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The Shifted Wave Equation on Non-flat Harmonic Manifolds

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Abstract

We solve the shifted wave equation

$$\frac{\partial^2}{\partial t^2}\varphi(x,t) = (\Delta_x + \rho^2)\varphi(x,t)$$

on a non-compact simply connected harmonic manifold with mean curvature of the horospheres $2\rho > 0$. We give an explicit representation of the solution as the inverse dual Abel transform of the spherical means of their initial conditions using the local injectivity of the Abel transform and symmetry properties of the spherical mean value operator. Furthermore, we investigate the shifted wave equation using the Fourier transform on harmonic manifolds of rank one. Additionally, we obtain a result analogous to the classical Paley–Wiener theorem and use it to show an asymptotic Huygens principle as well as asymptotic equidistribution of the energy of a solution of the shifted wave equation under assumptions on the Harish–Chandra type **c**-function.

Keywords Shifted wave equation · Harmonic manifolds · Fourier transform

Mathematics Subject Classification Primary 53C25, 53C23, 53C65

1 Introduction

In their paper [1], the authors solved the shifted wave equation on Damek–Ricci spaces explicitly. These spaces together with Euclidean and hyperbolic spaces provide all

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known examples of non-compact simply connected harmonic manifolds. A harmonic manifold is a complete Riemannian manifold (X, g) such that for all $p \in X$ the volume density function in geodesic polar coordinates $\sqrt{g_{ij}(p)} = \theta_q(p)$ only depends on the geodesic distance. The Euclidean and non-flat symmetric spaces of rank one are harmonic. It was a long-standing conjecture that all harmonic manifolds are of this type, referred to as the Lichnerowicz conjecture [36]. The conjecture was proven for compact simply connected spaces by Szabo[45] but shortly after this, in 1992, Damek and Ricci [18] provided for dimension 7 and higher a class of homogeneous harmonic spaces that are non-symmetric. These manifolds are called Damek–Ricci spaces. In 2006, Heber [24] showed that all homogeneous non-compact simply connected harmonic spaces are of the type mentioned above. Since homogeneous spaces have a rich algebraic structure one can use tools from harmonic analysis, see [29] and [42]. In [10], the authors showed that certain analytic properties of harmonic spaces can be obtained without the assumption of homogeneity only assuming purely exponential volume growth or equivalently rank one condition. Furthermore, in [40], the authors showed that important properties of the Abel transform and its dual are true for general non-compact harmonic manifolds. We now use their methods to generalise the results from [1]. The idea of the proof is identical: We use the symmetries of the mean value operator to express the solution of the shifted wave equation via the inverse dual Abel transform of spherical means of its initial conditions.

In Sect. 2, we provide all the generalities on harmonic manifolds needed for this discussion. In Sect. 3, we recall important properties of the Abel transform and its dual from [40], and in Sect. 4, we show Ásgeirsson's mean value theorem for harmonic manifolds (Lemma 4.1) before solving the shifted wave equation with smooth compactly supported initial conditions explicitly in Sect. 5 (Theorem 5.1). Up until this point, we assumed (X, g) to be a simply connected, non-compact and non-flat harmonic manifold. Starting with Sect. 6 we investigate the shifted wave equation under the additional assumption that X is of rank one and thereby obtain similar results to [5]. To conduct this investigation, we will use the Fourier transform on X. For this purpose, we give a brief overview of the Fourier transform on harmonic manifolds of rank one and study the action of the Laplacian under Fourier transform. Then in Sect. 7, we generalise the Paley–Wiener type theorem (Theorem 7.4) from [5] and use it to obtain bounds on the energy of a solution of the shifted wave equation on X under assumptions on the initial conditions. In Sect. 8, we improve the Paley–Wiener type theorem from the previous section by showing an analogous result of the classical Paley–Wiener theorem on harmonic manifolds of rank one (Theorem 8.1), generalising the results from [29] and [3] for symmetric and non-symmetric Damek–Ricci spaces, respectively. The main idea of the proof of this theorem is to use the Radon transform from [43] to translate the problem to the real line. We subsequently use this to obtain an asymptotic Huygens principle (Sect. 9, Theorem 9.2) and asymptotic equidistribution of energy (Sect. 10, Theorem 10.1). Under the assumption that the **c**-function of X has a polynomial holomorphic extension into a strip on the upper half plane in $\mathbb{C}$ with the first pole of multiplicity one. This generalises the results of symmetric spaces ([13–15, 28, 39]), non-symmetric Damek–Ricci spaces ([5]) and gives a non-radial version of the results in [21].

2 Preliminaries

In this Section, we give a brief introduction to non-compact simply connected harmonic manifolds. For more information, we refer the reader to the surveys [35] and [33]. Let (X, g) be a non-compact simply connected Riemannian manifold without conjugate points. Denote by $C^k(X)$ the space of k-times differentiable functions on X and by $C_c^k(X) \subset C^k(X)$ those with compact support. We use the usual conventions for continuous, smooth and analytic functions. Furthermore, for $x \in X$ denote by $C^k(X, x)$ resp. $C_c^k(X, x)$ the functions in $C^k(X)$ resp. C_c^k radial around x i.e. $f \in C^k(X, x)$ $(C_c^k(X, x))$ if there exists a even function $u \in C^k_{\text{even}}(\mathbb{R})$ on $\mathbb{R}$ (with compact support) such that $f = u \circ d(x, \cdot)$ where $d : X \times X \to \mathbb{R}_{\geq 0}$ is the distance induced by g. Furthermore, for $p \geq 1$, $L^p(X)$ refers to the L^p-space of X with regard to the measure induced by the metric and integration over a manifold is always interpreted as integration with respect to the canonical measure on this manifold unless stated otherwise. For $p \in X$ and $v \in S_pX$ denote by $c_v : \mathbb{R} \to X$ the unique unit speed geodesic with $c(0) = p$ and $\dot{c}(0) = v$. Define A_v to be the Jacobi tensor along c_v with initial conditions $A_v(0) = 0$ and $A_v'(0) = \text{id}$. For details on Jacobi tensors, see [31]. Then using the transformation formula and the Gauss lemma, the volume of the sphere of radius r around p is given by

$$\text{vol}\, S(p, r) = \int_{S_pX} \det A_v(r)\, dv. \tag{1}$$

The second fundamental form of $S(p, r)$ is given by $A_v'(r)A_v^{-1}(r)$ and the mean curvature by

$$\nu_p(r, v) = \text{trace}\, A_v'(r)A_v^{-1}(r). \tag{2}$$

Definition 2.1 Let (X, g) be a complete non-compact simply connected manifold without conjugate points and SX its unit tangent bundle. For $v \in SX$ let $A_v(r)$ be the Jacobi tensor with initial conditions $A_v(0) = 0$ and $A_v'(0) = \text{id}$. Then X is said to be harmonic if and only if there exists a function $A : [0, \infty) \to [0, \infty)$ such that

$$A(r) = \det(A_v(r)) \quad \forall v \in SX.$$

Hence the volume growth of a geodesic ball centred at $\pi(v)$ only depends on its radius.

From (2) one easily concludes that the definition above is equivalent to the mean curvature of geodesic spheres only depending on the radius. More precisely, the mean curvature of a geodesic sphere $S(x, r)$ of radius r around a point $x \in X$ is given by $\frac{A'(r)}{A(r)}$.

Using A_v, one can construct the Jacobi tensors $S_{v,r}$ and $U_{v,r}$ along c_v with $S_{v,r}(0) = \text{id}$, $S_{v,r}(r) = 0$, and $U_{v,r} = S_{v,-r}$.

Then the stable, respectively, unstable Jacobi tensor is obtained via the limiting process:

$$S_v = \lim_{r\to\infty} S_{v,r}$$
$$U_v = \lim_{r\to\infty} U_{v,r}.$$

Note that these limits exist [31].

Let $v \in S_pX$ and c_v be the unit speed geodesic with initial direction v. Now define for $x \in X$ the Busemann function $b_v(x) = \lim_{t\to\infty} b_{v,t}(x)$, where $b_{t,v}(x) = d(c_v(t), x) - t$. This limit exists and is a $C^{1,1}$ function on X, see for instance [30]. The level sets of the Busemann functions, $H_v^s := b_v^{-1}(s)$, are called horospheres, and in the case that $b_v \in C^2(X)$, their second fundamental form in $\pi(v) = p$ is given by $U_v'(0) =: U(v)$. Hence their mean curvature is given by the trace of $U(v)$. In the case of a harmonic manifold, $v \to \operatorname{trace} U(v)$ is independent of $v \in SX$, hence the mean curvature of horospheres is constant. Using this notion of stable and unstable Jacobi tensors, Knieper in [32] generalised the well-known notion of rank for general spaces of non-positive curvature introduced by Ballmann, Brin and Eberlein [6] to manifolds without conjugated points.

Define for $v \in SX$, $S(v) := S_v'(0)$ and $D(v) = U(v) - S(v)$. Then

$$\begin{aligned}\mathcal{L}(v) &:= \ker(D(v))\\ \operatorname{rank}(v) &:= \dim \mathcal{L}(v) + 1\\ \operatorname{rank}(X) &:= \min\{\operatorname{rank}(v) \mid v \in SM\}.\end{aligned}$$

Furthermore, Knieper showed that, for a non-compact harmonic manifold, $\operatorname{rank}(X) = 1$ is equivalent to other important notions in geometry, which are stated in Sect. 6.

For $f \in C^2(X)$, the Laplace–Beltrami operator is defined by

$$\Delta f := \operatorname{div} \operatorname{grad} f$$

and in local coordinates $\{x_i\}$ is given by

$$\Delta f = \sum_{i,j} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\Big(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j} f\Big),$$

where $g = (g_{ij})$ is the matrix which defines the metric tensor $g : TX \times TX \to [0, \infty)$ and (g^{ij}) its inverse. Δ is by definition linear on $C_c^\infty(X)$ and we have

$$\int_X -\Delta f(x) \cdot f(x)\, dx = \int_X \|\nabla f(x)\|_g^2\, dx \quad \forall f \in C_c^\infty(X),$$

where $\|\cdot\|_g$ is the norm induced by g. Hence, $-\Delta$ is a non-negative symmetric operator. Furthermore, $-\Delta$ is formally self adjoint. Hence, by the density of $C_c^\infty(X)$ in $L^2(X)$, we can extend Δ to an unbounded self adjoint operator with dense domain in $L^2(X)$

which in abuse of notation we will again denote by Δ. The above also implies that the spectrum of Δ is contained in the negative half line. From now on, assume that (X, g) is a non-compact simply connected harmonic manifold with mean curvature of the horosphere $h = 2\rho$. In this case, the authors showed in [41] that $\Delta b_v = h$, and hence, the Busemann functions as well as all eigenfunctions of Δ are analytic by elliptic regularity since harmonic manifolds are Einstein, see for instance [50, Sec. 6.8], and therefore analytic by the Kazdan-De Turck theorem [20]. Furthermore, the authors in [40, Corollary 5.2] showed that the top of the spectrum of Δ is given by $-\rho^2$.

Lemma 2.2 ([10], Lemma 3.1) *Let f be a C^2 function on (X, g) and u a C^∞ function on $\mathbb{R}$. Then we have*

$$\Delta(u \circ f) = (u'' \circ f)\|\text{grad}\, f\|_g^2 + (u' \circ f)\Delta f.$$

where $\|\cdot\|_g^2 = g(\cdot, \cdot)$.

With Lemma 2.2, we can calculate the spherical and horospherical part of the Laplacian, by choosing $f = d_x$ for some $x \in X$, where d_x is the distance function to x. We obtain with $\Delta d_x(r) = \frac{A'(r)}{A(r)} \circ d_x(r)$ using spherical coordinates around x

$$\Delta(u \circ d_x) = u'' \circ d_x + u' \circ d_x \cdot \frac{A'}{A} \circ d_x. \tag{3}$$

For the Busemann function $f = b_v$ with $\Delta b_v = h = 2\rho$, we obtain using horospherical coordinates

$$\Delta(u \circ b_v) = u'' \circ b_v + h \cdot u' \circ b_v. \tag{4}$$

From this, we have that the radial part of the Laplacian only depends on the radius and not on specific points. Therefore, we obtain

Lemma 2.3 *Let $f : X \to \mathbb{C}$ be a $C^\infty(X)$ function and $x \in X$. Then for the mean value operators*

$$M_x f(r) := \frac{1}{\text{vol}(S(x,r))} \int_{S(x,r)} f(z)\, dz$$

and

$$R_x(f)(y) := M_x f(d(x, y))$$

we have

$$\Delta R_x f(y) = R_x(\Delta f)(y).$$

Especially we have for

$$L_A := \frac{d^2}{dr^2} + \frac{A'(r)}{A(r)} \frac{d}{dr}$$

that

$$L_A M_x(f)(r) = M_x(\Delta f)(r).$$

Proof We can decompose the Laplacian

$$\Delta f(y) = \Delta_{S(x,d(x,y))} f(y) + \Delta_{\text{radial}} f(y).$$

Where $\Delta_{S(x,d(x,y))}$ denotes the Laplacian of $S(x, d(x, y))$ and Δ_{radial} is defined by

$$(\Delta_{\text{radial}} f)(c_v(r)) = L_A(f \circ c_v)(r),$$

where for $v \in SX$, c_v is the geodesic corresponding to the initial conditions $c_v(0) = \pi(v)$ and $\dot{c}_v(0) = v$. Since $S(x, d(x, y))$ is closed Green's first identity implies

$$\int_{S(x,d(x,y))} \Delta_{S(x,d(x,y))} f(z)\, dz = 0.$$

Now the radial part of the Laplacian only depends on radial derivatives and the mean curvature of the geodesic sphere which since X is harmonic also only depends on the radius. Therefore,

$$\begin{aligned} R_x(\Delta f)(y) &= R_x(\Delta_{\text{radial}} f)(y) \\ &\overset{X \text{ is harmonic}}{=} \Delta_{\text{radial}} R_x(f)(y) \\ &= \Delta R_x(f)(y). \end{aligned}$$

The second part of the Lemma follows now from (3). □

Remark 2.4 Note that X is harmonic if and only if the Laplace operator commutes with the mean value operator. See for instance [45, Lemma 1.1].

Lemma 2.5 *Let $x_0 \in X$. Then $R_{x_0} : C_c^\infty(X) \to C_c^\infty(X, x_0)$ is self adjoint with respect to the L^2-product on X, i.e.*

$$\int_X (R_{x_0} f)(x) g(x)\, dx = \int_X f(x)(R_{x_0} g)(x)\, dx \quad \forall f, g \in C_c^\infty(X).$$

Proof Let $f, g \in C_c^\infty(X)$ and $x_0 \in X$. We integrate in geodesic polar coordinates using equation (1) and the fact that X is harmonic:

$$\begin{aligned}\int_X (R_{x_0} f)(x) g(x)\,dx &= \frac{1}{\omega_{n-1}} \int_0^\infty \Big(\int_{S_{x_0}X} f(\exp(rv))\,dv \\ &\quad \cdot \int_{S_{x_0}X} g(\exp(rv))\,dv \Big) A(r)\,dr \\ &= \int_X f(x)(R_{x_0} g)(x)\,dx\end{aligned}$$

where $\omega_{n-1} = \operatorname{vol} S^{n-1}$. □

3 The Abel Transform and Its Dual

Peyerimhoff and Samion discussed the Abel transform and its dual for radial functions as well as its connection to the radial Fourier transform in [40]. We will use these to construct a solution to the shifted wave equation. Therefore, we recall the definition and some imported facts that we will need in the proof of our main theorems. For this purpose, we need the following version of the Co-area formula.

Theorem 3.1 ([17, p.160]) *Let M be a connected Riemannian manifold. Given a C^1-function $f : M \to \mathbb{R}$ such the gradient* $\operatorname{grad} f$ *never vanishes on M, let S_t denote the hypersurface defined by $S_t = \{x \in M \mid f(x) = t\}$, $t \in \mathbb{R}$. Then, for any $g \in C_c^0(M)$,*

$$\int_M g(x)\,dx = \int_{\mathbb{R}} \int_{S_t} \frac{g(y)}{\|\operatorname{grad} f(y)\|_g}\,dy\,dt.$$

Let $x_0 \in X$ and $v \in S_{x_0}X$. Then $H_v^s = b_v^{-1}(s)$ denote the horospheres and $N(x) = -\operatorname{grad} b_v(x)$. Then the map

$$\begin{aligned}\Psi_{v,s} : H_v^0 &\to H_v^s \\ x &\mapsto \exp(-sN(x))\end{aligned}$$

is a diffeomorphism and

$$\begin{aligned}\Psi_v : \mathbb{R} \times H_v^0 &\to X \\ \Psi_v(s, x) &= \Psi_{v,s}(x)\end{aligned} \tag{5}$$

is an orientation preserving diffeomorphism. Furthermore, the Jacobian of $\Psi_{v,s}$ is given by e^{hs} (see [40, Proposition 3.1]). Hence, for a measurable function $f : X \to \mathbb{R}$, we get

$$\int_{H_v^s} f(z)\,dz = e^{sh} \int_{H_v^0} f(\Psi_s(z))\,dz. \tag{6}$$

Definition 3.2 For $v \in S_{x_0}X$ and define

$$\begin{aligned} j &: C^\infty_{\text{even}}(\mathbb{R}) \to C^\infty(X) \\ (jf)(x) &= e^{-\rho b_v(x)} f(b_v(x)) \end{aligned}$$

and

$$a : C^\infty_{\text{even}}(\mathbb{R}) \to C^\infty(X, x_0)$$

by

$$a(f)(y) = M_{x_0}(j(f)) \circ d(x_0, y).$$

The dual with respect to the L^2-inner product of $\mathbb{R}$ and X is called the Abel transform and is denoted by $\mathcal{A}$. This means that for every $g \in C^\infty(X, x_0)$ and $f \in C^\infty_{\text{even}}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \mathcal{A}(g)(s) f(s)\, ds = \int_X g(x) a(f)(x)\, dx.$$

Furthermore, the authors in [40] showed in Proposition 3.5 that

Lemma 3.3 *For $f \in C^\infty_c(X, x_0)$, we have*

$$\begin{aligned} \mathcal{A}(f)(s) &= e^{-\rho s} \int_{H_v^s} f(z)\, dz \\ &= e^{\rho s} \int_{H_v^0} f(\Psi_{v,s}(z))\, dz. \end{aligned}$$

Furthermore, $\mathcal{A}(f)$ is smooth, has compact support and is even.

