



# Growth of Subsolutions of $\Delta_p u = V|u|^{p-2}u$ and of a General Class of Quasilinear Equations

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## Abstract

In this paper we prove some integral estimates on the minimal growth of the positive part  $u_+$  of subsolutions of quasilinear equations

$$\operatorname{div} A(x, u, \nabla u) = V|u|^{p-2}u$$

on complete Riemannian manifolds  $M$ , in the non-trivial case  $u_+ \not\equiv 0$ . Here  $A$  satisfies the structural assumption  $|A(x, u, \nabla u)|^{p/(p-1)} \leq k \langle A(x, u, \nabla u), \nabla u \rangle$  for some constant  $k > 0$  and for  $p > 1$  the same exponent appearing on the RHS of the equation, and  $V$  is a continuous positive function, possibly decaying at a controlled rate at infinity. We underline that the equation may be degenerate and that our arguments do not require any geometric assumption on  $M$  beyond completeness of the metric. From these results we also deduce a Liouville-type theorem for sufficiently slowly growing solutions.

**Keywords** Quasilinear equation ·  $p$ -Laplacian · Growth estimates · Liouville theorem

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### 1 Introduction

In the recent paper [4] (Lemma 8), the following theorem was established. Let  $M$  be a complete Riemannian manifold (without boundary),  $\lambda > 0$  a constant and  $u \in C^2(M)$ . If the superlevel set  $\Omega_+ := \{x \in M : u(x) > 0\}$  is not empty and  $u$  satisfies

$$\Delta u \geq \lambda u \quad \text{on } \Omega_+ \tag{1}$$

then for any fixed point  $x_0 \in M$  we have

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \log \int_{B_R(x_0)} u_+^2 > 0 \tag{2}$$

where  $u_+ := \max\{u, 0\}$  is the positive part of  $u$  and  $B_R(x_0)$  is the geodesic ball of radius  $R$  centered at  $x_0$ . Indeed, inspection of the proof also shows that there exists a constant  $C(\lambda)$ , not depending on  $M$  or  $u$ , such that

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \log \int_{B_R(x_0)} u_+^2 \geq C(\lambda) > 0 \tag{3}$$

and that the optimal value for  $C(\lambda)$  is not smaller than  $\frac{\log 2}{4} \sqrt{\lambda}$ . This can be regarded as a sort of “gap” theorem for subsolutions of  $\Delta u = \lambda u$ : if  $u \in C^2(M)$  satisfies

$$\Delta u \geq \lambda u \quad \text{on } M$$

then either  $u \leq 0$  or the positive part of  $u$  has to be sufficiently large in an integral sense (that is, its  $L^2$  norm on  $B_R(x_0)$  must grow at least exponentially with respect to  $R$ ). In fact, the result from [4] is more general and also covers the case of weighted Laplacians and locally Lipschitz weak solutions of (1).

In this paper we generalize the above theorem by considering differential inequalities for a wider class of (possibly degenerate) quasilinear elliptic operators in divergence form, including the  $p$ -Laplace operator

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad 1 < p < +\infty,$$

and also replacing the constant  $\lambda$  by a positive continuous function  $V$  possibly decaying at infinity at a controlled rate, namely, not faster than a negative power  $r(x)^{-\mu}$ ,  $\mu > 0$ , of the distance  $r(x) = \operatorname{dist}(x, o)$  from some fixed point  $o \in M$ . More precisely, for a given pair of parameters  $\lambda > 0$  and  $\mu \geq 0$  we shall assume that

$$\liminf_{x \rightarrow \infty} \begin{cases} V \geq \lambda & \text{if } \mu = 0 \\ [\operatorname{dist}(x, o)^\mu V(x)] \geq \lambda \text{ for some } o \in M & \text{if } \mu > 0. \end{cases} \tag{V_{\lambda, \mu}}$$

These conditions are clearly satisfied, for instance, if

$$V(x) \geq \frac{\lambda}{1 + \text{dist}(x, o)^\mu} \quad \text{on } M.$$

Also, in case  $\mu > 0$  the triangle inequality implies that the validity of  $(V_{\lambda, \mu})$  does not depend on the choice of the reference base point  $o \in M$ .

To give an example of our main result, we state it in the model case of the  $p$ -Laplace operator. To do so, we have to precise some terminology. For a function  $u \in W_{\text{loc}}^{1,p}(M)$ , we denote by  $\Omega_+ := \{x \in M : u(x) > 0\}$  its positivity set and for a given measurable function  $V \geq 0$  we say that  $u$  satisfies

$$\Delta_p u \geq V u^{p-1} \quad \text{weakly on } \Omega_+ \tag{4}$$

if

$$-\int_M \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \geq \int_M V u^{p-1} \varphi \quad \forall \varphi \in D^+(\Omega_+)$$

where

$$D^+(\Omega_+) := \{ \varphi \in W_c^{1,p}(M) : \varphi \geq 0 \text{ on } M, \\ \varphi = 0 \text{ and } \nabla \varphi = 0 \text{ a.e. on } M \setminus \Omega_+ \}.$$

(Note that if  $|\Omega_+| > 0$  then the space  $D^+(\Omega_+)$  of test functions is non-trivial because it contains at least elements of the form  $\varphi = u_+ \psi$ , with  $0 \leq \psi \in C_c^\infty(M)$ , so (4) is a meaningful condition.) In particular, (4) is always satisfied if

$$\Delta_p u \geq V |u|^{p-2} u \quad \text{weakly on } M \tag{5}$$

or even if

$$\Delta_p u_+ \geq V u_+^{p-1} \quad \text{weakly on } M \tag{6}$$

since  $\nabla u_+ = \mathbf{1}_{\Omega_+} \nabla u$  almost everywhere on  $M$ . Note that (6) is a weaker condition than (5), as follows from work of Le, [7].

**Theorem 1** *Let  $M$  be a complete Riemannian manifold,  $p \in (1, +\infty)$ ,  $\mu \in [0, p]$ ,  $\lambda > 0$ , and  $V : M \rightarrow (0, +\infty)$  a continuous function satisfying  $(V_{\lambda, \mu})$ .*

*Let  $u \in W_{\text{loc}}^{1,p}(M)$ . If  $\Omega_+ := \{x \in M : u(x) > 0\}$  is of positive measure and*

$$\Delta_p u \geq V u^{p-1} \quad \text{weakly on } \Omega_+$$

then for any  $x_0 \in M$  and  $q \in (p - 1, +\infty)$  we have

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{1-\frac{\mu}{p}}} \log \int_{B_R(x_0)} u_+^q \geq \frac{C_0}{1-\frac{\mu}{p}} > 0 \quad \text{if } \mu \in [0, p) \tag{7}$$

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_1 > p \quad \text{if } \mu = p \tag{8}$$

where  $C_0$  and  $C_1$  are explicitey given by

$$C_0 = \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} \lambda^{1/p}, \quad C_1^{1/p'}(C_1-p)^{1/p'} = C_0 \tag{9}$$

where  $p' = \frac{p}{p-1}$  is the exponent conjugate to  $p$ . Moreover, in case  $\mu = p$  we have

$$\lim_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_0 + p \tag{10}$$

whenever the limit on the LHS exists.

**Remark 2** Note that the value  $C_1 > p$  determined by (9) satisfies  $C_1 < C_0 + p$ , hence (10) gives a stronger estimate than (8) when its LHS is well defined.

The constants appearing in (7) and (10) are sharp, that is, for each combination of values of  $p, \mu, \lambda$  and  $q$  it is possible to find  $M$  and  $u$  for which the equality in (7) or (10) is attained. This is shown by explicit examples described at the end of Sect. 3. We don't know whether the value of  $C_1 > p$  in (9) is sharp or not for the validity of (8). It seems worth to underline that the case  $p = 2, q = 2, \mu = 0$  in the above theorem implies that the optimal value for  $C(\lambda)$  in (3) is  $C(\lambda) = 2\sqrt{\lambda}$ .

**Theorem 3** Let  $M$  be a complete Riemannian manifold,  $\mu \in [0, 2], \lambda > 0$  and  $V : M \rightarrow (0, +\infty)$  a continuous function satisfying  $(V_{\lambda, \mu})$ .

Let  $u \in W_{loc}^{1,2}(M)$ . If  $\Omega_+ := \{x \in M : u(x) > 0\}$  is of positive measure and

$$\Delta u \geq Vu \quad \text{weakly on } \Omega_+$$

then for any  $x_0 \in M$  and  $q \in (1, +\infty)$  we have

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{1-\frac{\mu}{2}}} \log \int_{B_R(x_0)} u_+^q \geq \frac{2\sqrt{q-1}\sqrt{\lambda}}{1-\frac{\mu}{2}} \quad \text{if } \mu \in [0, 2)$$

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq 1 + \sqrt{1+4(q-1)\lambda} > 2 \quad \text{if } \mu = 2$$

and in case  $\mu = 2$

$$\lim_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq 2(1 + \sqrt{q-1}\sqrt{\lambda})$$

provided the limit exists.

In full generality, in our main theorem we deal with differential inequalities involving quasilinear differential operators  $L$  formally defined by

$$Lu := \operatorname{div}(A(x, u, \nabla u)) \tag{11}$$

where  $A : \mathbb{R} \times TM \rightarrow TM$  is a continuous function (or, more generally, a Carathéodory-type function as specified in Sect. 2) satisfying

$$\langle A(x, s, \xi), \xi \rangle \geq 0 \quad \text{and} \quad |A(x, s, \xi)| \leq k \langle A(x, s, \xi), \xi \rangle^{\frac{p}{p-1}} \tag{12}$$

for all  $x \in M, s \in \mathbb{R}, \xi \in T_x M$  with some constant  $k > 0$ . If these conditions are satisfied, we say that the differential operator  $L$  defined by (11) is weakly  $p$ -coercive with coercivity constant  $k$ . The  $p$ -Laplace operator falls in this class since it can be expressed as in (11) for the choice  $A(x, s, \xi) = |\xi|^{p-2}\xi$ , which fulfills (12) with  $k = 1$ . In analogy with what we did above, we say that a function  $u \in W_{\text{loc}}^{1,p}(M)$  satisfies

$$Lu \geq Vu^{p-1} \quad \text{weakly on } \Omega_+ := \{u > 0\}$$

if

$$-\int_M \langle A(x, u, \nabla u), \nabla \varphi \rangle \geq \int_M Vu^{p-1}\varphi \quad \forall \varphi \in D^+(\Omega_+).$$

**Theorem 4** *Let  $M$  be a complete Riemannian manifold,  $p \in (1, +\infty)$ ,  $\mu \in [0, p]$  and  $\lambda > 0$ . Let  $L$  be a weakly  $p$ -coercive operator as in (11) with coercivity constant  $k > 0$  and  $V : M \rightarrow (0, +\infty)$  a continuous function satisfying  $(V_{\lambda,\mu})$ .*

*Let  $u \in W_{\text{loc}}^{1,p}(M)$ . If  $\Omega_+ := \{x \in M : u(x) > 0\}$  is of positive measure and*

$$Lu \geq Vu^{p-1} \quad \text{weakly on } \Omega_+$$

*then for any  $x_0 \in M$  and  $q \in (p - 1, +\infty)$  we have*

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{1-\frac{\mu}{p}}} \log \int_{B_R(x_0)} u_+^q \geq \frac{C_0}{1-\frac{\mu}{p}} \quad \text{if } \mu \in [0, p) \tag{13}$$

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_1 \quad \text{if } \mu = p \tag{14}$$

where  $C_0 > 0$  and  $C_1 > p$  are determined by

$$C_0 = \frac{p(q-p+1)^{1/p'} \lambda^{1/p}}{(p-1)^{1/p'} k}, \quad C_1^{1/p}(C_1-p)^{1/p'} = C_0$$

with  $p' = \frac{p}{p-1}$ . Moreover, in case  $\mu = p$  we have

$$\lim_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_0 + p \tag{15}$$

whenever the limit exists.

We point out that the RHS's of (14) and (15) both converge to  $p$  from above as  $\lambda \rightarrow 0^+$ . Hence, if  $u \in W_{loc}^{1,p}(M)$  satisfies

$$Lu \geq Vu^{p-1} \quad \text{weakly on } \Omega_+ = \{u > 0\}$$

with  $|\Omega_+| \neq 0$  and  $V$  a continuous positive function decaying to 0 faster than  $r(x)^{-p}$  as  $x \rightarrow \infty$ , then on arbitrary manifolds we couldn't expect the possible validity of an estimate stronger than

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R} u_+^q \geq p.$$

In fact, we are able to prove a weaker growth estimate (with  $\liminf$  replaced by  $\limsup$ ) holds more generally for any  $u \in W_{loc}^{1,p}(M)$  satisfying

$$Lu \geq f \quad \text{weakly on } \Omega_+ \tag{16}$$

for some measurable function  $f : M \rightarrow [0, +\infty]$  such that  $f > 0$  on a set  $E \subseteq \Omega_+$  of positive measure. Of course, by (16) we mean that

$$- \int_M \langle A(x, u, \nabla u), \nabla \varphi \rangle \geq \int_M f \varphi \quad \forall \varphi \in D^+(\Omega_+). \tag{17}$$

Note that if (16) holds with  $f$  as above then there exists  $\varphi \in D^+(\Omega_+)$  for which the LHS of (17) is strictly positive (this follows by considering a test function of the form  $\varphi = u_+ \psi$  for some  $0 \leq \psi \in C_c^\infty(M)$  strictly positive on a portion of  $E$  of positive measure), and then it must also be  $A(x, u, \nabla u) \neq 0$  on a subset  $E_0 \subseteq \Omega_+$  of positive measure.

**Theorem 5** *Let  $M$  be a complete Riemannian manifold,  $p \in (1, +\infty)$ ,  $L$  a weakly  $p$ -coercive operator as in (11) and  $u \in W_{loc}^{1,p}(M)$  such that  $\Omega_+ := \{x \in M : u(x) > 0\}$  has positive measure. If  $u$  satisfies*

$$Lu \geq 0 \quad \text{weakly on } \Omega_+$$

and further

$$A(x, u, \nabla u) \neq 0 \quad \text{on a set } E_0 \subseteq \Omega_+ \text{ of positive measure} \tag{18}$$

then for any  $q \in (p - 1, +\infty)$

$$\limsup_{R \rightarrow +\infty} \frac{1}{R^p} \int_{B_R} u_+^q = +\infty. \tag{19}$$

In particular,

$$\limsup_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R} u_+^q \geq p.$$

As said, (18) holds if  $u$  satisfies (16) for some measurable  $f : M \rightarrow [0, +\infty]$  with  $f$  not a.e. vanishing on  $\Omega_+$ . Alternatively, (18) is satisfied also when  $u$  is not constant on  $M$  and positive somewhere (so that  $|\Omega_+| > 0$ ) and  $A$  obeys the following mild non-degeneracy condition:

$$A(x, s, \xi) = 0 \quad \text{only if} \quad \xi = 0. \tag{20}$$

Theorem 5 is a consequence of the next Theorem 6, proved in the last part of the paper where we extend some arguments from [8] to general weakly  $p$ -coercive operators  $L$  of the form (11).

