The Journal of Geometric Analysis (2024) 34:9
https://doi.org/10.1007/s12220-023-01452-1

Smooth Maps Minimizing the Energy and the Calibrated Geometry

Kota Hattori[1]

Received: 21 May 2023 / Accepted: 25 September 2023 / Published online: 24 October 2023

Abstract

We generalize the notion of calibrated submanifolds to smooth maps and show that several kinds of smooth maps appearing in the differential geometry are applicable to our situation. Moreover, we apply this notion to give the lower bound to some energy functionals of smooth maps in the given homotopy class between Riemannian manifolds and consider the energy functional which is minimized by the identity maps on the Riemannian manifolds with special holonomy groups.

Keywords Calibrated geometry · Energy of maps · Special holonomy groups

Mathematics Subject Classification 53C38

Contents

✉ Kota Hattori
hattori@math.keio.ac.jp

1 Keio University, 3-14-1 Hiyoshi, Kohoku, Yokohama 223-8522, Japan

1 Introduction

In this article, we introduce the notion of calibrated geometry for smooth maps between Riemannian manifolds and consider the lower bound or the minimizers of several energy of smooth maps. Let (X, g) and (Y, h) be compact Riemannian manifolds and $f : X \to Y$ be a smooth map. Then the p-energy of f is defined by

$$\mathcal{E}_p(f) := \int_X |\mathrm{d}f|^p \mathrm{d}\mu_g$$

for $p \geq 1$, where μ_g is the volume measure of g. A harmonic map is a critical point of $\mathcal{E}_2$ and it is studied well by many researchers in differential geometry. In 1964, Eells and Sampson [4] have shown that there is a harmonic map f' homotopic to f if the sectional curvature of h is nonpositive. Moreover, Hartman [5] showed that such harmonic maps minimize $\mathcal{E}_2|_{[f]}$, where $[f]$ is the homotopy class represented by f.

In general, harmonic maps need not minimize the energy. For example, although the identity maps on any Riemannian manifolds are always harmonic, it is known that there is a family of smooth maps $\{f_\varepsilon\}_{\varepsilon>0}$ homotopic to the identity map of the n-sphere S^n with the standard metric such that $\lim_{\varepsilon\to 0} \mathcal{E}_2(f_\varepsilon) = 0$, if $n \geq 3$. By the result shown by White [10], if $\pi_l(X)$ is trivial for all $1 \leq l \leq k$, then $\inf \mathcal{E}_k|_{[1_X]} = 0$, where 1_X is the identity map of X.

One of the motivations of this article is to give the lower bound to the energy restricting to a given homotopy class $[f]$ and the minimizer of them. Such a lower bound was first obtained by Lichnerowicz [8] in the case of (X, g) and (Y, h) are Kähler manifolds, then it was shown that any holomorphic maps between Kähler manifolds minimize $\mathcal{E}_2$ in their homotopy classes. Moreover, Croke [3] showed that the identity map on the real projective space with the standard metric minimize $\mathcal{E}_2$ in its homotopy class, then Croke and Fathi [2] introduced the new homotopy invariant called the intersection, which gives the lower bound to $\mathcal{E}_2|_{[f]}$ for a given homotopy class $[f]$. Recently, Hoisington [7] give the lower bound to $\mathcal{E}_p$ for an appropriate p in the case of X is real, complex, or quaternionic projective spaces with the standard metrics.

In this article, we generalize the notion of calibrated geometry to smooth maps between smooth manifolds, which give the lower bound to several energies. The origin of calibrated geometry is the Wirtinger's inequality for the even-dimensional subspaces in Hermitian inner product spaces [11], then it refined or generalized by many researchers. In [6], Harvey and Lawson defined calibrated submanifolds in the Calabi–Yau, G_2 or $\mathrm{Spin}(7)$ manifolds which minimize the volume in their homology classes. Similarly, we define the new class of smooth maps, called calibrated maps, and show that they minimize the appropriate energy for the given situation. Moreover, we obtain the next results as applications.