Proof Let $f \in C^\infty_c(X, x_0)$ and define

$$g(s) := e^{-\rho s} \int_{H_v^s} f(z)\, dz.$$

The bottom equality in Lemma 3.3 follows immediately from (6). Therefore, we only need to show that

$$\int_{\mathbb{R}} g(s) h(s)\, ds = \int_X f(x) a(h)(x)\, dx \quad \forall h \in C^\infty_{\text{even}}(\mathbb{R}) \tag{7}$$

and that $g(s)$ is even since the smoothness follows after showing the equality from the smoothness of $\Psi_{s,v}$ in s. Now we prove (7)

$$\begin{aligned}
\int_{\mathbb{R}} g(s)h(s)\,ds &= \int_{\mathbb{R}} h(s)e^{-\rho s}\int_{H_v^s} f(z)\,dz\,ds \\
&= \int_{\mathbb{R}}\int_{H_v^s} h(b_v(z))e^{-\rho s} f(z)\,dz\,ds \\
&\overset{\text{Co-area formula}}{=} \int_X f(x)e^{-\rho b_v(x)}h(b_v(x))\,dx \\
&= \int_X f(x)j(h)(x)\,dx \\
&= \int_X R_{x_0}(f)(x)j(h)(x)\,dx \\
&\overset{\text{Lemma 2.5}}{=} \int_X f(x)R_{x_0}(j(h))(x)\,dx \\
&= \int_X f(x)a(h)(x)\,dx.
\end{aligned}$$

Let for $\lambda \in \mathbb{C}$, φ_{λ,x_0} be the eigenfunction of the Laplacian with eigenvalue $-(\lambda^2+\rho^2)$ radial around x_0 with $\varphi_{\lambda,x_0}(x_0) = 1$. Now evenness follows similar to (7) if we observe that since the Laplacian commutes with R_{x_0} and by (4) $e^{(i\lambda-\rho)b_v(x)}$ is for all $\lambda \in \mathbb{C}$ a eigenfunction of Δ with eigenvalue $-(\lambda^2+\rho^2)$,

$$R_{x_0}\left(e^{(i\lambda-\rho)b_v(\cdot)}\right)(x) = \varphi_{\lambda,x_0}(x). \tag{8}$$

Using this and integration in horospherical coordinates yields

$$\begin{aligned}
\int_{\mathbb{R}} g(s)e^{i\lambda s}\,ds &= \int_{\mathbb{R}} e^{i\lambda s}e^{-\rho s}\int_{H_v^s} f(z)\,dz\,ds \\
&= \int_{\mathbb{R}}\int_{H_v^s} e^{i\lambda b_v(z)}e^{-\rho s} f(z)\,dz\,ds \\
&\overset{\text{horospherical coordinates}}{=} \int_X f(x)e^{(i\lambda-\rho)b_v(x)}\,dx \\
&\overset{\text{f radial +Lemma 2.5}}{=} \int_X f(x)R_{x_0}(e^{(i\lambda-\rho)b_v(\cdot)})(x)\,dx \\
&\overset{(8)}{=} \int_X f(x)\varphi_{\lambda,x_0}(x)\,dx.
\end{aligned}$$

Now we have that $\varphi_{\lambda,x_0} = \varphi_{-\lambda,x_0}$, hence

$$\int_{\mathbb{R}} g(s)e^{i\lambda s}\,ds = \int_{\mathbb{R}} g(s)e^{-i\lambda s}\,ds.$$

This in turn implies that

$$\int_{\mathbb{R}} e^{i\lambda s}(g(s)-g(-s))\,ds = 0 \quad \forall \lambda \in \mathbb{C}.$$

By taking $\lambda \in \mathbb{R}$, this implies that g is even. □

Furthermore, the authors showed in [40, Proposition 3.10] that the Euclidean Fourier transform of the Abel transform is equal to the radial Fourier transform, given for a function radial around x_0 with compact support by

$$\hat{f}^{x_0}(\lambda) = \int_X f(x)\varphi_{\lambda,x_0}(x)\,dx,$$

where φ_{λ,x_0} is the radial eigenfunction of the Laplacian around x_0 with eigenvalue $-(\lambda^2+\rho^2)$ and $\varphi_{\lambda,x_0}(x_0)=1$. This means that

$$\hat{f}^{x_0}(\lambda) = \mathcal{F}(\mathcal{A}(f))(\lambda), \tag{9}$$

where $\mathcal{F}(u)(\lambda) = \int_{\mathbb{R}} e^{i\lambda s}u(s)\,ds$ for $u:\mathbb{R}\to\mathbb{R}$ sufficiently regular and decaying is the Euclidean Fourier transform.

Remark 3.4 Applying $\mathcal{F}^{-1}$ to both sides in equation (9) yields that the Abel transform and thereby its dual are independent of the choice of $v \in S_{x_0}X$. See also Lemma 8.5.

Theorem 3.5 ([40], Theorem 3.8) *The dual Abel transform is a topological isomorphism between the spaces of smooth even functions on $\mathbb{R}$ and smooth radial functions around x_0, where both function spaces are equipped with suitable topologies, see [26, Chap. II, Sect. 2] for instance.*

This fact is going to be exploited to characterise solutions of the shifted wave equation on X with smooth initial conditions with compact support.

4 Symmetry of the Mean Value Operator

From now onwards, we will consider complex-valued functions $u : X \to \mathbb{C}$, where the Laplacian of u is given via the decomposition of u in real and imaginary part $u = u_1 + iu_2$ by $\Delta u = \Delta u_1 + i\Delta u_2$. The proof of the following lemma follows the lines of the proof of Theorem 17 in [27] which in turn follows the proof in [2, p.334]. The lemma below is a generalisation of Ásgeirsson's mean value theorem to harmonic manifolds.

Lemma 4.1 *Let (X, g) be a non-compact simply connected harmonic manifold, and $u : X \times X \to \mathbb{C}$ a twice continuous differentiable function with*

$$\Delta_1 u(x,y) = \Delta_2 u(x,y) \quad \forall x, y \in X,$$

where Δ_i denotes to Laplacian with respect to the i-th variable. Then for each $(x_0, y_0) \in X \times X$, we have

$$\frac{1}{\mathrm{vol}(S(x_0,r))}\frac{1}{\mathrm{vol}(S(y_0),s)}\int_{S(x_0,r)}\int_{S(y_0,s)} u(z_1,z_2)\,dz_2\,dz_1$$
$$=\frac{1}{\mathrm{vol}(S(x_0,s))}\frac{1}{\mathrm{vol}(S(y_0,r))}\int_{S(x_0,s)}\int_{S(y_0,r)} u(z_1,z_2)\,dz_2\,dz_1$$

for all $r, s \geq 0$.

Proof Let $(x_0, y_0) \in X \times X$ be arbitrary points define

$$U(x,y) := \frac{1}{\mathrm{vol}(S(x_0,r))}\frac{1}{\mathrm{vol}(S(y_0,s))}\int_{S(x_0,r)}\int_{S(y_0,s)} u(z_1,z_2)\,dz_2\,dz_1$$

with $r = d(x_0, x)$ and $s = d(y_0, y)$. Then U can both be viewed as a function on $X \times X$ and $\mathbb{R}^+ \times \mathbb{R}^+$, since by definition $U(x, y)$ depends only on $r = d(x_0, x)$ and $s = d(y_0, y)$.

Since the Laplacian Δ commutes with the mean value operator (see Lemma 2.3) and u is twice continuous differentiable, and therefore, so is U (see [45, p.5]), we have

$$\begin{aligned}
\Delta_1 U(x,y) &= \Delta_1 R_{x_0}\big((z,y) \to R_{y_0}(u(z,\cdot))(y)\big)(x)\\
&= R_{x_0}\big((z,y) \to \Delta_1 R_{y_0}(u(z,\cdot))(y)\big)(x)\\
&= R_{x_0}\big((z,y) \to R_{y_0}(\Delta_1 u(z,\cdot))(y)\big)(x)\\
&= R_{x_0}\big((z,y) \to R_{y_0}(\Delta_2 u(z,\cdot))(y)\big)(x)\\
&= R_{x_0}\big((z,y) \to \Delta_2 R_{y_0}(u(z,\cdot))(y)\big)(x)\\
&= \Delta_2 R_{x_0}\big((z,y) \to R_{y_0}(u(z,\cdot))(y)\big)(x)\\
&= \Delta_2 U(x,y).
\end{aligned}$$

Then with the representation of the Laplacian in radial coordinates (see(3)), we have

$$\frac{\partial^2 U}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial s^2} + \frac{A'(s)}{A(s)}\frac{\partial U}{\partial s}.$$

If we set $F(r, s) = U(r, s) - U(s, r)$, we obtain

$$\frac{\partial^2 F}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial F}{\partial r} - \Big(\frac{\partial^2 F}{\partial s^2} + \frac{A'(s)}{A(s)}\frac{\partial F}{\partial s}\Big) = 0, \tag{10}$$

$$F(r,s) = -F(s,r). \tag{11}$$

Our goal is now to show that $F \equiv 0$. Since $F(r,r) = 0$ it is sufficient to show that all partial derivatives of F vanish. We have

$$\begin{aligned} A'(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s} =& \frac{\partial}{\partial r}\Big(A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\Big) - A(r)\frac{\partial^2 F}{\partial^2 r}\frac{\partial F}{\partial s} \\ & - A(r)\frac{\partial F}{\partial r}\frac{\partial^2 F}{\partial s \partial r}, \end{aligned}$$

and

$$\frac{\partial}{\partial s}\Big(\frac{\partial F}{\partial r}\Big)^2 = 2\frac{\partial F}{\partial r}\frac{\partial^2 F}{\partial s\partial r}, \quad \frac{\partial}{\partial s}\Big(\frac{\partial F}{\partial s}\Big)^2 = 2\frac{\partial F}{\partial s}\frac{\partial^2 F}{\partial s^2}.$$

Therefore, multiplying (10) by $2A(r)\frac{\partial F}{\partial s}$, we obtain

$$\begin{aligned} -A(r)\frac{\partial}{\partial s}\Big(\Big(\frac{\partial F}{\partial r}\Big)^2 + \Big(\frac{\partial F}{\partial s}\Big)^2\Big) + 2\frac{\partial}{\partial r}\Big(A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\Big) \\ -2\frac{A'(s)A(r)}{A(s)}\Big(\frac{\partial F}{\partial s}\Big)^2 = 0. \end{aligned} \tag{12}$$

Now set

$$L_1 := A(r)\Big(\Big(\frac{\partial F}{\partial r}\Big)^2 + \Big(\frac{\partial F}{\partial s}\Big)^2\Big)$$

and

$$L_2 := 2\Big(A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\Big).$$

Let $C > 0$ be arbitrary and consider the line $r + s = C$. We want to integrate the formula (12) over the triangle D with oriented boundary $\partial D = OMN$ (see Fig. 1), where $O = (0,0)$, $M = (\frac{C}{2}, \frac{C}{2})$ and $N = (0, C)$, using Stokes theorem. With this, we then show F vanishes on D. For this, we first need the check that the expressions in (12) have no singularities in D. The critical term is $2\frac{A'(s)A(r)}{A(s)}$. To rule out such a singularity, let $r \le s$. Since A is monotonously increasing, we have $\frac{A'(s)A(r)}{A(s)} \le A'(s)$ and $A'(0) = 1$ hence we have no singularity at O. Using Stokes theorem and equation (12), we get

$$\begin{aligned} \iint_D \frac{2A(r)A'(s)}{A(s)}\Big(\frac{\partial F}{\partial s}\Big)^2 dr\, ds &= \iint_D \frac{\partial L_2}{\partial r} - \frac{\partial L_1}{\partial s}\, dr \wedge ds \\ &= \int_D d(L_1 dr + L_2 ds) \\ &= \int_{\partial D} L_1 dr + L_2 ds. \end{aligned} \tag{13}$$

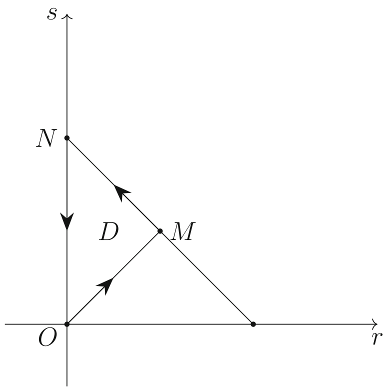

Fig. 1 The triangle D with oriented boundary $\partial D = OMN$

We have to break the path along the boundary into the three lines. First consider the line $r = s$ parameterised by the curve $\gamma_1(t) = (t, t)$ ending at M denoted by OM. Then we have $\dot{\gamma}_1 = (1, 1)$, and therefore,

$$\int_{OM} L_1 dr + L_2 ds = \int_0^{C/2} A(t)\Big(\big(\frac{\partial F}{\partial r}(t,t)\big)^2 + \big(\frac{\partial F}{\partial s}(t,t)\big)^2 + 2\big(\frac{\partial F}{\partial r}(t,t)\frac{\partial F}{\partial s}(t,t)\big)\Big)\, dt. \tag{14}$$

Since $F(\gamma_1(t)) = F(t, t) = 0$ for all $t \geq 0$, we have

$$0 = DF(\gamma_1(t)) \cdot \dot{\gamma}_1(t) = \frac{\partial F}{\partial r}(t,t) + \frac{\partial F}{\partial s}(t,t) \quad \forall t \geq 0, \tag{15}$$

hence

$$\big(\frac{\partial F}{\partial r}(t,t) + \frac{\partial F}{\partial s}(t,t)\big)^2 = 0.$$

From this, we conclude that the integral (14) vanishes.

Next, we consider the line ON. We have that $A(r) = 0$. Therefore, $L_1 = 0 = L_2$ on ON and

$$\int_{ON} L_1 dr + L_2 ds = 0.$$

Lastly we consider the curve jointing N and M given by $\gamma_2(t) = (t, C - t)$. Then we have $\dot{\gamma}_2(t) = (1, -1)$ and obtain

$$\begin{aligned}\int_{MN} L_1 dr + L_2 ds &= \int_{C/2}^{0} 2\Big(A(t)\frac{\partial F}{\partial r}(t, C-t)\frac{\partial F}{\partial s}(t, C-t)\Big) \\ &\quad - A(t)\Big(\big(\frac{\partial F}{\partial r}(t, C-t)\big)^2 + \big(\frac{\partial F}{\partial s}(t, C-t)\big)^2\Big)\, dt \\ &= \int_0^{C/2} A(t)\Big(\frac{\partial F}{\partial r}(t, C-t) - \frac{\partial F}{\partial s}(t, C-t)\Big)^2 dt.\end{aligned}$$

Now we have using (13)

$$\begin{aligned}&\int_0^{C/2} A(t)\Big(\frac{\partial F}{\partial r}(t, C-t) - \frac{\partial F}{\partial s}(t, C-t)\Big)^2 dt \\ &\quad + \iint_D \frac{2A(r)A'(s)}{A(s)}\big(\frac{\partial F}{\partial r}\big)^2 dr\, ds = 0.\end{aligned}$$

Since $A'(s) \geq 0$ both integrals are non-negative. This implies that

$$0 = \frac{\partial F}{\partial r}(t, C-t) - \frac{\partial F}{\partial s}(t, C-t) = DF(\gamma_2(t)) \cdot \dot{\gamma}_2(t) \quad \forall t \geq 0. \tag{16}$$

Now since $C > 0$ is arbitrary (15) together with (16) implies that all partial derivatives of F vanish and therefore that F is constant on the left side of the line (t, t). Since $F(r, r) = 0$, we conclude $F(s, r) = 0$ on the left side of the line (t, t). Since F is antisymmetric, see equation (11), the same holds true for the rest of $\mathbb{R}^2_+$ hence the claim follows. □

Corollary 4.2 *Under the conditions and with the notations of the proof of Lemma 4.1, we have that $U(r, 0) = U(0, r)$ for all $r \geq 0$. Hence, we obtain*

$$M_{y_0}(u(x_0, \cdot))(r) = M_{x_0}(u(\cdot, y_0))(r). \tag{17}$$

With a classical Lemma by Willmore [50, p. 249], one can deduce a near equivalence in Corollary 4.2.