**Theorem 6** *Let  $M$  be a complete, non-compact Riemannian manifold,  $p \in (1, +\infty)$ ,  $L$  a weakly  $p$ -coercive operator as in (11) and  $u \in W_{\text{loc}}^{1,p}(M)$ . If  $\{u > 0\}$  has positive measure,  $u$  satisfies*

$$Lu \geq 0 \quad \text{weakly on } \{u > 0\} \tag{21}$$

and for some  $q > p - 1$  it holds

$$\lim_{R \rightarrow +\infty} \int_r^R \left( \int_{\partial B_s} u_+^q \right)^{-\frac{1}{p-1}} ds = +\infty \quad \forall r > 0, \tag{22}$$

then  $A(x, u, \nabla u) = 0$  almost everywhere on  $\{u > 0\}$ . In particular, if the structural condition (20) holds, then  $u$  is constant on  $M$ .

We remark that condition (22) amounts to saying that the function  $\varphi : (0, +\infty) \rightarrow [0, +\infty]$  given by

$$\varphi(s) = \left( \int_{\partial B_s} u_+^q \right)^{-\frac{1}{p-1}} \quad \forall s > 0$$

is not in  $L^1((r, +\infty))$  for any  $r > 0$ . In fact, as proved in Lemma 19 below, in the assumptions of Theorem 6 there exists  $r_0 \geq 0$  such that  $\varphi$  is finite a.e. on  $(r_0, +\infty)$  and  $\varphi \in L^1((r, R))$  for any  $r_0 < r < R < +\infty$ , so that (22) is satisfied if and only if  $\varphi$  is not integrable in a neighborhood of  $+\infty$ . Note that in general  $\varphi$  may be integrable

at  $+\infty$  and still satisfy  $\varphi = +\infty$  on  $(0, r_0)$  for some  $r_0 > 0$ . For instance, for fixed  $n \in \mathbb{N}$  and  $p > n$ , the function

$$u(x) := |x|^{\frac{p-n}{p-1}} - 1 \quad \text{on } \mathbb{R}^n$$

satisfies  $\Delta_p u = 0$  on  $\Omega_+ = \mathbb{R}^n \setminus \overline{B_1}$ , and for any  $q > p - 1$

$$\varphi(s) = \begin{cases} +\infty & \text{for } 0 < s \leq 1 \\ \left[ C s^{n-1} \left( s^q \frac{p-n}{p-1} - 1 \right) \right]^{-\frac{1}{p-1}} & \text{for } s > 1 \end{cases}$$

(with  $C = |\partial B_1|$ ) is integrable at  $+\infty$ : indeed,

$$\varphi(s) \sim C^{-\frac{1}{p-1}} s^{-\frac{(n-1)(p-1)+q(p-n)}{(p-1)^2}} \quad \text{as } s \rightarrow +\infty$$

and (under the assumption  $p > n$ ) we have  $-\frac{(n-1)(p-1)+q(p-n)}{(p-1)^2} < -1$  if and only if  $q > p - 1$ . This shows that the clause “ $\forall r > 0$ ” in (22) cannot in general be replaced by “for some  $r > 0$ ”.

Note that (22) is a condition about the growth of the integral of  $u_+^q$  on geodesic spheres  $\partial B_s$ . This can be related to the growth of the integral of  $u_+^q$  on balls  $B_s$ . More precisely, (22) is implied (see Proposition 1.3 in [8]) by the stronger condition

$$\lim_{R \rightarrow +\infty} \int_r^R \left( \frac{s}{\int_{B_s} u_+^q} \right)^{\frac{1}{p-1}} ds = +\infty \quad \forall r > 0$$

which in turn is satisfied, for instance, when

$$\int_{B_R} u_+^q = O(R^p) \quad \text{as } R \rightarrow +\infty.$$

Since this last condition is exactly the negation of condition (19) above, Theorem 5 follows at once from Theorem 6.

As hinted at the beginning of this Introduction, our main Theorem 4 can be also interpreted as a “gap” theorem for functions  $u \in W_{\text{loc}}^{1,p}(M)$  satisfying

$$Lu \geq V|u|^{p-2}u \quad \text{on } M.$$

Namely, if  $u$  satisfies the above differential inequality, then either  $u \leq 0$  a.e. on  $M$  or the positive part of  $u$  must grow sufficiently fast. As an easy consequence we have the following Liouville-type result (for its proof it is enough to apply Theorem 4 to both  $u$  and  $-u$ ). For the sake of simplicity, we only state it in case  $V$  is a positive constant, but the interested reader can immediately generalize it to the case where  $V$  is a function satisfying  $(V_{\lambda,\mu})$  for some  $\lambda > 0$  and  $\mu \in [0, p]$ .



**Theorem 7** *Let  $M$  be a complete Riemannian manifold,  $p \in (1, +\infty)$ ,  $\lambda > 0$  and  $L$  a weakly  $p$ -coercive operator as in (11) with coercivity constant  $k > 0$ . Let  $u \in W_{\text{loc}}^{1,p}(M)$  satisfy*

$$Lu = \lambda|u|^{p-2}u \quad \text{on } M.$$

*If for some  $x_0 \in M$  and  $q \in (p - 1, +\infty)$*

$$\int_{B_R(x_0)} |u|^q \leq e^{CR} \quad \text{for all sufficiently large } R$$

*for some constant  $C < \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} \frac{\lambda^{1/p}}{k}$ , then  $u \equiv 0$ .*

We conclude this introduction with a few comments on some technical points. First, in all the results stated above, except for Theorem 6,  $M$  is not explicitly assumed to be non-compact. Indeed, if  $M$  is compact (without boundary) and  $u$  satisfies

$$Lu \geq f \geq 0 \quad \text{on } \Omega_+$$

for some measurable  $f$ , then necessarily  $f = 0$  and  $A(x, u, \nabla u) = 0$  a.e. on  $\Omega_+$  (see Lemma 8 in Sect. 2). Hence, in the assumptions of Theorems 1, 3, 4 and 5,  $M$  is necessarily non-compact. Secondly, in all our results we do not make additional regularity assumptions on the subolutions beside their belonging to the appropriate Sobolev class  $W_{\text{loc}}^{1,p}(M)$ . Since we do not know if Sobolev subolutions of possibly degenerate equations of the form

$$\text{div}A(x, u, \nabla u) = V|u|^{p-2}u$$

are always locally essentially upper bounded (that is, if they necessarily satisfy  $u_+ \in L_{\text{loc}}^\infty(M)$ ), in some of our arguments we have to follow more winding roads using approximation procedures.

The paper is organized as follows. In Sect. 2 we collect the notation and all relevant definitions. In Sect. 3 we prove the main Theorem 4 and we provide examples showing sharpness of the constants in the statements. Section 4 is devoted to the proof of Theorem 6, from which Theorem 5 can be easily deduced (see Corollary 22 and Remark 23).

Comparison results and the case  $p = 1$  will appear in a forthcoming paper.

We recently learned that on arXiv has just appeared a paper by Bisterzo, Farina and Pigola [2] which is somehow related to our work, at least where  $L$  is the Laplace–Beltrami operator. However, even in the above overlapping case, the two papers are different in setting, scope and sharpness of the results.

## 2 Definitions and Notation

Throughout this paper,  $M$  will always be a connected Riemannian manifold without boundary. We denote by  $TM$  its tangent bundle and by  $\langle \cdot, \cdot \rangle$  its Riemannian metric. For any  $p \in (1, +\infty)$  we also denote by  $W_{loc}^{1,p}(M)$  the space of Sobolev functions  $u$  whose restrictions to any relatively compact set  $\Omega \subseteq M$  belong to  $W^{1,p}(\Omega)$ . This is equivalent to requiring that  $u \circ \psi^{-1} \in W_{loc}^{1,p}(\psi(U))$  for any local chart  $\psi : U \subseteq M \rightarrow \mathbb{R}^m$ , where  $m = \dim M$ . We also denote by  $W_c^{1,p}(M)$  the subspace of  $W_{loc}^{1,p}(M)$  consisting of functions with compact support.

We consider quasilinear differential operators  $L$  in divergence form weakly defined on functions  $u \in W_{loc}^{1,p}(M)$  by

$$Lu(x) = \operatorname{div}A(x, u, \nabla u). \tag{23}$$

Here  $A : \mathbb{R} \times TM \rightarrow TM$  is a function such that

$$A(x, s, \xi) \in T_x M \quad \forall x \in M, s \in \mathbb{R}, \xi \in T_x M$$

and whose local representation  $\tilde{A} : \psi(U) \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  in any chart  $\psi : U \subseteq M \rightarrow \mathbb{R}^m$  satisfies the Carathéodory conditions

- $\tilde{A}(y, \cdot, \cdot)$  is continuous for a.e.  $y \in \psi(U)$
- $\tilde{A}(\cdot, s, v)$  is measurable for every  $(s, v) \in \mathbb{R} \times \mathbb{R}^m$ .

(The representation  $\tilde{A}$  is defined by

$$\tilde{A}(\psi(x), s, v) := A\left(x, s, \sum_{i=1}^m v^i \frac{\partial}{\partial y^i} \Big|_x\right) \quad \forall x \in U, s \in \mathbb{R}, v = (v^1, \dots, v^m) \in \mathbb{R}^m$$

where  $y^1, \dots, y^m$  are the coordinates induced by  $\psi$ .) In particular, these conditions on  $\tilde{A}$  are satisfied whenever  $A$  is a continuous function of its arguments. Following terminology from [5, Definition 2.1], we say that  $A$  and the corresponding operator  $L$  given by (23) are *weakly- $p$ -coercive* for some  $p \in (1, +\infty)$  if

$$\langle A(x, s, \xi), \xi \rangle \geq 0 \quad \forall x \in M, s \in \mathbb{R}, \xi \in T_x M \tag{24}$$

$$|A(x, s, \xi)| \leq k \langle A(x, s, \xi), \xi \rangle^{\frac{p-1}{p}} \quad \forall x \in M, s \in \mathbb{R}, \xi \in T_x M \tag{25}$$

for some constant  $k > 0$  that we will call the *coercivity constant* of  $A$ . Note that the above conditions imply that

$$|A(x, s, \xi)| \leq k^p |\xi|^{p-1} \quad \forall x \in M, s \in \mathbb{R}, \xi \in T_x M. \tag{26}$$

Indeed, this is clearly true when  $A(x, s, \xi) = 0$ ; otherwise, by Cauchy–Schwarz inequality and (25) we have  $|A(x, s, \xi)|^p \leq k^p |A(x, s, \xi)|^{p-1} |\xi|^{p-1}$ , and then (26)

follows dividing both sides by  $|A(x, s, \xi)|^{p-1}$ . In particular, we have

$$A(x, s, 0) = 0 \quad \forall x \in M, s \in \mathbb{R}. \tag{27}$$

On the other hand, in general we do not assume non-degeneracy of  $A$ , that is, we do not assume that  $A(x, s, \xi) \neq 0$  when  $\xi \neq 0$ .

Let  $A$  be a weakly  $p$ -coercive function for some  $p \in (1, +\infty)$ . For any given  $u \in W_{loc}^{1,p}(M)$  and any  $s_0 \in \mathbb{R}$  we set

$$\Omega_{s_0} := \{x \in M : u(x) > s_0\}$$

and for any non-negative measurable  $f : M \rightarrow [0, +\infty]$  we say that  $u$  satisfies

$$Lu \geq f \quad (\text{weakly}) \text{ on } \Omega_{s_0} \tag{28}$$

if

$$-\int_M \langle A(x, u, \nabla u), \nabla \varphi \rangle \geq \int_M f \varphi \quad \forall \varphi \in D^+(\Omega_{s_0}) \tag{29}$$

where

$$D^+(\Omega_{s_0}) := \{\varphi \in W_c^{1,p}(M) : \varphi \geq 0 \text{ on } M, \\ \varphi = 0 \text{ and } \nabla \varphi = 0 \text{ a.e. on } M \setminus \Omega_{s_0}\}.$$

We remark that our assumptions on  $A$  and  $u$  imply that  $|A(x, u, \nabla u)| \in L_{loc}^{p'}(M)$ , with  $p' = \frac{p}{p-1}$  the exponent conjugate to  $p$ , and that  $\langle A(x, u, \nabla u), \nabla \varphi \rangle$  is measurable for each  $\varphi \in D^+(\Omega_{s_0})$  (see for instance [9, Lemma 2.4]). Hence, the LHS of (29) is well defined and finite for each  $\varphi \in D^+(\Omega_{s_0})$ .

The next lemma justifies our focus on complete, non-compact manifolds in the introduction and in the following sections.

**Lemma 8** *Let  $M$  be a compact manifold without boundary,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23). If  $u \in W^{1,p}(M)$  satisfies*

$$Lu \geq f \geq 0 \quad \text{on } \Omega_{s_0} := \{u > s_0\}$$

for some measurable  $f : M \rightarrow \mathbb{R}$  and some  $s_0 \in \mathbb{R}$ , then

$$f = 0 \text{ and } A(x, u, \nabla u) = 0 \quad \text{a.e. on } \Omega_{s_0}. \tag{30}$$

**Proof** Considering the test function  $\varphi = (u - s_0)_+ \in D^+(\Omega_{s_0})$  we have

$$\int_{\Omega_{s_0}} \langle A(x, u, \nabla u), \nabla u \rangle \leq - \int_{\Omega_{s_0}} (u - s_0)_+ f \leq 0$$

and by the weak coercivity condition (25) we obtain

$$\int_{\Omega_{s_0}} |A(x, u, \nabla u)|^{\frac{p}{p-1}} \leq 0.$$

By non-negativity of  $f$  and of  $|\cdot|$ , this immediately yields (30). □

Lastly, we precise the following terminology. For an open interval  $I \subseteq \mathbb{R}$  we say that a function  $F : I \rightarrow \mathbb{R}$  is piecewise  $C^1$  if  $F$  is continuous on  $I$  and there exists a discrete (possibly empty) set  $E \subseteq I$  such that

- (i)  $F'$  exists and is continuous on  $I \setminus E$
- (ii)  $\forall a \in E \lim_{x \rightarrow a^-} F'(x)$  and  $\lim_{x \rightarrow a^+} F'(x)$  exist and are finite.

