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The John–Nirenberg Space: Equality of the Vanishing Subspaces VJN_p and CJN_p

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Abstract
The John–Nirenberg spaces JN_p are generalizations of the space of bounded mean oscillation BMO with $JN_\infty = BMO$. Their vanishing subspaces VJN_p and CJN_p are defined in similar ways as VMO and CMO, which are subspaces of BMO. As our main result, we prove that VJN_p and CJN_p coincide by showing that certain Morrey type integrals of JN_p functions tend to zero for small and large cubes. We also show that $JN_{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^n)/\mathbb{R}$, if $p = q$.

Keywords John–Nirenberg space · Vanishing subspace · Morrey type integral · Euclidean space · Bounded mean oscillation · John–Nirenberg inequality

Mathematics Subject Classification 42B35 · 46E30

1 Introduction

In 1961, John and Nirenberg studied the well-known space of bounded mean oscillation BMO and proved the profound John–Nirenberg inequality for BMO functions [10]. The space BMO plays a vital role in harmonic analysis and it has been studied very extensively. For example, a celebrated result of Fefferman and Stein states that BMO can be characterized as the dual space of the real Hardy space H^1 [7]. In [10], John and Nirenberg also defined a generalization of BMO, which is now known as the John–Nirenberg space, or JN_p, with a parameter $1 < p < \infty$, see Definition 2.3 below. In addition, they proved the John–Nirenberg inequality for JN_p functions, see Theorem 2.4 below. From this theorem, it follows that $JN_p(Q_0) \subset L^{p,\infty}(Q_0)$, where $Q_0 \subset \mathbb{R}^n$ is a bounded cube. It is also easy to see that $L^p(Q_0) \subset JN_p(Q_0)$. Both of

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these inclusions are strict; however, this is far from trivial. An example of a function in $JN_p \setminus L^p$ was discovered in 2018 [4]. Thus, the space JN_p is a nontrivial space between L^p and $L^{p,\infty}$. However, there are still many unanswered questions related to the study of John–Nirenberg spaces.

Various John–Nirenberg type spaces have attracted attention in recent years, including the dyadic John–Nirenberg space [11], the congruent John–Nirenberg space [9, 22], the John–Nirenberg–Campanato space [17, 19], and the sparse John–Nirenberg space [5]. The John–Nirenberg space can also be defined with medians instead of using integral averages [13]. Hurri-Syrjänen et al. established a local-to-global result for the space $JN_p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set [8] and Marola and Saari found similar results for JN_p in the setting of metric measure spaces [12].

The spaces VMO and CMO are well-known vanishing subspaces of BMO. They were introduced by Sarason [16] and Neri [14], respectively. The aforementioned duality phenomenon of Fefferman and Stein was later complemented by Coifman and Weiss who showed that H^1 is the dual space of CMO [3]. Recently, there has been research on the JN_p counterparts of these spaces, which are denoted by VJN_p and CJN_p [20]. Brudnyi and Brudnyi showed that VJN_p is a predual to the Hardy-type space $HK_{p'}$ among other duality results for a family of related spaces $V_{pq}^{k\lambda}$ on a bounded cube [2]. The space $HK_{p'}$ was first introduced by Dafni et al. as a predual to the space JN_p [4]. Here, p' is the conjugate index of p, i.e., $1/p + 1/p' = 1$. It follows directly from the definitions of VJN_p and CJN_p that $L^p \subseteq CJN_p \subseteq VJN_p \subseteq JN_p$. Moreover, examples in [18] demonstrate that $L^p \neq CJN_p$ and $VJN_p \neq JN_p$. However, it has been an open question whether the set $VJN_p \setminus CJN_p$ is nonempty, see [18, 20]. As our main result, we show that VJN_p and CJN_p coincide.

Our method is to study Morrey or weak L^p type integrals

$$|Q|^{\frac{1}{p}-1} \int_Q |f|, \tag{1.1}$$

where Q is a cube. We prove that if $f \in JN_p$, then these integrals tend to zero both when $|Q| \to 0$ and when $|Q| \to \infty$. See Theorems 3.5 and 3.8 below for precise statements of these results. Note that L^p functions have this property, but weak L^p functions do not. From Theorem 3.5, it follows easily that $CJN_p = VJN_p$, see Corollary 3.7.

In Sect. 2 we briefly study the more general version of the John–Nirenberg type spaces $JN_{p,q}(X)$, where the L^1-norm of the oscillation term is replaced with the L^q-norm where $q \geq 1$. This generalization has been studied in [4, 21, 22]. It has turned out that in case X is a bounded cube, the $JN_{p,q}$ norm is equivalent with the JN_p norm (for $q < p$) or L^q norm (for $q \geq p$). In case $X = \mathbb{R}^n$, the $JN_{p,q}$ norm is equivalent with the JN_p norm (for $q < p$), and if $q > p$, the space contains only functions that are constant almost everywhere. We complete this picture by showing that in the borderline case $p = q$ and $X = \mathbb{R}^n$, this space is equivalent with the space $L^p(\mathbb{R}^n)/\mathbb{R}$, i.e., the space of functions f for which there is a constant b such that $f - b \in L^p(\mathbb{R}^n)$. The result answers a question raised in [22, Remark 2.9].

2 Preliminaries

Throughout this paper by a cube, we mean an open cube with edges parallel to the coordinate axes. We let $X \subseteq \mathbb{R}^n$ be either a bounded cube or the entire space $\mathbb{R}^n$. If Q is a cube, we denote by $l(Q)$ its side length. For any $r > 0$, we denote by rQ the cube with the same center as Q but with side length $r \cdot l(Q)$. For any measurable set $E \subset \mathbb{R}^n$, such that $0 < |E| < \infty$, we denote the integral average of a function f over E by

$$f_E := \fint_E f := \frac{1}{|E|} \int_E f.$$

Definition 2.1 (Weak L^p-spaces) Let $1 \leq p < \infty$. For a measurable function f, we define

$$\|f\|_{L^{p,\infty}(X)} := \sup_{t>0} t|\{x \in X : |f(x)| > t\}|^{1/p}.$$

We say that f is a weak L^p function, or $f \in L^{p,\infty}(X)$, if $\|f\|_{L^{p,\infty}(X)}$ is finite. We define

$$\|f\|_{L^{p,w}(X)} := \sup_{\substack{E \subseteq X \\ 0<|E|<\infty}} |E|^{1/p} \fint_E |f(x)|dx,$$

where E is any measurable set. We say that $f \in L^{p,w}(X)$, if $\|f\|_{L^{p,w}(X)}$ is finite.

Remark 2.2 The expression $\|\cdot\|_{L^{p,\infty}(X)}$ is not a norm, since the triangle inequality fails to hold. However, $\|\cdot\|_{L^{p,w}(X)}$ does define a norm. Additionally, if $p > 1$, $\|f\|_{L^{p,\infty}(X)}$ and $\|f\|_{L^{p,w}(X)}$ are comparable and therefore $L^{p,w}(X) = L^{p,\infty}(X)$, see Chap. 2.8.3 in [6].

Definition 2.3 (JN_p) Let $1 < p < \infty$. A function f is in $JN_p(X)$ if $f \in L^1_{loc}(X)$ and there is a constant $K < \infty$ such that

$$\sum_{i=1}^{\infty} |Q_i| \left(\fint_{Q_i} |f - f_{Q_i}| \right)^p \leq K^p$$

for all countable collections of pairwise disjoint cubes $(Q_i)_{i=1}^{\infty}$ in X. We denote the smallest such number K by $\|f\|_{JN_p(X)}$.

