, the associated eigenfunctions. Setting for $x = (x', x_N) \in \Omega_\omega$

$$\phi(x', x_N) := z(x_N)\psi(x'),$$

we note that

$$\begin{aligned} -\Delta\phi &= -z''\psi - \Delta_{\mathbb{R}^{N-1}}\psi z \\ &= f'(u_\omega)z\psi + \alpha z\psi + \lambda z\psi \\ &= f'(u_\omega)\phi + (\alpha + \lambda)\phi. \end{aligned} \tag{4.25}$$

Finally, by construction, we easily check that ϕ satisfies the boundary conditions of (4.23). As a consequence, we deduce that

$$\alpha + \lambda \in \sigma(L_{u_\omega})$$

and this concludes the proof. □

Corollary 4.6 *The problem (4.17) admits zero as an eigenvalue if and only if there exist $i \in \mathbb{N}^+$ and $j \in \mathbb{N}$ such that*

$$\alpha_i + \lambda_j(\omega) = 0$$

holds.

Proof It follows immediately from Lemma 4.5. □

Corollary 4.7 *A one-dimensional solution of (4.2) is nondegenerate if both the following conditions are satisfied:*

(i) the eigenvalue problem (4.18) in $(0, 1)$ *does not admit zero as an eigenvalue;*
(ii) $\lambda_1(\omega) > -\alpha_1$.

Proof Analogous to the proof of Corollary 3.7. □

4.3 Stability/Instability of the Pair $(\Omega_\omega, u_\omega)$

In this subsection, we prove a general stability/instability theorem for the pair $(\Omega_\omega, u_\omega)$. We begin with some preliminary results.

Firstly, we recall that when $\varphi \equiv 0$ the tangent space $T_0 M$ is given by

$$T_0 M = \left\{ v \in C^2(\overline{\omega}) \ : \ \int_\omega v \, dx' = 0 \right\}. \tag{4.26}$$

Since u_ω depends on x_N only, in order to simplify the notations, we denote with a prime the derivative with respect to x_N, and thus we write

$$u'_\omega(x_N) = u'_\omega(x', x_N) := \frac{\partial u_\omega}{\partial x_N}(x', x_N).$$

Then, for $v \in T_0 M$, we have that the function $\widetilde{u}$ (see (4.5)), which belongs to $H^1(\Omega_\omega)$, is a weak solution of

$$\begin{cases} -\Delta \widetilde{u} = f'(u_\omega)\widetilde{u} & \text{in } \Omega_\omega \\ \widetilde{u} = -u'_\omega(1) v & \text{on } \Gamma_0 \\ \dfrac{\partial \widetilde{u}}{\partial \nu} = 0 & \text{on } \Gamma_{1,0} \end{cases} \tag{4.27}$$

As before, by elliptic regularity we know that $\widetilde{u}$ is regular in $\overline{\Omega}_\omega$, and thus it is a classical solution. We also note that, by the nondegeneracy of u_ω, there exists a unique solution of (4.27).

Lemma 4.8 *Let* $\lambda_j > 0$ *be any positive eigenvalue for the Neumann problem (4.20) and let* ψ_j *be any normalized eigenfunction associated to* λ_j. *Let* $\widetilde{u}_j \in H^1(\Omega_\omega)$ *be* *the solution of (4.27) with $v = \psi_j$. Then the function*

$$h_j(x_N) := \int_\omega \widetilde{u}_j(x', x_N)\psi_j(x')\, dx', \quad x_N \in (0, 1] \tag{4.28}$$

satisfies

$$\begin{cases} -h_j'' - f'(u_\omega)h_j = -\lambda_j h_j & \text{in} \quad (0, 1) \\ h_j(1) = -u_\omega'(1) \\ h_j'(0) = 0 \end{cases} \tag{4.29}$$

Proof For simplicity of notation we drop the index j and simply write $\tilde{u}$, h, ψ and λ instead of $\tilde{u}_j$, h_j, ψ_j and λ_j.