If  $u \in W_{loc}^{1,p}(M)$  with  $u(M) \subseteq I$  and  $F'$  is bounded on  $I \setminus E$ , then by Stampacchia’s lemma the function  $v = F(u)$  is also in  $W_{loc}^{1,p}(M)$  and

$$\nabla v = \begin{cases} F'(u)\nabla u & \text{a.e. on } M \setminus u^{-1}(E) \\ 0 & \text{a.e. on } u^{-1}(E), \end{cases}$$

see for instance Theorem 7.8 in [6]. (Here and in the following statements, “a.e.” always refers to the  $m$ -dimensional Riemannian volume measure of  $M$ .) Since  $\nabla u = 0$  a.e. on each level set of  $u$ , we can further write

$$\nabla v = F'(u)\nabla u \quad \text{a.e. on } M.$$

### 3 Proof of the Main Theorem

The aim of this section is to prove the main Theorem 13 below, which is slightly more general than Theorem 4 from the Introduction. To do so, we have to collect some preliminary lemmas about functions  $u$  satisfying  $Lu \geq 0$  on some superlevel set  $\Omega_{s_0} := \{x \in M : u(x) > s_0\}$ ,  $s_0 \in \mathbb{R}$ . Note that for the validity of the following lemmas it is not necessary to assume that  $|\Omega_{s_0}| > 0$ , that is,  $s_0$  may be a priori larger than or equal to  $\text{ess sup}_M u$  (in which case it is clearly true that  $Lu \geq 0$  on  $\Omega_{s_0}$  in the sense of (29), and the thesis of each lemma holds trivially).

**Lemma 9** *Let  $M$  be a Riemannian manifold,  $p > 1$  and  $L$  a weakly  $p$ -coercive operator as in (23) with coercivity constant  $k > 0$ . Let  $u \in W_{loc}^{1,p}(M)$  satisfy*

$$Lu \geq f \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{31}$$

for some  $s_0 \in \mathbb{R}$  and some measurable  $f : M \rightarrow \mathbb{R}$ . Let  $F$  be a non-negative, non-decreasing, piecewise  $C^1$  function on  $(0, +\infty)$ . Then for every  $0 \leq \eta \in C_c^\infty(M)$

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla \eta| \geq k^{-p'} \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta F(w)f, \tag{32}$$

where  $w := (u - s_0)_+$ ,  $A_u := A(x, u, \nabla u)$  and  $p' = \frac{p}{p-1}$ .

**Proof** Let  $0 \leq \eta \in C_c^\infty(M)$  be given and let

$$w := (u - s_0)_+ \in W_{\text{loc}}^{1,p}(M), \quad A_u := A(x, u, \nabla u)$$

as in the statement. Let  $\lambda \in C^\infty(\mathbb{R})$  be such that

$$\lambda(s) = 0 \text{ if } s \leq 1, \quad \lambda(s) = 1 \text{ if } s \geq 2, \quad \lambda' \geq 0 \text{ on } \mathbb{R} \tag{33}$$

and for any  $\varepsilon > 0$  define  $\lambda_\varepsilon \in C^\infty(\mathbb{R})$  by

$$\lambda_\varepsilon(s) := \lambda(s/\varepsilon). \tag{34}$$

Clearly we have

$$0 \leq \lambda_\varepsilon \leq \mathbf{1}_{(0,+\infty)} \quad \forall \varepsilon > 0 \quad \text{and} \quad \lambda_\varepsilon \nearrow \mathbf{1}_{(0,+\infty)} \text{ as } \varepsilon \rightarrow 0^+, \tag{35}$$

where  $\mathbf{1}$  denotes the indicator function and  $\nearrow$  denotes monotone convergence from below. Let  $h > 0$  be fixed and for any  $\varepsilon \in (0, h/2)$  let  $F_{\varepsilon,h} : \mathbb{R} \rightarrow [0, +\infty)$  be given by

$$F_{\varepsilon,h}(s) = \begin{cases} 0 & \text{if } s < 0 \\ \lambda_\varepsilon(s)F(s) & \text{if } 0 \leq s < h \\ F(h) & \text{if } s \geq h. \end{cases}$$

By our choice of  $\lambda$  and our assumptions on  $F$ , the function  $F_{\varepsilon,h}$  is non-negative, non-decreasing, piecewise  $C^1$  on  $\mathbb{R}$  (with an additional corner point at  $s = h$ ) and globally Lipschitz, so  $F_{\varepsilon,h}(w) \in W_{\text{loc}}^{1,p}(M)$  with

$$\nabla F_{\varepsilon,h}(w) = F'_{\varepsilon,h}(w)\nabla u \quad \text{a.e. on } M.$$

In particular we have

$$F'_{\varepsilon,h}(s) = \begin{cases} \lambda'_\varepsilon(s)F(s) + \lambda_\varepsilon(s)F'(s) \geq \lambda_\varepsilon(s)F'(s) & \text{if } \varepsilon < s < h \\ 0 & \text{if } s \leq \varepsilon \text{ or } s > h. \end{cases}$$

Set

$$\varphi = \varphi_{\varepsilon,h} := \eta F_{\varepsilon,h}(w).$$

We have  $0 \leq \varphi \in W_c^{1,p}(M)$  and by the choice of  $\lambda_\varepsilon$  we also have that  $\varphi$  vanish outside  $\{w > 0\} \equiv \Omega_{s_0}$ . So  $\varphi$  is an admissible test function for (29) and we have

$$-\int_M \langle A_u, \nabla \varphi \rangle \geq \int_M f \varphi. \tag{36}$$

By direct computation and using that  $\eta F(w)\lambda'_\varepsilon(w)\langle A_u, \nabla u \rangle \geq 0$  by our assumptions on  $\lambda_\varepsilon, F, \eta$  and  $A$ , together with weak  $p$ -coercivity (25) of  $A$  and Cauchy–Schwarz inequality we have

$$\begin{aligned} \langle A_u, \nabla \varphi \rangle &= F_{\varepsilon,h}(w)\langle A_u, \nabla \eta \rangle + \eta F'_{\varepsilon,h}(w)\langle A_u, \nabla u \rangle \\ &\geq F_{\varepsilon,h}(w)\langle A_u, \nabla \eta \rangle + \eta F'(w)\langle A_u, \nabla u \rangle \lambda_\varepsilon(w) \mathbf{1}_{\{\varepsilon < w < h\}} \\ &\geq -F_{\varepsilon,h}(w)|A_u||\nabla \eta| + k^{-p'} \eta F'(w)|A_u|^{p'} \lambda_\varepsilon(w) \mathbf{1}_{\{\varepsilon < w < h\}}. \end{aligned}$$

We substitute into (36) and rearrange terms to get

$$\int_{\Omega_{s_0}} F_{\varepsilon,h}(w)|A_u||\nabla \eta| \geq k^{-p'} \int_{\{\varepsilon < w < h\}} \eta \lambda_\varepsilon(w) F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta F_{\varepsilon,h}(w) f.$$

Using non-negativity of  $F, F', f, \eta$ , monotonicity of  $F$  and (35), by the monotone convergence theorem we get

$$\begin{aligned} \lim_{\substack{\varepsilon \rightarrow 0^+ \\ h \rightarrow +\infty}} \int_{\Omega_{s_0}} F_{\varepsilon,h}(w)|A_u||\nabla \eta| &= \int_{\Omega_{s_0}} F(w)|A_u||\nabla \eta| \\ \lim_{\substack{\varepsilon \rightarrow 0^+ \\ h \rightarrow +\infty}} \int_{\{\varepsilon < w < h\}} \eta \lambda_\varepsilon(w) F'(w)|A_u|^{p'} &= \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} \\ \lim_{\substack{\varepsilon \rightarrow 0^+ \\ h \rightarrow +\infty}} \int_{\Omega_{s_0}} \eta F_{\varepsilon,h}(w) f &= \int_{\Omega_{s_0}} \eta F(w) f \end{aligned}$$

and then we obtain (85). □

We underline that the LHS of (32) can be further estimated from above via Young’s inequality in two different ways, both useful in what will follow.

(1) Suppose that  $F' > 0$  on  $(0, +\infty)$ . By Hölder’s and Young’s inequalities with conjugate exponents  $p$  and  $p'$ , for any  $\sigma > 0$  we get

$$\begin{aligned} \int_{\Omega_{s_0}} F(w)|A_u||\nabla \eta| &\leq \left( \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla \eta| \right)^{1/p} \left( \int_{\Omega_{s_0}} F'(w)|A_u|^{p'} |\nabla \eta| \right)^{1/p'} \\ &\leq \frac{\sigma^p}{p} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla \eta| + \frac{\sigma^{-p'}}{p'} \int_{\Omega_{s_0}} F'(w)|A_u|^{p'} |\nabla \eta|. \end{aligned} \tag{37}$$

(2) If  $0 \leq \psi \in C_c^\infty(M)$ , then applying (32) with  $\eta := \psi^p \in C_c^\infty(M)$  we get

$$p \int_{\Omega_{s_0}} \psi^{p-1} F(w)|A_u||\nabla \psi| \geq k^{-p'} \int_{\Omega_{s_0}} \psi^p F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \psi^p F(w) f \tag{38}$$

and by Young’s inequality we have, again for any  $\sigma > 0$ ,

$$\begin{aligned}
 p \int_{\Omega_{s_0}} \psi^{p-1} F(w) |A_u| |\nabla \psi| &\leq \frac{p^p \sigma^p}{p} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla \psi|^p \\
 &+ \frac{\sigma^{-p'}}{p'} \int_{\Omega_{s_0}} \psi^p F'(w) |A_u|^{p'}. \tag{39}
 \end{aligned}$$

By suitably choosing  $\sigma$  in (39) and rearranging terms we deduce the following

**Lemma 10** *In the assumptions of Lemma 9, if*

$$F'(w) |A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M) \tag{40}$$

then for any  $\varepsilon > 0$  and for any  $0 \leq \eta \in C^\infty_c(M)$  we have

$$\begin{aligned}
 &\frac{k^p (p-1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla \eta|^p \\
 &\geq (1-\varepsilon) k^{-p'} \int_{\Omega_{s_0}} \eta^p F'(w) |A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F(w) f. \tag{41}
 \end{aligned}$$

In particular, (40) holds under one of the following assumptions:

- (a)  $F(s) = O(s)$  as  $s \rightarrow +\infty$
- (b)  $u_+ \in L^r_{\text{loc}}(M)$  and  $F(s) = O(s^{r/p})$  as  $s \rightarrow +\infty$ , for some  $r > p$
- (c)  $u_+ \in L^\infty_{\text{loc}}(M)$ .

**Proof** If  $\varepsilon > 0$  is given then for  $\sigma = (\varepsilon p')^{-1/p'} k$  we have

$$\frac{\sigma^{-p'}}{p'} = \varepsilon k^{-p'}, \quad \frac{p^p \sigma^p}{p} = \frac{k^p (p-1)^{p-1}}{\varepsilon^{p-1}}$$

and then from (39) we get

$$\begin{aligned}
 &\frac{k^p (p-1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla \eta|^p + \varepsilon k^{-p'} \int_{\Omega_{s_0}} \eta^p F'(w) |A_u|^{p'} \\
 &\geq k^{-p'} \int_{\Omega_{s_0}} \eta^p F'(w) |A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F(w) f. \tag{42}
 \end{aligned}$$

In the assumption (40) we can rearrange terms to obtain (41). In view of (32) and since  $f \geq 0$  on  $\Omega_{s_0}$ , condition (40) is automatically satisfied if  $F(w) |A_u| \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M)$ . In particular this is always the case if  $F(w) \mathbf{1}_{\Omega_{s_0}} \in L^p_{\text{loc}}(M)$ , because then  $F(w) |A_u| \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M)$  by Hölder inequality (recall that  $u \in W^{1,p}_{\text{loc}}(M)$ , so  $|A_u| \leq k^p |\nabla u|^{p-1} \in L^{p'}_{\text{loc}}(M)$ ), and condition  $F(w) \mathbf{1}_{\Omega_{s_0}} \in L^p_{\text{loc}}(M)$  is in turn satisfied in either one of the cases (a), (b) or (c). □

A case that will be relevant for our subsequent discussion is where  $u_+ \in L^q_{\text{loc}}(M)$  and  $F(s) = s^{q-p+1}$  for some  $q \in (p - 1, +\infty)$ . In this setting the desired inequality takes the form

$$\frac{k^p(p - 1)^{p-1}}{\varepsilon^{p-1}\gamma^{p-1}} \int_{\Omega_{s_0}} w^q |\nabla \eta|^p \geq (1 - \varepsilon)k^{-p'} \int_{\Omega_{s_0}} \eta^p w^{q-p} |A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p w^p f$$

where  $\gamma := q - p + 1 \in (0, +\infty)$ . Note that for  $p - 1 < q \leq p$  we have  $0 < \gamma \leq 1$ , hence  $F(s) = s^{q-p+1} = s^\gamma = O(s)$  and this scenario is covered by alternative (a) in Lemma 10, while for  $q > p$  (and without assuming  $u_+ \in L^\infty_{\text{loc}}(M)$ ) we cannot refer to (b) or (c).

**Lemma 11** *Let  $M$  be a Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23) with coercivity constant  $k > 0$ . Let  $u \in W^{1,p}_{\text{loc}}(M)$  satisfy*

$$Lu \geq f \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{43}$$

for some  $s_0 \in \mathbb{R}$  and some measurable  $f : M \rightarrow \mathbb{R}$ . Let  $w := (u - s_0)_+$  and  $A_u := A(x, u, \nabla u)$ . Then for any  $q \in (p - 1, +\infty)$  and for every  $0 \leq \eta \in C^\infty_c(M)$

$$\frac{k^p(p - 1)^{p-1}}{\varepsilon^{p-1} \min\{1, \gamma^{p-1}\}} \int_{\Omega_{s_0}} w^q |\nabla \eta|^p \geq (1 - \varepsilon)\gamma k^{-p'} \int_{\Omega_{s_0}} \eta^p w^{q-p} |A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p w^{q-p+1} f \tag{44}$$

where  $\gamma := q - p + 1$ . If  $u_+ \in L^q_{\text{loc}}(M)$ , this can be strengthened to

$$\frac{k^p(p - 1)^{p-1}}{\varepsilon^{p-1}\gamma^{p-1}} \int_{\Omega_{s_0}} w^q |\nabla \eta|^p \geq (1 - \varepsilon)\gamma k^{-p'} \int_{\Omega_{s_0}} \eta^p w^{q-p} |A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p w^{q-p+1} f. \tag{45}$$

In particular, if  $u_+ \in L^\infty_{\text{loc}}(M)$  then this holds for any  $q \in (p - 1, +\infty)$ .

**Proof** Let  $0 \leq \eta \in C^\infty_c(M)$ ,  $q \in (p - 1, +\infty)$  be given and set  $F(s) = s^\gamma$  for  $s > 0$ , where  $\gamma := q - p + 1$  as in the statement of the Lemma.

If  $p - 1 < q \leq p$  then  $0 < \gamma \leq 1$  and by Lemma 10 we have the validity of (45) for any  $\varepsilon \in (0, 1]$ . (Note that in this case (44) and (45) coincide.)