Corollary 4.3 *Let $u : X \times X \to \mathbb{R}$ be a smooth function such that equation (17) holds for a small neighbourhood of $(x_0, y_0) \in X \times X$ and all small $r > 0$. Then*

$$\Delta_1 u(x_0, y_0) = \Delta_2 u(x_0, y_0).$$

Proof We have by [50, p. 249] for $f \in C^\infty(X)$, $x \in X$ and $r > 0$:

$$M_x(f)(r) = f(x) + \frac{1}{2n}\Delta f(x) r^2 + O(r^4) \quad \text{for } r \to 0,$$

where $n = \dim X$. Applying this to u yields

$$M_{x_0}(u(\cdot, y_0))(r) = u(x_0, y_0) + \frac{1}{2n}\Delta_1 u(x_0, y_0)r^2 + O(r^4) \quad \text{for } r \to 0,$$
$$M_{y_0}(u(x_0, \cdot))(r) = u(x_0, y_0) + \frac{1}{2n}\Delta_2 u(x_0, y_0)r^2 + O(r^4) \quad \text{for } r \to 0,$$

Since the terms on the left-hand side coincide, we obtain the claim. □

5 The Shifted Wave Equation

In this section, we solve the shifted wave equation:

$$\varphi : X \times \mathbb{R} \to \mathbb{C}$$
$$\frac{\partial^2}{\partial t^2}\varphi(x,t) = (\Delta_x + \rho^2)\varphi(x,t)$$

on X with initial conditions

$$\varphi(x, 0) = f(x) \in C_c^\infty(X)$$

and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(x,t) = g(x) \in C_c^\infty(X),$$

via the inverse Abel transform. This is analogous to Ásgeirsson characterisation of the solutions of the wave equation on $\mathbb{R}^n$ [2] and generalises work on non-compact symmetric spaces and Damek–Ricci spaces by [27, 38] and [1], respectively. The methods used are to a large part identical and rely heavily on [40, Theorem 3.8] and Corollary 4.2. Where our approach differs is in that we do not have an explicit formula for the inverse dual Abel transform and hence need to rely on the local injectivity of the dual Abel transform shown in [40, Theorem 3.8] to obtain the existence of solutions and that they possess finite speed of propagation.

Theorem 5.1 *$\varphi : X \times \mathbb{R} \to \mathbb{C}$ is a C^∞ solution of the shifted wave equation*

$$\frac{\partial^2}{\partial t^2}\varphi(x,t) = (\Delta_x + \rho^2)\varphi(x,t)$$

on X with initial conditions $\varphi(x, 0) = f(x) \in C_c^\infty(X)$ and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(x,t) = g(x) \in C_c^\infty(X)$$

if and only if

$$\varphi(x,t) = (a)^{-1}((M_x f)\circ d(x_0,\cdot))(|t|) + \int_0^t (a)^{-1}((M_x g)\circ d(x_0,\cdot))(s)\,ds,$$

where a is the dual Abel transform on X based at a point $x_0 \in X$.

Proof The proof will be conducted via Theorems 5.3 and 5.7. □

The first step in proving Theorem 5.1 is to show the if part. Hence, we have to show if a solution to the shifted wave equation with compactly supported initial conditions exists it is explicitly given by the inverse dual Abel transform of its initial conditions (Theorem 5.3). In the second step, we show in Theorem 5.7, the existence of solutions by proving that the function given by the expression in Theorem 5.1 is a solution to the shifted wave equation with the prescribed initial conditions.

Lemma 5.2 *Let $x_0 \in X$, $v \in S_{x_0}X$ and $u : X \times \mathbb{R} \to \mathbb{C}$ be a $C^2(X \times \mathbb{R})$ function. Then for the function $U : X \times X \to \mathbb{C}$ defined by $U(x,y) = e^{-\rho b_v(y)} u(x, b_v(y))$, the Laplacian Δ_2 of U with respect to the second variable is given by*

$$\Delta_2 U(x,y) = e^{-\rho b_v(y)}\left(\frac{\partial^2}{\partial t^2} - \rho^2\right)u(x,\cdot)) \circ b_v(y).$$

Proof Define $h : X \times \mathbb{R} \to \mathbb{C}$ by $h(x,t) = e^{-\rho t} u(x,t)$. Then by the representation of the Laplacian in horospherical coordinates (4), the Laplacian with respect to the second variable can be expressed by

$$\Delta_2 U(x,y) = \left(\frac{\partial^2}{\partial t^2} h(x,\cdot) + 2\rho \frac{\partial}{\partial t} h(x,\cdot)\right) \circ b_v(y). \tag{18}$$

With

$$\frac{\partial}{\partial t} h(x,t) = -\rho e^{-\rho t} u(x,t) + e^{-\rho t} \frac{\partial}{\partial t} u(x,t),$$
$$\frac{\partial^2}{\partial t^2} h(x,t) = \rho^2 e^{-\rho t} u(x,t) - 2\rho e^{-\rho t} \frac{\partial}{\partial t} u(x,t) + e^{-\rho t} \frac{\partial^2}{\partial t^2} u(x,t).$$

We get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}h(x,t)+2\rho\frac{\partial}{\partial t}h(x,t) &= \rho^2 e^{-\rho t}u(x,t)-2\rho e^{-\rho t}\frac{\partial}{\partial t}u(x,t)\\ &\quad + e^{-\rho t}\frac{\partial^2}{\partial t^2}u(x,t)-2\rho^2 e^{-\rho t}u(x,t)\\ &\quad + 2\rho e^{-\rho t}\frac{\partial}{\partial t}u(x,t)\\ &= e^{-\rho t}\big(\frac{\partial^2}{\partial t^2}u(x,t)-\rho^2 u(x,t)\big)\\ &= e^{-\rho t}(\frac{\partial^2}{\partial t^2}-\rho^2)u(x,t). \end{aligned} \tag{19}$$

Now plugging (19) into (18) yields the claim. □

Theorem 5.3 *Let $\varphi : X \times \mathbb{R} \to \mathbb{C}$ be a C^∞ solution of the shifted wave equation*

$$\frac{\partial^2}{\partial t^2}\varphi(x,t) = (\Delta_x + \rho^2)\varphi(x,t)$$

on X with initial conditions $\varphi(x,0) = f(x) \in C_c^\infty(X)$ and

$$\frac{\partial}{\partial t}\bigg|_{t=0}\varphi(x,t) = g(x) \in C_c^\infty(X).$$

Then

$$\varphi(x,t) = (a)^{-1}((M_x f)\circ d(x_0,\cdot))(|t|) + \int_0^t (a)^{-1}((M_x g)\circ d(x_0,\cdot))(s)\,ds,$$

where a is the dual Abel transform on X based at a point $x_0 \in X$.

Proof Let $x_0 \in X$ and $v \in S_{x_0}X$. First consider a solution to the shifted wave equation $\varphi_1(x,t)$ with initial conditions $\varphi_1(x,0) = f(x)$ and $\frac{\partial}{\partial t}\varphi_1(x,0) = 0$ for all $x \in X$. Because of the reversibility and uniqueness of solutions of the shifted wave equation, φ_1 is even in t. Define the function

$$\Phi_1 : X \times X \to \mathbb{C}$$

by

$$\Phi_1(x,y) := e^{-\rho b_v(y)}\varphi_1(x, b_v(y)).$$

Then since $\varphi_1(x,t)$ is a solution of the shifted wave equation, we have

$$\begin{aligned}\Delta_1\Phi_1(x,y) &= e^{-\rho b_v(y)}\Delta_1\varphi_1(x,b_v(y)) \\ &= e^{-\rho b_v(y)}\Big(\big((\frac{\partial^2}{\partial t^2}-\rho^2)\varphi_1(x,\cdot)\big)\circ b_v(y)\Big).\end{aligned}$$

Furthermore, by Lemma 5.2, we have that

$$\Delta_2\Phi_1(x,y) = e^{-\rho b_v(y)}\Big(\big((\frac{\partial^2}{\partial t^2}-\rho^2)\varphi_1(x,\cdot)\big)\circ b_v(y)\Big).$$

Therefore,

$$\Delta_1\Phi_1 = \Delta_2\Phi_1.$$

Now we can apply Corollary 4.2 above and obtain that for every pair $x, y \in X$

$$\begin{aligned}a(t\mapsto\varphi_1(x,t))(y) &= M_{x_0}(e^{-\rho b_v(\cdot)}\varphi_1(x,b_v(\cdot)))\circ d(x_0,y) \\ &= M_{x_0}(\Phi_1(x,\cdot))\circ d(x_0,y) \\ &= M_x(\Phi_1(\cdot,x_0))\circ d(x_0,y) \\ &= M_x(e^{-\rho b_v(x_0)}\varphi_1(\cdot,b_v(x_0))\circ d(x_0,y) \\ &= M_x(f)\circ d(x_0,y),\end{aligned}$$

where $a : C^\infty_{\text{even}}(\mathbb{R}) \to C^\infty(X, x_0)$ denotes the dual Abel transform with the choice of $v \in S_{x_0}X$ as above. Hence, by Theorem 3.8 in [40], we get for every $t \in \mathbb{R}$ and $x \in X$:

$$\varphi_1(x,t) = a^{-1}\big(M_x(f)\circ d(x_0,\cdot)\big)(|t|).$$

Now let φ_2 be a solution of the shifted wave equation on X with $\varphi_2(x,0) = 0$ and $\frac{\partial}{\partial t}\varphi_2(x,0) = g(x)$ for all $x \in X$. Then the initial conditions imply

$$\frac{\partial^2}{\partial t^2}\varphi_2(x,0) = (\Delta+\rho^2)\varphi_2(x,0) = 0,$$

hence by the same arguments as above $\frac{\partial}{\partial t}\varphi_2(x,t)$ is for all $x \in X$ a smooth and even function in t. Define

$$\Phi_2(x,y) := e^{-\rho b_v(y)}\frac{\partial}{\partial t}\varphi_2(x,b_v(y)).$$

Since φ_2 is a solution of the wave equation

$$\Delta_1\Phi_2(x,y) = e^{-\rho b_v(y)}\Big(\big((\frac{\partial^2}{\partial t^2}-\rho^2)\frac{\partial}{\partial t}\varphi_2(x,\cdot)\big)\circ b_v(y)\Big)$$

and by Lemma 5.2,

$$\Delta_2 \Phi_2(x, y) = e^{-\rho b_v(y)} \Big(\big(\frac{\partial^2}{\partial t^2} - \rho^2\big) \frac{\partial}{\partial t} \varphi_2(x, \cdot) \Big) \circ b_v(y) \Big).$$

Hence,

$$\Delta_1 \Phi_2 = \Delta_2 \Phi_2.$$

Now we can again apply Corollary 4.2 and obtain that for every pair $x, y \in X$

$$\begin{aligned} a(t \mapsto \frac{\partial}{\partial t} \varphi_2(x, t))(y) &= M_{x_0}(e^{-\rho b_v(\cdot)} \frac{\partial}{\partial t} \varphi_2(x, b_v(\cdot))) \circ d(x_0, y) \\ &= M_{x_0}(\Phi_2(x, \cdot)) \circ d(x_0, y) \\ &= M_x(\Phi_2(\cdot, x_0)) \circ d(x_0, y) \\ &= M_x(e^{-\rho b_v(x_0)} \frac{\partial}{\partial t} \varphi_2(\cdot, b_v(x_0)) \circ d(x_0, y) \\ &= M_x(g) \circ d(x_0, y). \end{aligned}$$

Now by Theorem 3.8 in [40] and integrating with respect to time, we have for $t \in \mathbb{R}$

$$\varphi_2(x, t) = \int_0^t a^{-1}(M_x(g) \circ d(x_0, \cdot))(s)\, ds.$$

Since the shifted wave equation is linear, we obtain a solution to the shifted wave equation with $\varphi(x, 0) = f(x)$ and $\frac{\partial}{\partial t}\varphi(x, t) = g(x)$ by $\varphi = \varphi_1 + \varphi_2$. This yields the claim. □

Corollary 5.4 *From the characterisation in Theorem 5.3, it follows now that φ is a unique solution to the initial data f, g as above.*

Next, we are going to show that a solution of the shifted wave equation has finite speed of propagation.

Corollary 5.5 *Under the assumption of Theorem 5.3, assume that f, g have support in a geodesic ball of radius R around $x_0 \in X$. Then*

$$\operatorname{supp} \varphi \subset \{(x, t) \in X \times \mathbb{R} \mid d(x_0, x) \le R + |t|\}.$$

Proof By Theorem 5.3, it is sufficient to prove that for $h \in C_c^\infty(X)$ with support $B(x_0, R)$ and $d(x_0, x) > R + |t|$

$$v_x(t) := a^{-1}\big(M_x(h) \circ d(x_0, \cdot)\big) = 0. \tag{20}$$

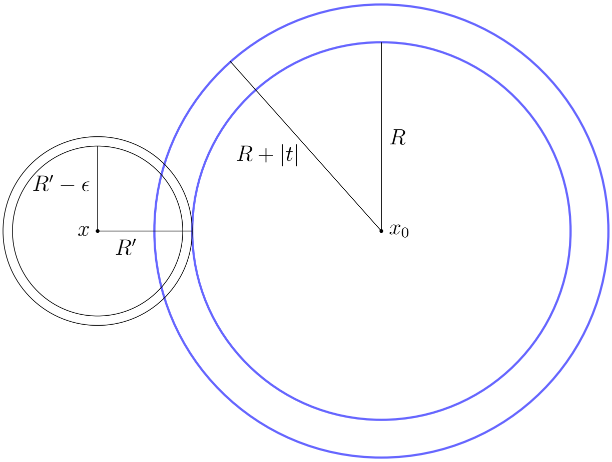

Fig. 2 A sketch for the proof of Corollary 5.5

By the local injectivity of the dual Abel transform [40, proof of Theorem 3.8], we have that for $u : \mathbb{R} \to \mathbb{R}$ smooth and even

$$a(u)|_{B(x_0,R)} = 0 \Rightarrow u|_{[-R,R]} = 0. \tag{21}$$

Now let $\epsilon > 0$ arbitrary, $d(x_0, x) > R + |t|$ and $R' = d(x_0, x) - R$ (Fig. 3). Then (see Figure 2 for a visualisation)

$$a(v_x)(y) \overset{(20)}{=} M_x(h) \circ d(x_0, y) = 0 \quad \forall y \in B(x_0, R' - \epsilon). \tag{22}$$

Furthermore, we have $R' = d(x_0, x) - R > |t|$ hence since $\epsilon > 0$ is arbitrary we obtain from (21) and (22)

$$v_x(t) = 0.$$

for all $(x, t) \in X \times \mathbb{R}$ with $d(x_0, x) > R + |t|$. □

Remark 5.6 The finite speed of propagation also follows from the general theory in [22, Chap. 5] or [46, Chap. 2, Proposition 8.1] by choosing the canonical space-time structure on $\mathbb{R} \times X$. See also [15, Lemma 1.1].

Next, we provide an intrinsic proof of the existence of a solution to the shifted wave equation without using general existence results mentioned in Remark 5.9 below.

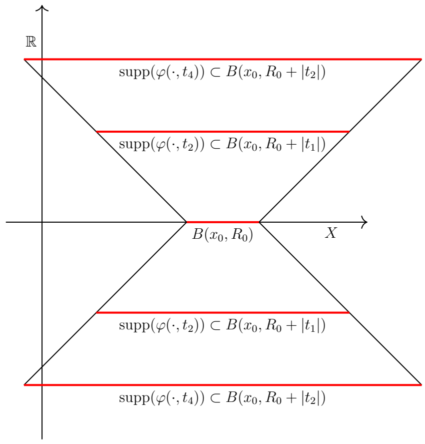

Fig. 3 Finite propagation speed of a solution of the shifted wave equation with initial conditions supported in $B(x_0, R_0)$

Theorem 5.7 *Let $f, g \in C_c^\infty(X)$. Then the functions*

$$\varphi_1(x,t) = a^{-1}\big(M_x(f) \circ d(x_0, \cdot)\big)(|t|)$$

and

$$\varphi_2(x,t) = \int_0^t a^{-1}(M_x(g) \circ d(x_0, \cdot))(s)\, ds$$

are solutions of the shifted wave equation with the initial condition

$$\varphi_1(x,0) = f(x)$$

$$\frac{\partial}{\partial t}\bigg|_{t=0} \varphi_1(x,t) = 0$$

and

$$\varphi_2(x,0) = 0$$

$$\frac{\partial}{\partial t}\bigg|_{t=0} \varphi_2(x,t) = g(x)$$

respectively. Consequently, $\varphi = \varphi_1 + \varphi_2$ is a solution of the shifted wave equation with initial conditions $\varphi(x,0) = f(x)$ and $\frac{\partial}{\partial t}\big|_{t=0} \varphi(x,t) = g(x)$.