First observe that, as $\tilde{u} = -u_\omega'(1)\psi$ on Γ_0, we have

$$h(1) = \int_\omega -u_\omega'(1)\psi^2\, dx' = -u_\omega'(1).$$

Now, differentiating with respect to x_N under the integral sign and using Green's formula, taking into account the boundary conditions, we have

$$\begin{aligned} -h'' &= \int_\omega -\frac{\partial^2 \widetilde{u}}{\partial x_N^2}\psi\, dx' = \int_\omega (-\Delta\widetilde{u} + \Delta_{\mathbb{R}^{N-1}}\widetilde{u})\psi\, dx' \\ &= \int_\omega f'(u_\omega)\widetilde{u}\psi\, dx' + \int_\omega \Delta_{\mathbb{R}^{N-1}}\widetilde{u}\psi\, dx' \\ &= f'(u_\omega)h + \int_\omega \widetilde{u}\Delta_{\mathbb{R}^{N-1}}\psi\, dx' \\ &= f'(u_\omega)h - \lambda\int_\omega \widetilde{u}\psi\, dx' = f'(u_\omega)h - \lambda h. \end{aligned}$$

Finally, exploiting the Neumann condition for $\widetilde{u}$ on $\Gamma_{1,0}$, we check that $h'(0) = 0$. □

Remark 4.9 Note that for $\widetilde{u}_j$, h_j as in Lemma 4.8 we have that

$$\widetilde{u}_j(x', x_N) = h_j(x_N)\psi_j(x').$$

Indeed:

$$\begin{aligned} -\Delta(h_j(x_N)\psi_j(x')) &= -h_j(x_N)\Delta_{\mathbb{R}^{N-1}}\psi_j(x') - h_j''(x_N)\psi_j(x') \\ &= \lambda_j h_j(x_N)\psi_j(x') + f'(u_\omega)h_j(x_N)\psi_j(x') - \lambda_j h_j(x_N)\psi_j(x') \\ &= f'(u_\omega)\widetilde{u}_j. \end{aligned}$$

Moreover, by (4.29) and (4.20), the function $h_j\psi_j$ satisfies the boundary conditions in (4.27), so that $h_j\psi_j$ is the unique solution of (4.27) and thus coincides with $\widetilde{u}_j$.

Proposition 4.10 *Let $j \geq 1$, λ_j be a positive Neumann eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ in ω, and let h_j be the solution of (4.29). Assume that $-\alpha_1 < \lambda_j$, where α_1 is the smallest eigenvalue of (4.18). Then it holds that*

$$h_j > 0 \quad \textit{in } [0, 1].$$

Proof We can reflect h_j by parity with respect to 0 to have a solution of the linear problem

$$\begin{cases} -h_j'' - f'(u_\omega)h_j + \lambda_j h_j = 0 & \text{in } (-1, 1) \\ h_j(-1) = h_j(1) = -u_\omega'(1) > 0. \end{cases} \tag{4.30}$$

By reflection and (4.18), the first eigenvalue of the linear operator

$$z'' - f'(u_\omega)z \quad \text{in } (0, 1)$$

with the boundary condition $z(-1) = z(1) = 0$ is exactly α_1. Therefore the first eigenvalue of the linear operator

$$\widetilde{L}_{u_\omega} g = -g'' - f'(u_\omega)g + \lambda_j g$$

with zero boundary condition in $(-1, 1)$ is $\beta_1 = \alpha_1 + \lambda_j$.

It is well-known that $\widetilde{L}_{u_\omega}$ satisfies the maximum principle whenever $\beta_1 > 0$, i.e., when $\lambda_j > -\alpha_1$. Therefore, by (4.30), the function h_j satisfies $h_j \geq 0$ in $(-1, 1)$, and by the strong maximum principle we conclude that $h_j > 0$ in $(-1, 1)$. □

We can now state and prove the main result of this section.

Theorem 4.11 *Let $\omega \subset \mathbb{R}^{N-1}$ be a smooth bounded domain. Let $f \in C^{1,\alpha}_{loc}(\mathbb{R})$ such that there exists a positive one-dimensional non-degenerate solution u_ω of (1.1) in Ω_ω, and let h_1 be the solution to (4.29) with $j = 1$. Let $\lambda_1 = \lambda_1(\omega)$ be the first non-trivial eigenvalue of $-\Delta_{\mathbb{R}^{N-1}}$ with homogeneous Neumann conditions, let α_1 be the first-eigenvalue of (1.10) and let ρ be the number defined by*

$$\rho := -f(u_\omega(0))h_1(0) - \lambda_1 \int_0^1 h_1 u_\omega' \, dx_N. \tag{4.31}$$

Assume that $\lambda_1 > -\alpha_1$. Then

(i) if $\rho < 0$, then $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair;
(ii) if $\rho > 0$, then $(\Omega_\omega, u_\omega)$ is a stable energy stationary pair.