The space JN_p is related to BMO in the sense that the BMO norm of a function is the limit of the function's JN_p norm when p tends to infinity. It is easy to see that

$$\||f|\|_{JN_p(X)} \leq 2\|f\|_{JN_p(X)}. \tag{2.1}$$

Likewise, it is clear that $L^p(X) \subset JN_p(X)$, as we get from Hölder's inequality that $\|f\|_{JN_p(X)} \leq 2\|f\|_{L^p(X)}$. If X is a bounded cube, then $JN_p(X) \subset L^{p,\infty}(X)$. This is known as the John–Nirenberg inequality for JN_p functions.

Theorem 2.4 (John–Nirenberg inequality for JN_p) *Let* $1 < p < \infty$, $Q_0 \subset \mathbb{R}^n$ *a bounded cube and* $f \in JN_p(Q_0)$. *Then* $f \in L^{p,\infty}(Q_0)$ *and*

$$\|f - f_{Q_0}\|_{L^{p,\infty}(Q_0)} \leq c\|f\|_{JN_p(Q_0)}$$

with some constant $c = c(n, p)$.

The proof can be found in [1, 10], for example. In [18], this result was extended to the space $JN_p(\mathbb{R}^n)$. Also a more general John–Nirenberg space $JN_{p,q}$ has been studied, for example, in [4, 21, 22].

Definition 2.5 ($JN_{p,q}$) Let $1 \leq p < \infty$ and $1 \leq q < \infty$. A function f is in $JN_{p,q}(X)$ if $f \in L^1_{loc}(X)$, and there is a constant $K < \infty$ such that

$$\sum_{i=1}^{\infty} |Q_i| \left(\fint_{Q_i} |f - f_{Q_i}|^q \right)^{p/q} \leq K^p$$

for all countable collections of pairwise disjoint cubes $(Q_i)_{i=1}^{\infty}$ in X. We denote the smallest such number K by $\|f\|_{JN_{p,q}(X)}$.

It was shown in [4, Proposition 5.1] that if X is a bounded cube, then $JN_{p,q}(X) = JN_p(X)$, if $1 \leq q < p$ and $JN_{p,q}(X) = L^q(X)$ if $p \leq q < \infty$. The same proof also shows us that $JN_{p,q}(\mathbb{R}^n) = JN_p(\mathbb{R}^n)$, if $1 \leq q < p$. It was shown in [22, Corollary 2.8] that the space $JN_{p,q}(\mathbb{R}^n)$ contains only functions that are constant almost everywhere, if $p < q < \infty$. However, it was stated in [22, Remark 2.9] that the situation is unclear if $q = p$. We complete the picture by showing that $JN_{p,p}(\mathbb{R}^n)$ is equal to $L^p(\mathbb{R}^n)$ up to a constant. This also answers [21, Question 15].

Proposition 2.6 *Let* $1 \leq p < \infty$. *Then* $JN_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)/\mathbb{R}$ *and there is a constant* $c = c(p)$ *such that for any function* $f \in L^1_{loc}(\mathbb{R}^n)$, *we have*

$$\frac{1}{c}\|f\|_{JN_{p,p}(\mathbb{R}^n)} \leq \inf_{b \in \mathbb{R}} \|f - b\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{JN_{p,p}(\mathbb{R}^n)}.$$

Proof First assume that $f \in L^p/\mathbb{R}$, that is there is a constant b such that $f - b \in L^p$. Then for any set of pairwise disjoint cubes Q_i,

$$\begin{aligned}\sum_{i=1}^{\infty} \int_{Q_i} |f - f_{Q_i}|^p &= \sum_{i=1}^{\infty} \int_{Q_i} |(f - b) - (f - b)_{Q_i}|^p \\ &\leq \sum_{i=1}^{\infty} 2^p \int_{Q_i} |f - b|^p \leq 2^p \int_{\mathbb{R}^n} |f - b|^p\end{aligned}$$

and therefore $f \in JN_{p,p}(\mathbb{R}^n)$.

Now assume that $f \in JN_{p,p}(\mathbb{R}^n)$. Clearly

$$\int_Q |f - f_Q|^p \leq \|f\|^p_{JN_{p,p}(\mathbb{R}^n)}$$

for every cube $Q \subset \mathbb{R}^n$. Let $(Q_k)_{k=1}^\infty$ be a sequence of cubes such that the center of every cube is the origin and $|Q_k| = 2^k$. Then $Q_1 \subset Q_2 \subset \ldots$ and $\cup_{k=1}^\infty Q_k = \mathbb{R}^n$. We shall prove that the sequence of integral averages $(f_{Q_k})_{k=1}^\infty$ is a Cauchy sequence. For any integer i, we have

$$|f_{Q_i} - f_{Q_{i+1}}| \leq \fint_{Q_i} |f - f_{Q_{i+1}}| \leq 2 \fint_{Q_{i+1}} |f - f_{Q_{i+1}}|.$$

This means that

$$\begin{aligned} |f_{Q_i} - f_{Q_{i+1}}|^p &\leq 2^p \left(\fint_{Q_{i+1}} |f - f_{Q_{i+1}}| \right)^p \leq 2^p \fint_{Q_{i+1}} |f - f_{Q_{i+1}}|^p \\ &\leq 2^{p-i-1} \|f\|^p_{JN_{p,p}(\mathbb{R}^n)}. \end{aligned}$$

Then for any positive integers m and k, we get

$$\begin{aligned} |f_{Q_m} - f_{Q_k}| &\leq \sum_{i=\min(m,k)}^{\max(m,k)-1} |f_{Q_{i+1}} - f_{Q_i}| \leq \sum_{i=\min(m,k)}^{\infty} 2^{1-\frac{1}{p}} \|f\|_{JN_{p,p}(\mathbb{R}^n)} 2^{-i/p} \\ &= c\|f\|_{JN_{p,p}(\mathbb{R}^n)} 2^{-\min(m,k)/p}, \end{aligned} \tag{2.2}$$

where the constant c depends only on p. Therefore, $(f_{Q_k})_{k=1}^\infty$ is a Cauchy sequence. Then by using (2.2), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \left| f - \lim_{k\to\infty} f_{Q_k} \right|^p &= \lim_{m\to\infty} \int_{Q_m} \left| f - \lim_{k\to\infty} f_{Q_k} \right|^p \\ &\leq \lim_{m\to\infty} \int_{Q_m} 2^{p-1} \left(|f - f_{Q_m}|^p + \left| f_{Q_m} - \lim_{k\to\infty} f_{Q_k} \right|^p \right) \\ &\leq \lim_{m\to\infty} 2^{p-1} \left(\|f\|^p_{JN_{p,p}(\mathbb{R}^n)} + 2^m \lim_{k\to\infty} \left| f_{Q_m} - f_{Q_k} \right|^p \right) \\ &\leq 2^{p-1} \|f\|^p_{JN_{p,p}(\mathbb{R}^n)} \left(1 + \lim_{m\to\infty} 2^m \lim_{k\to\infty} c 2^{-\min(m,k)} \right) \\ &= 2^{p-1} \|f\|^p_{JN_{p,p}(\mathbb{R}^n)} \left(1 + \lim_{m\to\infty} 2^m c 2^{-m} \right) = c\|f\|^p_{JN_{p,p}(\mathbb{R}^n)}, \end{aligned}$$

where the constant c depends only on p. This means that $f - \lim_{k\to\infty} f_{Q_k} \in L^p$ and therefore $f \in L^p/\mathbb{R}$. This completes the proof. □

The spaces VJN_p and CJN_p were studied in [18, 20]. These spaces are JN_p counterparts of the spaces VMO and CMO, which are subspaces of BMO. The spaces VJN_p and CJN_p can also be defined in a bounded cube Q_0 instead of $\mathbb{R}^n$ as in the following definitions. However, in that case, it is clear that the spaces coincide [18, 20]. In this paper, we only define the spaces in $\mathbb{R}^n$ and we write $VJN_p = VJN_p(\mathbb{R}^n)$ and $CJN_p = CJN_p(\mathbb{R}^n)$ to simplify the notation.