Proof Let $T \in \mathbb{R}$. Because f and g have compact support there exists an $R_0 > 0$ such that the support of f and of g is contained in the closed ball $B(x_0, R_0)$. Now choose $R \geq |T| + R_0$. Then by Lemma 5.5, $\varphi_i(x,t)$ is supported in $B(x_0, R)$ for all $|t| \leq |T|$. We choose an orthonormal basis of eigenfunctions of the Dirichlet Laplacian on $B(x_0, 2R)$, with respect to the L^2 norm on $B(x_0, 2R)$, $\{\phi_k\}_{k\in\mathbb{N}}$ with $\Delta\phi_k = -\mu_k\phi_k$, $0 \leq \mu_1 \leq \mu_2 \leq \cdots < \infty$ and $\mu_k = (\lambda_k^2 + \rho^2)$ for some $\lambda_k \in \pm i[0,\rho] \cup \mathbb{R}$. First we observe that by Lemma 2.3 for $x \in B(x_0, R)$

$$M_x\phi_k(r) = \phi_k(x)\varphi_{\lambda_k}(r) \quad \forall r \leq R \tag{23}$$

where φ_{λ_k} is a eigenfunction of the operator L_A (see Lemma 2.3 for the definition) with $L_A\varphi_{\lambda_k} = -(\lambda_k^2 + \rho^2)\varphi_{\lambda_k}$, $\varphi_{\lambda_k}(0) = 1$ and $\lambda_k \in \pm i[0, \rho] \cup \mathbb{R}$. Now we can represent f and g by a series in ϕ_k:

$$f(y) = \sum_{k=0}^{\infty} a_k \phi_k(y) \text{ and } g(y) = \sum_{k=0}^{\infty} b_k \phi_k(y), \forall y \in B(x_0, 2R), \ a_k, b_k \in \mathbb{C}.$$

Using (23), we obtain for all $r \leq R$ and $x \in B(x_0, R)$

$$M_x f(r) = \sum_{k=0}^{\infty} a_k \phi_k(x) \varphi_{\lambda_k}(r) \text{ and } M_x g(r) = \sum_{k=0}^{\infty} b_k \phi_k(x) \varphi_{\lambda_k}(r).$$

Applying the inverse dual Abel transform a^{-1} yields, using that

$$\begin{aligned} a^{-1}(\varphi_{\lambda_k} \circ d(x_0, \cdot))(|t|) &= a^{-1}(\varphi_{\lambda_k, x_0})(|t|) \\ &= \cos(\lambda_k t) \end{aligned}$$

(see [40, Proposition 3.4]) and that a^{-1} is linear, that

$$a^{-1}\big(M_x(f) \circ d(x_0, \cdot)\big)(t) = \sum_{k=0}^{\infty} a_k \phi_k(x) \cos(\lambda_k t) \tag{24}$$

$$a^{-1}\big(M_x(g) \circ d(x_0, \cdot)\big)(s) = \sum_{k=0}^{\infty} b_k \phi_k(x) \cos(\lambda_k s). \tag{25}$$

Therefore, if we can show that (24) converges uniformly in x and t, we get

$$\begin{aligned} \Delta \sum_{k=0}^{\infty} a_k \phi_k(x) \cos(\lambda_k t) &= \sum_{k=0}^{\infty} a_k \Delta\phi_k(x) \cos(\lambda_k t) \\ &= -\sum_{k=0}^{\infty} (\lambda_k^2 + \rho^2) a_k \phi_k(x) \cos(\lambda_k t) \end{aligned}$$

and

$$\frac{\partial^2}{\partial t^2} \sum_{k=0}^{\infty} a_k \phi_k(x) \cos(\lambda_k t) = -\sum_{k=0}^{\infty} \lambda_k^2 a_k \phi_k(x) \cos(\lambda_k t).$$

Hence, φ_1 solves the shifted wave equation and satisfies the initial conditions $\varphi_1(x, 0) = f$ and $\frac{\partial}{\partial t}\big|_{t=0} \varphi_1(x, t) = 0$ as one sees by (24). Now suppose that (25) converges uniformly in x and s. Then by integration, we obtain

$$\varphi_2(x,t) = \sum_{k=0}^{\infty} b_k \phi_k(x) \sin(\lambda_k t) \cdot \frac{1}{\lambda_k}$$

where we interpret $\sin(\lambda_j t) \cdot \frac{1}{\lambda_j} = t$ if $\lambda_j = 0$. Now applying the Laplacian yields

$$\Delta\varphi_2(x,t) = -\sum_{k=0}^{\infty} (\lambda_k^2 + \rho^2) b_k \phi_k(x) \cdot \sin(\lambda_k t) \frac{1}{\lambda_k}$$

and we also get

$$\frac{\partial^2}{\partial t^2}\varphi_2(x,t) = -\sum_{k=0}^{\infty} \lambda_k^2 b_k \phi_k(x) \cdot \sin(\lambda_k t) \frac{1}{\lambda_k}.$$

Therefore, φ_2 satisfies the shifted wave equation, with the required initial conditions, as one can see by (25). Hence, the proof would be complete if we show that (24) and (25) converge uniformly in both variables. This will follow from Lemma 5.8 below. Under these assumptions, we have shown that φ_1 and φ_2 satisfy the theorem locally on the ball $B(x_0, R)$. If we now take $R' > R$ and repeat the construction above, we have by the local infectivity of the dual Abel transform [40, proof of Theorem 3.8] that the series above coincides on $B(x_0, R)$. Therefore, using the finite speed of propagation of the solution, we can repeat the argument for a series $R_n \to \infty$ (see Figur 3) to obtain the theorem. □

The lemma that finishes the proof of the theorem above is already contained in the proof of Theorem 3.8 in [40].

Lemma 5.8 *Let $x_0 \in X$, $R > 0$ and $f \in C_c^\infty(X)$ be such that the support of f is contained in the closed ball $B(x_0, R)$ and $\{\phi_k\}_{k\in\mathbb{N}}$ an orthonormal basis of eigenfunctions of the Dirichlet Laplacian on $B(x_0, R)$, with respect to the L^2 norm on $B(x_0, r)$ with $\Delta\phi_k = -\mu_k\phi_k$, $0 \le \mu_1 \le \mu_2 \le \cdots < \infty$ and $\mu_k = (\lambda_k^2 + \rho^2)$ for some $\lambda_k \in \pm i[0, \rho] \cup \mathbb{R}$. Furthermore, let for $a_k \in \mathbb{C}$ the Fourier decomposition of f be given by $f = \sum_{k=0}^{\infty} a_k \phi_k$. Then the series*

$$\sum_{k=0}^{\infty} a_k \phi_k(x) |\lambda_k|^m$$

converges uniformly in $x \in B(x_0, R)$. Consequently, all series in the proof of Theorem 5.7 converge uniformly.

Proof First, we observe that by the Sobolev embedding theorem (see for instance [25, Chap. 3]) there exists a constant $C_0 > 0$, such that for every function u in the Sobolev space $H^2_{2n}(B(x_0, R))$, we have

$$\|u\|_{\sup} \leq C_o\big(\|u\|_{L^2(B(x_0,R))} + \|\Delta^n u\|_{L^2(B(x_0,R))}\big), \tag{26}$$

where $\|\cdot\|_{\sup}$ is the sup norm on $C^0(B(x_0, R))$ and $n = \dim X$. Now since ϕ_k is an orthonormal basis with respect to the L^2 norm on $B(x_0, R)$, we have

$$|\phi_k(x)| \leq \|\phi_k\|_{\sup} \overset{(26)}{\leq} C_0(1 + \mu_k^n), \quad \forall x \in B(x_0, R).$$

By Weyl's law (see for instance [16, p.155]), we obtain that $k \sim \mu_k^{n/2}$, meaning that for $k > 0$ there is a constant $C \geq 1$ such that $\frac{1}{C} \leq \frac{\mu_k^{n/2}}{k} \leq C$. Therefore, there is a $k_0 \in \mathbb{N}$ such that for some $C_1 > 0$

$$C_1(1 + \mu_k^n) \leq C_1 k^2 \quad \forall k > k_0.$$

This yields

$$|\phi_k(x)| \leq \|\phi_k\|_{\sup} \leq C_1 k^2 \quad \forall k > k_0. \tag{27}$$

Now observe that $f \in C_c^\infty(X)$ with support contained in $B(x_0, R)$. Hence, $\Delta^j f \in C_c^\infty(X)$ for every $j \in \mathbb{N}$ and has support in $B(x_0, R)$. Therefore

$$\Delta^j f = \sum_{k=0}^{\infty} a_k \mu_k^j \phi_k$$

converges uniformly on $B(x_0, R)$ and $\Delta^j f \in L^2(B(x_0, R))$. This yields since $\{\phi_k\}_{k\in\mathbb{N}}$ is a orthonormal basis with respect to the L^2 norm

$$\infty > \|\Delta^j f\|_2^2 = \sum_{k=0}^{\infty} |a_k|^2 \mu_k^{2j}.$$

Now $\mu_k = \lambda_k^2 + \rho^2$, hence

$$\infty > \sum_{k=0}^{\infty} |a_k|^2 (\lambda_k^2 + \rho^2)^{2j} \geq \sum_{k=0}^{\infty} |a_k|^2 (\lambda_k)^{4j} \quad \forall j \in \mathbb{N}. \tag{28}$$

With this we obtain for $l \in \mathbb{N}$ arbitrarily and any $x \in B(x_0, R)$:

$$\begin{aligned}\sum_{k=0}^{\infty}|a_k||\phi_k(x)|\lambda_k|^m &\overset{(27)}{\leq} C_1 \sum_{k=0}^{\infty}|a_k|k^2|\lambda_k|^m \\ &= C_1 \sum_{k=0}^{\infty}|a_k|k^2|\lambda_k|^{m+l}|\lambda_k|^{-l} \\ &\overset{\text{Cauchy Schwarz}}{\leq} C_1\Big(\sum_{k=0}^{\infty}|a_k|^2k^2|\lambda_k|^{2m+2l}\Big)^{1/2} \\ &\qquad \cdot \Big(\sum_{k=0}^{\infty}|\lambda_k|^{-2l}\Big)^{1/2}.\end{aligned}$$

Now using Weyl's law and $\mu_k = \lambda_k^2 + \rho^2$, we conclude

$$\begin{aligned}&C_1\Big(\sum_{k=0}^{\infty}|a_k|^2k^2|\lambda|^{2m+2l}\Big)^{1/2} \cdot \Big(\sum_{k=0}^{\infty}|\lambda_k|^{-2l}\Big)^{1/2} \\ &\quad \leq C_1\Big(\sum_{k=0}^{\infty}|a_k|^2|\lambda_k|^{2(m+l+2n)}\Big)^{1/2} \cdot \Big(\sum_{k=0}^{\infty}|\lambda_k|^{-2l}\Big)^{1/2}.\end{aligned}$$

Now with $l = n$, we have

$$\sum_{k=0}^{\infty}|a_k|^2|\lambda_k|^{2(m+4n)} \overset{(28)}{<} \infty$$

and using Weyl's law there is a constant C_2 such that

$$\sum_{k=0}^{\infty}|\lambda_k|^{-2n} \leq C_2 \cdot \sum_{k=0}^{\infty}\frac{1}{k^2} < \infty.$$

This yields the claim. □

Remark 5.9 It also follows from the abstract theory of PDE,s that the solution of the shifted wave equation exists. See for instance [46, Chap. 2+6], [22, Chap. 5+6], [7, Chap. 3] and [23]. In their context, one would consider the product manifold $\mathbb{R} \times X$ with the canonical space-time structure where the shifted wave equation corresponds to a lower-order perturbation of the ordinary wave equation.

This closes our investigation of the shifted wave equation on general simply connected, non-compact and non-flat harmonic manifolds. To obtain further results, we need tools that require us to assume that (X, g) is of rank one or equivalently has purely exponential volume growth. It should be noted by the reader that all results past this point require this assumption on (X, g) unless stated otherwise.

6 The Rank One Case

A non-compact simply connected harmonic manifold X is said to be of purely exponential volume growth if there exists some constants $C \geq 1$ and $\rho > 0$ such that

$$\frac{1}{C} \leq \frac{A(r)}{e^{2\rho r}} \leq C.$$

This property is by [32] equivalent to

- The Geodesic Flow in SX is Anosov with respect to the Sasaki metric
- Gromov Hyperbolicity
- Rank one.

Note that the author in [32] showed that non-positive curvature implies purely exponential volume growth.

From now on, let (X, g) be a non-compact simply connected harmonic manifold of rank one. The geometric boundary ∂X is defined by equivalence classes of geodesic rays. Where two rays are equivalent if their distance is bounded. The topology on ∂X is the cone topology with the property that for $\overline{X} = X \cup \partial X$ and $B_1(x) = \{v \in T_x X | \ \|v\| \leq 1\}$ the map, $pr_x : B_1(x) \to \overline{X}$

$$pr_x(v) = \begin{cases} \gamma_v(\infty) & \text{if } \|v\| = 1 \\ \exp(\frac{1}{1-\|v\|} v) & \text{if } \|v\| < 1 \end{cases}$$

is a homeomorphism. It turns out that since the geodesic flow is Anosov the Busemann function only depends on the direction of the ray up to a constant (See [10, Lemma 2.1]). Hence, for $x \in X$ and $\xi \in \partial X$ being the point at infinity of the geodesic γ, we can alternatively define the Busemann function $B_{\xi,x} : X \to \mathbb{R}$ by

$$B_{\xi,x}(y) = \lim_{t\to\infty} (d(y, \gamma(t)) - d(x, \gamma(t))).$$

Furthermore, we obtain a cocycle property for $x, \sigma \in X$ and $\xi \in \partial X$:

$$B_{\xi,x} = B_{\xi,\sigma} - B_{\xi,\sigma}(x). \tag{29}$$

If $v \in S_\sigma X$ defines the unique geodesic ray such that $c_v(\infty) = \xi$. Then by the considerations above, the two representations of the Busemann function are related as follows:

$$b_v(x) = B_{\xi,\sigma}(x) \quad \forall x \in X.$$

For a proof of this, see [10, Lemma 2.2]. We conclude that $\Delta B_{\xi,\sigma} = 2\rho$, where 2ρ is the mean curvature of the horospheres. Hence, $g(y) = e^{(i\lambda-\rho)B_{\xi,x}(y)}$ is a eigenfunction of the Laplacian with $g(x) = 1$ and $\Delta g = -(\lambda^2 + \rho^2)g$ for $\lambda \in \mathbb{C}$. Furthermore, by pushing forward the canonical probability measure θ_x induced by the metric of the Euclidean unit sphere $S_x X$ under pr_x, we obtain a probability measure μ_x on ∂X.

Hence, we have a family of probability measures $\{\mu_x\}_{x\in X}$, that are pairwise absolutely continuous with Radon–Nikodym derivative

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-2\rho B_{\xi,x}(y)}. \tag{30}$$

For a detailed proof, see [34, Theorem 1.4].

6.1 Fourier Transform and Plancherel Theorem on Rank One Harmonic Manifolds

The main tool in defining the Fourier transform on rank one harmonic manifolds is the theory of hypergroups. This was first presented for harmonic manifolds with pinched negative curvature in [9] and subsequently extended in [10] to rank one harmonic manifold. Since we refrain from details, we refer the reader to [12] for a thorough discussion of the topic of hypergroups and their definition. In [10, Sect. 4.2], the authors showed that the density function $A(r)$ of a harmonic manifold of rank one satisfies the following conditions

(C1) A is increasing and $A(r) \to \infty$ for $r \to \infty$.
(C2) $\frac{A'}{A}$ is decreasing and $\rho = \frac{1}{2}\lim\limits_{r\to\infty}\frac{A'(r)}{A(r)} > 0$.
(C3) For $r > 0$, $A(r) = r^{2\alpha+1}B(r)$ for some $\alpha > -\frac{1}{2}$ and some even C^∞ function $B(r)$ on $\mathbb{R}$ with $B(0) = 1$.
(C4)

$$G(r) = \frac{1}{4}\left(\frac{A'}{A}(r)\right)^2 + \frac{1}{2}\left(\frac{A'}{A}(r)\right)' - \rho^2$$

is bounded on $[r_0, \infty)$ for all $r_0 > 0$ and

$$\int_{r_1}^{\infty} r|G(r)|\,dr < \infty \quad \text{for some } r_1 > 0.$$

Therefore, $A(r)$ defines a Chébli-Triméche hypergroup. The structure of the so-defined hypergroup is related to the second-order differential operator given by the radial part of the Laplacian:

$$L_A = \frac{d^2}{dr^2} + \frac{A'(r)}{A(r)}\frac{d}{dr}. \tag{31}$$

Let

$$\varphi_\lambda : \mathbb{R}^+ \to \mathbb{R}, \quad \lambda \in [0,\infty) \cup [0, i\rho] \tag{32}$$

be the eigenfunction of L_A with

$$L_A\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda \tag{33}$$

and which admits a smooth extension to zero with $\varphi_\lambda(0) = 1$. Under conditions (C1)-(C4), it was shown in [11] that there is a complex function $\mathbf{c}$ on $\mathbb{C} \setminus \{0\}$ such that for the two linear independent solutions of

$$L_A u = -(\lambda^2 + \rho^2)u$$

Φ_λ and $\Phi_{-\lambda}$ which are asymptotic to exponential functions, i.e.

$$\Phi_{\pm\lambda}(r) = e^{(\pm i\lambda - \rho)r}(1 + o(1)) \text{ as } r \to \infty \tag{34}$$

we have

$$\varphi_\lambda = \mathbf{c}(\lambda)\Phi_\lambda + \mathbf{c}(-\lambda)\Phi_{-\lambda} \quad \forall \lambda \in \mathbb{C} \setminus \{0\}. \tag{35}$$

Imposing the additional condition that $|\alpha| > \frac{1}{2}$, the authors in [11] showed that the $\mathbf{c}$-function does not have zeros on the closed lower half plane. This at first excludes the case that $\dim X = 3$. This is due to the following consideration used by the authors in [10] to obtain (C3): Let $p \in X$, $n = \dim X$ and $v \in S_p X$. The Jacobi tensor $A_v(r)$ along the geodesic $c_v : \mathbb{R} \to X$ with initial conditions $A_v(0) = 0$ and $A_v'(0) = \mathrm{id}$ is given by

$$D \exp_p(rv)(tw) = A_v(r)w(r),$$

where $w \in T_p X$ and $w(r) \in T_{c_v(r)} X$ is the parallel transport of w along c_v. Then $A(r) = \det A_v(r)$ is the Jacobian of the map $v \to \exp(rv) = \exp \circ (v \to rv)$. Hence,

$$A(r) = r^{n-1} \det(D \exp_p)_{rv} \quad \forall v \in S_p X. \tag{36}$$

Observe that since X is harmonic, $B(rv) := \det(D \exp_p)_{rv}$ is independent of the choice of v, and therefore, $B(v) = B(-v)$. Hence, B can be seen as the restriction of an even function on $\mathbb{R}$. Therefore, $\alpha = (n-2)/2$. To satisfy the condition $|\alpha| > \frac{1}{2}$, we therefore need to assume that $\dim X \neq 3$. But by [8, 36, 49] and [37], every non-compact simply connected harmonic manifold with $\dim X < 6$ is a symmetric space of rank one, hence one can apply the theory by Helgason [29], using Harish–Chandra's $\mathbf{c}$-function. We therefore do not need to exclude the case $\dim X = 3$ since we can default to the theory of Helgason. We can define the radial Fourier transform by

Definition 6.1 Let $f : X \to \mathbb{C}$ be radial, i.e. $f = u \circ d_\sigma$ for some $\sigma \in X$. The radial Fourier transform of f is given by

$$\widehat{f}(\lambda) := \widehat{u}(\lambda) = \int_0^\infty u(r)\varphi_\lambda(r)A(r)\, dr.$$

Note that in the following we will omit to mention the base point σ unless there is the possibility of confusion. For f radial around $\sigma \in X$, we will use σ as a base point for the radial Fourier transform unless stated otherwise. Now observe that we obtain the radial eigenfunctions of the Laplace operator with eigenvalue $-(\lambda^2+\rho^2)$ by

$$\varphi_{\lambda,\sigma}(y) = \varphi_\lambda \circ d(\sigma, y) \quad \forall x, y \in X. \tag{37}$$

Using the results from [11], the authors in [10] showed that there is a constant C_0 such that for $f \in L^1(X)$ radial, i.e. $f = u \circ d_\sigma$ for some $\sigma \in X$ and $u : [0,\infty) \to \mathbb{R}$ such that $\widehat{u} \in L^1((0,\infty), C_0|\mathbf{c}(\lambda)|^{-2}\, d\lambda)$, we have

$$f(y) = C_0 \int_0^\infty \widehat{f}(\lambda)\varphi_{\lambda,\sigma}(y)|\mathbf{c}(\lambda)|^{-2}\, d\lambda. \tag{38}$$

Moreover, the radial Fourier transform extends to an isometry between the space $L^2(X,\sigma)$ of L^2-radial functions around σ and

$$L^2((0,\infty), C_0|\mathbf{c}(\lambda)|^{-2}\, d\lambda).$$

For more details, see [10, Theorem 4.7]. In the same fashion as in the case of the Helgason Fourier transform on symmetric spaces, we can extend the Fourier transform to non-radial functions, by using the radial symmetry if the Poisson kernel. Again the main reference for this is [10].