If  $q > p$  then we proceed by approximating  $F$  from below with globally Lipschitz functions. For any  $h > 0$  let  $F_h : (0, +\infty) \rightarrow (0, +\infty)$  be defined by

$$F_h(s) = \begin{cases} s^\gamma & \text{if } 0 < s \leq h \\ h^{\gamma-1}s & \text{if } s > h. \end{cases}$$

Then  $F_h$  is piecewise smooth with a corner point at  $s = h$  and satisfies  $F_h(s) = O(s)$  as  $s \rightarrow +\infty$ , so by Lemma 10 we have

$$\begin{aligned} & \frac{k^p(p - 1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} |\nabla \eta|^p \\ & \geq (1 - \varepsilon)k^{-p'} \int_{\Omega_{s_0}} \eta^p F'_h(w) |A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F_h(w) f. \end{aligned}$$



By direct computation we have

$$\begin{aligned}
 F'_h(w)|A_u|^{p'} &= \gamma w^{q-p}|A_u|^{p'} \mathbf{1}_{\{0 < w \leq h\}} + h^{q-p}|A_u|^{p'} \mathbf{1}_{\{w > h\}} \quad \text{a.e. on } \Omega_{s_0} \\
 \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} &\leq \frac{w^q}{\gamma^{p-1}} \mathbf{1}_{\{0 < w \leq h\}} + h^{q-p} w^p \mathbf{1}_{\{w > h\}} \leq w^q \quad \text{on } \Omega_{s_0}
 \end{aligned}$$

We substitute the second estimate into the previous inequality to obtain

$$\begin{aligned}
 &\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} w^q |\nabla \eta|^p \\
 &\geq (1-\varepsilon)k^{-p'} \int_{\Omega_{s_0}} \eta^p F'_h(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F_h(w) f
 \end{aligned}$$

and then letting  $h \rightarrow +\infty$  we get, by the monotone convergence theorem,

$$\begin{aligned}
 &\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} w^q |\nabla \eta|^p \\
 &\geq (1-\varepsilon)\gamma k^{-p'} \int_{\Omega_{s_0}} \eta^p w^{q-p}|A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p w^{q-p+1} f
 \end{aligned}$$

proving (44).

If additionally  $u_+ \in L^q_{\text{loc}}(M)$ , then for any given  $0 \leq \eta \in C^\infty_c(M)$

$$\int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla \eta|^p \equiv \frac{1}{\gamma^{p-1}} \int_{\Omega_{s_0}} w^q |\nabla \eta|^p < +\infty$$

and from (44) applied for any  $\varepsilon \in (0, 1)$  we deduce (since  $f \geq 0$ ) that also

$$\int_{\Omega_{s_0}} \eta^p F'(w)|A_u|^{p'} \equiv \gamma \int_{\Omega_{s_0}} \eta^p w^{q-p}|A_u|^{p'} < +\infty.$$

Since this holds for any  $0 \leq \eta \in C^\infty_c(M)$  we have that  $F'(w)|A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M)$ , that is, the hypothesis (40) in Lemma 10 is satisfied, and then (45) directly follows from that lemma. □

We briefly comment on the condition  $u_+ \in L^\infty_{\text{loc}}(M)$ . If the function  $A$  satisfies the additional coercivity condition

$$|A(x, s, \xi)| \geq k_2 |\xi|^{p-1} \quad \forall x \in M, s \in \mathbb{R}, \xi \in T_x M \tag{46}$$

for some constant  $k_2 > 0$  (note that this is the case for the  $p$ -Laplacian  $L = \Delta_p$ ) then subsolutions of  $Lu = 0$  on  $M$  are locally essentially bounded above, that is, condition  $u_+ \in L^\infty_{\text{loc}}(M)$  is automatically satisfied for any  $u \in W^{1,p}_{\text{loc}}(M)$  satisfying

$$Lu \geq 0 \quad \text{weakly on } M. \tag{47}$$

More generally,  $u_+ \in L^\infty_{\text{loc}}(M)$  holds for functions  $u \in W^{1,p}_{\text{loc}}(M)$  such that, for some  $s_0 \in \mathbb{R}$ , the truncation  $w := (u - s_0)_+$  satisfies  $Lw \geq 0$  weakly on  $M$ .

**Proposition 12** *Let  $M$  be a Riemannian manifold,  $p > 1$  and  $L$  as in (23) a weakly  $p$ -coercive operator for which (46) holds. Let  $u \in W^{1,p}_{\text{loc}}(M)$  satisfy*

$$L(u - s_0)_+ \geq 0 \quad \text{weakly on } M \tag{48}$$

for some  $s_0 \in \mathbb{R}$ . Then  $u_+ \in L^\infty_{\text{loc}}(M)$ .

**Sketch of proof** For  $p > \dim M$  the thesis holds because  $W^{1,p}_{\text{loc}}(M) \subseteq C(M)$  by (local) Sobolev embeddings, while for  $1 < p \leq \dim M$  the statement can be proved by Moser iteration technique, using the Caccioppoli-type inequality

$$\frac{2^p(p-1)^{p-1}k^{pp'}}{\gamma \min\{1, \gamma^{p-1}\}} \int_M |\nabla \eta|^p (u - s_0)_+^q \geq k_2^{p'} \int_M \eta^p (u - s_0)_+^{q-p} |\nabla u|^p$$

obtained by (44) (with the choices  $\varepsilon = 1/2$  and  $f = 0$ ) and (46), together with the fact that every point  $x \in M$  has a relatively compact neighborhood  $U \subseteq M$  on which a Sobolev inequality holds. In fact, the Moser technique can be used to prove that  $(u - s_0)_+ \in L^\infty_{\text{loc}}(M)$ , from which  $u_+ \in L^\infty_{\text{loc}}(M)$  immediately follows.  $\square$

Since the argument above is of local nature, clearly it also applies in case (46) is satisfied with  $k_2 : M \rightarrow (0, +\infty)$  a continuous function possibly decaying to zero at infinity. However, in our analysis we are not assuming coercivity conditions of the form (46), and in fact we don't know whether a function  $u \in W^{1,p}_{\text{loc}}(M)$  such that  $Lu \geq 0$  on some superlevel set  $\{u > s_0\}$ , with  $L$  only satisfying assumptions (24)–(25) from Sect. 2, is necessarily locally upper bounded.

We are now ready to state and prove the main theorem of this section.

**Theorem 13** *Let  $M$  be a complete Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23) with coercivity constant  $k > 0$ . Let  $\lambda > 0$ ,  $\mu \in [0, p]$  and  $V : M \rightarrow (0, +\infty)$  be a continuous function satisfying*

$$\begin{aligned} V &\geq \lambda && \text{if } \mu = 0 \\ \liminf_{x \rightarrow \infty} [\text{dist}(x, o)^\mu V(x)] &\geq \lambda \text{ for some } o \in M && \text{if } \mu \in (0, p]. \end{aligned} \tag{49}$$

Let  $u \in W^{1,p}_{\text{loc}}(M)$  satisfy, for some  $0 \leq s_0 < \text{ess sup}_M u$ ,

$$Lu \geq Vu^{p-1} \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\}.$$

Then for any  $x_0 \in M$  and  $q \in (p - 1, +\infty)$  we have

$$\liminf_{R \rightarrow +\infty} \frac{1 - \frac{\mu}{p}}{R^{1 - \frac{\mu}{p}}} \log \int_{B_R} (u - s_0)_+^q \geq C_0 > 0 \quad \text{if } \mu \in [0, p) \tag{50}$$

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q \geq C_1 > p \quad \text{if } \mu = p \tag{51}$$

where  $C_0$  and  $C_1$  are determined by

$$C_0 := \frac{p(q - p + 1)^{1/p'} \lambda^{1/p}}{(p - 1)^{1/p'} k}, \quad C_1^{1/p} (C_1 - p)^{1/p'} = C_0.$$

Moreover, in case  $\mu = p$  we have

$$\lim_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q \geq C_0 + p \tag{52}$$

whenever the limit on the LHS exists.

**Remark 14** Note that  $C_0 + p > C_1 > C_0$  always.

**Proof** Let us set  $w := (u - s_0)_+$  and  $A_u := A(x, u, \nabla u)$ . Let  $x_0 \in M$  and  $q \in (p - 1, +\infty)$  be given. For the sake of brevity, for any  $R > 0$  we shall write  $B_R$  to denote the geodesic ball  $B_R(x_0)$ . Without loss of generality we can assume  $w^q \in L^1_{\text{loc}}(M)$ , since otherwise  $\int_{B_R} w^q = +\infty$  for each sufficiently large  $R > 0$  and the conclusion is trivial. Note that under this assumption we also have  $w^{q-p} |A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M)$ , as a consequence of (45) in Lemma 11. Let  $G, H : (0, +\infty) \rightarrow [0, +\infty)$  be defined by

$$G(t) := \int_{B_t} w^q, \quad H(t) := \int_{\Omega_{s_0} \cap B_t} w^{q-p} |A_u|^{p'}. \tag{53}$$

By the previous observation, the functions  $G$  and  $H$  are well defined, non-decreasing and absolutely continuous on any compact interval contained in  $(0, +\infty)$ . In particular, they are differentiable a.e. on  $(0, +\infty)$ .

Since  $s_0 \geq 0$ , we have  $u^{p-1} \geq w^{p-1}$  on  $\Omega_{s_0}$ . Then by applying Lemma 9 with the choices  $F(s) = s^{q-p+1}$  and  $f = V w^{p-1}$  we have

$$\int_M w^{q-p+1} |A_u| |\nabla \eta| \geq \gamma k^{-p'} \int_{\Omega_{s_0}} \eta w^{q-p} |A_u|^{p'} + \int_M V \eta w^q \tag{54}$$

for any  $0 \leq \eta \in C_c^\infty(M)$ , where  $\gamma := q - p + 1 > 0$ , and applying Young’s inequality as in (37) we have, for any  $\sigma > 0$ ,

$$\int_M w^{q-p+1} |A_u| |\nabla \eta| \leq \frac{\sigma^p}{p} \int_M w^q |\nabla \eta| + \frac{\sigma^{-p'}}{p'} \int_{\Omega_{s_0}} w^{q-p} |A_u|^{p'} |\nabla \eta|. \tag{55}$$

Let  $\varepsilon \in (0, \lambda)$  be given. By condition (49) and continuity and (strict) positivity of  $V$ , there exists  $R_0 = R_0(x_0, \varepsilon) > 0$  large enough so that

$$V(x) \geq \frac{\lambda - \varepsilon}{\text{dist}(x, x_0)^\mu} \quad \text{for all } x \in M \setminus B_{R_0} \tag{56}$$

and

$$\inf_{B_R} V \geq \frac{\lambda - \varepsilon}{R^\mu} \quad \forall R > R_0. \tag{57}$$

Indeed, for  $\mu = 0$  this is clearly true since  $V \geq \lambda$  everywhere on  $M$  by assumption (49). In case  $\mu > 0$ , note that it is possible to first find  $r_0 > 0$  such that

$$V(x) \geq \frac{\lambda - \varepsilon}{\text{dist}(x, x_0)^\mu} \quad \text{for all } x \in M \setminus B_{r_0} \tag{58}$$

since from (49) and the triangle inequality we have

$$\liminf_{x \rightarrow \infty} [\text{dist}(x, x_0)^\mu V(x)] \geq \lambda,$$

and then for any  $R > r_0$  we get

$$\inf_{B_R} V \geq \min \left\{ \inf_{B_{r_0}} V, \frac{\lambda - \varepsilon}{R^\mu} \right\}. \tag{59}$$

From the assumption that  $V$  is continuous and strictly positive on  $M$  we have  $\inf_{B_{r_0}} V > 0$ , so we can find  $R_0 \geq r_0$  such that  $\inf_{B_{r_0}} V \geq (\lambda - \varepsilon)/R_0^\mu$ . Then for any  $R > R_0$  the RHS in (59) is just  $(\lambda - \varepsilon)/R^\mu$ , and so (56)–(57) hold for such  $R_0$ .

Let  $t > R_0$  be a value for which  $G'(t)$  and  $H'(t)$  both exist. For any  $0 < \delta < t$  choose  $\eta_\delta \in C_c^\infty(M)$  satisfying

- (i)  $\eta_\delta \equiv 1$  on  $B_{t-\delta}$ ,
- (ii)  $\eta_\delta \equiv 0$  on  $M \setminus B_t$ ,
- (iii)  $0 \leq \eta_\delta \leq 1$  on  $B_t \setminus B_{t-\delta}$
- (iv)  $|\nabla \eta_\delta| \leq \frac{1}{\delta} + 1$  on  $M$ .

Since  $|\nabla \eta_\delta| \leq (1 + \delta^{-1})\mathbf{1}_{B_R \setminus B_{R-\delta}}$  we have

$$\int_M w^q |\nabla \eta_\delta| \leq \left(\frac{1}{\delta} + 1\right) \int_{B_t \setminus B_{t-\delta}} w^q = (1 + \delta) \frac{G(t) - G(t - \delta)}{\delta}$$

and letting  $\delta \searrow 0$  we get

$$\limsup_{\delta \rightarrow 0^+} \int_M w^q |\nabla \eta_\delta| \leq G'(t).$$

Similarly, we have

$$\limsup_{\delta \rightarrow 0^+} \int_{\Omega_{s_0}} w^{q-p} |A_u|^{p'} |\nabla \eta_\delta| \leq H'(t).$$

On the other hand, since  $\eta_\delta = 0$  on  $M \setminus B_t$  and  $\eta_\delta \rightarrow \mathbf{1}_{B_t}$  pointwise as  $\delta \rightarrow 0$ , by the dominated convergence theorem and also using (57) we get

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \int_{\Omega_{s_0}} \eta_\delta w^{q-p} |A_u|^{p'} &= \int_{\Omega_{s_0} \cap B_t} w^{q-p} |A_u|^{p'} = H(t) \\ \lim_{\delta \rightarrow 0^+} \int_M V \eta_\delta w^q &= \int_{B_t} V w^q. \end{aligned}$$

Thus, in view of (54)–(55) we have, for any  $\sigma > 0$ ,

$$\frac{\sigma^p}{p} G'(t) + \frac{\sigma^{-p'}}{p'} H'(t) \geq \int_{B_t} V w^q + \gamma k^{-p'} H(t) \tag{60}$$

and using (57) to further estimate

$$\int_{B_t} V w^q \geq \frac{\lambda - \varepsilon}{t^\mu} \int_{B_t} w^q = \frac{\lambda - \varepsilon}{t^\mu} G(t)$$

we obtain

$$\frac{\sigma^p}{p} G'(t) + \frac{\sigma^{-p'}}{p'} H'(t) \geq \frac{\lambda - \varepsilon}{t^\mu} G(t) + \gamma k^{-p'} H(t).$$

We apply the above reasoning to each value  $t > R_0$  for which  $G$  and  $H$  are simultaneously differentiable to deduce that for any  $\sigma : (0, +\infty) \rightarrow (0, +\infty)$

$$\frac{[\sigma(t)]^p}{p} G'(t) + \frac{[\sigma(t)]^{-p'}}{p'} H'(t) \geq \frac{\lambda - \varepsilon}{t^\mu} G(t) + \gamma k^{-p'} H(t) \quad \text{for a.e. } t > R_0$$

that is, multiplying everything by  $p[\sigma(t)]^{-p}$  and recalling that  $p + p' = pp'$ ,

$$G'(t) + \frac{p - 1}{[\sigma(t)]^{pp'}} H'(t) \geq \frac{p(\lambda - \varepsilon)}{[\sigma(t)]^p t^\mu} \left( G(t) + \frac{\gamma}{(\lambda - \varepsilon) k^{p'}} t^\mu H(t) \right) \tag{61}$$

for a.e.  $t > R_0$ . We now consider separately the cases  $\mu \in [0, p)$  and  $\mu = p$ .