The first application is to obtain the lower bound to p-energy restricting to the given homotopy class. We assume X is oriented. The pullback of f induces a linear map $[f^*]^k : H^k(Y, \mathbb{R}) \to H^k(X, \mathbb{R})$. By fixing basis of $H^k(X, \mathbb{R})$, $H^k(Y, \mathbb{R})$, we obtain the matrix $P([f^*]^k)$ of $[f^*]^k$ and put $|P([f^*]^k)| := \sqrt{\mathrm{tr}({}^t P([f^*]^k) \cdot P([f^*]^k))}$.

Theorem 1.1 *Let (X, g) and (Y, h) be as above. For any $1 \leq k \leq \dim X$, there is a positive constant C depending only on k, (X, g), (Y, h) and the basis of $H^k(X, \mathbb{R})$, $H^k(Y, \mathbb{R})$ such that for any $f \in C^\infty(X, Y)$, we have*

$$\mathcal{E}_k(f) \geq C|P([f^*]^k)|.$$

In particular, if $[f^*]^k$ is nonzero, then $\inf(\mathcal{E}_k|_{[f]})$ is positive.

In the above theorem, the compactness of Y is not essential. See Theorem 4.2.

The second application is to show that the identity maps of some Riemannian manifolds with special holonomy groups minimize the appropriate energy. As we have already mentioned, the identity map on the real or complex projective space minimizes $\mathcal{E}_2$ in its homotopy class by [3] and [8], respectively. It was shown by Wei [9] that the identity map on the quaternionic projective space $\mathbb{HP}^n$ with the standard metric is an unstable critical point of $\mathcal{E}_p$ for $1 \leq p < 2 + 4n/(n+1)$. Moreover, Hoisington gave the nontrivial lower bound of $\mathcal{E}_p|_{[1_{\mathbb{HP}^n}]}$ for $p \geq 4$. Here, the quaternionic projective space is a typical example of quaternionic Kähler manifolds, which are Riemannian manifolds of dimension $4n$ whose holonomy group is contained in $Sp(n) \cdot Sp(1)$. Now, let A be an $n \times m$ real-valued matrix and denote by $a_1, \ldots, a_m \in \mathbb{R}$ the nonnegative eigenvalues of ${}^tAA$, then put $|A|_p := (\sum_{i=1}^m a_i^{p/2})^{1/p}$. Moreover, we define an energy $\mathcal{E}_{p,q}$ by

$$\mathcal{E}_{p,q}(f) := \int_X |\mathrm{d}f|_p^q \mathrm{d}\mu_g,$$

then we have $\mathcal{E}_p = \mathcal{E}_{2,p}$.

Theorem 1.2 *Let (X, g) be a compact quaternionic Kähler manifold of dimension $4n \geq 8$. Then the identity map of X minimizes $\mathcal{E}_{4,4}$ in its homotopy class.*

We can also show the similar theorem in the case of other holonomy groups. If (X, g) is a compact G_2 manifold, then 1_X minimizes $\mathcal{E}_{3,3}|_{[1_X]}$ and if (X, g) is a compact Spin(7) manifold, then 1_X minimizes $\mathcal{E}_{4,4}|_{[1_X]}$ (see Theorem 5.6). Moreover, it is easy to see that if the identity map minimizes $\mathcal{E}_{p,q}$, then it also minimizes $\mathcal{E}_{p',q'}$ for all $p' \geq p$ and $q' \geq q$ by the Hölder's inequality. Of course, we can also consider the case of Kähler, Calabi–Yau, and hyper-Kähler manifolds, respectively; however, the results in these cases also follow from [8].

This paper is organized as follows. In Sect. 2, we define the notion of calibrated maps, which is the analogy of the calibrated submanifolds. In Sect. 3, we explain some examples of calibrated maps. We show that holomorphic maps between Kähler manifolds and the inclusion maps of calibrated submanifolds can be regarded as calibrated maps. Moreover, we can also show that the fibration whose regular fibers are calibrated submanifolds are calibrated maps. We prove Theorem 1.1 in Sect. 4, and Theorem 1.2 in Sect. 5. In Sect. 6, we compare the homotopy invariant introduced in [2] with the invariants defined in this paper.