Definition 6.2 Let $\sigma \in X$. For $f : X \to \mathbb{C}$ measurable, the Fourier transform of f based at σ is given by

$$\tilde{f}^\sigma(\lambda,\xi) = \int_X f(y)e^{(-i\lambda-\rho)B_{\xi,\sigma}(y)}\, dy$$

for $\lambda \in \mathbb{C}$, $\xi \in \partial X$, provided the integral above converges.

We note that the cocycle property (29) of the Busemann function implies the following:

Lemma 6.3 *Let $f \in C_c^\infty(X)$ and $x, \sigma \in X$. Then, we have*

$$\tilde{f}^x(\lambda,\xi) = e^{(i\lambda+\rho)B_{\xi,\sigma}(x)}\tilde{f}^\sigma(\lambda,\xi). \tag{39}$$

Proof Let $x, \sigma \in X$ and $f \in C_c^\infty(X)$. Then, we have for $\lambda \in \mathbb{C}$ and $\xi \in \partial X$ that

$$\begin{aligned}
\tilde{f}^x(\lambda,\xi) &= \int_X f(y)e^{(-i\lambda-\rho)B_{\xi,x}(y)}\, dy \\
&\overset{(29)}{=} \int_X f(y)e^{(-i\lambda-\rho)B_{\xi,\sigma}(y)} \cdot e^{(i\lambda+\rho)B_{\xi,\sigma}(x)}\, dy \\
&= e^{(i\lambda+\rho)B_{\xi,\sigma}(x)} \int_X f(y)e^{(-i\lambda-\rho)B_{\xi,\sigma}(y)}\, dy \\
&= e^{(i\lambda+\rho)B_{\xi,\sigma}(x)}\tilde{f}^\sigma(\lambda,\xi).
\end{aligned}$$

□

Furthermore, the Fourier transform coincides with the radial Fourier transform on radial functions. For details, see [10, Lemma 5.2]. The inversion formula for the non-radial Fourier transform follows now from the representation of the radial eigenfunctions via a convex combination of non-radial eigenfunctions, [10, Theorem 5.6],:

$$\varphi_{\lambda,\sigma}(y) = \int_{\partial X} e^{(i\lambda-\rho)B_{\xi,\sigma}(y)}\, d\mu_\sigma(\xi) \quad \forall\sigma \in X. \tag{40}$$

This is analogous to the well-known formula on a rank one symmetric space G/K and harmonic NA groups. See for the symmetric case [29, Chap. III, Sect. 11] and for the harmonic NA group [19] and [42]. Using equation (40), the authors obtain

$$f(x) = C_0 \int_0^\infty \int_{\partial X} \tilde{f}^\sigma(\lambda,\xi) e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\, d\mu_\sigma(\xi) |\mathbf{c}(\lambda)|^{-2}\, d\lambda, \tag{41}$$

where C_0 is the same constant given in (38). Additionally, the authors obtain a Plancherel theorem:

Theorem 6.4 ([10]) *Let $\sigma \in X$ and $f, g \in C_c^\infty(X)$. Then, we have*

$$\int_X f(x)\overline{g(x)}\, dx = C_0 \int_0^\infty \int_{\partial X} \tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma(\lambda,\xi)} |\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi) d\lambda$$

and the Fourier transform extends to an isometry between

$$L^2(X)$$

and

$$L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda).$$

6.2 Wave Equation Under Fourier Transform and Conservation of Energy

Using the Fourier transform, we can obtain the conservation of energy for solutions of the wave equation similar to the result in [5] for Damek–Ricci spaces. For this, we first need to study the action of the Laplacian under the Fourier transform.

Lemma 6.5 *Let $f \in L^2(X)$ such that $\Delta f \in L^2(X)$, where Δf is meant in the sense of distributions i.e. Δf is defined by*

$$\int_X \Delta f(x) g(x)\, dx := \int_X f(x)\Delta g(x)\, dx \quad \forall g \in C_c^\infty(x),$$

and $\sigma \in X$. Then

$$\widetilde{\Delta f}^\sigma(\lambda,\xi) = -(\lambda^2+\rho^2)\widetilde{f}^\sigma(\lambda,\xi)$$

for almost every $(\lambda,\xi) \in (0,\infty)\times\partial X$.

Proof Let $\sigma \in X$. Since $C_c^\infty(X)$ is dense in $L^2(X)$ and by using the Plancherel theorem, it is sufficient to prove the assertion for $f \in C_c^\infty(X)$. To be more precise, If $f, \Delta f \in L^2(X)$ then there is a sequence $f_n \in C_c^\infty(X)$ such that $f_n \to f$ and $\Delta f_n \to \Delta f$ in $L^2(X)$. For this, see [44, Corollary 2.5]. Let $\sigma \in X$. Then the above implies by the Plancherel theorem that $\tilde{f_n}^\sigma \to \tilde{f}^\sigma$ and $\widetilde{\Delta f_n}^\sigma \to \widetilde{\Delta f}^\sigma$ in $L^2((0,\infty) \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)$. Therefore, we find subsequences such that both converge point-wise almost everywhere.

Then since the Laplacian is essentially self adjoint and

$$\Delta e^{(-i\lambda-\rho)B_{\xi,\sigma}(y)} = -(\lambda^2+\rho^2)e^{(-i\lambda-\rho)B_{\xi,\sigma}(y)} \quad \forall y \in X$$

we have almost everywhere

$$\begin{aligned} \widetilde{\Delta f_n}^\sigma(\lambda,\xi) &= \int_X \Delta f_n(x) e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx \\ &= \int_X f_n(x) \Delta e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx \\ &= -(\lambda^2+\rho^2)\int_X f_n(x) e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx \\ &= -(\lambda^2+\rho^2)\tilde{f_n}^\sigma(\lambda,\xi). \end{aligned}$$

Therefore, we have after passing to a subsequence, if necessary, that

$$\begin{aligned} -(\lambda^2+\rho^2)\tilde{f}^\sigma(\lambda,\xi) &= \lim_{n\to\infty} -(\lambda^2+\rho^2)\tilde{f_n}^\sigma(\lambda,\xi) \\ &= \lim_{n\to\infty} \widetilde{\Delta f_n}^\sigma(\lambda,\xi) \\ &= \widetilde{\Delta f}^\sigma(\lambda,\xi) \end{aligned}$$

almost everywhere. □

Theorem 6.6 *Suppose (X, g) is a harmonic manifold of rank one. Let $\sigma \in X$. Then the Fourier transform of a C^∞ solution to the shifted wave equation $\varphi : X \times \mathbb{R} \to \mathbb{C}$ with initial conditions*

$$\begin{aligned} \varphi(x,0) &= f(x) \in C_c^\infty(X), \\ \frac{\partial}{\partial t}\bigg|_{t=0} \varphi(x,t) &= g(x) \in C_c^\infty(X) \end{aligned}$$

is given by

$$\tilde{\varphi}^\sigma((\lambda,\xi);t) = \tilde{f}^\sigma(\lambda,\xi)\cos(\lambda t) + \tilde{g}^\sigma(\lambda,\xi)\frac{\sin(\lambda t)}{\lambda}.$$

Consequently,

$$\varphi(x,t) = C_0 \int_0^\infty \int_{\partial X} \left(\tilde{f}^\sigma(\lambda,\xi)\cos(\lambda t) + \tilde{g}^\sigma(\lambda,\xi)\frac{\sin(\lambda t)}{\lambda}\right) \cdot e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\, d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\, d\lambda.$$

Proof Since by Remark 5.6 $\varphi(\cdot,t)$ and all its derivatives in t have compact support for every $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\widetilde{\varphi}^\sigma((\lambda,\xi);t) &= \frac{\partial^2}{\partial t^2}\int_X \varphi(x)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx \\
&= \int_X \frac{\partial^2}{\partial t^2}\varphi(x)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx \\
&= \widetilde{\frac{\partial^2}{\partial t^2}\varphi}^\sigma((\lambda,\xi);t) \\
&= \widetilde{\Delta\varphi}^\sigma((\lambda,\xi);t) + \rho^2\widetilde{\varphi}^\sigma((\lambda,\xi);t) \\
&\overset{\text{Lemma 6.5}}{=} -(\lambda^2-\rho^2)\widetilde{\varphi}^\sigma((\lambda,\xi);t) + \rho^2\widetilde{\varphi}^\sigma((\lambda,\xi);t) \\
&= -\lambda^2\widetilde{\varphi}^\sigma((\lambda,\xi);t).
\end{aligned}$$

Now the shifted wave equation becomes

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\widetilde{\varphi}^\sigma((\lambda,\xi);t) &= -\lambda^2\widetilde{\varphi}^\sigma((\lambda,\xi);t) \\
\widetilde{\varphi}^\sigma((\lambda,\xi);0) &= \tilde{f}^\sigma(\lambda,\xi) \\
\frac{\partial}{\partial t}\widetilde{\varphi}^\sigma((\lambda,\xi);0) &= \tilde{g}^\sigma(\lambda,\xi)
\end{aligned}$$

Hence analogous to solving the equation given by the Fourier transform of the wave equation on $\mathbb{R}$, see [46, Chap. 3, Sect. 5], we obtain

$$\widetilde{\varphi}^\sigma((\lambda,\xi);t) = \tilde{f}^\sigma(\lambda,\xi)\cos(\lambda t) + \tilde{g}^\sigma(\lambda,\xi)\frac{\sin(\lambda t)}{\lambda},$$

therefore applying the inverse Fourier transform yields the claim. □

Remark 6.7 While the representation of the solutions of the shifted wave equation from Theorem 5.3 corresponds to the classical representation of the solutions of the wave equation on $\mathbb{R}^n$ by Ásgeirsson [2], the representation obtained in Theorem 6.6 corresponds to the operator expression for the operator $\Delta_\rho := \Delta + \rho^2$:

$$\varphi(x,t) = \cos(\sqrt{-\Delta_\rho}t) f(x) + \frac{\sin(\sqrt{-\Delta_\rho}t)}{\sqrt{-\Delta_\rho}} g(x).$$

In turn, this again corresponds to the expression of the solution as a power series in the proof Theorem 5.7.

Definition 6.8 Let $\varphi : X \times \mathbb{R} \to \mathbb{C}$ be a solution of the shifted wave equation. We define its kinetic energy $\mathcal{K}(\varphi)$ by

$$\mathcal{K}(\varphi)(t) := \frac{1}{2}\int_X \left|\frac{\partial}{\partial t}\varphi(x,t)\right|^2 dx$$

and its potential energy $\mathcal{P}(\varphi)(t)$ by

$$\mathcal{P}(\varphi)(t) := \frac{1}{2}\int_X \varphi(x,t)(-\Delta - \rho^2)\overline{\varphi}(x,t)\, dx.$$

The total energy is defined by

$$\mathcal{E}(\varphi)(t) := \mathcal{K}(\varphi)(t) + \mathcal{P}(\varphi)(t).$$

Lemma 6.9 *Suppose (X, g) is a harmonic manifold of rank one. Let $\sigma \in X$ and $\varphi : X \times \mathbb{R} \to \mathbb{C}$ be a solution to the shifted wave equation with initial conditions*

$$\varphi(x,0) = f(x) \in C_c^\infty(X)$$
$$\left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(x,t) = g(x) \in C_c^\infty(X).$$

Then, we have

$$\begin{aligned} 2\mathcal{K}(\varphi)(t) =& C_0 \int_0^\infty \int_{\partial X} |-\lambda \widetilde{f}^\sigma(\lambda,\xi)\sin(\lambda t) \\ &+ \widetilde{g}^\sigma(\lambda,\xi)\cos(\lambda t)|^2\, d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\, d\lambda \end{aligned} \tag{42}$$

and

$$\begin{aligned} 2\mathcal{P}(\varphi)(t) =& C_0 \int_0^\infty \int_{\partial X} |\lambda \widetilde{f}^\sigma(\lambda,\xi)\cos(\lambda t) \\ &+ \widetilde{g}^\sigma(\lambda,\xi)\sin(\lambda t)|^2\, d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\, d\lambda. \end{aligned} \tag{43}$$

Proof Using the Plancherel theorem for the Fourier transform and Theorem 6.6, we obtain for the kinetic energy

$$
\begin{aligned}
2\mathcal{K}(\varphi)(t) &= \int_X \left|\frac{\partial}{\partial t}\varphi(x,t)\right|^2 dx \\
&\overset{\text{Plancherel theorem}}{=} C_0 \int_0^\infty \int_{\partial X} \left|\frac{\partial}{\partial t}\widetilde{\varphi}^\sigma(\lambda,\xi;t)\right|^2 d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\
&\overset{\text{Theorem 6.6}}{=} C_0 \int_0^\infty \int_{\partial X} |-\lambda \widetilde{f}^\sigma(\lambda,\xi)\sin(\lambda t) \\
&\qquad + \tilde{g}^\sigma(\lambda,\xi)\cos(\lambda t)|^2\, d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\,d\lambda.
\end{aligned}
$$

For the potential energy, we use the Plancherel theorem for the Fourier transform, Theorem 6.6 and Lemma 6.5:

$$
\begin{aligned}
2\mathcal{P}(\varphi)(t) &= \int_X \varphi(x,t)(-\Delta-\rho^2)\overline{\varphi}(x,t)\,dx \\
&\overset{\text{Plancherel theorem}}{=} C_0 \int_0^\infty \int_{\partial X} \widetilde{\varphi}^\sigma(\lambda,\xi;t) \\
&\qquad \cdot \big(-\overline{\widetilde{\Delta\varphi}^\sigma}(\lambda,\xi;t) - \overline{\widetilde{\rho^2\varphi}^\sigma}(\lambda,\xi;t)\big)\,d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\
&\overset{\text{Lemma 6.5}}{=} C_0 \int_0^\infty \int_{\partial X} \widetilde{\varphi}^\sigma(\lambda,\xi;t) \\
&\qquad \cdot \big((\lambda^2+\rho^2)\overline{\widetilde{\varphi}^\sigma}(\lambda,\xi;t) - \overline{\widetilde{\rho^2\varphi}^\sigma}(\lambda,\xi;t)\big)\,d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\,d\lambda \\
&\overset{\text{Theorem 6.6}}{=} C_0 \int_0^\infty \int_{\partial X} |\lambda \widetilde{f}^\sigma(\lambda,\xi)\cos(\lambda t) \\
&\qquad + \widetilde{g}^\sigma(\lambda,\xi)\sin(\lambda t)|^2\, d\mu_\sigma(\xi)|\mathbf{c}(\lambda)|^{-2}\,d\lambda.
\end{aligned}
$$

□

Theorem 6.10 *Suppose (X,g) is a harmonic manifold of rank one. Let $\sigma \in X$ and $\varphi : X \times \mathbb{R} \to \mathbb{C}$ a solution to the shifted wave equation with initial conditions $f, g \in C_c^\infty(X)$. Then, the total energy $\mathcal{E}(\varphi)(t)$ is independent of t. In particular,*

$$
\begin{aligned}
2\mathcal{E}(\varphi)(t) =& \|\lambda \tilde{f}^\sigma\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)} \\
&+ \|\tilde{g}^\sigma\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)}.
\end{aligned}
$$