Definition 2.7 (VJN_p) Let $1 < p < \infty$. Then the vanishing subspace VJN_p of JN_p is defined by setting

$$VJN_p := \overline{D_p(\mathbb{R}^n) \cap JN_p}^{JN_p},$$

where

$$D_p(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n)\}.$$

Definition 2.8 (CJN_p) Let $1 < p < \infty$. Then the subspace CJN_p of JN_p is defined by setting

$$CJN_p := \overline{C_c^\infty(\mathbb{R}^n)}^{JN_p},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the set of smooth functions with compact support in $\mathbb{R}^n$.

As in the case of vanishing subspaces of BMO, there exist characterizations of VJN_p and CJN_p as JN_p functions for which certain integrals vanish, see [20, Theorems 3.2 and 4.3].

Theorem 2.9 *Let $1 < p < \infty$. Then $f \in VJN_p$ if and only if $f \in JN_p$ and*

$$\lim_{a \to 0} \sup_{\substack{Q_i \subset \mathbb{R}^n \\ l(Q_i) \le a}} \sum_{i=1}^\infty |Q_i| \left(\fint_{Q_i} |f - f_{Q_i}| \right)^p = 0,$$

where the supremum is taken over all collections of pairwise disjoint cubes $(Q_i)_{i=1}^\infty$ in $\mathbb{R}^n$, such that the side length of each Q_i is at most a.

Theorem 2.10 *Let $1 < p < \infty$. Then $f \in CJN_p$ if and only if $f \in VJN_p$ and*

$$\lim_{a \to \infty} \sup_{\substack{Q \subset \mathbb{R}^n \\ l(Q) \ge a}} |Q|^{1/p} \fint_Q |f - f_Q| = 0,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that the side length of Q is at least a.

From the definitions, we can see that $L^p/\mathbb{R} \subseteq CJN_p \subseteq VJN_p \subseteq JN_p$. It was shown in [18] that $L^p/\mathbb{R} \ne CJN_p$ and $VJN_p \ne JN_p$. However, the question of whether CJN_p and VJN_p coincide remained open.

3 Equality of VJN_p and CJN_p

Let $1 < p < \infty$. In this section, we prove the equality of VJN_p and CJN_p by showing that for any JN_p function f, integrals of the type

$$|Q|^{\frac{1}{p}} \fint_Q |f|$$

tend to zero both when $|Q| \to 0$ and when $|Q| \to \infty$. This type of integral appears in the Morrey norm, see for example [15]. Compare it also to the weak L^p norm (Definition 2.1), where the supremum is taken over such integrals with the cube Q replaced with an arbitrary measurable set.

The aforementioned results follow from Proposition 3.1.

Proposition 3.1 *Let $X \subseteq \mathbb{R}^n$ be either a bounded cube or the entire space $\mathbb{R}^n$. Let $Q \subset X$ be a cube such that $3Q \subseteq X$. Let $1 < p < \infty$, $0 < A < \infty$ and $0 < \epsilon \leq \epsilon_0(n, p)$. Suppose that $f \in L^1_{loc}(X)$ is a nonnegative function such that*

$$|Q|^{\frac{1}{p}} \fint_Q f \geq A(1-\epsilon) \tag{3.1}$$

and for any cube $Q' \subset X$ with $l(Q') = \frac{2}{3}l(Q)$ or $l(Q') = \frac{4}{3}l(Q)$ we have

$$|Q'|^{\frac{1}{p}} \fint_{Q'} f \leq A(1+\epsilon). \tag{3.2}$$

Then there exist two cubes $Q_1 \subset 3Q$ and $Q_2 \subset 3Q$ such that for $i \in \{1, 2\}$, we have

$$l(Q_i) = \frac{2}{3}l(Q) \text{ or } l(Q_i) = \frac{4}{3}l(Q), \tag{3.3}$$

$$\operatorname{dist}(Q_1, Q_2) \geq \frac{1}{3}l(Q), \text{ and} \tag{3.4}$$

$$|Q_i|^{\frac{1}{p}} \fint_{Q_i} |f - f_{Q_i}| \geq c \cdot A, \tag{3.5}$$

where $c = c(n, p)$ is a positive constant.

To prove this proposition, we first need to prove Lemmas 3.2 and 3.4.

Lemma 3.2 *Let $0 < \alpha < 1 < \beta$. Let $Q' \subset Q \subset \tilde{Q} \subset \mathbb{R}^n$ be cubes such that $l(Q') = \alpha l(Q)$ and $l(\tilde{Q}) = \beta l(Q)$. Suppose that $f \in L^1(\tilde{Q})$ is a nonnegative function, $1 < p < \infty$, $0 < A < \infty$ and $0 < \epsilon \leq \epsilon_0(n, p, \alpha, \beta)$. Assume also that (3.1) holds for Q and (3.2) holds for Q' and $\tilde{Q}$. Then we have*

$$|\tilde{Q} \setminus Q'|^{\frac{1}{p}} \fint_{\tilde{Q} \setminus Q'} \left| f - f_{\tilde{Q} \setminus Q'} \right| \geq c_1 \cdot A,$$

where $c_1 = c_1(n, p, \alpha, \beta)$ is a positive constant.

Proof From the assumptions of the lemma, we get directly

$$\begin{aligned}
\int_{\tilde{Q}\setminus Q'} \left|f - f_{\tilde{Q}\setminus Q'}\right| &\geq \left|\int_{\tilde{Q}\setminus Q} f - \frac{|\tilde{Q}\setminus Q|}{|\tilde{Q}\setminus Q'|}\int_{\tilde{Q}\setminus Q'} f\right| + \left|\int_{Q\setminus Q'} f - \frac{|Q\setminus Q'|}{|\tilde{Q}\setminus Q'|}\int_{\tilde{Q}\setminus Q'} f\right| \\
&= 2\left|\int_Q f - \frac{1-\alpha^n}{\beta^n-\alpha^n}\int_{\tilde{Q}} f - \frac{\beta^n-1}{\beta^n-\alpha^n}\int_{Q'} f\right| \\
&\geq 2\bigg(A(1-\epsilon)|Q|^{1-\frac{1}{p}} - \frac{1-\alpha^n}{\beta^n-\alpha^n}A(1+\epsilon)|\tilde{Q}|^{1-\frac{1}{p}} \\
&\quad - \frac{\beta^n-1}{\beta^n-\alpha^n}A(1+\epsilon)|Q'|^{1-\frac{1}{p}}\bigg) \\
&= 2A|Q|^{1-\frac{1}{p}}\bigg(1-\epsilon - \frac{1-\alpha^n}{\beta^n-\alpha^n}(1+\epsilon)\left(\beta^n\right)^{1-\frac{1}{p}} \\
&\quad - \frac{\beta^n-1}{\beta^n-\alpha^n}(1+\epsilon)\left(\alpha^n\right)^{1-\frac{1}{p}}\bigg).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&|\tilde{Q}\setminus Q'|^{\frac{1}{p}} \fint_{\tilde{Q}\setminus Q'} \left|f - f_{\tilde{Q}\setminus Q'}\right| \\
&\geq 2\left(\beta^n-\alpha^n\right)^{\frac{1}{p}-1}\left(1-\epsilon - \frac{1-\alpha^n}{\beta^n-\alpha^n}(1+\epsilon)\beta^{n-\frac{n}{p}} - \frac{\beta^n-1}{\beta^n-\alpha^n}(1+\epsilon)\alpha^{n-\frac{n}{p}}\right)A \\
&= C(n,p,\alpha,\beta,\epsilon)A.
\end{aligned}$$