2 Calibrated Maps

Let X, Y be smooth manifold of $\dim X = m$ and $\dim Y = n$. Throughout of this paper, we suppose X is compact and oriented. We fix a volume form $\mathrm{vol} \in \Omega^m(X)$ on X, namely, a nowhere vanishing m-form which determines an orientation and a measure of X. For m-forms $v_1, v_2 \in \Omega^m(X)$, there are $\varphi_i \subset C^\infty(X)$ with $v_i = \varphi_i \mathrm{vol}$. Then we write $v_1 \leq v_2$ if $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in X$.

If a map $\sigma \colon C^\infty(X, Y) \to L^1(X)$ is given, then we can define an energy $\mathcal{E} \colon C^\infty(X, Y) \to \mathbb{R}$ by

$$\mathcal{E}(f) := \int_X \sigma(f)\mathrm{vol}.$$

Now, $f_0, f_1 \in C^\infty(X, Y)$ are said to be *homotopic* if there is a smooth map $F \colon [0, 1] \times X \to Y$ such that $F(0, \cdot) = f_0$ and $F(1, \cdot) = f_1$. By Whitney approximation theorem, it is equivalent to the existence of the continuous homotopy joining f_0 and f_1. For $f \in C^\infty(X, Y)$, denote by $[f] \subset C^\infty(X, Y)$ the homotopy equivalent class represented by f. In this paper, we consider the lower bound to $\mathcal{E}|_{[f]}$ or the minimum of $\mathcal{E}|_{[f]}$.

Denote by $1_X \colon X \to X$, the identity map on X. We define a smooth map $(1_X, f) \colon X \to X \times Y$ by

$$(1_X, f)(x) := (x, f(x)).$$

The next definition is the analogy of [6].

Definition 2.1 $\Phi \in \Omega^m(X \times Y)$ is a *σ-calibration* if $d\Phi = 0$ and

$$(1_X, f)^*\Phi \leq \sigma(f)\mathrm{vol}$$

for any smooth map $f \colon X \to Y$. Moreover, f is a *(σ, Φ)-calibrated map* if

$$(1_X, f)^*\Phi = \sigma(f)\mathrm{vol}.$$

Theorem 2.2 *Let σ be an energy density and Φ be a σ-calibration.*

(i) *The constant $\int_X (1_X, f)^*\Phi$ is determined by the homotopy class $[f]$. In other words, $\int_X (1_X, f_0)^*\Phi = \int_X (1_X, f_1)^*\Phi$ if $[f_0] = [f_1]$.*
(ii) *We have* $\inf \mathcal{E}|_{[f]} \geq \int_X (1_X, f)^*\Phi$ *for any $f \in C^\infty(X, Y)$.*
(iii) *We have $\mathcal{E}(f) = \int_X (1_X, f)^*\Phi$ iff f is (σ, Φ)-calibrated map. In particular, any (σ, Φ)-calibrated map minimizes $\mathcal{E}$ in its homotopy class.*

Proof (i) If f_0, f_1 are homotopic, then $(1_X, f_0)$ and $(1_X, f_1)$ are homotopic, accordingly $(1_X, f_0)^*\Phi$ and $(1_X, f_1)^*\Phi$ represent the same cohomology class by [1, Corollary 4.1.2].

(ii) follows from the definition of σ-calibration.

(iii) By the point-wise inequality $(1_X, f)^*\Phi \leq \sigma(f)\mathrm{vol}$, we have $\mathcal{E}(f) = \int_X (1_X, f)^*\Phi$ iff $(1_X, f)^*\Phi = \sigma(f)\mathrm{vol}$. □

3 Examples

One of the typical example of the energy of maps is p-energy defined for the smooth maps between Riemannian manifolds. Let (X, g) and (Y, h) be Riemannian manifolds and $f : X \to Y$ be a smooth map. Then the pullback f^*h is a section of $T^*X \otimes T^*X$, and we can take the trace $\mathrm{tr}_g(f^*h)$. For $p \geq 1$, put $\sigma_p(f) := \{\mathrm{tr}_g(f^*h)\}^{p/2}$. We assume that X is oriented and denoted by vol_g the volume form of g. The p-energy $\mathcal{E}_p(f)$ is defined by

$$\mathcal{E}_p(f) := \int_X \sigma_p(f)\mathrm{vol}_g.$$

Now, the differential $\mathrm{d}f_x$ is an element of $T_x^*X \otimes T_{f(x)}Y$ for every $x \in X$. Since g_x and $h_{f(x)}$ induce the natural inner product and the norm on $T_x^*X \otimes T_{f(x)}Y$, then we may also write $\sigma_p(f)(x) = |\mathrm{d}f_x|^p$.