Proof We first observe that since $\frac{\partial u_\omega}{\partial \nu}$ is constant on Γ_0 then, by the analogous of Proposition 3.3 for cylinders, we infer that the pair $(\Omega_\omega, u_\omega)$ is an energy-stationary pair.

Let $w \in T_0 M$ and assume without loss of generality that $\int_\omega w^2\,dx' = 1$. In order to prove (i)-(ii) we first determine a suitable expression for $I''(0)[w, w]$. To this end, for each $j \in \mathbb{N}^+$, let $\widetilde{u}_j$ be the solution of (4.27) with $v = \psi_j$ and let h_j be the solution of (4.29). Then we can write

$$w = \sum_{j=1}^{\infty} (w, \psi_j)\psi_j$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\omega)$. Moreover, we can check that

$$\widetilde{u} = \sum_{j=1}^{\infty} (w, \psi_j)\widetilde{u}_j$$

is the solution of (4.27) corresponding to w. Then, taking $\varphi = 0$ in Lemma 4.2, exploiting Lemma 4.4, taking into account that by Proposition 4.3 the Lagrange multiplier μ is given by

$$\mu = -\frac{1}{2}(u'_\omega(1))^2,$$

by Remark 4.9 and observing that $\nabla u_\omega \perp (\nabla_{\mathbb{R}^{N-1}} w, 0)$, we infer that

$$\begin{aligned}
I''(0)[w, w] &= -\frac{1}{2}\int_\omega (u'_\omega(1))^2 w^2\,dx' \\
&\quad - \int_\omega u'_\omega(1)\left(\sum_{j=1}^{\infty}(w, \psi_j)h'_j(1)\psi_j\right)\left(\sum_{k=1}^{\infty}(w, \psi_k)\psi_k\right)dx' \\
&\quad - \int_\omega u'_\omega(1)u''_\omega(1)w^2\,dx' + \frac{1}{2}(u'_\omega(1))^2\int_\omega w^2\,dx' \\
&= -u'_\omega(1)\int_\omega\left(\sum_{j=1}^{\infty}(w, \psi_j)^2 h'_j(1)\psi_j^2\right)dx' - u'_\omega(1)u''_\omega(1)
\end{aligned}$$

Finally, since u_ω is a solution to (4.16) we deduce that

$$I''(0)[w, w] = -u'_\omega(1)\int_\omega\left(\sum_{j=1}^{\infty}(w, \psi_j)^2 h'_j(1)\psi_j^2\right)dx' + u'_\omega(1)f(0). \tag{4.32}$$

In particular, choosing $w = \psi_1$ and plugging it into (4.32) we infer that

$$I''(0)[\psi_1, \psi_1] = -u'_\omega(1)h'_1(1) + u'_\omega(1)f(0). \tag{4.33}$$

Multiplying the equation in (4.29) (with $j = 1$) by u'_ω and integrating by parts we get

$$-(h_1' u_\omega')\big|_0^1 + \int_0^1 h_1' u_\omega''\, dx_N = \int_0^1 (f'(u_\omega) - \lambda_1) h_1 u_\omega'\, dx_N.$$

Exploiting (4.16), integrating by parts and taking into account that $h_1(1) = -u'_\omega(1)$ we obtain

$$\begin{aligned}
&-h_1'(1)u_\omega'(1) - \int_0^1 h_1' f(u_\omega)\, dx_N \\
&= \int_0^1 f'(u_\omega)u_\omega' h_1\, dx_N - \lambda_1 \int_0^1 h_1 u_\omega'\, dx_N \\
&= (f(u_\omega)h_1)\big|_0^1 - \int_0^1 f(u_\omega)h_1'\, dx_N - \lambda_1 \int_0^1 h_1 u_\omega'\, dx_N \\
&= -f(0)u_\omega'(1) - f(u_\omega(0))h_1(0) - \int_0^1 f(u_\omega)h_1'\, dx_N - \lambda_1 \int_0^1 h_1 u_\omega'\, dx_N
\end{aligned} \tag{4.34}$$

Hence, we deduce that

$$-h_1'(1)u_\omega'(1) = -f(0)u_\omega'(1) - f(u_\omega(0))h_1(0) - \lambda_1 \int_0^1 h_1 u_\omega'\, dx_N \tag{4.35}$$

In the end, from (4.33), (4.35) and recalling (4.31), we obtain

$$I''(0)[\psi_1, \psi_1] = -f(u_\omega(0))h_1(0) - \lambda_1 \int_0^1 h_1 u_\omega'\, dx_N = \rho.$$

Therefore, if $\rho < 0$ then $I''(0)[\psi_1, \psi_1] < 0$, i.e., $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair, and this proves (i).