Notice that

$$\begin{aligned}
\lim_{\epsilon\to 0} C(n,p,\alpha,\beta,\epsilon) &= C(n,p,\alpha,\beta,0) \\
&= 2\left(\beta^n-\alpha^n\right)^{\frac{1}{p}-1}\left(1 - \frac{1-\alpha^n}{\beta^n-\alpha^n}\beta^{n-\frac{n}{p}} - \frac{\beta^n-1}{\beta^n-\alpha^n}\alpha^{n-\frac{n}{p}}\right).
\end{aligned}$$

This is positive for every n, p, α, and β. Indeed, we have $C(n,p,\alpha,\beta,0) = 2(\beta^n - \alpha^n)^{\frac{1}{p}-1}h(\frac{1}{p})$ with

$$h(x) = 1 - \frac{1-\alpha^n}{\beta^n-\alpha^n}\beta^{n-nx} - \frac{\beta^n-1}{\beta^n-\alpha^n}\alpha^{n-nx}.$$

We notice that $h(0) = h(1) = 0$ and the second derivative of h is strictly negative. Thus h is concave and $h(x) > 0$ for every $0 < x < 1$. In conclusion, if ϵ is small enough, we have

$$C(n,p,\alpha,\beta,\epsilon) \geq \frac{1}{2}C(n,p,\alpha,\beta,0) := c_1(n,p,\alpha,\beta) > 0.$$

This completes the proof. □

For the reader's convenience, we start by giving a proof of Proposition 3.1 in the special case $n = 1$ as it is technically much simpler. The idea of the proof is the same also in the multidimensional case.

Proof of Proposition 3.1 in the case $n = 1$ Let us assume that $Q = [a, a + L]$. Define $\tilde{Q} = \left[a, a + \frac{4}{3}L\right]$ and $Q' = \left[a, a + \frac{2}{3}L\right]$. We set $Q_1 := \tilde{Q}\backslash Q' = \left[a + \frac{2}{3}L, a + \frac{4}{3}L\right]$ and we get from Lemma 3.2 and the assumptions in Proposition 3.1 that

$$|Q_1|^{\frac{1}{p}} \fint_{Q_1} |f - f_{Q_1}| \geq c_1 A,$$

if ϵ is small enough. Here $c_1 = c_1(n, p, \alpha, \beta)$ with $n = 1$, $\alpha = \frac{2}{3}$ and $\beta = \frac{4}{3}$.

On the other hand, if we set $\tilde{Q} = \left[a - \frac{1}{3}L, a + L\right]$ and $Q' = \left[a + \frac{1}{3}L, a + L\right]$ and define $Q_2 := \tilde{Q}\backslash Q' = \left[a - \frac{1}{3}L, a + \frac{1}{3}L\right]$, then we get from Lemma 3.2 that

$$|Q_2|^{\frac{1}{p}} \fint_{Q_2} |f - f_{Q_2}| \geq c_1 A,$$

if ϵ is small enough. Finally we notice that the distance between the cubes Q_1 and Q_2 is $\frac{1}{3}L$. This completes the proof. □

The case $n \geq 2$ is more complicated as the set $\tilde{Q}\backslash Q'$ is usually not a cube. Before the actual proof, we fix some notation about directions and projections.

Definition 3.3 Let $\{v_1, \ldots, v_n\}$ denote the standard orthonormal basis for $\mathbb{R}^n$. Let $Q_1 \subset \mathbb{R}^n$ and $Q_2 \subset \mathbb{R}^n$ be cubes. The cubes can be presented as Cartesian products of intervals as

$$Q_1 = I_1^1 \times I_2^1 \times \ldots \times I_n^1 \text{ and}$$
$$Q_2 = I_1^2 \times I_2^2 \times \ldots \times I_n^2.$$

We say that $P_k(Q_1) := I_k^1$ is the projection of cube Q_1 to the subspace spanned by the base vector v_k. Fix an index k with $1 \leq k \leq n$. If for every $x_k \in P_k(Q_1)$ and $y_k \in P_k(Q_2)$ we have $x_k \leq y_k$, then we say that Q_2 is located in direction $\uparrow^k$ from Q_1 and Q_1 is located in direction $\downarrow^k$ from Q_2.

Lemma 3.4 *Let $Q \subset \mathbb{R}^n$ be a cube. Let $Q' \subset Q \subset \tilde{Q}$ be cubes with $l(Q') = \frac{2}{3}l(Q)$ and $l(\tilde{Q}) = \frac{4}{3}l(Q)$ such that all the cubes share a corner. By symmetry, we may assume that $Q = [0, L]^n$, $Q' = \left[0, \frac{2}{3}L\right]^n$ and $\tilde{Q} = \left[0, \frac{4}{3}L\right]^n$. Let $f \in L^1([0, 2L]^n)$ be a nonnegative function, $1 < p < \infty$, $0 < A < \infty$ and $0 < \epsilon \leq \epsilon_0$ from Lemma 3.2 with $\alpha = \frac{2}{3}$ and $\beta = \frac{4}{3}$. Suppose also that (3.1) holds for Q and (3.2) holds for Q' and $\tilde{Q}$. Then there exists a cube $\bar{Q} \subset [0, 2L]^n \setminus Q'$ such that either*

(a) *$l(\bar{Q}) = \frac{2}{3}L$ and $\bar{Q} \subset \tilde{Q}$*
or
(b) *$l(\bar{Q}) = \frac{4}{3}L$ and $P_k(\bar{Q}) = P_k(\tilde{Q}) = [0, \frac{4}{3}L]$ for every $1 \leq k \leq n$ except one*

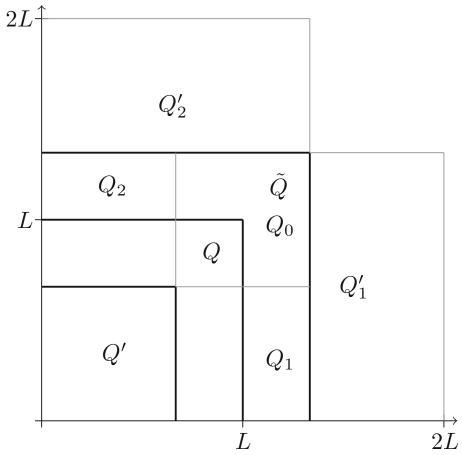

Fig. 1 The cubes Q, Q', $\tilde{Q}$, $(Q_i)_{i=0}^{2^n-2}$ and $(Q'_j)_{j=1}^n$, when $n = 2$. We have $\tilde{Q} = Q' \cup Q_0 \cup Q_1 \cup Q_2$, $Q_0 \cup Q_1 \subset Q'_1$ and $Q_0 \cup Q_2 \subset Q'_2$

and in addition

$$|\bar{Q}|^{\frac{1}{p}} \fint_{\bar{Q}} |f - f_{\bar{Q}}| \geq c_2 \cdot A,$$

where $c_2 = c_2(n, p)$ is a positive constant.

Proof Let us divide the cube $\tilde{Q}$ dyadically into 2^n subcubes. Then one of them is Q'. Let us define $Q_0 = \left[\frac{2}{3}L, \frac{4}{3}L\right]^n$ and let us denote the rest of the subcubes by $(Q_i)_{i=1}^{2^n-2}$. Notice that Q' does not have an index unlike all the other dyadic subcubes. For any $1 \leq j \leq n$, we define

$$Q'_j := I_1 \times I_2 \times ... \times I_n,$$

where

$$I_k = \begin{cases} [\frac{2}{3}L, 2L], & k = j, \\ [0, \frac{4}{3}L], & k \neq j. \end{cases}$$

See Fig. 1 to see how these cubes are located with respect to each other. It is simple to check that then

$$Q_0 \subset Q'_j \subset [0, 2L]^n \setminus Q' \quad \text{and} \quad l(Q'_j) = 2l(Q_0)$$

for every j. Also for every Q_i with $1 \leq i \leq 2^n - 2$, there exists at least one cube Q'_j such that $Q_i \subset Q'_j$. For every Q_i, let us denote by Q'_{j_i} one of these cubes Q'_j.