Proof If we look at the terms under the integrals in Lemma 6.9 separately. For the integrand in (42), we obtain

$$\begin{aligned}&|-\lambda \widetilde{f}^{\sigma}(\lambda,\xi)\sin(\lambda t)+\widetilde{g}^{\sigma}(\lambda,\xi)\cos(\lambda t)|^2\\&\quad=\lambda^2|\widetilde{f}^{\sigma}(\lambda,\xi)|^2\sin^2(\lambda t)+|\widetilde{g}^{\sigma}(\lambda,\xi)|^2\cos^2(\lambda t)\\&\qquad-\lambda \widetilde{f}^{\sigma}(\lambda,\xi)\sin(\lambda t)\cdot\overline{\widetilde{g}^{\sigma}}(\lambda,\xi)\cos(\lambda t)\\&\qquad-\lambda \overline{\widetilde{f}^{\sigma}}(\lambda,\xi)\sin(\lambda t)\cdot\widetilde{g}^{\sigma}(\lambda,\xi)\cos(\lambda t)\end{aligned}$$

and for the integrand in (43):

$$\begin{aligned}&|\lambda \widetilde{f}^{\sigma}(\lambda,\xi)\cos(\lambda t)+\widetilde{g}^{\sigma}(\lambda,\xi)\sin(\lambda t)|^2\\&\quad=\lambda^2|\widetilde{f}^{\sigma}(\lambda,\xi)|^2\cos^2(\lambda t)+|\widetilde{g}^{\sigma}(\lambda,\xi)|^2\sin^2(\lambda t)\\&\qquad+\lambda \widetilde{f}^{\sigma}(\lambda,\xi)\cos(\lambda t)\cdot\overline{\widetilde{g}^{\sigma}}(\lambda,\xi)\sin(\lambda t)\\&\qquad+\lambda \overline{\widetilde{f}^{\sigma}}(\lambda,\xi)\cos(\lambda t)\cdot\widetilde{g}^{\sigma}(\lambda,\xi)\sin(\lambda t).\end{aligned}$$

Therefore, we obtain for the sum of the integrands in (42) and (43):

$$\begin{aligned}&\lambda^2|\widetilde{f}^{\sigma}(\lambda,\xi)|^2\sin^2(\lambda t)+|\widetilde{g}^{\sigma}(\lambda,\xi)|^2\cos^2(\lambda t)\\&\qquad+\lambda^2|\widetilde{f}^{\sigma}(\lambda,\xi)|^2\cos^2(\lambda t)+|\widetilde{g}^{\sigma}(\lambda,\xi)|^2\sin^2(\lambda t)\\&\quad=\lambda^2|\widetilde{f}^{\sigma}(\lambda,\xi)|^2+|\widetilde{g}^{\sigma}(\lambda,\xi)|^2.\end{aligned}$$

Therefore, the total energy is given by

$$\begin{aligned}2\mathcal{E}(\varphi)(t)=&\|\lambda \tilde{f}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)}\\&+\|\tilde{g}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)}\end{aligned}$$

and is independent of the time. □

Note that using a different method one can prove the conservation of energy of solutions of the shifted wave equation on an arbitrarily oriented Riemannian manifold (see [29, Lemma V.5.12]). However, via this proof, one does not obtain the explicit expression for the total energy above. Using Theorem 6.10, Green's identity and the fact that f has compact support, we obtain that

$$2\mathcal{E}(\varphi)=\|g\|^2_{L^2(X)}+\|\nabla f\|^2_{L^2(X)}-\rho^2\|f\|^2_{L^2(X)}.$$

Hence comparing the above with the expression for the energy from Theorem 6.10, we obtain using the Plancherel theorem and Lemma 6.5

$$\begin{aligned}&\|\nabla f\|^2_{L^2(X)}-\rho^2\|f\|^2_{L^2(X)}\\&\qquad=\|\lambda \tilde{f}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)}.\end{aligned}\tag{44}$$

In the next section, we are going to investigate the term on the right-hand side to obtain bounds on the energy just using the L^2 norms of the initial conditions.

7 A Paley–Wiener Type Theorem on Harmonic Manifolds of Rank One

The classical Paley–Wiener theorem (see for instance [51, p.161]) gives sharp bounds on the decay of the Fourier transform of a compactly supported function on $\mathbb{R}^n$:

Theorem 7.1 *A holomorphic function $F : \mathbb{C}^n \to \mathbb{C}$ is the Fourier transform of a smooth function with support in the ball $\{x \in \mathbb{R}^n \mid \|x\| \leq R\}$ if and only if for every $N \in \mathbb{N}_{>0}$ there exists a constant $C_N > 0$ such that*

$$|F(\lambda)| \leq C_N(1 + |\lambda|)^{-N} e^{R|\operatorname{Im}\lambda|} \quad \forall \lambda \in \mathbb{C}.$$

In this section, we want to show a weaker statement (Theorem 7.4) namely that a sufficient decay of the derivatives of a function forces their Fourier transform to have support within a bounded set. Using mainly Lemma 6.5 and the Plancherel theorem, this is an extension of a Paley–Wiener type theorem from [5] to harmonic manifolds of rank one. The proof follows the lines in [5] closely with the addition of some details, but the statement of the Paley–Wiener type theorem is weaker than the one in [5] since it is still not known if the Fourier transform on harmonic manifolds is surjective. Furthermore, we use this result to show that the total energy of a solution to the shifted wave equation with specific initial conditions is bounded by bounds only depending on the L^2 norm of the initial conditions and bounds on the support of the Fourier transform of the initial conditions. Let $g : \mathbb{R}^+ \times \partial X \to \mathbb{C}$ be a measurable function with respect to the measure $C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda$. Then, we define

$$R_g := \sup_{(\lambda,\xi)\in \operatorname{supp} g} |\lambda|.$$

Note that this might be infinite.

Lemma 7.2 *Let g be a function on $\mathbb{R}^+ \times \partial X$ such that $(\lambda, \xi) \to \lambda^j g(\lambda, \xi)$ belongs to $L^2(\mathbb{R}^+ \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)$ for all integers j. Then,*

$$R_g = \lim_{j\to\infty} \left(C_0 \int_0^\infty \int_{\partial X} \lambda^{2j} |g(\lambda, \xi)|^2 |\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda \right)^{1/(2j)}$$

Proof First, we assume $R_g < \infty$. Let $0 < \epsilon < R_g$ and we get for some $\delta > 0$ that

$$C_0 \int_0^{R_g-\epsilon} \int_{\partial X} \lambda^{2j} |g(\lambda, \xi)|^2 |\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda \geq (R_g - \epsilon)^{2j+1}\delta.$$

Hence, on one hand,

$$\begin{aligned}
&\liminf_{j\to\infty}\Big(C_0\int_0^\infty\int_{\partial X}\lambda^{2j}|g(\lambda,\xi)|^2|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda\Big)^{1/(2j)}\\
&\qquad\geq \liminf_{j\to\infty}\Big(C_0\int_0^{R_g-\epsilon}\int_{\partial X}\lambda^{2j}|g(\lambda,\xi)|^2|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda\Big)^{1/(2j)}\\
&\qquad\geq R_g-\epsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\limsup_{j\to\infty}\Big(C_0\int_0^\infty\int_{\partial X}\lambda^{2j}|g(\lambda,\xi)|^2|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda\Big)^{1/(2j)}\\
&\qquad\leq R_g \limsup_{j\to\infty}\|g\|^{1/j}_{L^2(\mathbb{R}^+\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)}\\
&\qquad= R_g.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary this completes the case $R_g<\infty$. Now suppose $R_g=\infty$. Then, for every $M>0$, we have,

$$C_0\int_M^\infty\int_{\partial X}\lambda^{2j}|g(\lambda,\xi)|^2|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda>0$$

and

$$\begin{aligned}
&\liminf_{j\to\infty}\Big(C_0\int_0^\infty\int_{\partial X}\lambda^{2j}|g(\lambda,\xi)|^2|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda\Big)^{1/(2j)}\\
&\qquad\geq \liminf_{j\to\infty}\Big(C_0\int_M^\infty\int_{\partial X}\lambda^{2j}|g(\lambda,\xi)|^2C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda\Big)^{1/(2j)}\\
&\qquad\geq M.
\end{aligned}$$

□

Definition 7.3 Let $R>0$. We define

$$\begin{aligned}
&L^2_R(\mathbb{R}^+\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)\\
&\quad:=\{g\in L^2(\mathbb{R}^+\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\,d\mu_\sigma(\xi)\,d\lambda)\mid R_g=R\}
\end{aligned}$$

and

$$\begin{aligned}
PW^2_R(X):=\{f\in C^\infty(X)\ |\Delta^j f\in L^2(X)\,\forall j\in\mathbb{N}\\
\text{and}\ \lim_{j\to\infty}\|(\Delta+\rho^2)^j f\|_2^{1/(2j)}=R\}.
\end{aligned}$$

Theorem 7.4 *Let $R > 0$. Then, if it exists, the inverse Fourier transform of a function in $L^2_R(\mathbb{R}^+ \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)$ belongs to $PW^2_R(X)$ and the Fourier transform maps $PW^2_R(X)$ to*

$$L^2_R(\mathbb{R}^+ \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda).$$

Proof Let $g \in L^2_R(\mathbb{R}^+ \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)$ and denote its inverse Fourier transform with respect to $\sigma \in X$ by f. f is smooth by the Lebesgue's dominant convergence theorem and $\Delta^j f \in L^2(X)$ for all $j \in \mathbb{N}$ since by Lemma 6.5, we have

$$\Delta^j f = (-1)^j C_0 \int_0^\infty \int_{\partial X} (\lambda^2 + \rho^2)^j \tilde{f}^\sigma(\lambda, \xi) \cdot e^{(i\lambda - \rho)B_{\xi,\sigma}(x)} |\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda$$

and $g = \tilde{f}^\sigma \in L^2_R(\mathbb{R}^+ \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)$ with $R = \sup_{(\lambda,\xi)\in \operatorname{supp} g} |\lambda|$. Using the Plancherel theorem, Lemma 6.5 and Lemma 7.2, we have

$$\begin{aligned}
&\lim_{j\to\infty} \|(\Delta + \rho^2)^j f\|_2^{1/(2j)} \\
&\quad = \lim_{j\to\infty} \left(C_0 \int_0^\infty \int_{\partial X} \lambda^{2j} |\tilde{f}^\sigma(\lambda, \xi)|^2 |\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda \right)^{1/(2j)} \\
&\quad = \lim_{j\to\infty} \left(C_0 \int_0^\infty \int_{\partial X} \lambda^{2j} |g(\lambda, \xi)|^2 |\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda \right)^{1/(2j)} \\
&\quad = R.
\end{aligned}$$

Now if $f \in PW^2_R(X)$, then by the Plancherel theorem and Lemma 6.5, we have $\Delta^{2j} \tilde{f}^\sigma$ is in $L^2_R(\mathbb{R}^+ \times \partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)$, and by Lemma 7.2, we have $R_g = R$. □

Corollary 7.5 *Let $\sigma \in X$ and $R > 0$. Then, for a smooth solution of the shifted wave equation $\varphi : X \times \mathbb{R} \to \mathbb{C}$ with initial conditions*

$$\varphi(x, 0) = f(x) \in PW^2_R(X)$$
$$\left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(x, t) = g(x) \in C_c^\infty(X)$$

we have

$$2\mathcal{E}(\varphi)(t) \le R^2 \|f\|^2_{L^2(X)} + \|g\|^2_{L^2(X)}.$$

Furthermore, we obtain

$$\|\nabla f\|^2_{L^2(X)} \le (R^2 + \rho^2) \|f\|^2_{L^2(X)}.$$

Proof We have by Theorem 6.10 that

$$2\mathcal{E}(\varphi)(t) = \|\lambda \tilde{f}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)} + \|\tilde{g}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)}$$

and since $f \in PW^2_R(X)$, we obtain

$$\|\lambda \tilde{f}^{\sigma}\|^2{}_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)} \leq R^2 \|\tilde{f}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)}. \tag{45}$$

Therefore, applying the Plancherel theorem yields

$$2\mathcal{E}(\varphi)(t) \leq R^2 \|f\|^2_{L^2(X)} + \|g\|^2_{L^2(X)}.$$

Now using equation (44), equation (45) and the Plancherel theorem, we conclude

$$\begin{aligned}\|\nabla f\|^2_{L^2(X)} &\overset{(44)}{=} \|\lambda \tilde{f}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)} + \rho^2 \|f\|^2_{L^2(X)} \\ &\overset{(45)}{\leq} R^2 \|\tilde{f}^{\sigma}\|^2_{L^2((0,\infty)\times\partial X, C_0|\mathbf{c}(\lambda)|^{-2}\, d\mu_\sigma(\xi)\, d\lambda)} + \rho^2 \|f\|^2_{L^2(X)} \\ &\overset{\text{Plancherel theorem}}{=} R^2 \|f\|^2_{L^2(X)} + \rho^2 \|f\|^2_{L^2(X)} \\ &= (R^2 + \rho^2)\|f\|^2_{L^2(X)}.\end{aligned}$$

□

8 The Paley–Wiener Theorem for Harmonic Manifolds of Rank One

Theorem 8.1 *Let $f : X \to \mathbb{C}$ be a smooth function with compact support in the ball $B(\sigma, R)$ for some $\sigma \in X$ and $R > 0$. Then, the Fourier transform of f based at σ*

$$\tilde{f}^{\sigma}(\lambda, \xi) = \int_X f(x) e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx$$

is a holomorphic function in λ, and we have

$$\sup_{\lambda\in\mathbb{C},\, \xi\in\partial X} e^{-R|\mathrm{Im}(\lambda)|}(1 + |\lambda|)^N |\tilde{f}^{\sigma}(\lambda, \xi)| < \infty \quad \forall N \in \mathbb{N}_{>0}.$$

The above is a generalisation of Theorem 4.5 in [3] but our method differs from theirs which relies on the homogeneity of Damek–Ricci spaces. Furthermore, the boundary structure of the Damek–Ricci space NA used consists of the non-compact group N where we use the geometric boundary which is equivalent to using the one-point compactification of N, for an explanation of this correspondence see for example [4, Sect. 3]. The idea of the proof: We first show that for $f \in C_c^\infty(X)$ the Radon transform $\mathcal{R}_\sigma(f)(s,\xi)$, a modification of the one introduced in [43], is smooth in s. Then, we argue that it vanishes for $|s| > R$ and all $\xi \in \partial X$, if $\operatorname{supp} f \subset B(\sigma, R)$. Using the connection of the Radon transform and the Fourier transform via the Euclidean Fourier transform, we apply the classical Paley–Wiener theorem to show the claim. This approach is also used by Helgason to show the Paley–Wiener theorem for non-compact symmetric space (see [29, p. 278]). We begin by introducing the Radon transform, a generalisation of the Abel transform to non-radial functions.

8.1 The Radon Transform

We define the Radon transform $\mathcal{R}_\sigma(f) : \mathbb{R} \times \partial X \to \mathbb{C}$ at $\sigma \in X$ for $f \in C_c^\infty(X)$ by

$$\mathcal{R}_\sigma(f)(s,\xi) := e^{-\rho s} \int_{H_{\xi,\sigma}(s)} f(z)\, dz$$

for all $s \in \mathbb{R}$ and $\xi \in \partial X$. Note that this definition differs from the one given in [43] by the factor $e^{-\rho s}$. Furthermore, all signs are swapped compared to Rouvière's work since he chooses the Busemann function to be defined with the opposite sign to ours. We choose this factor deliberately to have a direct correspondence to the Fourier transform via the Euclidean Fourier transform in Lemma 8.5 and obtain the Abel transform on radial functions.

Lemma 8.2 *Let $f \in C_c^\infty(X)$. Then $\mathcal{R}_\sigma(f)(s,\xi)$ is smooth in s.*

Proof In coordinates given by the diffeomorphism (5) and by (6), the regularity of $\mathcal{R}_\sigma(f)(s,\xi)$ in s is given by the minimum of the regularity of f and Ψ_s. But since the Busemann functions and the metric are analytic, Ψ_s is analytic in s. Hence, $\mathcal{R}_\sigma(f)(s,\xi)$ is smooth in s. $\square$

The lemma is a version of the projection slice theorem for harmonic manifolds.

Lemma 8.3 *Let $f \in C_c^\infty(X)$ have support in the ball $B(\sigma, R)$ for some $\sigma \in X$ and $R > 0$. Then, $\mathcal{R}_\sigma(f)(s,\xi) = 0$ for $|s| > R$ and all $\xi \in \partial X$.*

Proof Let $|s| > R$. Since the Busemann function is Lipschitz with Lipschitz constant 1, we have that $|B_{\xi,\sigma}(x)|$ is a lower bound of $d(\sigma, x)$. Hence, for all $x \in H^s_{\xi,\sigma}$ we have that $d(\sigma, x) > R$ hence $f = 0$ on $H^s_{\xi,\sigma}$, and therefore,

$$\mathcal{R}_\sigma(f)(s,\xi) = e^{-\rho s} \int_{H_{\xi,\sigma}(s)} f(z)\, dz = 0$$

for all $\xi \in \partial X$. $\square$

Remark 8.4 Since the gradient of the Busemann function $B_{\xi,\sigma}$ in $\sigma \in X$ coincides up to a sign with the initial condition of the unique geodesic emitting from σ and ending in ξ, the distance from $H^s_{\xi,\sigma}$ is given by $|s|$.