**Case  $\mu \in [0, p)$ .** Assume that  $\mu \in [0, p)$ . Choosing

$$\begin{aligned} c_1 &= c_{1,\varepsilon} = (p - 1)^{\frac{1}{pp'}} (\lambda - \varepsilon)^{\frac{1}{pp'}} \gamma^{-\frac{1}{pp'}} k^{1/p} \\ c_2 &= c_{2,\varepsilon} = \frac{(p - 1)}{c_1^{pp'}} \equiv \gamma (\lambda - \varepsilon)^{-1} k^{-p'} \\ c_3 &= c_{3,\varepsilon} = \frac{p(\lambda - \varepsilon)}{c_1^p} \equiv \frac{p\gamma^{1/p'} (\lambda - \varepsilon)^{1/p}}{(p - 1)^{1/p'} k} \\ \sigma(t) &= c_1 t^{-\frac{\mu}{pp'}} \end{aligned}$$

we get

$$G'(t) + c_2 t^\mu H'(t) \geq c_3 t^{-\frac{\mu}{p}} (G(t) + c_2 t^\mu H(t)) \quad \text{for a.e. } t > R_0.$$

Let  $\Phi : (0, +\infty) \rightarrow [0, +\infty)$  be defined by

$$\Phi(t) = G(t) + c_2 t^\mu H(t).$$

The function  $\Phi$  is absolutely continuous on each compact subset of  $(0, +\infty)$  with

$$\Phi'(t) = G'(t) + c_2 t^\mu H'(t) + \mu c_2 t^{\mu-1} H(t) \quad \text{for a.e. } t \in (0, +\infty). \tag{62}$$

Then, in view of the previous inequality and since  $\mu c_2 t^{\mu-1} H(t) \geq 0$ , we get

$$\Phi'(t) \geq c_{3,\varepsilon} t^{-\frac{\mu}{p}} \Phi(t) \quad \text{for a.e. } t > R_0. \tag{63}$$

We have  $|\Omega_{s_0}| > 0$  because  $s_0 < \text{ess sup}_M u$ , so there exists  $R_1 > R_0$  such that  $G(R_1) > 0$ . Let  $R > R_1$  be given. By monotonicity of  $G$  and since  $c_2 t^\mu H(t) \geq 0$ , we have  $\Phi(t) \geq G(t) \geq G(R_1) > 0$  for all  $t \in [R_1, R]$ . Since  $[G(R_1), +\infty) \ni s \mapsto \log s$  is Lipschitz, the function  $\log \Phi$  is absolutely continuous on  $[R_1, R]$  with

$$(\log \Phi)'(t) = \frac{\Phi'(t)}{\Phi(t)} \quad \text{for a.e. } t \in [R_1, R].$$

Thus, integrating (63) and using that  $\Phi(R_1) \geq G(R_1) > 0$  we get

$$\log \Phi(R) \geq \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}} R^{1 - \frac{\mu}{p}} + \log G(R_1) - \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}} R_1^{1 - \frac{\mu}{p}} \quad \forall R > R_1. \tag{64}$$

Note that dividing both sides by  $R^{1 - \frac{\mu}{p}}$ , letting  $R \rightarrow +\infty$  and then  $\varepsilon \rightarrow 0^+$  we would obtain

$$\liminf_{R \rightarrow +\infty} \frac{1 - \frac{\mu}{p}}{R^{1 - \frac{\mu}{p}}} \log \Phi(R) \geq \lim_{\varepsilon \rightarrow 0^+} c_{3,\varepsilon} = \frac{p(q - p + 1)^{1/p'} \lambda^{1/p}}{(p - 1)^{1/p'k}},$$

which is (formally) weaker than (50) since  $\Phi(R) \geq G(R)$ . To show that the same inequality holds with  $\log G(R)$  in place of  $\Phi(R)$ , we proceed as follows. Let  $R > R_1$  and  $h > 0$  be given. By inequality (45) in Lemma 11 applied with the choice  $\varepsilon = \frac{1}{2}$  and with a cut-off function  $0 \leq \eta \in C_c^\infty(M)$  satisfying

- (i)  $\eta \equiv 1$  on  $B_R$ ,
- (ii)  $\eta \equiv 0$  on  $M \setminus B_{R+h}$ ,
- (iii)  $0 \leq \eta \leq 1$  on  $B_{R+h} \setminus B_R$
- (iv)  $|\nabla \eta| \leq \frac{2}{h}$  on  $M$

we get

$$\frac{k^{pp'}(p-1)^{p-1}4^p}{\gamma \min\{1, \gamma^{p-1}\}}G(R+h) \geq h^p H(R) \tag{65}$$

and thus, choosing  $h = R^{\mu/p}$ ,

$$\begin{aligned} \Phi(R) &= G(R) + c_2 R^\mu H(R) \\ &\leq G(R) + \frac{c_2 k^{pp'}(p-1)^{p-1}4^p}{\gamma \min\{1, \gamma^{p-1}\}}G(R + R^{\mu/p}) \\ &\leq \left(1 + \frac{c_2 k^{pp'}(p-1)^{p-1}4^p}{\gamma \min\{1, \gamma^{p-1}\}}\right)G(R + R^{\mu/p}) =: C_2 G(R + R^{\mu/p}) \end{aligned}$$

where in the last inequality we used monotonicity of  $G$ . Then, from (64) we get

$$\log G(R + R^{\mu/p}) \geq \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}}R^{1-\frac{\mu}{p}} + \log \frac{G(R_1)}{C_2} - \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}}R_1^{1-\frac{\mu}{p}} \quad \forall R > R_1. \tag{66}$$

Dividing both sides by  $(R + R^{\mu/p})^{1-\frac{\mu}{p}}$  and then letting  $R \rightarrow +\infty$  we get

$$\liminf_{R \rightarrow +\infty} \frac{\log G(R + R^{\mu/p})}{(R + R^{\mu/p})^{1-\frac{\mu}{p}}} \geq \lim_{R \rightarrow +\infty} \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}} \left(\frac{R}{R + R^{\mu/p}}\right)^{1-\frac{\mu}{p}} = \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}}$$

that is,

$$\lim_{R \rightarrow +\infty} \frac{1 - \frac{\mu}{p}}{R^{1-\frac{\mu}{p}}} \log G(R) \geq c_{3,\varepsilon}$$

and letting  $\varepsilon \rightarrow 0^+$  we obtain (50).

**Case  $\mu = p$ .** Assume now that  $\mu = p$ . We first prove (51), and then (52) in the assumption that its LHS is well defined.

**Proof of (51).** Choosing

$$\sigma(t) = c_4 t^{-1/p'}$$

for a suitable constant  $c_4 = c_{4,\varepsilon}$  to be suitably selected later, from (61) we get

$$G'(t) + \frac{p-1}{c_4^{pp'}}t^p H'(t) \geq \frac{p(\lambda - \varepsilon)}{c_4^p t} \left( G(t) + \frac{\gamma}{(\lambda - \varepsilon)k^{p'}}t^p H(t) \right) \tag{67}$$

for a.e.  $t > R_0$ . In analogy with the previous case, we aim at using this to deduce an inequality of the form

$$\Phi'(t) \geq c_5 t^{-1} \Phi(t) \quad \text{for a.e. } t > R_0 \tag{68}$$

with

$$\Phi(t) = G(t) + c_6 t^p H(t) \tag{69}$$

for suitable constants  $c_5 = c_{5,\varepsilon}$  and  $c_6 = c_{6,\varepsilon}$ . Computing  $\Phi'$  and rearranging terms we see that the desired inequality takes the form

$$\begin{aligned} G'(t) + c_6 t^p H'(t) &\geq c_5 t^{-1} (G(t) + c_6 t^p H(t)) - p c_6 t^{p-1} H(t) \\ &= c_5 t^{-1} \left( G(t) + c_6 \left( 1 - \frac{p}{c_5} \right) t^p H(t) \right) \end{aligned} \tag{70}$$

so we want to choose  $c_4, c_5$  and  $c_6$  matching the following relations:

$$\frac{p-1}{c_4^{pp'}} = c_6, \quad \frac{p(\lambda-\varepsilon)}{c_4^p} = c_5, \quad c_6 \left( 1 - \frac{p}{c_5} \right) = \frac{\gamma}{(\lambda-\varepsilon)k^{p'}}.$$

Expressing everything in terms of  $c_5$  this amounts to

$$\begin{aligned} c_4 &= \frac{p^{1/p}(\lambda-\varepsilon)^{1/p}}{c_5^{1/p}}, \quad c_6 = \frac{(p-1)c_5^{p'}}{p^{p'}(\lambda-\varepsilon)^{p'}}, \\ \frac{\gamma}{(\lambda-\varepsilon)k^{p'}} &= \frac{c_6}{c_5}(c_5-p) = \frac{(p-1)c_5^{p'-1}(c_5-p)}{p^{p'}(\lambda-\varepsilon)^{p'}}. \end{aligned} \tag{71}$$

that is, raising everything to the power  $1/p'$  in the last relation, we choose  $c_5 = c_{5,\varepsilon}$  as the unique value in  $(p, +\infty)$  satisfying

$$c_5^{1/p}(c_5-p)^{1/p'} = \frac{p\gamma^{1/p'}(\lambda-\varepsilon)^{1/p}}{(p-1)^{1/p'}k} (= c_{3,\varepsilon})$$

and then we let  $c_4$  and  $c_6$  be defined accordingly by (71). Summarizing, for these choices of  $c_4, c_5$  and  $c_6$  we have that (67) and (70) coincide, and each of them is equivalent to (68) for  $\Phi$  defined as in (69). Then choosing  $R_1 > R_0$  such that  $G(R_1) > 0$  and reasoning as in the previous case we see that

$$\log \Phi(R) \geq c_{5,\varepsilon} \log R + \log G(R_1) - c_{5,\varepsilon} \log R_1 \quad \forall R > R_1$$

and then by applying (65) with  $h = R$  we obtain

$$\log G(2R) \geq c_{5,\varepsilon} \log R + \log G(R_1) - c_{5,\varepsilon} \log R_1 - \log C_2 \quad \forall R > R_1.$$

Dividing both sides by  $\log(2R)$  and using that  $\log(2R) \sim \log R$  as  $R \rightarrow +\infty$  we get (after relabeling)

$$\liminf_{R \rightarrow +\infty} \frac{\log G(R)}{\log R} \geq c_{5,\varepsilon}$$



and then letting  $\varepsilon \rightarrow 0$  we get (51).

**Proof of (52).** Assume that

$$\ell := \lim_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q = \lim_{R \rightarrow +\infty} \frac{\log G(R)}{\log R}$$

exists. From (51) we already know that  $\ell \geq C_1 > p$ . If  $\ell = +\infty$  then (52) is trivially satisfied, so let us assume that  $\ell < +\infty$ . Let  $\varepsilon > 0$  be as above and small enough so that  $\ell - \varepsilon > p$ . Then there exists  $R_2 > R_0$  such that

$$R^p < R^{\ell-\varepsilon} < G(R) < R^{\ell+\varepsilon} \quad \forall R > R_2. \tag{72}$$

We recall, from the discussion preceding the treatment of case  $\mu < p$ , that for each  $t > R_2$  such that  $G'(t)$  and  $H'(t)$  exist we have (60), that is,

$$\frac{\sigma^p}{p} G'(t) + \frac{\sigma^{-p'}}{p'} H'(t) \geq \int_{B_t} V w^q + \gamma k^{-p'} H(t)$$

for any  $\sigma > 0$ . Using the co-area formula twice together with (56) we get

$$\begin{aligned} \int_{B_t} V w^q &\geq \int_{B_t \setminus B_{R_2}} V w^q = \int_{R_2}^t \left( \int_{\partial B_s} V w^q \, d\mathcal{H}^{m-1} \right) \, ds \\ &\geq \int_{R_2}^t \frac{\lambda - \varepsilon}{s^p} \left( \int_{\partial B_s} w^q \, d\mathcal{H}^{m-1} \right) \, ds \\ &= \int_{R_2}^t \frac{\lambda - \varepsilon}{s^p} G'(s) \, ds \end{aligned}$$

where  $m = \dim M$  and  $\mathcal{H}$  is the Hausdorff measure induced by the Riemannian structure. Substituting into the above inequality and multiplying both sides by  $p\sigma^{-p}t^{-p}$  we get

$$\frac{G'(t)}{t^p} + \frac{p-1}{\sigma^{pp'}} \frac{H'(t)}{t^p} \geq \frac{(\lambda - \varepsilon)p}{\sigma^p t^p} \left[ \int_{R_2}^t \frac{G'(s)}{s^p} \, ds + \frac{\gamma}{(\lambda - \varepsilon)k^{-p'}} H(t) \right]$$

and then choosing

$$\begin{aligned} c_1 &= c_{1,\varepsilon} = (p-1)^{\frac{1}{pp'}} (\lambda - \varepsilon)^{\frac{1}{pp'}} \gamma^{-\frac{1}{pp'}} k^{1/p} \\ c_2 &= c_{2,\varepsilon} = \frac{p-1}{c_1^{pp'}} \equiv \gamma (\lambda - \varepsilon)^{-1} k^{-p'} \\ c_3 &= c_{3,\varepsilon} = \frac{p(\lambda - \varepsilon)}{c_1^p} \equiv \frac{p\gamma^{1/p'} (\lambda - \varepsilon)^{1/p}}{(p-1)^{1/p'} k} \\ \sigma &= \sigma(t) = c_1 t^{-1/p'} \end{aligned}$$

this yields

$$\frac{G'(t)}{t^p} + c_2H(t) \geq \frac{c_3}{t} \left[ \int_{R_2}^t \frac{G'(s)}{s^p} ds + c_2H(t) \right] \quad \text{for a.e. } t > R_2. \tag{73}$$