Let us prove (ii). Let $w \in T_0 M$ such that $\int_\omega w^2\, dx' = 1$. From (4.32) we know that $I''(0)[w, w] = -u_\omega'(1) \int_\omega \left(\sum_{j=1}^\infty (w, \psi_j)^2 h_j'(1) \psi_j^2 \right) dx' + u_\omega'(1) f(0)$. Thanks to the assumption $\lambda_1 > -\alpha_1$ the following holds true.

Claim: if $k > j$, then

$$h_k'(1) \geq h_j'(1), \tag{4.36}$$

and actually $h_k'(1) > h_j'(1)$ if $\lambda_k > \lambda_j$.

Indeed, by definition h_k, h_j satisfy, respectively, the following:

$$-h_k'' - f'(u_\omega)h_k = -\lambda_k h_k, \tag{4.37}$$

$$-h_j'' - f'(u_\omega)h_j = -\lambda_j h_j. \tag{4.38}$$

Multiplying (4.37) by h_j and integrating on $(0, 1)$ we obtain

$$\int_0^1 -h_k'' h_j\, dx_N = \int_0^1 h_k' h_j'\, dx_N - (h_k' h_j)\big|_0^1$$

$$= \int_0^1 f'(u_\omega) h_j h_k \, dx_N - \lambda_k \int_0^1 h_j h_k \, dx_N \tag{4.39}$$

Similarly, multiplying (4.38) by h_k, integrating on $(0, 1)$ and then subtracting the result from (4.39), we obtain

$$-(h_k' h_j - h_j' h_k)(1) = (\lambda_j - \lambda_k) \int_0^1 h_j h_k \, dx_N \le 0, \tag{4.40}$$

because $h_j > 0$ and $h_k > 0$ (see Proposition 4.10, which holds true for any $j \in \mathbb{N}^+$ because $\lambda_1 > -\alpha_1$). Now, since $h_j(1) = h_k(1) = -u_\omega(1)$, then by (4.40) we deduce that

$$u_\omega'(1)(h_k'(1) - h_j'(1)) \le 0.$$

Hence, as $u_\omega'(1) < 0$, Claim (4.36) easily follows.

Now, thanks to (4.32) and Claim (4.36), recalling again that $u_\omega'(1) < 0$ and exploiting (4.35) it follows that

$$\begin{aligned} I''(0)[w, w] &\ge -u_\omega'(1) h_1'(1) \int_\omega \left(\sum_{j=1}^{\infty} (w, \psi_j)^2 \psi_j^2 \right) dx' + u_\omega'(1) f(0) \\ &= -u_\omega'(1) h_1'(1) + u_\omega'(1) f(0) \\ &= -f(u_\omega(0)) h_1(0) - \lambda_1 \int_0^1 h_1 u_\omega' \, dx_N = \rho. \end{aligned} \tag{4.41}$$

Hence, if $\rho > 0$ we have that $I''(0)[w, w] > 0$ for all $w \in T_0 M$, i.e., $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair, and this proves (ii). The proof is complete. □

As a simple corollary of Theorem 4.11 we can now prove the stability/instability result of Theorem 1.4, which concerns the case of the torsional energy, i.e. when $f \equiv 1$.