Let us prove that for at least one of the cubes Q_i or Q'_j, we have

$$|\bar{Q}|^{\frac{1}{p}} \fint_{\bar{Q}} |f - f_{\bar{Q}}| \geq \left(2^n - 1\right)^{-\frac{1}{p}} \left(1 + 2^{1+n-\frac{n}{p}} \left(\frac{2^n - 2}{2^n - 1}\right)^2\right)^{-1} c_1 A, \qquad (3.6)$$

where $c_1 = c_1(n, p, \alpha, \beta)$ is the constant from Lemma 3.2 with $\alpha = \frac{2}{3}$ and $\beta = \frac{4}{3}$. We prove this by contradiction. Assume that (3.6) does not hold for any Q_i or Q'_j. We get from Lemma 3.2 that

$$\begin{aligned} c_1 A &\leq |\tilde{Q} \setminus Q'|^{\frac{1}{p}} \fint_{\tilde{Q}\setminus Q'} \left| f - f_{\tilde{Q}\setminus Q'} \right| \\ &= \left((2^n - 1)|Q'|\right)^{\frac{1}{p}-1} \cdot \sum_{i=0}^{2^n-2} \int_{Q_i} \left| f - \frac{1}{(2^n-1)|Q'|} \sum_{k=0}^{2^n-2} \int_{Q_k} f \right|. \end{aligned} \tag{3.7}$$

We continue estimating one of the integrals in the sum above

$$\begin{aligned} &\int_{Q_i} \left| f - \frac{1}{(2^n-1)|Q'|} \sum_{k=0}^{2^n-2} \int_{Q_k} f \right| \\ &= \int_{Q_i} \left| f - f_{Q_i} + \frac{1}{(2^n-1)|Q'|} \sum_{k=0}^{2^n-2} \left(\int_{Q_i} f - \int_{Q_k} f \right) \right| \\ &\leq \int_{Q_i} |f - f_{Q_i}| + \frac{1}{2^n-1} \sum_{k=0}^{2^n-2} \left| \int_{Q_i} f - \int_{Q_k} f \right|. \end{aligned}$$

Assume that $k \geq 1$. Then because $Q_k \cup Q_0 \subset Q'_{j_k}$ and $Q_k \cap Q_0 = \varnothing$, we have

$$\left| \int_{Q_k} f - \int_{Q_0} f \right| \leq \left| \int_{Q_k} \left(f - f_{Q'_{j_k}} \right) \right| + \left| \int_{Q_0} \left(f_{Q'_{j_k}} - f \right) \right| \leq \int_{Q'_{j_k}} \left| f - f_{Q'_{j_k}} \right|.$$

Thus if $i = 0$, we get

$$\sum_{k=0}^{2^n-2} \left| \int_{Q_i} f - \int_{Q_k} f \right| \leq \sum_{k=1}^{2^n-2} \int_{Q'_{j_k}} \left| f - f_{Q'_{j_k}} \right|.$$

On the other hand, if $i \geq 1$, then

$$\begin{aligned} \sum_{\substack{k=0 \\ k\neq i}}^{2^n-2} \left| \int_{Q_i} f - \int_{Q_k} f \right| &\leq (2^n - 2) \left| \int_{Q_i} f - \int_{Q_0} f \right| + \sum_{\substack{k=0 \\ k\neq i}}^{2^n-2} \left| \int_{Q_0} f - \int_{Q_k} f \right| \\ &\leq (2^n - 2) \int_{Q'_{j_i}} \left| f - f_{Q'_{j_i}} \right| + \sum_{\substack{k=1 \\ k\neq i}}^{2^n-2} \int_{Q'_{j_k}} \left| f - f_{Q'_{j_k}} \right|. \end{aligned}$$

We continue by estimating the sum in (3.7) and we get

$$\begin{aligned}
&\sum_{i=0}^{2^n-2}\int_{Q_i}\left|f-\frac{1}{(2^n-1)|Q'|}\sum_{k=0}^{2^n-2}\int_{Q_k}f\right| \\
&\quad\le\sum_{i=0}^{2^n-2}\left(\int_{Q_i}|f-f_{Q_i}|+\frac{1}{2^n-1}\sum_{k=0}^{2^n-2}\left|\int_{Q_i}f-\int_{Q_k}f\right|\right) \\
&\quad\le\sum_{i=0}^{2^n-2}\int_{Q_i}|f-f_{Q_i}|+\frac{1}{2^n-1}\sum_{k=1}^{2^n-2}\int_{Q'_{j_k}}\left|f-f_{Q'_{j_k}}\right| \\
&\qquad+\sum_{i=1}^{2^n-2}\left(\frac{1}{2^n-1}\left((2^n-2)\int_{Q'_{j_i}}\left|f-f_{Q'_{j_i}}\right|+\sum_{\substack{k=1\\k\neq i}}^{2^n-2}\int_{Q'_{j_k}}\left|f-f_{Q'_{j_k}}\right|\right)\right) \\
&\quad=\sum_{i=0}^{2^n-2}\int_{Q_i}|f-f_{Q_i}|+\sum_{k=1}^{2^n-2}\int_{Q'_{j_k}}\left|f-f_{Q'_{j_k}}\right|+\frac{1}{2^n-1}\sum_{i=1}^{2^n-2}\sum_{\substack{k=1\\k\neq i}}^{2^n-2}\int_{Q'_{j_k}}\left|f-f_{Q'_{j_k}}\right| \\
&\quad=\sum_{i=0}^{2^n-2}\int_{Q_i}|f-f_{Q_i}|+\left(1+\frac{2^n-3}{2^n-1}\right)\sum_{k=1}^{2^n-2}\int_{Q'_{j_k}}\left|f-f_{Q'_{j_k}}\right|.
\end{aligned}$$

Then finally from our assumption, that (3.6) does not hold, we get

$$\begin{aligned}
&\sum_{i=0}^{2^n-2}\int_{Q_i}|f-f_{Q_i}|+\left(1+\frac{2^n-3}{2^n-1}\right)\sum_{k=1}^{2^n-2}\int_{Q'_{j_k}}\left|f-f_{Q'_{j_k}}\right| \\
&\quad<\sum_{i=0}^{2^n-2}(2^n-1)^{-\frac{1}{p}}\left(1+2^{1+n-\frac{n}{p}}\left(\frac{2^n-2}{2^n-1}\right)^2\right)^{-1}c_1A|Q_i|^{1-\frac{1}{p}} \\
&\qquad+2\cdot\frac{2^n-2}{2^n-1}\sum_{k=1}^{2^n-2}(2^n-1)^{-\frac{1}{p}}\left(1+2^{1+n-\frac{n}{p}}\left(\frac{2^n-2}{2^n-1}\right)^2\right)^{-1}c_1A|Q'_{j_k}|^{1-\frac{1}{p}} \\
&\quad=\left(2^n-1\right)^{1-\frac{1}{p}}\left(1+2^{1+n-\frac{n}{p}}\left(\tfrac{2^n-2}{2^n-1}\right)^2\right)^{-1}c_1A\left(|Q'|^{1-\frac{1}{p}}+\tfrac{2(2^n-2)^2}{(2^n-1)^2}\left(2^n|Q'|\right)^{1-\frac{1}{p}}\right) \\
&\quad=(2^n-1)^{1-\frac{1}{p}}c_1A|Q'|^{1-\frac{1}{p}}.
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}
c_1A&\le|\tilde{Q}\setminus Q'|^{\frac{1}{p}}\fint_{\tilde{Q}\setminus Q'}\left|f-f_{\tilde{Q}\setminus Q'}\right| \\
&<\left((2^n-1)|Q'|\right)^{\frac{1}{p}-1}\cdot(2^n-1)^{1-\frac{1}{p}}c_1A|Q'|^{1-\frac{1}{p}}=c_1A,
\end{aligned}$$