Let  $\Psi : (R_2, +\infty) \rightarrow [0, +\infty)$  be defined by

$$\Psi(t) = \int_{R_2}^t \frac{G'(s)}{s^p} ds + c_2H(t).$$

The function  $\Psi$  is absolutely continuous on each compact interval contained in  $(R_2, +\infty)$  and inequality (73) can be restated as

$$\Psi'(t) \geq \frac{c_{3,\varepsilon}}{t} \Psi(t) \quad \text{for a.e. } t > R_2. \tag{74}$$

Reasoning as in the previous cases, since  $\Psi \not\equiv 0$  we reach the conclusion

$$\liminf_{R \rightarrow +\infty} \frac{\log \Psi(R)}{\log R} \geq c_{3,\varepsilon}. \tag{75}$$

We now use this to deduce (52). Let  $R > R_2$  be given. Applying (65) with  $h = R$ , integrating by parts and then using (72) twice we get

$$\begin{aligned} \Psi(R) &\leq \int_{R_2}^R \frac{G'(s)}{s^p} ds + C_2 \frac{G(2R)}{R^p} \\ &= \frac{G(R)}{R^p} - \frac{G(R_2)}{R_2^p} + p \int_{R_2}^R \frac{G(s)}{s^{p+1}} ds + C_2 \frac{G(2R)}{R^p} \\ &\leq \frac{G(R)}{R^p} - \frac{G(R_2)}{R_2^p} + p \int_{R_2}^R s^{\ell+\varepsilon-p-1} ds + C_2 \frac{G(2R)}{R^p} \\ &= \frac{G(R)}{R^p} - \frac{G(R_2)}{R_2^p} + \frac{pR^{\ell+\varepsilon-p}}{\ell + \varepsilon - p} - \frac{pR_2^{\ell+\varepsilon-p}}{\ell + \varepsilon - p} + C_2 \frac{G(2R)}{R^p} \\ &\leq \frac{G(R)}{R^p} + \frac{pR^{2\varepsilon}}{\ell + \varepsilon - p} \frac{G(R)}{R^p} + C_2 \frac{G(2R)}{R^p} - \frac{G(R_2)}{R_2^p} - \frac{pR_2^{\ell+\varepsilon-p}}{\ell + \varepsilon - p}. \end{aligned}$$

Since  $G$  is non-decreasing, we have  $G(R) \leq G(2R)$  and then

$$\Psi(R) \leq \left( \frac{p}{\ell + \varepsilon - p} + (1 + C_2)R^{-2\varepsilon} \right) R^{-p+2\varepsilon} G(2R) + O(1)$$

as  $R \rightarrow +\infty$ . By (72) we see that  $R^{-p+2\varepsilon} G(2R) > 2^p R^{2\varepsilon} \rightarrow +\infty$ , so

$$\log \left[ \left( \frac{p}{\ell + \varepsilon - p} + (1 + C_2)R^{-2\varepsilon} \right) R^{-p+2\varepsilon} G(2R) + O(1) \right] \sim \log(R^{-p+2\varepsilon} G(2R))$$

as  $R \rightarrow +\infty$ , and then

$$\begin{aligned} \liminf_{R \rightarrow +\infty} \frac{\log \Psi(R)}{\log R} &\leq \liminf_{R \rightarrow +\infty} \frac{\log(R^{-p+2\varepsilon} G(2R))}{\log R} \\ &= -p + 2\varepsilon + \liminf_{R \rightarrow +\infty} \frac{\log(G(2R))}{\log R} \end{aligned}$$

and then, using that  $\log R \sim \log(2R)$ , after relabeling we get

$$\liminf_{R \rightarrow +\infty} \frac{\log \Psi(R)}{\log R} \leq -p + 2\varepsilon + \lim_{R \rightarrow +\infty} \frac{\log G(R)}{\log R}.$$

Substituting this into (75) yields

$$\lim_{R \rightarrow +\infty} \frac{\log G(R)}{R} \geq c_{3,\varepsilon} + p - 2\varepsilon$$

and then letting  $\varepsilon \rightarrow 0^+$  we finally obtain (52). □

**Remark 15** As a byproduct of the previous proof (namely, inequality (66) above), we showed that if  $u \in W_{loc}^{1,p}(M)$  satisfies

$$Lu \geq Vu^{p-1} \quad \text{on } \Omega_{s_0} = \{u > s_0\}$$

with  $V : M \rightarrow (0, +\infty)$  continuous and matching (49) for some  $\lambda > 0$  and  $\mu \in [0, p]$ , then for each  $\varepsilon \in (0, \lambda)$  and  $R_0 > 0$  large enough (so that (56)–(57) are satisfied) and for each  $R_1 > R_0$  such that

$$I_1 := \int_{B_{R_1}} (u - s_0)_+^q > 0$$

we have

$$\log \int_{B_{R+R^{\mu/p}}} (u - s_0)_+^q \geq C_{0,\varepsilon} \int_{R_1}^R t^{-\mu/p} dt + \log \frac{I_1}{C_2} \quad \forall R > R_1 \quad (76)$$

where

$$C_{0,\varepsilon} = \frac{p(q - p + 1)^{1/p'}}{ (p - 1)^{1/p'} } \frac{(\lambda - \varepsilon)^{1/p}}{k}, \quad C_{2,\varepsilon} = 1 + \frac{k^p (p - 1)^{p-1}}{(\lambda - \varepsilon) \min\{1, \gamma^{p-1}\}}$$

do not depend on  $u$ . Inequality (76) only involves the integrals of  $w = (u - s_0)_+^q$  on geodesic balls, so it would still hold for functions  $u \in L_{loc}^q(M)$  that can be approximated pointwise and in  $L^q$  norm on balls  $B$  of arbitrary large radii by Sobolev functions  $\tilde{u} \in W_{loc}^{1,p}(B)$  satisfying

$$L\tilde{u} \geq V|\tilde{u}|^{p-2}\tilde{u} \quad \text{on } B.$$

For instance, when  $L = \Delta$  is the Laplace–Beltrami operator and  $V \equiv 1$ , a non-trivial result concerning local smooth monotone approximation of distributional  $L^1_{loc}$  subsolutions of  $\Delta u = u$  (namely, Theorem D in [3]) allows to extend the estimate

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} u_+^q \geq 2\sqrt{q-1}$$

to distributional and not everywhere negative  $L^1_{loc}$  subsolutions of  $\Delta u = u$ .

The following examples are aimed at showing the sharpness of the constant appearing in (50) and (52). Let  $M$  be a model surface, that is, a complete Riemannian manifold diffeomorphic to  $\mathbb{R}^2$  and radially symmetric around some point  $o \in M$  so that in global polar coordinates  $(r, \theta)$  centered at  $o$  the metric takes the form

$$\langle , \rangle = dr^2 + g(r)^2 d\theta^2$$

for a smooth  $g : (0, +\infty) \rightarrow (0, +\infty)$  satisfying  $g'(0^+) = 1$  and  $g^{(2k)}(0^+) = 0$  for each  $k \in \{0\} \cup \mathbb{N}$ . Let  $v : [0, +\infty) \rightarrow \mathbb{R}$  be smooth and such that

$$v^{(k)}(0) = 0 \quad \forall k \in \mathbb{N} \quad \text{and} \quad v'(t) > 0 \quad \forall t > 0.$$

Then  $u := v \circ r \in C^\infty(M)$ ,  $|\nabla u| \neq 0$  on  $M \setminus \{o\}$  and for any  $p > 1$  we have

$$\Delta_p u = \left[ (p-1)(v')^{p-2}v'' + \frac{g'}{g}(v')^{p-1} \right] \circ r \quad \text{on } M \setminus \{o\}. \tag{77}$$

**Case  $\mu \in [0, p)$ .** Let  $p > 1$  and  $\mu \in [0, p)$  be given. Consider  $a, c \in \mathbb{R}$  satisfying

$$c > 0, \quad (p-1)c + a > 0 \tag{78}$$

and set

$$\beta := 1 - \frac{\mu}{p} \in (0, 1].$$

Choose  $g$  and  $v$  satisfying the above requirements and such that

$$g(t) = \begin{cases} t & \text{for } 0 < t \leq 1/2 \\ \exp(at^\beta) & \text{for } t \geq 1 \end{cases}$$

and

$$v(t) = \exp(ct^\beta) \quad \text{for } t \geq 1.$$

By (77) we have

$$\Delta_p u = Vu^{p-1} \quad \text{on } \Omega := M \setminus \overline{B_1} \tag{79}$$

where

$$V = \left( (p - 1) \left( 1 + \frac{\beta - 1}{c\beta r^\beta} \right) c + a \right) \frac{\beta^p c^{p-1}}{r^\mu}. \tag{80}$$

Let  $s_0 > e^c$ . Since  $v$  is non-decreasing, the set  $\Omega_{s_0} := \{u > s_0\}$  coincides with  $M \setminus \overline{B_{t_0}}$ , where  $t_0 = [(\log s_0)/c]^{1/\beta} > 1$ , so in particular  $\Omega_{s_0} \subseteq \Omega$ . Also, for any  $q > p - 1$  we have

$$\int_{B_R} (u - s_0)_+^q = \int_{t_0}^R g(s)(v(s) - s_0)^q ds \sim \int_{t_0}^R \exp((a + qc)s^\beta) ds \quad \text{as } R \rightarrow +\infty$$

where the asymptotic equivalence between the integrals holds because

$$g(s)(v(s) - s_0)^q \sim \exp((a + qc)s^\beta) \rightarrow +\infty \quad \text{as } s \rightarrow +\infty.$$

(Recall that  $a + qc > a + (p - 1)c > 0$  due to our assumptions on  $a$  and  $c$ .) Integrating by parts yields

$$\begin{aligned} \int_{t_0}^R \exp((a + qc)s^\beta) ds &= \int_{t_0}^R \frac{\frac{d}{ds} \exp((a + qc)s^\beta)}{(a + qc)\beta s^{\beta-1}} ds \\ &= \frac{1}{(a + qc)\beta} \left( \frac{\exp((a + qc)R^\beta)}{R^{\beta-1}} - \frac{\exp((a + qc)t_0^\beta)}{t_0^{\beta-1}} \right) \\ &\quad - \frac{1 - \beta}{(a + qc)\beta} \int_{t_0}^R s^{-\beta} \exp((a + qc)s^\beta) ds \\ &\geq \frac{1}{(a + qc)\beta} \left( \frac{\exp((a + qc)R^\beta)}{R^{\beta-1}} - \frac{\exp((a + qc)t_0^\beta)}{t_0^{\beta-1}} \right) \\ &\quad - \frac{(1 - \beta)t_0^{-\beta}}{(a + qc)\beta} \int_{t_0}^R \exp((a + qc)s^\beta) ds \end{aligned}$$

hence, rearranging terms and using that  $\beta \in (0, 1]$ , we get

$$\frac{\exp((a + qc)R^\beta)}{a_1 R^{\beta-1}} + O(1) \geq \int_{t_0}^R \exp((a + qc)s^\beta) ds \geq \frac{\exp((a + qc)R^\beta)}{a_2 \beta R^{\beta-1}} + O(1)$$

for  $R \rightarrow +\infty$ , with

$$a_1 = (a + qc)\beta, \quad a_2 = (a + qc)\beta + (1 - \beta)t_0^{-\beta}.$$

Passing to logarithms, we obtain

$$\log \int_{B_R} (u - s_0)_+^q \sim \log \int_{t_0}^R \exp((a + qc)s^\beta) ds \sim (a + qc)R^\beta$$

as  $R \rightarrow +\infty$ , that is, multiplying both sides by  $\beta R^{-\beta}$  and recalling that  $\beta = 1 - \frac{\mu}{p}$ ,

$$\lim_{R \rightarrow +\infty} \frac{1 - \frac{\mu}{p}}{R^{1 - \frac{\mu}{p}}} \log \int_{B_R} (u - s_0)_+^q = (a + qc)\beta .$$

On the other hand, from (80) we clearly have

$$\lim_{x \rightarrow \infty} r(x)^\mu V(x) = \lambda \quad \text{with} \quad \lambda = \beta^p c^{p-1} ((p - 1)c + a) . \tag{81}$$

Since the  $p$ -Laplacian is weakly  $p$ -coercive with coercivity constant  $k = 1$ , to prove that estimate (50) is sharp it is enough to show that for any  $p$  and  $q > p - 1$  there exist  $a$  and  $c$  satisfying (78) and such that

$$a + qc = \frac{p(q - p + 1)^{1/p'}}{(p - 1)^{1/p'}} c^{1/p'} ((p - 1)c + a)^{1/p} . \tag{82}$$

This can be done by picking any  $a$  and  $c > 0$  such that

$$(p - 1)a = (q - p(p - 1))c$$

since this would yield

$$a + qc = p((p - 1)c + a) = \frac{p(q - p + 1)}{p - 1} c > 0$$

and then

$$\begin{aligned} a + qc &= (a + qc)^{1/p'} (a + qc)^{1/p} = \left( \frac{p(q - p + 1)}{p - 1} c \right)^{1/p'} (p((p - 1)c + a))^{1/p} \\ &= \frac{p(q - p + 1)^{1/p'}}{(p - 1)^{1/p'}} c^{1/p'} ((p - 1)c + a)^{1/p} \end{aligned}$$

as desired. For instance, a feasible choice for  $a$  and  $c$  would be the following:

$$\begin{cases} a = -1 \quad \text{and} \quad c = \frac{p - 1}{p(p - 1) - q} & \text{if } p - 1 < q < p(p - 1) \\ a = 0 \quad \text{and} \quad c = 1 & \text{if } q = p(p - 1) \\ a = 1 \quad \text{and} \quad c = \frac{p - 1}{q - p(p - 1)} & \text{if } q > p(p - 1) . \end{cases} \tag{83}$$

**Case  $\mu = p$ .** Let  $p > 1$  be given, consider  $a, c \in \mathbb{R}$  satisfying (78) and choose  $g$  and  $v$  satisfying the general requirements and such that

$$g(t) = \begin{cases} t & \text{for } 0 < t \leq 1/2 \\ t^{a+p-1} & \text{for } t \geq 1 \end{cases}$$

and

$$v(t) = t^c \quad \text{for } t \geq 1.$$

By (77) we have

$$\Delta_p u = V u^{p-1} \quad \text{on } \Omega = M \setminus \overline{B_1}$$

with

$$V = \frac{c^{p-1}((p-1)c + a)}{r^p}.$$

Let  $s_0 > 1$  be given. Then  $\Omega_{s_0} := \{u > s_0\}$  is contained in  $\{u > 1\} = M \setminus \overline{B_1}$  and for any  $q > p - 1$  we have

$$\log \int_{B_R} (u - s_0)_+^q \sim \log \int_{s_0}^R s^{a+p-1+qc} ds \sim (a + p + qc) \log R \quad \text{as } R \rightarrow +\infty$$

that is,

$$\lim_{R \rightarrow +\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q = (a + qc) + p$$

and then again to prove sharpness of (52) we need to show that for any  $p > 1$  and  $q > p - 1$  we can choose  $a$  and  $c$  satisfying (78) and

$$a + qc = \frac{p(q - p + 1)^{1/p'} c^{1/p'} ((p - 1)c + a)^{1/p}}{(p - 1)^{1/p'}},$$

but this is precisely what we did in the previous case.