In the next lemma, the choice of the factor $e^{-\rho s}$ will become apparent. A version without the factor can be found in [43, Proposition 9].

Lemma 8.5 *Let $\mathcal{F}$ be the Euclidean Fourier transform given for a smooth complex-valued function u on $\mathbb{R}$ with compact support by*

$$\mathcal{F}(u)(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} u(t)\, dt \quad \lambda \in \mathbb{C}.$$

Then, for $f \in C_c^\infty(X)$, we have

$$\tilde{f}^\sigma(\lambda, \xi) = \mathcal{F}\big(\mathcal{R}_\sigma(f)(\cdot, \xi)\big)(\lambda).$$

Proof We have for $f \in C_c^\infty(X)$ using the Co-area formula:

$$\begin{aligned}
\tilde{f}^\sigma(\lambda, \xi) &= \int_X f(x) e^{-(i\lambda+p)B_{\xi,\sigma}(x)}\, dx \\
&= \int_{-\infty}^{\infty} \int_{H_{s,\xi}} f(z) e^{-(i\lambda+p)s}\, dz\, ds \\
&= \int_{-\infty}^{\infty} e^{-i\lambda s} e^{-ps} \int_{H_{s,\xi}} f(z)\, dz, ds \\
&= \int_{-\infty}^{\infty} e^{-i\lambda s} \mathcal{R}_\sigma(f)(s, \xi)\, ds \\
&= \mathcal{F}(\mathcal{R}_\sigma(f)(s, \xi))(\lambda).
\end{aligned}$$

Where we get the existence of the Euclidean Fourier transform above from Lemma 8.3. □

Remark 8.6 In [43, Theorem 11], Rouvière uses Lemma 8.5 to prove an inversion formula for the Radon transform. The idea is to apply the inverse Fourier transform on X to the result of the lemma.

Proof of Theorem 8.1 First we note that $e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}$ is for all $x \in X$ holomorphic in $\lambda \in \mathbb{C}$ and since

$$\tilde{f}^\sigma(\lambda, \xi) = \int_X f(x) e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, dx,$$

it is sufficient to show that

$$\int_X \left| f(x) e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)} \right| dx < \infty \quad \forall \lambda \in \mathbb{C}.$$

But this is given by the fact that f has compact support. Hence, $\tilde{f}^\sigma(\lambda, \xi)$ is holomorphic in $\lambda \in \mathbb{C}$ for all $\xi \in \partial X$ by Morera's theorem. Now by Lemma 8.2, $\mathcal{R}_\sigma(f)(s, \xi)$

is smooth in s and, by Lemma 8.3, $\mathcal{R}_\sigma(f)(s,\xi)$ has support in $[-R, R]$. Furthermore, by Lemma 8.5,

$$\tilde{f}^\sigma(\lambda,\xi) = \mathcal{F}\big(\mathcal{R}_\sigma(f)(s,\xi)\big)(\lambda).$$

Hence, by the classical Paley–Wiener theorem (see Theorem 7.1), we have that for every $\xi \in \partial X$ and $N \in \mathbb{N}_{>0}$ there exists a constant $C_{N,\xi} > 0$ such that

$$|\tilde{f}^\sigma(\lambda,\xi)| \le C_{N,\xi}(1+|\lambda|)^{-N} e^{R|\mathrm{Im}\,\lambda|} \quad \forall \lambda \in \mathbb{C}.$$

Now ∂X is compact and $\tilde{f}^\sigma(\lambda,\xi)$ is continuous in ξ, since the Busemann boundary and the geometric boundary coincide, hence there exists a $C_N > 0$ such that for all $\xi \in \partial X$:

$$|\tilde{f}^\sigma(\lambda,\xi)| \le C_N(1+|\lambda|)^{-N} e^{R|\mathrm{Im}\,\lambda|} \quad \forall \lambda \in \mathbb{C}.$$

This yields the claim. □

Proposition 8.7 *Let $f \in C_c^\infty(X)$. Then, we have*

$$\int_{\partial X} \tilde{f}^\sigma(-\lambda,\xi) e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, d\mu_\sigma(\xi) = \int_{\partial X} \tilde{f}^\sigma(\lambda,\xi) e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\, d\mu_\sigma(\xi).$$

The proof follows from the following lemma with the relation

$$\varphi_{-\lambda,\sigma} = \varphi_{\lambda,\sigma}.$$

Lemma 8.8 *Let $f \in C_c^\infty(X)$. Then, we have*

$$\begin{aligned} f * \varphi_{\lambda,\sigma}(x) &:= \int_X f(y)\cdot \varphi_{\lambda,x}(y)\, dy \\ &= \int_{\partial X} \tilde{f}^\sigma(-\lambda,\xi)\cdot e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\, d\mu_\sigma(\xi). \end{aligned}$$

Proof Recall the relations (30), (39) and (40). Then, we obtain for $x, \sigma \in X$:

$$\begin{aligned} f * \varphi_{\lambda,\sigma}(x) &= \int_X f(y)\cdot \varphi_{\lambda,x}(y)\, dy \\ &\overset{(40)}{=} \int_X f(y)\cdot \int_{\partial X} e^{(i\lambda-\rho)B_{\xi,x}(y)}\, d\mu_x(\xi)\, dy \\ &= \int_X \int_{\partial X} f(y) e^{(i\lambda-\rho)B_{\xi,x}(y)}\, d\mu_x(\xi)\, dy \\ &= \int_{\partial X} \int_X f(y) e^{(i\lambda-\rho)B_{\xi,x}(y)}\, dy\, d\mu_x(\xi) \\ &= \int_{\partial X} \tilde{f}^x(-\lambda,\xi)\, d\mu_x(\xi) \end{aligned}$$

$$\overset{(39)}{=} \int_{\partial X} \tilde{f}^{\sigma}(-\lambda, \xi) \cdot e^{(-i\lambda+\rho)B_{\xi,\sigma}(x)} \, d\mu_x(\xi)$$
$$\overset{(30)}{=} \int_{\partial X} \tilde{f}^{\sigma}(-\lambda, \xi) \cdot e^{(-i\lambda+\rho)B_{\xi,\sigma}(x)} e^{-2\rho B_{\xi,\sigma}(x)} \, d\mu_\sigma(\xi)$$
$$= \int_{\partial X} \tilde{f}^{\sigma}(-\lambda, \xi) \cdot e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)} \, d\mu_\sigma(\xi).$$

The interchange of integrals is justified by the Fubini-Tonelli theorem and the facts that f has compact support and ∂X has finite measure ($d\mu_\sigma(\xi)$ is a probability measure). □

Corollary 8.9 *Let $R > 0$ and denote by PW_R^0 all functions $F : \mathbb{C} \times \partial X \to \mathbb{C}$ holomorphic on $\mathbb{C}$ which satisfy*

$$\sup_{\lambda \in \mathbb{C},\, \xi \in \partial X} e^{-R|\mathrm{Im}(\lambda)|}(1 + |\lambda|)^N |F(\lambda, \xi)| < \infty \quad \forall N \in \mathbb{N}_{>0}.$$

and for $\sigma \in X$:

$$\int_{\partial X} F(-\lambda, \xi) \cdot e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)} \, d\mu_\sigma(\xi) = \int_{\partial X} F(\lambda, \xi) \cdot e^{(i\lambda-\rho)B_{\xi,\sigma}(x)} \, d\mu_\sigma(\xi).$$

Then, the image of $C_c^\infty(X)$ under the Fourier transform based at σ is contained in

$$\bigcup_{R \geq 0} PW_R^0.$$

9 Huygens' Principle

In this section, we want to prove an asymptotic Huygens' principle along the lines of the proof of [13]. For this we need to make assumptions on the **c**-function, namely we need that the function η defined by $\eta(\lambda)^{-1} := \mathbf{c}(\lambda)\overline{\mathbf{c}(\lambda)}$ on the lower half plane of $\mathbb{C}$ has a holomorphic extension up to $\mathrm{Im}(\lambda) = \epsilon_{max} > 0$ where it has a singular pole and is a polynomial with real coefficients up to this point such that $\eta(\lambda) = \lambda^{n-1}\eta_0(\lambda)$ where all poles of η are also poles of η_0 with the same multiplicity. We will call this as the C-condition. The C-condition is satisfied in the case of symmetric spaces of rank one and Damek–Ricci spaces whose nilpotent part has a centre of even dimension as well as on the hyperbolic spaces of odd dimension. For this, see [21]. For more detail on the **c**-function of Damek–Ricci space, see [48], especially Proposition 4.7.13−4.7.15 and Theorem 6.3.4.

Remark 9.1 Note that $\eta(\lambda) = |\mathbf{c}(\lambda)|^{-2}$ and that by [11, Lemma 3.4 and Proposition 3.17] (alternatively one can observe this from (33) combined with (34) and (35)), we have

$$\mathbf{c}(\lambda) = \overline{\mathbf{c}(-\lambda)} \quad \forall \lambda \in \mathbb{R}.$$

From this, we get that for all $\lambda \in \mathbb{R}$

$$\eta(-\lambda) = (\mathbf{c}(-\lambda)\overline{\mathbf{c}(-\lambda)})^{-1} = (\overline{\mathbf{c}(\lambda)}\mathbf{c}(\lambda))^{-1} = \eta(\lambda)$$

hence η is even in λ.

Theorem 9.2 *Let (X, g) be a non-compact simply connected harmonic manifold of rank one of dimension bigger than one, such that the $\mathbf{c}$-function satisfies the C-condition. Let φ be a solution of the shifted wave equation with initial conditions f, g supported in a ball of radius R around $\sigma \in X$. Let ϵ_{max} be as above and $0 < \epsilon < \epsilon_{max} < \infty$. Then, there is a constant $C > 0$ such that*

$$|\varphi(x,t)| \leq C(\epsilon_{max} - \epsilon)^{-1} \cdot e^{-\epsilon(|t| - d(x,\sigma) - R)} \quad \forall (x,t) \in X \times \mathbb{R}.$$

If $\epsilon_{max} = \infty$, we get

$$|\varphi(x,t)| \leq C \cdot e^{-\epsilon(|t| - d(x,\sigma) - R)} \quad \forall \epsilon > 0, \forall (x,t) \in X \times \mathbb{R}.$$

In particular, we have in this case,

$$\varphi(x,t) = 0 \quad \textit{for } |t| - d(x,\sigma) \geq R.$$

The proof of this statement will be conducted via a series of lemma occupying the remainder of the section. We will always require the assumptions of the theorem.

Lemma 9.3 *Let $h : \mathbb{C} \to \mathbb{C}$ be a function holomorphic on the strip $P = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z \leq \epsilon\}$ such that there is a $C > 0$ with $|h(z)| \leq C(1 + |z|)^{-N}$ for some $N > 0$ on P. Then*

$$\int_{-\infty}^{\infty} h(z)\,dz = \int_{-\infty}^{\infty} h(a + i\epsilon)\,da.$$

Proof Consider the contour in Fig. 4. Let $\gamma_1 : [0,1] \to \mathbb{C}$ be given by $\gamma_1(s) = r + is\epsilon$ and $\gamma_2 : [0,1] \to \mathbb{C}$ be given by $\gamma_2(s) = -r + i(1-s)\epsilon$. Then, by the bounds on h on the strip P, there are constants $C_1, C_2 > 0$ such that

$$\left|\int_{\gamma_1} h\,ds\right| = \left|\int_0^1 h(r + is\epsilon) \cdot i\theta\,ds\right| \leq C_1(1 + |r|)^{-N},$$
$$\left|\int_{\gamma_2} h\,ds\right| = \left|\int_0^1 h(-r + (1 - is)\epsilon) \cdot -i\theta\,ds\right| \leq C_2(1 + |r|)^{-N}.$$

Therefore, since both integrals tend to zero for $r \to \pm\infty$ and we get the assertion. □

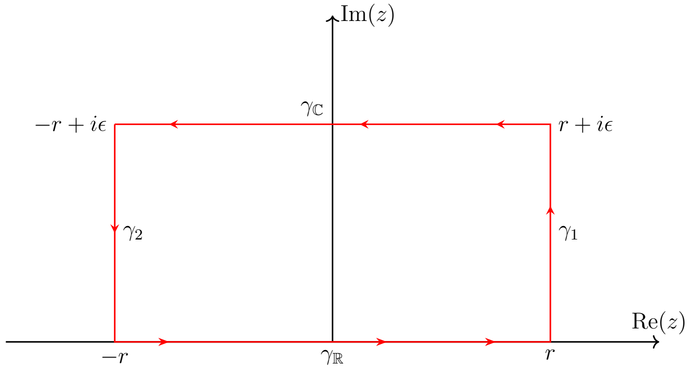

Fig. 4 Contour of Lemma 9.3, for $r \to \infty$ the integral along γ_1 and γ_2 vanishes because of the bounds on h

Lemma 9.4 *Let $f, g \in C_c^\infty(X)$. Then the functions*

$$F(\lambda, x) := \int_{\partial X} \tilde{f}^\sigma(\lambda, \xi) e^{(i\lambda-\rho)B_{\xi,\sigma}(x)} \eta(\lambda)\, d\mu_\sigma(\xi)$$

and

$$G(\lambda, x) := \int_{\partial X} \tilde{g}^\sigma(\lambda, \xi) e^{(i\lambda-\rho)B_{\xi,\sigma}(x)} \eta(\lambda)\, d\mu_\sigma(\xi)$$

are even in λ and

$$\int_0^\infty F(\lambda, x)\cos(\lambda t) + G(\lambda, \xi)\frac{\sin(\lambda t)}{\lambda}\, d\lambda = \frac{1}{2}\int_{-\infty}^\infty \big(F(\lambda, x) + \frac{G(\lambda, x)}{i\lambda}\big) e^{i\lambda t}\, d\lambda.$$

Proof Note that η is even in λ by Remark 9.1, and $F(\lambda, x)$ and $G(\lambda, x)$ are also even in λ by Proposition 8.7. Now using this and

$$2\cos(\lambda t) = e^{i\lambda t} + e^{-i\lambda t}$$

we get

$$\begin{aligned}\int_0^\infty F(\lambda, x)\cos(\lambda t)\, d\lambda &= \frac{1}{2}\Big(\int_0^\infty F(\lambda, x)e^{i\lambda t}\, d\lambda + \int_0^\infty F(\lambda, x)e^{-i\lambda t}\, d\lambda\Big)\\ &= \frac{1}{2}\Big(\int_0^\infty F(\lambda, x)e^{i\lambda t}\, d\lambda + \int_{-\infty}^0 F(\lambda, x)e^{i\lambda t}\, d\lambda\Big)\\ &= \frac{1}{2}\int_{-\infty}^\infty F(\lambda, x)e^{i\lambda t}\, d\lambda.\end{aligned}$$

Since $2i\sin(\lambda t) = e^{i\lambda t} - e^{-i\lambda t}$ and $G(\lambda, x)$ is even in λ, we obtain

$$\begin{aligned}\int_0^\infty G(\lambda, x)\frac{\sin(\lambda t)}{\lambda}\,d\lambda =& \frac{1}{2i}\Big(\int_0^\infty G(\lambda, x)\frac{e^{i\lambda t}}{\lambda}d\lambda - \int_0^\infty G(\lambda, x)\frac{e^{-i\lambda t}}{\lambda}d\lambda\Big)\\ =& \frac{1}{2i}\Big(\int_0^\infty G(\lambda, x)\frac{e^{i\lambda t}}{\lambda}\,d\lambda + \int_{-\infty}^0 G(\lambda, x)\frac{e^{i\lambda t}}{\lambda}\,d\lambda\Big)\\ =& \frac{1}{2}\int_{-\infty}^\infty G(\lambda, x)\frac{e^{i\lambda t}}{i\lambda}d\lambda.\end{aligned}$$

□

By [47, Prop. 6.1.1 and Prop. 6.1.4] and (37), we have the following bounds for the radial eigenfunctions of the Laplacian:

Lemma 9.5 *For all $x, \sigma \in X$ and $\lambda \in \mathbb{C}$, we have*

(1) $|\varphi_{\lambda,\sigma}(x)| \leq \varphi_{i\,\mathrm{Im}(\lambda),\sigma}(x) \leq \varphi_{0,\sigma}(x) \cdot e^{|\mathrm{Im}(\lambda)|d(\sigma,x)}$,
(2) $|\mathrm{Im}(\lambda)| \leq \rho \Rightarrow e^{(|\mathrm{Im}(\lambda)|-\rho)d(\sigma,x)} \leq \varphi_{i\,\mathrm{Im}(\lambda),\sigma}(x) \leq 1$,
(3) $|\mathrm{Im}(\lambda)| \geq \rho \Rightarrow 1 \leq \varphi_{i\,\mathrm{Im}(\lambda),\sigma}(x) \leq e^{(|\mathrm{Im}(\lambda)|-\rho)d(\sigma,x)}$.

Furthermore, we have

$$\varphi_{i\,\mathrm{Im}(\lambda),\sigma}(x) \leq k(1 + d(\sigma, x))e^{(|\mathrm{Im}(\lambda)|-\rho)d(\sigma,x)}$$

for some positive constant $k > 0$.