which is a contradiction. Hence, there is at least one cube $\bar{Q} \subset [0, 2L]^n \setminus Q'$ that satisfies the conditions of the lemma and

$$|\bar{Q}|^{\frac{1}{p}} \fint_{\bar{Q}} |f - f_{\bar{Q}}| \geq \left(2^n - 1\right)^{-\frac{1}{p}} \left(1 + 2^{1+n-\frac{n}{p}} \left(\frac{2^n - 2}{2^n - 1}\right)^2\right)^{-1} c_1 A = c_2 A,$$

where $c_2 = c_2(n, p)$ is a positive constant. This completes the proof. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1 in the case $n \geq 2$ Let $0 < \epsilon \leq \epsilon_0$ from Lemma 3.2 with $\alpha = \frac{2}{3}$ and $\beta = \frac{4}{3}$. We apply Lemma 3.4 2^n times to each corner of Q and obtain 2^n sets of cubes Q', $\tilde{Q}$ and $\bar{Q}$. Here, every $\bar{Q}$ satisfies (3.3) and (3.5). No matter which corner the cubes Q', Q and $\tilde{Q}$ share, we always have $\frac{1}{3}Q \subset Q'$. Since $\bar{Q}$ and Q' are disjoint, we get that $\bar{Q}$ and $\frac{1}{3}Q$ are also disjoint and thus each $\bar{Q}$ is located in at least one direction from $\frac{1}{3}Q$ in the sense of Definition 3.3.

Because Q' is in the corner of $\tilde{Q}$, there is one direction for each $k \in \{1, 2, ..., n\}$ such that $\bar{Q}$ cannot be located in that direction from $\frac{1}{3}Q$. For example, if $Q = [0, L]^n$, $Q' = [0, \frac{2}{3}L]^n$ and $\tilde{Q} = [0, \frac{4}{3}L]^n$, then the possible directions where $\bar{Q}$ may be located in from $\frac{1}{3}Q$ are $\uparrow^1$, $\uparrow^2$,... and $\uparrow^n$. The cube $\bar{Q}$ cannot be located in any of the directions $\downarrow^1$, $\downarrow^2$,... and $\downarrow^n$ from $\frac{1}{3}Q$. Thus, for each cube $\bar{Q}$, there are n possible directions and for any two cubes $\bar{Q}$, the sets of possible directions do not coincide.

If one cube $\bar{Q}$ is located in direction $\uparrow^k$ from $\frac{1}{3}Q$ and another is located in direction $\downarrow^k$, then the distance between those two cubes is at least $\frac{1}{3}l(Q)$ – thus the proposition is true. Therefore let us assume by contradiction that no two cubes $\bar{Q}$ are located in opposite directions.

Let S be the set of all directions in which all the cubes $\bar{Q}$ are located from $\frac{1}{3}Q$. Then we clearly have $|S| \leq n$, because by our assumption, there is at most one direction in S for each $k \in \{1, 2, ..., n\}$. However, there is always at least one cube $\bar{Q}$ for which the possible directions are all exactly opposite to the directions in S. If for example the directions in S are $\uparrow^1$, $\uparrow^2$,... and $\uparrow^m$ for some $m \leq n$, then there is no direction in S for the cube $\bar{Q}$ for which the possible directions are only $\downarrow^1$, $\downarrow^2$,... and $\downarrow^n$. This contradicts with the assumption that the directions of all $\bar{Q}$ are represented in S. Thus, we conclude that there must be two cubes $\bar{Q}$ in opposite directions. This completes the proof. □

Now we can show that the Morrey type integral (1.1) vanishes as the measure of the cube tends to infinity.

Theorem 3.5 *Let* $1 < p < \infty$ *and suppose that* $f \in JN_p(\mathbb{R}^n)$*. Then there is a constant* b *such that*

$$\lim_{a \to \infty} \sup_{\substack{Q \subset \mathbb{R}^n \\ l(Q) \geq a}} |Q|^{1/p} \fint_Q |f - b| = 0, \tag{3.8}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that the side length of Q is at least a.

Proof Assume first that $f \in JN_p(\mathbb{R}^n) \cap L^{p,\infty}(\mathbb{R}^n)$ and f is nonnegative. Let

$$\lim_{a\to\infty} \sup_{\substack{Q\subset\mathbb{R}^n\\ l(Q)\geq a}} |Q|^{1/p} \fint_Q f = A,$$

where $A \geq 0$. The limit exists as this sequence is decreasing, the elements are finite – this follows from Remark 2.2 because $f \in L^{p,\infty}(\mathbb{R}^n)$ – and the sequence is bounded from below by 0.

Let us assume that $A > 0$. Let $0 < \epsilon \leq \epsilon_0(n, p)$ as in Proposition 3.1. Then there exists a number $N < \infty$ such that (3.2) holds for every cube Q where $l(Q) \geq N$. Also for any number $M < \infty$, there exists a cube Q such that $l(Q) \geq M$ and (3.1) holds. This means that we can find a sequence of cubes $(Q_i)_{i=1}^{\infty}$ such that $l(Q_1) \geq \frac{3}{2}N$, $l(Q_{i+1}) > l(Q_i)$, $\lim_{i\to\infty} l(Q_i) = \infty$ and

$$|Q_i|^{1/p} \fint_{Q_i} f \geq A(1-\epsilon)$$

for every $i \in \mathbb{N}$.

Let Q_i be one of these cubes. According to Proposition 3.1, there exist two cubes $Q_{i,1} \subset 3Q_i$ and $Q_{i,2} \subset 3Q_i$ that satisfy (3.3), (3.4) and (3.5). The cubes $(Q_{i,1})_{i=1}^{\infty}$ may not be pairwise disjoint. However, for every cube Q_i, we have two cubes to choose from.

Let us construct a new sequence of cubes $(Q'_{i_j})_{j=1}^{\infty}$ iteratively. We start with $Q_{i_1} := Q_1$ and choose $Q'_{i_1} := Q_{1,1}$. Let Q_{i_2}, $i_2 > 1$, be the smallest cube in the sequence $(Q_i)_{i=1}^{\infty}$ such that $\frac{1}{3}l(Q_{i_2}) \geq l(Q'_{i_1})$. Then at least one of the cubes $Q_{i_2,1}$ and $Q_{i_2,2}$ is pairwise disjoint with Q'_{i_1}. Let's say that $Q_{i_2,1}$ is the disjoint one and set $Q'_{i_2} := Q_{i_2,1}$.

Let us denote by Q the smallest cube such that $Q'_{i_1} \cup Q'_{i_2} \subset Q$. Let Q_{i_3}, $i_3 > i_2$, be the smallest cube in the sequence $(Q_i)_{i=1}^{\infty}$ such that $\frac{1}{3}l(Q_{i_3}) \geq l(Q)$. Then at least one of the cubes $Q_{i_3,1}$ and $Q_{i_3,2}$ is pairwise disjoint with both Q'_{i_1} and Q'_{i_2}.

By repeating this process and taking a subsequence of $(Q_i)_{i=1}^{\infty}$, if necessary, we get infinitely many pairwise disjoint cubes $(Q'_{i_j})_{j=1}^{\infty}$. Then we get

$$\sum_{j=1}^{\infty} |Q'_{i_j}| \left(\fint_{Q'_{i_j}} \left| f - f_{Q'_{i_j}} \right| \right)^p \geq \sum_{j=1}^{\infty} c(n,p)A^p = \infty.$$

This contradicts with the assumption that $f \in JN_p(\mathbb{R}^n)$. Thus, we conclude that $A = 0$.