### 4 The Case $Lu \geq 0$

In this section we are concerned with lower bounds on the growth of functions  $u$  satisfying the differential inequality  $Lu \geq 0$  on a non-empty superlevel set. The main result of this section is Theorem 20 below, corresponding to Theorem 6 from the Introduction. The starting point in this case is again Lemma 9. For ease of the reader we point out that in this case it takes the following form.

**Lemma 16** *Let  $M$  be a Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23). Let  $u \in W_{\text{loc}}^{1,p}(M)$  satisfy*

$$Lu \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{84}$$

for some  $s_0 \in \mathbb{R}$ . Then for any  $0 \leq \eta \in C_c^\infty(M)$  and for any non-negative, non-decreasing, piecewise  $C^1$  function on  $(0, +\infty)$  we have

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta| \geq \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} \tag{85}$$

where  $w := (u - s_0)_+$  and  $A_u := A(x, u, \nabla u)$ .

The main tool to prove Theorem 20 is the next proposition.

**Proposition 17** *Let  $M$  be a complete, non-compact Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23). Let  $u \in W_{loc}^{1,p}(M)$  satisfy*

$$Lu \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{86}$$

for some  $s_0 \in \mathbb{R}$ .

(a) For any  $q > p - 1$  and for any  $x_0 \in M$  and  $0 < r < R$

$$\int_{B_r(x_0) \cap \Omega_{s_0}} w^{q-p}|A_u|^{p'} \leq \frac{(p-1)^{p-1}}{\min\{1, \gamma^p\}} \left( \int_r^R \left( \int_{\partial B_s(x_0)} w^q \right)^{1/(1-p)} ds \right)^{1-p} \tag{87}$$

where  $w := (u - s_0)_+$ ,  $A_u := A(x, u, \nabla u)$  and  $\gamma := q - p + 1$ .

(b) If  $u_+ \in L_{loc}^\infty(M)$  and  $F$  is a non-negative, piecewise  $C^1$  function on  $(0, +\infty)$  such that  $F' > 0$  everywhere on  $(0, +\infty)$ , then

$$\int_{B_r(x_0) \cap \Omega_{s_0}} F'(w)|A_u|^{p'} \leq (p-1)^{p-1} \left( \int_r^R \left( \int_{\Omega_{s_0} \cap \partial B_s(x_0)} \frac{[F(w)]^p}{[F'(w)]^{p-1}} \right)^{1/(1-p)} ds \right)^{1-p} \tag{88}$$

for every  $x_0 \in M$  and  $0 < r < R$ , with  $w$  and  $A_u$  as above.

**Remark 18** We remark that the exponents  $1 - p$  and  $1/(1 - p)$  appearing on the RHS's of (87) and (88) are negative. With the agreement that  $0^a = +\infty$  and  $(+\infty)^a = 0$  for any  $a \in (-\infty, 0)$ , the inequalities make sense also in case one or more of the integrals on the RHS's are either vanishing or diverging.

**Proof** Let  $w$  and  $A_u$  be as in the statement. We first prove (b), since the proof of (a) relies on the same idea coupled with suitable approximation arguments.

**Proof of (b).** Suppose that  $u_+ \in L_{loc}^\infty(M)$  and let  $F$  be as in the statement. The function  $F$  satisfies all the requirements in Lemma 16 and therefore

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta| \geq \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} \tag{89}$$

for any  $0 \leq \eta \in C_c^\infty(M)$ . Note that both integrals are finite since  $F(w), F'(w) \in L^\infty(\Omega_{s_0})$  and  $|A_u|\mathbf{1}_{\Omega_{s_0}} \in L_{loc}^{p'}(M)$ . Applying Hölder inequality with conjugate expo-



nents  $p$  and  $p'$  as in (37) we further obtain

$$\left( \int_{\Omega_{s_0}} F'(w)|A_u|^{p'}|\nabla\eta| \right)^{1/p'} \left( \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta| \right)^{1/p} \geq \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} \tag{90}$$

where the middle integral is again finite since  $[F(w)]^p/[F'(w)]^{p-1} \in L^\infty(\Omega_{s_0})$ . Let  $x_0 \in M$  be fixed and let us write  $B_s$  for the geodesic ball  $B_s(x_0)$ , for any  $s > 0$ . Let  $G, H : (0, +\infty) \rightarrow [0, +\infty)$  be defined by

$$G(s) := \int_{\Omega_{s_0} \cap B_s} F'(w)|A_u|^{p'} , \quad H(s) := \int_{\Omega_{s_0} \cap B_s} \frac{[F(w)]^p}{[F'(w)]^{p-1}} . \tag{91}$$

Since  $F'(w)|A_u|^{p'}\mathbf{1}_{\Omega_{s_0}} \in L^1_{loc}(M)$  and  $[F(w)]^p/[F'(w)]^{p-1}\mathbf{1}_{\Omega_{s_0}} \in L^\infty(M) \subseteq L^1_{loc}(M)$ , the functions  $G$  and  $H$  are well defined, non-decreasing and absolutely continuous on any compact interval contained in  $(0, +\infty)$ . In particular, they are differentiable a.e. on  $(0, +\infty)$ . Let  $s > 0$  be a value for which  $G'(s)$  and  $H'(s)$  both exist. For any  $\varepsilon > 0$  choose  $\eta_\varepsilon \in C^\infty(M)$  satisfying

- (i)  $\eta_\varepsilon \equiv 1$  on  $B_s$ ,
- (ii)  $\eta_\varepsilon \equiv 0$  on  $M \setminus B_{s+\varepsilon}$ ,
- (iii)  $0 \leq \eta_\varepsilon \leq 1$  on  $B_{s+\varepsilon} \setminus B_s$
- (iv)  $|\nabla\eta_\varepsilon| \leq \frac{1}{\varepsilon} + 1$  on  $M$ .

Then

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta_\varepsilon| \leq \left( \frac{1}{\varepsilon} + 1 \right) \int_{\Omega_{s_0} \cap B_{s+\varepsilon} \setminus B_s} F(w)|A_u| \leq (1 + \varepsilon) \frac{G(s + \varepsilon) - G(s)}{\varepsilon}$$

and passing to limits as  $\varepsilon \rightarrow 0^+$  we get

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta_\varepsilon| \leq G'(s) \in [0, +\infty) .$$

Similarly, we obtain

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta_\varepsilon| \leq H'(s)$$

and by dominated convergence theorem we also have

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega_{s_0}} \eta_\varepsilon F'(w)|A_u|^{p'} = G(s) .$$

Then by (90) we deduce

$$[H'(s)]^{p'/p} G'(s) \geq [G(s)]^{p'} \quad \text{for a.e. } s > 0. \tag{92}$$

Moreover, by the co-area formula we have

$$H'(s) = \int_{\Omega_{s_0} \cap \partial B_s} \frac{[F(w)]^p}{[F'(w)]^{p-1}} =: \varphi(s) \quad \text{for a.e. } s > 0. \tag{93}$$

Let  $0 < r < R$  be given. If  $G(r) = 0$  then (88) is trivially satisfied. If  $G(r) > 0$  then by monotonicity of  $G$  we have that  $G(s) \geq G(r)$  for all  $s \in [r, R]$ . Since  $G'(s)$  is finite for a.e.  $s \in [r, R]$ , from (92) and (93) we infer that  $\varphi(s) > 0$  for a.e.  $s \in [r, R]$  and then

$$\frac{G'(s)}{[G(s)]^{p'}} \geq [\varphi(s)]^{-p'/p} \quad \text{for a.e. } s \in [r, R]. \tag{94}$$

Since  $G(s) \geq G(r) > 0$  for all  $s \in [r, R]$  and  $[G(r), +\infty) \ni t \mapsto t^{1/(1-p)}$  is Lipschitz, the function  $G^{1/(1-p)} \equiv G^{1-p'}$  is absolutely continuous on  $[r, R]$  with

$$\frac{d}{ds} [G(s)]^{1-p'} = \frac{1}{1-p} \frac{G'(s)}{[G(s)]^{p'}} \quad \text{for a.e. } s \in [r, R].$$

Thus, integrating (94) we get (noting that  $p'/p = 1/(p-1)$ )

$$(p-1) \left[ G(r)^{-1/(p-1)} - G(R)^{-1/(p-1)} \right] = \int_r^R \frac{G'(s)}{G(s)^{p'}} ds \geq \int_r^R [\varphi(s)]^{1/(1-p)} ds.$$

Discarding the term containing  $G(R)$  and raising everything to  $1-p$  we get

$$G(r) \leq (p-1)^{p-1} \left( \int_r^R [\varphi(s)]^{1/(1-p)} ds \right)^{1-p}$$

that is, (88).

**Proof of (a).** We observe that the argument developed above can be applied straightforwardly, without the assumption  $u_+ \in L^\infty_{\text{loc}}(M)$ , as long as we consider a piecewise  $C^1$  function  $F : (0, +\infty) \rightarrow (0, +\infty)$  with  $F' > 0$  such that

$$F'(w)|A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M), \quad \frac{[F(w)]^p}{[F'(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M). \tag{95}$$

Indeed, if the conditions in (95) are satisfied then all the integrals appearing in (89) and (90) are finite and the functions  $G$  and  $H$  defined as in (91) are again finite-valued, non-decreasing and absolutely continuous on every compact interval contained in  $(0, +\infty)$ .

**Case**  $q \geq p$ . Set  $\gamma := q - p + 1 \geq 1$ . For any  $h > 0$  define  $F_h$  by

$$F_h(s) := \begin{cases} s^\gamma & \text{if } 0 < s < h \\ \frac{\gamma}{h^\gamma} + (s - h)h^{\gamma-1} & \text{if } s \geq h. \end{cases} \tag{96}$$

Note that  $F_h$  is positive and  $C^1$  on  $(0, +\infty)$  with

$$F'_h(s) = \begin{cases} s^{\gamma-1} & \text{if } 0 < s < h \\ h^{\gamma-1} & \text{if } s \geq h. \end{cases} \tag{97}$$

We have  $F'_h > 0$  everywhere on  $(0, +\infty)$  and  $F'_h(w) \in L^\infty(\Omega_{s_0})$ , therefore also  $F'_h(w)|A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M)$ , due to (26) and  $u \in W^{1,p}_{\text{loc}}(M)$ . Moreover,

$$\begin{aligned} \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} &= \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + h^{\gamma-1} \left( w - \frac{\gamma-1}{\gamma} h \right)^p \mathbf{1}_{\{w \geq h\}} \\ &\leq \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + h^{\gamma-1} w^p \mathbf{1}_{\{w \geq h\}} \end{aligned} \tag{98}$$

so in particular

$$\frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} \leq h^{\gamma-1} w^p \in L^1_{\text{loc}}(M)$$

since  $\gamma \geq 1$  and  $u \in W^{1,p}_{\text{loc}}(M)$ . Hence, conditions (95) are satisfied for  $F = F_h$  and we can repeat the argument in the proof of (a) up to obtaining

$$[\varphi_h(s)]^{p'/p} G'_h(s) \geq [G_h(s)]^{p'} \quad \text{for a.e. } s > 0$$

with

$$G_h(s) = \int_{\Omega_{s_0} \cap B_s} F'_h(w)|A_u|^{p'}, \quad \varphi_h(s) = \int_{\Omega_{s_0} \cap \partial B_s} \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}}.$$

From (98) and recalling that  $\gamma = q - p + 1$  we also have

$$\frac{[F_h(w)]^p}{[f_h(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} \leq w^q \quad \text{on } M$$

hence

$$\varphi_h(s) \leq \varphi(s) := \int_{\partial B_s} w^q \quad \forall s > 0.$$

Reasoning again as in the proof of (a) we deduce that either  $G_h(r) = 0$  or

$$\begin{cases} G_h(s) \geq G_h(r) > 0 & \forall s \in [r, R] \\ \frac{G'_h(s)}{[G_h(s)]^{p'}} \geq [\varphi_h(s)]^{1/(1-p)} \geq [\varphi(s)]^{1/(1-p)} & \text{for a.e. } s \in [r, R]. \end{cases}$$

In any case we get

$$\int_{B_r} \min\{w, h\}^{\gamma-1} |A_u|^{p'} = G_h(r) \leq (p-1)^{p-1} \left( \int_r^R [\varphi(s)]^{1/(1-p)} ds \right)^{1-p}$$

and the conclusion follows by the monotone convergence theorem letting  $h \rightarrow +\infty$ .