Proof of Theorem 1.4

When $f \equiv 1$ the eigenvalue problem (4.18) has only positive eigenvalues and therefore the condition $\lambda_1 > -\alpha_1$ is automatically satisfied. The only solution of

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega_\omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{1,0} \end{cases} \tag{4.42}$$

is the one-dimensional positive function given by

$$u_\omega(x_N) = \frac{1 - x_N^2}{2}. \tag{4.43}$$

Clearly, as $u'_\omega(1) = -1$ and $f \equiv 1$, then for any $j \in \mathbb{N}^+$ (4.29) reduces to

$$\begin{cases} -h''_j + \lambda_j h_j = 0 \quad \text{in} \quad (0, 1) \\ h_j(1) = -u'_\omega(1) \\ h'_j(0) = 0 \end{cases}$$

whose unique solution is given by

$$h_j(x_N) = \frac{1}{\cosh(\sqrt{\lambda_j})} \cosh(\sqrt{\lambda_j} x_N).$$

In particular, taking $j = 1$ and exploiting (4.43) we can compute explicitly the number ρ in (4.31), namely

$$\rho = -\frac{1}{\cosh(\sqrt{\lambda_1})} + \frac{\lambda_1}{\cosh(\sqrt{\lambda_1})} \int_0^1 \cosh(\sqrt{\lambda_1} x_N) x_N \, dx_N.$$

Integrating by parts we readily check that

$$\int_0^1 \cosh(\sqrt{\lambda_1} x_N) x_N \, dx_N = \frac{\sinh(\sqrt{\lambda_1})}{\sqrt{\lambda_1}} - \frac{\cosh(\sqrt{\lambda_1})}{\lambda_1} + \frac{1}{\lambda_1},$$

and thus we obtain

$$\rho = \sqrt{\lambda_1} \tanh(\sqrt{\lambda_1}) - 1. \tag{4.44}$$

Let us consider the function $g : [0, +\infty[\to \mathbb{R}$, defined by $g(t) = \sqrt{t} \tanh(\sqrt{t}) - 1$. Clearly $g(0) = -1$ and $g(t) \to +\infty$ as $t \to +\infty$ and by monotonicity we infer that g has a unique zero in $]0, +\infty[$. We denote it by β and from the previous argument and (4.44) we infer that $\rho < 0$ if and only if $\lambda_1 < \beta$. Then, by Theorem 4.11-(i) we get that $(\Omega_\omega, u_\omega)$ is an unstable energy-stationary pair, and this proves (i).

Analogously, as $\rho > 0$ if and only if $\lambda_1 > \beta$, from Theorem 4.11-(ii) we obtain that $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair. The proof is complete. □

We conclude this section with the proof of Theorem 1.5.

Proof of Theorem 1.5

Let $w \in T_0 M$ such that $\int_\omega w^2 \, dx' = 1$. Since $\lambda_1 > -\alpha_1$, we can argue as in the proof of Theorem 4.11-(ii), in particular, from the first two lines of (4.41), taking into account that, by assumption, $f(0) = 0$, we have

$$I''(0)[w, w] \geq -u'_\omega(1)h'_1(1). \tag{4.45}$$

Now, since $h''_1 = (\lambda_1 - f'(u_\omega))h_1$ in $(0, 1)$ and $h_1 > 0$ in $[0, 1]$ by Proposition 4.10, then, thanks to the assumption $\lambda_1 > \sup_{x_N \in (0,1)} |f'(u_\omega(x_N))|$ we infer that $h''_1 > 0$ in $[0, 1]$. In particular, as $h'_1(0) = 0$ we deduce that

$$h'_1(1) > 0. \tag{4.46}$$

Finally, combining (4.45) and (4.46) we obtain that $I''(0)[w, w] > 0$ for all $w \in T_0M$, which means that $(\Omega_\omega, u_\omega)$ is a stable energy-stationary pair. □

Remark 4.12 We notice that, if f is a non-negative monotone increasing function, as in the case of the Lane-Emden nonlinearity (1.3), then by the Gidas-Ni-Nirenberg theorem ([15]) and by the monotonicity of f we infer that $\sup_{x_N \in (0,1)} |f'(u_\omega(x_N))| = f'(u_\omega(0))$. Thus the stability condition of Theorem 1.5 reduces to

$$\lambda_1 > f'(u_\omega(0)).$$

Remark 4.13 In the case of the Lane-Emden nonlinearity $f(u) = u^p$, at least for some integer values of p, it is possible to compute the solution u_ω numerically, as well as the eigenvalue α_1 and the function h_1 for different values of $\lambda_1(\omega)$. This allows to compute ρ numerically, so that, plotting the result for ρ as a function of $\lambda_1(\omega)$, we obtain a region of instability for $\lambda_1(\omega)$ close to $-\alpha_1$.

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