Now assume only that $f \in JN_p(\mathbb{R}^n)$. From [18, Theorem 5.2], we get that there is a constant b such that $f - b \in L^{p,\infty}(\mathbb{R}^n)$. Obviously $f - b \in JN_p(\mathbb{R}^n)$. Then from (2.1), we get that $|f - b| \in JN_p(\mathbb{R}^n) \cap L^{p,\infty}(\mathbb{R}^n)$. Finally by the same reasoning as before, we conclude that

$$\lim_{a\to\infty} \sup_{\substack{Q\subset\mathbb{R}^n \\ l(Q)\geq a}} |Q|^{1/p} \fint_Q |f-b| = 0.$$

This completes the proof. □

Remark 3.6 Naturally Theorem 3.5 implies that the result holds also for L^p functions with $p > 1$, but this can also be seen easily by considering the Hardy-Littlewood maximal function. Indeed assume by contradiction that

$$\lim_{a\to\infty} \sup_{\substack{Q\subset\mathbb{R}^n \\ l(Q)\geq a}} |Q|^{\frac{1}{p}} \fint_Q |f| = A > 0.$$

Then for any number $a < \infty$, there exists a cube Q such that $l(Q) \geq a$ and

$$|Q|^{\frac{1}{p}} \fint_Q |f| \geq \frac{A}{2}.$$

Thus for every $x \in Q$, we have $Mf(x) \geq \frac{A}{2}|Q|^{-\frac{1}{p}}$, where Mf is the non-centered Hardy-Littlewood maximal function. Thus $Mf(x) \notin L^p$ and consequently $f \notin L^p$.

Clearly the theorem does not hold for weak L^p functions. For example let $n = 1$ and $f(x) = x^{-1/p}$, when $x > 0$. Then we have $f \in L^{p,\infty}(\mathbb{R})$ but (3.8) does not hold for any b. The same function shows us that Theorem 3.8 does not hold for weak L^p functions.

As a corollary from Theorem 3.5, we get that VJN_p and CJN_p coincide. This answers a question that was posed in [18] and it answers [20, Question 5.6] and [21, Question 17].

Corollary 3.7 *Let* $1 < p < \infty$. *Then* $CJN_p(\mathbb{R}^n) = VJN_p(\mathbb{R}^n)$.

Proof It is clear that $CJN_p \subseteq VJN_p$. Let $f \in VJN_p \subset JN_p$. Then we get from Theorem 3.5 that there is a constant b such that

$$\begin{aligned}\lim_{a\to\infty} \sup_{\substack{Q\subset\mathbb{R}^n \\ l(Q)\geq a}} |Q|^{1/p} \fint_Q |f-f_Q| &= \lim_{a\to\infty} \sup_{\substack{Q\subset\mathbb{R}^n \\ l(Q)\geq a}} |Q|^{1/p} \fint_Q |f-b-(f-b)_Q| \\ &\leq \lim_{a\to\infty} \sup_{\substack{Q\subset\mathbb{R}^n \\ l(Q)\geq a}} |Q|^{1/p} 2 \fint_Q |f-b| = 0.\end{aligned}$$

Then by Theorem 2.10, we get that $f \in CJN_p$. This completes the proof. □

From the following result, we can infer that the additional condition in [18, Lemma 5.8] is not necessary, answering the question posed in [18].

Theorem 3.8 *Let $X \subseteq \mathbb{R}^n$ be either a bounded cube or the entire space $\mathbb{R}^n$ and $1 < p < \infty$. Suppose that $f \in JN_p(X)$. Then*

$$\lim_{a\to 0} \sup_{\substack{Q\subseteq X\\ l(Q)\leq a}} |Q|^{1/p} \fint_Q |f| = 0, \tag{3.9}$$

where the supremum is taken over all cubes $Q \subseteq X$, such that the side length of Q is at most a.

Remark 3.9 The theorem implies that L^p functions also satisfy (3.9). However, in that case, the result is well known as it follows simply from Hölder's inequality and the dominated convergence theorem. Notice that for L^p functions, (3.9) holds also when $p = 1$, whereas in Remark 3.6, it is necessary that $p > 1$.

Remark 3.10 In particular, this result implies that for any $1 < p < \infty$ and bounded cube $X \subset \mathbb{R}^n$, the John–Nirenberg space $JN_p(X)$ is a subspace of the vanishing Morrey space $VL^{1,n-\frac{n}{p}}(X)$ as defined in [15].

Proof of Theorem 3.8 The proof is similar to the proof of Theorem 3.5, but we have to take into account the possibility that we might have $3Q \not\subseteq X$, even though $Q \subseteq X$.

Let $f \in JN_p(X)$. Let us assume first that f is nonnegative and $f \in L^{p,\infty}(X)$. Let

$$\lim_{a\to 0} \sup_{\substack{Q\subseteq X\\ l(Q)\leq a}} |Q|^{1/p} \fint_Q f = A. \tag{3.10}$$

The limit exists as the sequence is decreasing and bounded from below by 0.

Assume that $A > 0$. Let $0 < \epsilon \leq \epsilon_0(n, p)$ as in Proposition 3.1. Then there is a number $\delta > 0$ such that (3.2) holds for any cube Q where $l(Q) \leq \delta$. Also because of (3.10), we know that for every $a > 0$, there exists a cube $Q \subseteq X$ such that $l(Q) \leq a$ and (3.1) holds for Q. Therefore, we can find a sequence of cubes $(Q_i)_{i=1}^{\infty}$ such that $l(Q_1) \leq \frac{3}{4}\delta$, $l(Q_{i+1}) < l(Q_i)$, $\lim_{i\to\infty} l(Q_i) = 0$ and

$$|Q_i|^{1/p} \fint_{Q_i} f \geq A(1-\epsilon).$$

for every $i \geq 1$.

Case 1: There are infinitely many cubes Q_i in the sequence such that $3Q_i \subseteq X$.

By taking a subsequence, we may assume that $3Q_i \subseteq X$ for every i. Let Q_i be one of these cubes. Then according to Proposition 3.1, there exist two cubes $Q_{i,1} \subset 3Q_i$ and $Q_{i,2} \subset 3Q_i$ that satisfy (3.3), (3.4) and (3.5). The cubes $(Q_{i,1})_{i=1}^{\infty}$ may not be pairwise disjoint. However, for every cube Q_i, we have two cubes to choose from.

Let us construct a new sequence of cubes $(Q'_{i_j})_{j=1}^{\infty}$ iteratively. We start with $Q_{i_1} := Q_1$. Then there exist infinitely many cubes Q_i in the sequence such that $Q_{1,1} \cap 3Q_i = \varnothing$ for every i or $Q_{1,2} \cap 3Q_i = \varnothing$ for every i. This is true because $\operatorname{dist}(Q_{1,1}, Q_{1,2}) \geq l(Q_1)/3$ and $\lim_{i\to\infty} l(Q_i) = 0$. Without loss of generality, we may assume that

$Q_{1,1} \cap 3Q_i = \varnothing$ for infinitely many $i > 1$. We set $Q'_{i_1} := Q_{1,1}$. Let Q_{i_2}, $i_2 > 1$, be the first cube in the sequence $(Q_i)_{i=1}^{\infty}$ such that $Q'_{i_1} \cap 3Q_{i_2} = \varnothing$. Then the cubes Q'_{i_1}, $Q_{i_2,1}$ and $Q_{i_2,2}$ are all pairwise disjoint. Thus, we can continue by choosing one of the cubes $Q_{i_2,1}$ and $Q_{i_2,2}$ such that there are still infinitely many cubes $3Q_i$, $i > i_2$, that are pairwise disjoint with both the chosen cube and Q'_{i_1}.