Lemma 9.6 *Assume the assumptions of the Theorem 9.2. Let $f, g \in C_c^\infty(X)$ with support in the ball of radius $R > 0$ around $\sigma \in X$. Then F and G admit holomorphic extensions in λ up to $\epsilon_{\max}$ and for every $N \in \mathbb{N}$ we can find a constant C_N such that for all $\lambda \in \mathbb{C}$ with $0 \leq \mathrm{Im}\,\lambda \leq \epsilon < \epsilon_{\max}$ and $x \in X$*

$$|F(\lambda, x)| \leq C_N(\epsilon_{max} - \epsilon)^{-1}(1 + |\lambda|)^{-N}e^{\epsilon d(x,\sigma)+R\epsilon}$$

and

$$|G(\lambda, x)| \leq C_N(\epsilon_{max} - \epsilon)^{-1}(1 + |\lambda|)^{-N}e^{\epsilon d(x,\sigma)+R\epsilon}.$$

Furthermore, if $\dim X > 1$, we have that for every $N \in \mathbb{N}$, there is a constant D_N such that

$$|\lambda^{-1}G(\lambda, x)| \leq D_N(\epsilon_{max} - \epsilon)^{-1}(1 + |\lambda|)^{-N}e^{\epsilon d(x,\sigma)+R\epsilon}.$$

Proof That F, G are holomorphic up to ϵ_{max} in λ follows from the fact that all functions making up those are holomorphic up to this point. Let us begin with the estimate on

F. The one on G follows in the same manner.

$$\begin{aligned}|F(\lambda,x)| &\leq \Big|\int_{\partial X}\tilde{f}^{\sigma}(\lambda,\xi)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\eta(\lambda)\,d\mu_{\sigma}(\xi)\Big|\\ &\leq \sup_{\operatorname{Im}\lambda<\epsilon_{\max},\,\xi\in\partial X}|\tilde{f}^{\sigma}(\lambda,\xi)\eta(\lambda)|\Big|\int_{\partial X}e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_{\sigma}(\xi)\Big|.\end{aligned}$$

By Lemma 9.5 (1) and the integral representation of the radial eigenfunctions (40),

$$\begin{aligned}\Big|\int_{\partial X}e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_{\sigma}(\xi)\Big| &= |\varphi_{\lambda,\sigma}(x)|\\ &\leq |\varphi_{i\operatorname{Im}\lambda}(x)|\\ &\leq |\varphi_{0,\sigma}(x)|e^{|\operatorname{Im}\lambda|d(x,\sigma)}\\ &\leq e^{|\operatorname{Im}\lambda|d(x,\sigma)}.\end{aligned}$$

Now using Theorem 8.1, the assumption that η has a singular pole at ϵ_{max} and is a polynomial, and since ∂X is compact, we can conclude that for every $N\in\mathbb{N}$ there is a constant C_N such that for all $0\leq\operatorname{Im}\lambda\leq\epsilon<\epsilon_{\max}$

$$\begin{aligned}|F(\lambda,x)| &\leq C_N(\epsilon_{max}-\epsilon)^{-1}(1+|\lambda|)^{-N}e^{\epsilon d(x,\sigma)+R|\operatorname{Im}\lambda|}\\ &\leq C_N(\epsilon_{max}-\epsilon)^{-1}(1+|\lambda|)^{-N}e^{\epsilon d(x,\sigma)+R\epsilon}.\end{aligned}$$

For the last estimate on $|\lambda^{-1}G(\lambda,x)|$, one only needs to consider that $\eta(\lambda)=\lambda^{n-1}\eta_0(\lambda)$ where all poles of η are also poles of η_0 with the same multiplicity. Hence, one only needs to exclude the case where $\dim X=1$. Then, we get using the same lines as above:

$$\begin{aligned}|\lambda^{-1}G(\lambda,x)| &\leq \Big|\int_{\partial X}\lambda^{-1}\tilde{g}^{\sigma}(\lambda,\xi)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\eta(\lambda)\,d\mu_{\sigma}(\xi)\Big|\\ &\leq \sup_{\operatorname{Im}\lambda<\epsilon_{\max},\,\xi\in\partial X}|\lambda^{-1}\tilde{g}^{\sigma}(\lambda,\xi)\eta(\lambda)|\Big|\int_{\partial X}e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_{\sigma}(\xi)\Big|\\ &\leq \sup_{\operatorname{Im}\lambda<\epsilon_{\max},\,\xi\in\partial X}\Big(|\lambda^{n-2}\tilde{g}^{\sigma}(\lambda,\xi)\eta_0(\lambda)|\\ &\qquad\cdot\Big|\int_{\partial X}e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_{\sigma}(\xi)\Big|\Big)\end{aligned}$$

and unsing again the estimate

$$\Big|\int_{\partial X}e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_{\sigma}(\xi)\Big|\leq e^{|\operatorname{Im}\lambda|d(x,\sigma)}.$$

Hence, we obtain using the same arguments as above that for every $N \in \mathbb{N}$ there is a constant D_N such that for $0 \leq \operatorname{Im} \lambda \leq \epsilon < \epsilon_{\max}$

$$|\lambda^{-1} G(\lambda, x)| \leq D_N (\epsilon_{max} - \epsilon)^{-1} (1 + |\lambda|)^{-N} e^{\epsilon d(x,\sigma) + R\epsilon}.$$

□

Proof Theorem 9.2 First we note that $u(x, -t)$ solves the shifted wave equation with initial conditions $f, -g$. Hence, we only need to consider the case $t \geq 0$. Let $0 < \epsilon < \epsilon_{max}$. Then, using Lemma 9.3, we can move the integral defining u from $\mathbb{R}$ to $\mathbb{R} + i\epsilon$, hence,

$$\begin{aligned} 2|\varphi(x,t)| &= \left| C_0 \int_{-\infty}^{\infty} \big(F(\lambda, x) + \frac{G(\lambda, x}{i\lambda}\big) e^{i\lambda t}\, d\lambda \right| \\ &= \left| C_0 e^{-\epsilon t} \int_{-\infty}^{\infty} \big(F(a + i\epsilon, x) + \frac{G(a + i\epsilon)}{i(a + i\lambda)}\big) e^{iat}\, d\lambda \right|, \end{aligned}$$

now using Lemma 9.6, we obtain for $N \in \mathbb{N}$ a constant $C_N > 0$ such that

$$2|\varphi(x,t)| \leq C_N (\epsilon_{max} - \epsilon)^{-1} e^{-\epsilon(t - d(x,\sigma))} e^{R\epsilon} \int_{-\infty}^{\infty} (1 + |\lambda|)^{-N}\, d\lambda.$$

Since the last integral is bounded we obtain the claim. For the case that the **c**-function is an entire function and a polynomial, one notices that we can ignore the them $(\epsilon_{max} - \epsilon)^{-1}$ in all the estimates which yield the assertion in this case. □

10 Equidistribution of Energy

Under the same assumptions on the **c**-function as in the last section, we now want to prove an asymptotic equidistribution of the energy between the kinetic and potential energy of a wave on X.

Theorem 10.1 *Let (X, g) be a non-compact simply connected harmonic manifold of rank one, such that the* **c***-function satisfies the C-condition. Let φ be a solution of the shifted wave equation with smooth initial conditions f, g compactly supported within a ball of radius R around $\sigma \in X$. Let ϵ_{max} be as before and $0 < \epsilon < \epsilon_{max} < \infty$. Then, there is a constant $C > 0$ such that we have for the potential and kinetic energy $\mathcal{P}$ and $\mathcal{K}$*

$$|\mathcal{K}(\varphi)(t) - \mathcal{P}(\varphi)(t)| \leq C(\epsilon_{max} - \epsilon)^{-1} (e^{-2\epsilon(|t| - R)}) \quad \forall t \in \mathbb{R}$$

and if $\epsilon_{max} = \infty$, we have

$$\mathcal{K}(\varphi)(t) = \mathcal{P}(\varphi)(t) \quad \forall |t| \geq R.$$

The proof is similar to the proof of Theorem 9.2. Let us begin with calculating the difference between the kinetic and potential energy.

Lemma 10.2 *Let φ be a solution of the shifted wave equation with initial conditions $f, g \in C_c^\infty(X)$. Then*

$$\frac{2}{C_0}\Big(\mathcal{K}(\varphi)(t) - \mathcal{P}(\varphi)(t)\Big) =$$
$$\int_0^\infty \int_{\partial X} \Big(\big(-\lambda^2 \tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi) + \tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi)\big)\cos(2\lambda t)$$
$$-\big(\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi) + \tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\big)\cdot\lambda\sin(2\lambda t)\Big)d\mu_\sigma |c(\lambda)|^{-2} d\lambda.$$

Proof Subtracting the integrant in (43) from the integrand in (42) yields

$$\begin{aligned}\lambda^2 &\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\sin^2(\lambda t)\\ &+\tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi)\cos^2(\lambda t)\\ &-2\lambda\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi)\sin(\lambda t)\cos(\lambda t)\\ &-2\lambda\tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\sin(\lambda t)\cos(\lambda t)\\ &-\lambda^2\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\cos^2(\lambda t)\\ &-\tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi)\sin^2(\lambda t).\end{aligned}$$

Now using $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, we obtain that the above equates to

$$\begin{aligned}-\lambda^2 &\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\big(\cos^2(\lambda t) - \sin^2(\lambda t)\big)\\ &+\tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi)\big(\cos^2(\lambda t) - \sin^2(\lambda t)\big)\\ &-\lambda\big(\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi) + \tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\big)\sin(2\lambda t).\end{aligned}$$

Finally the claim follows from $\cos^2(x) - \sin^2(x) = \cos(2x)$. □

For us to be able to use the same arguments as in Sect. 9, the following lemma is essential.

Lemma 10.3 *Let $h_1, h_2 \in C_c^\infty(X)$ and $\sigma \in X$. Then for all $\lambda \in \mathbb{R}$ and $\xi \in \partial X$*

(1) $\overline{\widetilde{h}_1^\sigma}(\lambda,\xi) = \widetilde{\overline{h}}_1^\sigma(-\lambda,\xi)$.
(2) *We have*
$\int_{\partial X}\widetilde{h}_1^\sigma(\lambda,\xi)\overline{\widetilde{h}_2^\sigma}(\lambda,\xi)\,d\mu_\sigma(\xi) = \int_{\partial X}\widetilde{h}_1^\sigma(-\lambda,\xi)\overline{\widetilde{h}_2^\sigma}(-\lambda,\xi)\,d\mu_\sigma(\xi)$.

Proof For the first assertion, we only need to look at the definition of the Fourier transform:

$$\begin{aligned}
\overline{\widetilde{h_1^\sigma}}(\lambda,\xi) &= \overline{\int_X h_1(x)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\,dx} \\
&= \int_X \overline{h}_1(x)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,dx \\
&= \widetilde{\overline{h}}{}_1^\sigma(-\lambda,\xi).
\end{aligned}$$

The second assertion follows now from the first together with Proposition 8.7:

$$\begin{aligned}
&\int_{\partial X} \widetilde{h}_1^\sigma(\lambda,\xi)\overline{\widetilde{h}_2^\sigma}(\lambda,\xi)\,d\mu_\sigma(\xi) \\
&\overset{\text{Def. 6.2}}{=} \int_{\partial X}\Big(\int_X h_1(x)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\,dx\Big)\overline{\widetilde{h}_2^\sigma}(\lambda,\xi)\,d\mu_\sigma(\xi) \\
&= \int_{\partial X}\int_X h_1(x)\overline{\widetilde{h}_2^\sigma}(\lambda,\xi)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\,dx\,d\mu_\sigma(\xi) \\
&= \int_X\int_{\partial X} h_1(x)\overline{\widetilde{h}_2^\sigma}(\lambda,\xi)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_\sigma(\xi)\,dx \\
&= \int_X h_1(x)\int_{\partial X} \overline{\widetilde{h}_2^\sigma}(\lambda,\xi)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_\sigma(\xi)\,dx \\
&\overset{\text{(Lemma 10.3(i)}}{=} \int_X h_1(x)\int_{\partial X} \widetilde{\overline{h}}{}_2^\sigma(-\lambda,\xi)e^{(-i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_\sigma(\xi)\,dx \\
&\overset{\text{Lemma 8.7}}{=} \int_X h_1(x)\int_{\partial X} \widetilde{\overline{h}}{}_2^\sigma(\lambda,\xi)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_\sigma(\xi)\,dx \\
&= \int_X\int_{\partial X} h_1(x)\widetilde{\overline{h}}{}_2^\sigma(\lambda,\xi)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,d\mu_\sigma(\xi)\,dx \\
&= \int_{\partial X}\int_X h_1(x)\widetilde{\overline{h}}{}_2^\sigma(\lambda,\xi)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,dx\,d\mu_\sigma(\xi) \\
&= \int_{\partial X}\widetilde{\overline{h}}{}_2^\sigma(\lambda,\xi)\int_X h_1(x)e^{(i\lambda-\rho)B_{\xi,\sigma}(x)}\,dx\,d\mu_\sigma(\xi) \\
&\overset{\text{Def. 6.2}}{=} \int_{\partial X}\widetilde{h}_1^\sigma(-\lambda,\xi)\widetilde{\overline{h}}{}_2^\sigma(\lambda,\xi)(\lambda,\xi)\,d\mu_\sigma(\xi) \\
&\overset{\text{10.3(i)}}{=} \int_{\partial X}\widetilde{h}_1^\sigma(-\lambda,\xi)\overline{\widetilde{h}_2^\sigma}(-\lambda,\xi)\,d\mu_\sigma(\xi).
\end{aligned}$$

Here the interchange of integrals is justified by the Fubini-Tonelli theorem and the facts that h_1 and h_2 have compact support and ∂X has finite measure ($d\mu_\sigma(\xi)$ is a probability measure). □

Lemma 10.4 *Under the conditions of Theorem 10.1, define*

$$A(\lambda) := \int_{\partial X} \big(-\lambda^2 \tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi) + \tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi)\big)\eta(\lambda)\, d\mu_\sigma(\xi)$$

and

$$B(\lambda) := \int_{\partial X} \big(\tilde{f}^\sigma(\lambda,\xi)\overline{\tilde{g}^\sigma}(\lambda,\xi) + \tilde{g}^\sigma(\lambda,\xi)\overline{\tilde{f}^\sigma}(\lambda,\xi)\big)\eta(\lambda)\, d\mu_\sigma(\xi).$$

Then, for $\epsilon < \epsilon_{\max}$, we have

(1) *$A(\lambda)$ and $B(\lambda)$ admit a holomorphic extension up to* $\operatorname{Im}\lambda = \epsilon$.
(2) *$A(\lambda)$ and $B(\lambda)$ are even.*
(3) *For every $N \in \mathbb{N}$ there are constants A_N and B_N such that for every $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| \le \epsilon < \epsilon_{\max}$, we have*

$$(i)\ |A(\lambda)| \le A_N(\epsilon_{max} - \epsilon)^{-1}(1+|\lambda|)^{-N} e^{2R\epsilon},$$
$$(ii)\ |\lambda B(\lambda)| \le B_N(\epsilon_{max} - \epsilon)^{-1}(1+|\lambda|)^{-N} e^{2R\epsilon}.$$

(4) *We have for $|\operatorname{Im}\lambda| \le \epsilon$:*

$$\frac{4}{C_0}\Big(\mathcal{K}(\varphi)(t) - \mathcal{P}(\varphi)(t)\Big) = \int_{-\infty}^{\infty} \Big(A(\lambda) + i\lambda B(\lambda)\Big) e^{2i\lambda t}\, d\lambda.$$

Proof (1) is a direct consequence of the first assertion from Lemma 10.3 and Corollary 8.9. (3) also follows form Corollary 8.9 since the **c**-function satisfies the C-condition. If we have that A and B are even, then also (4) follows with the same arguments as in Lemma 9.4. Therefore, all that remains to show is (2) but this follows immediately from Lemma 10.3. □

Proof Theorem 10.1 With the same argument as in Theorem 9.2, we can restrict ourselves to the case $t \ge 0$. Let $0 < \epsilon < \epsilon_{max}$. Then, we have by using Lemma 9.3 and shifting the integral to $\mathbb{R} + i\epsilon$:

$$\begin{aligned} \left|\frac{4}{C_0}\Big(\mathcal{K}(\varphi)(t) - \mathcal{P}(\varphi)(t)\Big)\right| &= \left|\int_{-\infty}^{\infty} \Big(A(\lambda) + i\lambda B(\lambda)\Big) e^{2i\lambda t}\, d\lambda\right| \\ &= \left| e^{-2\epsilon t} \int_{-\infty}^{\infty} \Big(A(a+i\epsilon) + i(a+i\epsilon)B(a+i\epsilon)\Big) e^{2iat}\, da\right|. \end{aligned} \tag{46}$$

Now, by the bounds from Lemma 10.4, for every $N \in \mathbb{N}$ there is a constant C_N, such that for all $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| \le \epsilon < \epsilon_{max}$ (46) is bounded by

$$C_N(\epsilon_{max} - \epsilon)^{-1} e^{2R\epsilon} e^{-2\epsilon t} \int_{-\infty}^{\infty} (1+|\lambda|)^{-N}\, d\lambda \quad \forall t \ge 0. \tag{47}$$

Since the integral in (47) is bounded there is a constant $C > 0$ such that (46) bounded by

$$C(\epsilon_{max} - \epsilon)^{-1} e^{-2\epsilon(|t|-R)} \quad \forall t \geq 0.$$

For the case that the **c**-function is an entire function and a polynomial, one notices that we can ignore the term $(\epsilon_{max} - \epsilon)^{-1}$ in all the estimates. Taking the limit $\epsilon \to \infty$ yields the assertion. □

Remark 10.5 Note that the assumption on the pole of η to be of multiplicity one only affects the term $(\epsilon_{max} - \epsilon)^{-1}$, so one could restate Theorems 9.2 and 10.1 for η to have a pole of multiplicity $n \in \mathbb{N}$ by raising the power to $-n$. But there are no known examples for this case, even for **c**-functions on hypergroups. Hence, we state our theorems in the realistic setting.

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