**Case**  $p-1 < q < p$ . Set  $\gamma := q-p+1$  as in the previous case and note that now  $\gamma \in (0, 1)$ . For any  $h > 0$  let  $F_h$  be defined as in (96). We note that  $F_h$  is positive and  $C^1$  on  $(0, +\infty)$  in this case too, with  $F'_h > 0$  everywhere on  $(0, +\infty)$ . Then from Lemma 16 we get

$$\int_{\Omega_{s_0}} F_h(w) |A_u| |\nabla \eta| \geq \int_{\Omega_{s_0}} \eta F'_h(w) |A_u|^{p'} \quad \forall 0 \leq \eta \in C_c^\infty(M). \tag{99}$$

From the expression (96) we see that  $F_h(w) \leq C_{h,\gamma}(1+w)$ , hence  $F_h(w) |A_u| \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M)$  by Hölder inequality. By (99) this also yields

$$F'_h(w) |A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\text{loc}}(M).$$

On the other hand, we have

$$\begin{aligned} \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} &= \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + h^{\gamma-1} \left( w-h + \frac{h}{\gamma} \right)^p \mathbf{1}_{\{w \geq h\}} \\ &\leq \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + h^{\gamma-1} \left( \frac{w-h}{\gamma} + \frac{h}{\gamma} \right)^p \mathbf{1}_{\{w \geq h\}} \\ &= \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + \frac{h^{\gamma-1} w^p}{\gamma^p} \mathbf{1}_{\{w \geq h\}} \end{aligned} \tag{100}$$

where the inequality in the middle holds because  $w-h < (w-h)/\gamma$  on  $\{w > h\}$ , since  $0 < \gamma < 1$  in this case. From this estimate we get

$$\frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} \leq \max \left\{ \frac{h^q}{\gamma^p}, \frac{h^{\gamma-1}}{\gamma^p} w^p \right\} \in L^1_{\text{loc}}(M).$$

Hence, both conditions in (95) are satisfied. Setting again

$$G_h(s) = \int_{\Omega_{s_0} \cap B_s} F'_h(w) |A_u|^{p'}, \quad \varphi_h(s) = \int_{\Omega_{s_0} \cap \partial B_s} \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}}$$

we can repeat once more the general argument to get that either  $G_h(r) = 0$  or

$$\begin{cases} G_h(s) \geq G_h(r) > 0 & \forall s \in [r, R] \\ \frac{G'_h(s)}{[G_h(s)]^{p'}} \geq [\varphi_h(s)]^{1/(1-p)} & \text{for a.e. } s \in [r, R] \end{cases}$$

and in any case we get

$$\int_{B_r} F'_h(w)|A_u|^{p'} = G_h(r) \leq (p - 1)^{p-1} \left( \int_r^R [\varphi_h(s)]^{1/(1-p)} ds \right)^{1-p}. \tag{101}$$

We now let  $h \rightarrow +\infty$  in both sides of (101). By Fatou’s lemma we have

$$\liminf_{h \rightarrow +\infty} \int_{\Omega_{s_0} \cap B_r} F'_h(w)|A_u|^{p'} \geq \int_{\Omega_{s_0} \cap B_r} w^{\gamma-1}|A_u|^{p'} \equiv \int_{\Omega_{s_0} \cap B_r} w^{q-p}|A_u|^{p'}. \tag{102}$$

Concerning the RHS of (101), we aim at showing that

$$\lim_{h \rightarrow +\infty} \int_r^R [\varphi_h(s)]^{1/(1-p)} ds = \int_r^R [\varphi(s)]^{1/(1-p)} ds \tag{103}$$

with

$$\varphi(s) := \frac{1}{\gamma^p} \int_{\partial B_s} w^q.$$

From (100) and recalling that  $\gamma + p - 1 = q$  we have

$$0 \leq \varphi_h(s) - \frac{1}{\gamma^p} \int_{\partial B_s \cap \{w < h\}} w^q \leq \frac{h^{\gamma-1}}{\gamma^p} \int_{\partial B_s \cap \{w \geq h\}} w^p. \tag{104}$$

Since  $w \in W_{loc}^{1,p}(M)$ , for a.e.  $s \in [r, R]$  we have  $w \in L^p(\partial B_s)$  by the co-area formula. Then, using the monotone convergence theorem on the first integral in (104) together with the fact that  $h^{\gamma-1} \rightarrow 0$  as  $h \rightarrow +\infty$  (due to  $\gamma < 1$ ) we get

$$\lim_{h \rightarrow +\infty} \varphi_h(s) = \frac{1}{\gamma^p} \int_{\partial B_s} w^q = \varphi(s) \quad \text{for a.e. } s \in [r, R]. \tag{105}$$

If  $\varphi^{1/(1-p)} \notin L^1([r, R])$ , then by (105) and Fatou’s lemma we have

$$\liminf_{h \rightarrow +\infty} \int_r^R \varphi_h^{1/(1-p)} \geq \int_r^R \varphi^{1/(1-p)} = +\infty$$

so (103) holds with both sides equalling  $+\infty$ . Suppose, instead, that  $\varphi^{1/(1-p)} \in L^1([r, R])$ . From the first line in (100) we also deduce the reversed estimate

$$\begin{aligned} \frac{[F_h(w)]^p}{[F'_h(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} &\geq \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + h^{\gamma-1} w^p \mathbf{1}_{\{w \geq h\}} \\ &\geq \frac{w^{\gamma+p-1}}{\gamma^p} \mathbf{1}_{\{0 < w < h\}} + w^{\gamma-1} w^p \mathbf{1}_{\{w \geq h\}} \\ &\geq w^{\gamma+p-1} = w^q \end{aligned}$$

where in the second inequality we exploited again the fact that  $0 < \gamma < 1$ . Then, for every  $h > 0$  we also have  $\varphi_h \geq \gamma^p \varphi$  and therefore

$$\varphi_h^{1/(1-p)} \leq \gamma^{-p/(p-1)} \varphi^{1/(1-p)} \quad \text{on } [r, R].$$

Hence, if  $\varphi^{1/(1-p)} \in L^1([r, R])$  then (103) follows by the dominated convergence theorem. In any case, from the continuity of  $[0, +\infty) \ni t \mapsto t^{1-p} \in [0, +\infty]$  with the agreement that  $0^{1-p} = +\infty$  and  $(+\infty)^{1-p} = 0$  we get

$$\lim_{h \rightarrow +\infty} \left( \int_r^R [\varphi_h(s)]^{1/(1-p)} ds \right)^{1-p} = \left( \int_r^R [\varphi(s)]^{1/(1-p)} ds \right)^{1-p}. \tag{106}$$

By (101), (102) and (106) we obtain the desired conclusion. □

From Proposition 17 we easily deduce the following lemma.

**Lemma 19** *Let  $M$  be a complete, non-compact Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23). Let  $u \in W_{\text{loc}}^{1,p}(M)$  satisfy*

$$Lu \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{107}$$

for some  $s_0 \in \mathbb{R}$  and also suppose that

$$A(x, u, \nabla u) \neq 0 \quad \text{on a set } E_0 \subseteq \Omega_{s_0} \text{ of positive measure.} \tag{108}$$

Then there exists  $r_0 \geq 0$  such that for any  $q > p - 1$

$$\int_r^R \left( \int_{\partial B_s} (u - s_0)_+^q \right)^{1/(1-p)} ds < +\infty \quad \forall r_0 < r < R < +\infty. \tag{109}$$

In particular,

$$\mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_r) > 0 \quad \text{for a.e. } r > r_0 \tag{110}$$

where  $\mathcal{H}^{m-1}$  denotes the  $(m - 1)$ -dimensional Hausdorff measure. Moreover, if  $u_+ \in L_{\text{loc}}^\infty(M)$  then also

$$0 < \int_r^R \left( \mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_s) \right)^{1/(1-p)} ds < +\infty \quad \forall r_0 < r < R < +\infty. \tag{111}$$

**Proof** Choose  $r_0 \geq 0$  such that  $|B_r \cap E_0| > 0$  for every  $r > r_0$ , where  $E_0$  is as in (108). Then, for every  $r > r_0$

$$\int_{B_r \cap \Omega_{s_0}} w^{q-p} |A_u|^{p'} \geq \int_{B_r \cap E_0} w^{q-p} |A_u|^{p'} > 0$$

and then applying Proposition 17.(a) we see that the RHS of (87) must be strictly positive for any  $R > r$ , that is (since  $1 - p < 0$ ),

$$\int_r^R \left( \int_{\partial B_s} w^q \right)^{1/(1-p)} ds < +\infty \quad \forall R > r.$$

In particular,  $\left( \int_{\partial B_s} w^q \right)^{1/(1-p)}$  must be finite for a.e.  $s > r$ , hence for a.e.  $s > r_0$  by arbitrariness of  $r > r_0$ , and therefore it must be  $\int_{\partial B_s} w^q > 0$  for a.e.  $s > r_0$ , yielding (110). If  $u_+ \in L^\infty_{loc}(M)$ , to prove (111) we start from the two-sided estimate

$$\mathcal{H}^{m-1}(\partial B_s) \geq \mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_s) \geq \frac{1}{(1 + \text{ess sup}_{B_R} w)^p} \int_{\Omega_{s_0} \cap \partial B_s} (1 + w)^p,$$

holding for each  $s > r_0$ , from which we deduce

$$\begin{aligned} \left( \mathcal{H}^{m-1}(\partial B_s) \right)^{1/(1-p)} &\leq \left( \mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_s) \right)^{1/(1-p)} \\ &\leq (1 + \text{ess sup}_{B_R} w)^{p/(p-1)} \left( \int_{\Omega_{s_0} \cap \partial B_s} (1 + w)^p \right)^{1/(1-p)}. \end{aligned}$$

The function  $v(r) := \mathcal{H}^{m-1}(\partial B_r)$  satisfies

$$v(r) > 0 \quad \text{for } r > 0 \quad \text{and} \quad v, \frac{1}{v} \in L^\infty_{loc}((0, +\infty)) \tag{112}$$

see Proposition 1.6 in [1], so we have

$$\int_r^R \left( \mathcal{H}^{m-1}(\partial B_s) \right)^{1/(1-p)} ds > 0 \quad \forall 0 < r < R$$

and by Proposition 17.(b) applied with the choice  $f \equiv 1$  and  $F(s) = 1 + s$  we get

$$\int_r^R \left( \int_{\partial B_s} (1 + w)^p \right)^{1/(1-p)} ds < +\infty \quad \forall r_0 < r < R.$$

Putting together all inequalities above we obtain (111). □

We are now ready for the proof of the main result of this section.

**Theorem 20** *Let  $M$  be a complete, non-compact Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23). Let  $u \in W_{\text{loc}}^{1,p}(M)$  satisfy*

$$Lu \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\}$$

for some  $s_0 \in \mathbb{R}$  and suppose that for some  $x_0 \in M$  and  $q > p - 1$  it holds

$$\lim_{R \rightarrow +\infty} \int_r^R \left( \int_{\partial B_s(x_0)} (u - s_0)_+^q \right)^{-\frac{1}{p-1}} ds = +\infty \quad \forall r > 0. \tag{113}$$

Then  $A(x, u, \nabla u) = 0$  a.e. on  $\Omega_{s_0}$ . Thus, if  $A$  satisfies the structural condition

$$A(x, s, \xi) = 0 \quad \text{if and only if} \quad \xi = 0. \tag{114}$$

then either  $u \equiv c$  a.e. on  $M$  for some constant  $c > s_0$ , or  $u \leq s_0$  a.e. on  $M$ .

**Remark 21** Condition (113) can be stated, more briefly, as

$$\left( \int_{\partial B_s} (u - s_0)_+^q \right)^{-\frac{1}{p-1}} \notin L^1(+\infty)$$

with this notation meaning that the function  $\varphi : (0, +\infty) \rightarrow [0, +\infty]$  given by

$$\varphi(s) = \left( \int_{\partial B_s} (u - s_0)_+^q \right)^{-\frac{1}{p-1}} \quad \forall s > 0$$

is not in  $L^1((r, +\infty))$  for any  $r > 0$ . The previous Lemma 19 implies that this is a meaningful condition, since in general only two cases are possible:

- (i)  $\varphi = +\infty$  a.e. on  $(0, +\infty)$ , and then  $\Omega_{s_0}$  has zero measure while condition (113) is obviously satisfied, or
- (ii) there exists  $r_0 \geq 0$  such that  $\varphi < +\infty$  a.e. on  $(r_0, +\infty)$  and  $\varphi \in L^1((r, R))$  for any  $r_0 < r < R < +\infty$ , so that (113) is satisfied if and only if  $\varphi$  is not integrable in a neighborhood of  $+\infty$ .

Concerning case (ii), note that in general  $\varphi$  may be integrable at  $+\infty$  and still satisfy  $\varphi = +\infty$  on  $(0, r_0)$  for some  $r_0 > 0$  (for instance, on  $\mathbb{R}^n$  this may happen if  $u$  satisfies  $u \leq s_0$  on  $B_{r_0}$  and  $u(x) \geq |x|^a$  as  $x \rightarrow \infty$  for some  $a > (p - n)/q$ ), so the clause “ $\forall r > 0$ ” in (113) cannot in general be replaced by “for some  $r > 0$ ”.

**Proof of Theorem 20** Suppose, by contradiction, that  $A_u := A(x, u, \nabla u)$  is non-zero on a set  $E_0 \subseteq \Omega_{s_0}$  of positive measure. Then reasoning as in the proof of Lemma 19 we see that there exists  $r > 0$  such that

$$\int_{\Omega_{s_0} \cap B_r} w^{q-p} |A_u|^{p'} > 0$$



and by Proposition 17 this implies that

$$\int_r^R \left( \int_{\partial B_s} (u - s_0)_+^q \right)^{1/(1-p)} ds \leq \left( \frac{\min\{1, \gamma^p\}}{(p - 1)^{p-1}} \int_{\Omega_{s_0} \cap B_r} w^{q-p} |A_u|^{p'} \right)^{1/(1-p)}$$

for all  $R > r$ , with  $\gamma = q - p + 1$ . Since the RHS of this inequality is finite, letting  $R \rightarrow +\infty$  in the LHS we reach the desired contradiction. So, we conclude that  $A(x, u, \nabla u) = 0$  a.e. on  $\Omega_{s_0}$ .

If  $A$  satisfies the non-degeneracy condition (114) then we further deduce that  $\nabla u = 0$  a.e. on  $\Omega_{s_0}$ , and since the function  $w := (u - s_0)_+ \in W_{loc}^{1,p}(M)$  has weak gradient  $\nabla w = \mathbf{1}_{\Omega_{s_0}} \nabla u$  this yields  $\nabla w \equiv 0$  a.e. on  $M$ . By connectedness of  $M$  this implies that  $w = a$  a.e. on  $M$  for some constant  $a \geq 0$ . If  $a > 0$  then  $u = c := s_0 + a$  a.e. on  $M$  (and  $\Omega_{s_0}$  is of full measure), while if  $a = 0$  then  $u \leq s_0$  a.e. on  $M$  (and  $\Omega_{s_0}$  has zero measure). □

As a consequence of Theorem 20 we have the following Liouville-type theorem.

**Corollary 22** *Let  $M$  be a complete, non-compact Riemannian manifold,  $p \in (1, +\infty)$  and  $L$  a weakly  $p$ -coercive operator as in (23). Let  $u \in W_{loc}^{1,p}(M)$  satisfy*

$$Lu \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\}$$

for some  $s_0 \in \mathbb{R}$  and suppose that for some  $x_0 \in M$  and  $q > p - 1$  it holds

$$\lim_{R \rightarrow +\infty} \int_r^R \left( \frac{s}{\int_{B_s} (u - s_0)_+^q} \right)^{\frac{1}{p-1}} ds = +\infty \quad \forall r > 0. \tag{115}$$

Then  $A(x, u, \nabla u) = 0$ , and if  $A$  satisfies the structural condition (114) then either  $u \equiv c$  a.e. on  $M$  for some  $c > s_0$  or  $u \leq s_0$  a.e. on  $M$ .

**Proof** The corollary is a direct consequence of Theorem 20 since (115) implies (113). For the details, see the proof of Proposition 1.3 in [8] (the parameter  $\delta$  there corresponds to  $p - 1$  in our setting). □

**Remark 23** Note, in particular, that (115) holds if

$$\int_{B_R} (u - s_0)_+^q = O(R^p) \quad \text{as } R \rightarrow +\infty \tag{116}$$

or even if, for some  $n \in \mathbb{N}$

$$\int_{B_R} (u - s_0)_+^q = O(R^p g_n^{p-1}(R)) \quad \text{as } R \rightarrow +\infty. \tag{117}$$

where

$$g_n(t) = (\log t)(\log \log t) \cdots \underbrace{(\log \log \cdots \log t)}_{n \text{ iterations}} \quad \text{for } t \gg 1.$$

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