By repeating this process and taking a subsequence of $(Q_i)_{i=1}^{\infty}$, if necessary, we get infinitely many pairwise disjoint cubes $(Q'_{i_j})_{j=1}^{\infty}$ in X that all satisfy (3.5).

Case 2: There are only finitely many cubes Q_i in the sequence such that $3Q_i \subseteq X$. This can only happen if X is a bounded cube. In this case, we shall also construct a new sequence of pairwise disjoint cubes $(Q'_{i_j})_{j=1}^{\infty}$ iteratively. By taking a subsequence of $(Q_i)_{i=1}^{\infty}$, we may assume that $l(Q_1) \leq \frac{1}{4}l(X)$ and $l(Q_{i+1}) \leq \frac{2}{9}l(Q_i)$. Assume that $3Q_i \not\subset X$. Then we know that for any $k \in \{1, 2, ..., n\}$, the projection $P_k(3Q_i)$ can only intersect one of the endpoints of $P_k(X)$. Let m be the number of base vectors v_k such that $P_k(3Q_i) \subset P_k(X)$. Then $0 \leq m < n$ and there exist 2^m cubes $\bar{Q}_i \subset X \cap 3Q_i \setminus \frac{1}{3}Q_i$ as they are defined in Lemma 3.4.

Every such cube $\bar{Q}_i$ is located in some direction from the cube $\frac{1}{3}Q_i$, in a similar way as in the proof of Proposition 3.1. Without loss of generality, we may assume that the possible directions for these cubes are $\uparrow^1, \downarrow^1, \uparrow^2, \downarrow^2, ..., \uparrow^m, \downarrow^m, \downarrow^{m+1}, \downarrow^{m+2}, ..., \downarrow^n$, but for each $\bar{Q}_i$ there is naturally only one possible direction for each $k \in \{1, 2, ..., n\}$. If one of the cubes is located in direction $\uparrow^k$ and another is in direction $\downarrow^k$, $k \leq m$, then the distance between these two cubes is at least $\frac{1}{3}l(Q_i)$. Thus, we can choose one of these cubes into the sequence $(Q'_{i_j})_{j=1}^{\infty}$ the same way as in the first case and there are still infinitely many cubes Q_l, $l > i$, such that $3Q_l \cap \bar{Q}_i = \varnothing$.

Let us assume that there are no two cubes $\bar{Q}_i$ that are located in opposite directions from $\frac{1}{3}Q_i$. Then by similar reasoning as in the proof of Proposition 3.1, we know that at least one of the cubes $\bar{Q}_i$ must be located in direction $\downarrow^k$ for some $k > m$. Let us denote such a cube by $Q_{i,1}$. If there are infinitely many cubes Q_l, $l > i$, such that $3Q_l \cap Q_{i,1} = \varnothing$, then we may choose $Q_{i,1}$ into the sequence $(Q'_{i_j})_{j=1}^{\infty}$.

Assume that there are only finitely many cubes Q_l, $l > i$, such that $3Q_l \cap Q_{i,1} = \varnothing$. Then there are infinitely many cubes Q_l, $l > i$, such that $3Q_l \cap Q_{i,1} \neq \varnothing$. Let Q_l be one of these cubes. Because $l(Q_l) \leq \frac{2}{9}l(Q_i)$, we know that $P_k(3Q_l) \subset P_k(X)$. This is because the distance from $P_k(Q_{i,1})$ to $\partial P_k(X)$ is at least $\frac{2}{3}l(Q_i)$.

In addition for every $1 \leq s \leq m$, we have $P_s(3Q_l) \subset P_s(X)$. This is because of how the cube $\bar{Q}$ was chosen in Lemma 3.4. Because the cube $Q_{i,1}$ is located in direction $\downarrow^k$ from the cube $\frac{1}{3}Q_i$, we have either $Q_{i,1} \subset \tilde{Q}_i$ (where $\tilde{Q}_i$ is as in Lemma 3.4) or the projection of $\tilde{Q}_i$ and the projection of $Q_{i,1}$ coincide for every base vector except v_k. Thus $P_s(Q_{i,1}) \subset P_s(\tilde{Q}_i)$. Because $P_s(3Q_i) \subset P_s(X)$ for every $1 \leq s \leq m$, we get that the distance from $P_s(\tilde{Q}_i)$ to $\partial P_s(X)$ is at least $\frac{2}{3}l(Q_i)$. See Fig. 2 to get a better understanding of the situation.

In conclusion, if $3Q_i \not\subset X$ and there are m base vectors v_s such that $P_s(3Q_i) \subset P_s(X)$, then we can find a cube $Q_{i,1} \subset X \cap 3Q_i$ such that (3.5) holds and there are infinitely many cubes Q_l, $l > i$, such that $Q_{i,1} \cap 3Q_l = \varnothing$, or, if the first option is not possible, there exist infinitely many cubes Q_l, $l > i$, such that $P_s(3Q_l) \subset P_s(X)$ for at least $m + 1$ base vectors v_s. Because there are only n base vectors in total, this

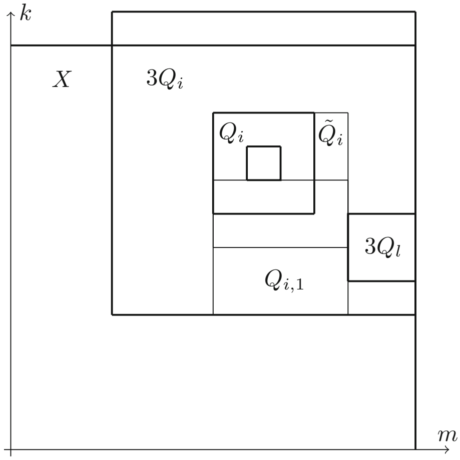

Fig. 2 An example of how the cubes X, Q_i, $\tilde{Q}_i$, $Q_{i,1}$ and Q_l may be situated with respect to each other. In the picture, we have the projections of these cubes to the mk-plane. The cube $Q_{i,1}$ is located in direction $\downarrow^k$ from $\frac{1}{3}Q_i$. Therefore, we have either $Q_{i,1}$ as shown in the picture or $Q_{i,1} \subset \tilde{Q}_i$

latter option can only happen at most n times. Thus we can always find infinitely many pairwise disjoint cubes $(Q'_{i_j})_{j=1}^{\infty}$ in X that all satisfy (3.5).

In both cases, we get

$$\sum_{j=1}^{\infty} |Q'_{i_j}| \left(\fint_{Q'_{i_j}} \left| f - f_{Q'_{i_j}} \right| \right)^p \geq \sum_{j=1}^{\infty} c(n,p) A^p = \infty.$$

This contradicts with the fact that $f \in JN_p(X)$. Therefore, we conclude that $A = 0$.

Now assume only that $f \in JN_p(X)$. Then there is a constant b such that $f - b \in L^{p,\infty}(X)$. Also from (2.1), we get that $|f - b| \in JN_p(X) \cap L^{p,\infty}(X)$. Then using the same argument as earlier, we get

$$\begin{aligned}\lim_{a\to 0} \sup_{\substack{Q\subseteq X\\ l(Q)\leq a}} |Q|^{1/p} \fint_Q |f| &\leq \lim_{a\to 0} \sup_{\substack{Q\subseteq X\\ l(Q)\leq a}} |Q|^{1/p} \fint_Q |f - b| \\ &\quad + \lim_{a\to 0} \sup_{\substack{Q\subseteq X\\ l(Q)\leq a}} |Q|^{1/p} \fint_Q |b| = 0.\end{aligned}$$

This completes the proof. □

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