By the Hölder's inequality, we have

$$\mathcal{E}_p(f) \leq \mathrm{vol}_g(X)^{1-p/q}\mathcal{E}_q(f)^{p/q}$$

for $1 \leq p \leq q$. Thus, we have the following proposition.

Proposition 3.1 *Let $\Phi \in \Omega^m(X \times Y)$ be a σ_p-calibration. Then*

$$\mathrm{vol}_g(X)^{-1+p/q} \int_X (1_X, f)^*\Phi \leq \mathcal{E}_q(f)^{p/q}$$

for any $q \geq p$ and $f \in C^\infty(X, Y)$.

3.1 Holomorphic Maps

Here, assume that X, Y are complex manifolds and g, h are Kähler metrics. Let $m = \dim_{\mathbb{C}} X$ and $n = \dim_{\mathbb{C}} Y$. Then we have the decomposition

$$T^*X \otimes \mathbb{C} = \Lambda^{1,0}T^*X \oplus \Lambda^{0,1}T^*X,$$
$$TY \otimes \mathbb{C} = T^{1,0}Y \oplus T^{0,1}Y,$$

accordingly the derivative $\mathrm{d}f \in \Gamma(T^*X \otimes f^*TY)$ is decomposed into

$$\begin{aligned} \mathrm{d}f &= (\partial f)^{1,0} + (\partial f)^{0,1} + (\overline{\partial} f)^{1,0} + (\overline{\partial} f)^{0,1} \\ &\in (\Lambda^{1,0}T^*X \otimes T^{1,0}Y) \oplus (\Lambda^{1,0}T^*X \otimes T^{0,1}Y) \\ &\quad \oplus (\Lambda^{0,1}T^*X \otimes T^{1,0}Y) \oplus (\Lambda^{0,1}T^*X \otimes T^{0,1}Y). \end{aligned}$$

Since $\mathrm{d}f$ is real, we have

$$\overline{(\partial f)^{1,0}} = (\overline{\partial} f)^{0,1}, \quad \overline{(\partial f)^{0,1}} = (\overline{\partial} f)^{1,0}.$$

Denote by ω_g, ω_h the Kähler form of g, h, respectively, then the volume form is given by $\mathrm{vol}_g = \frac{1}{m!}\omega_g^m$. The following observation was given by Lichnerowicz.

Theorem 3.2 [8] *For any smooth map $f : X \to Y$, we have*

$$\omega_g^{m-1} \wedge f^*\omega_h = (m-1)!(|(\partial f)^{1,0}|^2 - |(\overline{\partial} f)^{1,0}|^2)\mathrm{vol}_g,$$
$$|\mathrm{d}f|^2 = 2|(\partial f)^{1,0}|^2 + 2|(\overline{\partial} f)^{1,0}|^2.$$

In particular, we have

$$\mathcal{E}_2(f) \geq \frac{2}{(m-1)!}\int_X \omega_g^{m-1} \wedge f^*\omega_h$$

and the equality holds iff f is holomorphic.

Now, we consider $\omega_g^{m-1} \wedge \omega_h \in \Omega^m(X \times Y)$. The first two equalities in Theorem 3.2 implies that $\frac{2}{(m-1)!}\omega_g^{m-1} \wedge \omega_h$ is a σ_2-calibration. Moreover, the second statement implies that f is a $(\sigma_2, \frac{2}{(m-1)!}\omega_g^{m-1} \wedge \omega_h)$-calibrated map iff f is holomorphic. One can also see that f is $(\sigma_2, -\frac{2}{(m-1)!}\omega_g^{m-1} \wedge \omega_h)$-calibrated map iff f is anti-holomorphic.

3.2 Calibrated Submanifolds

In this subsection, we see the relation between the calibrated submanifolds in the sense of [6] and the calibrated maps. We assume (Y^n, h) is a Riemannian manifold.

Definition 3.3 [6] For an integer $0 < m < n$, $\psi \in \Omega^m(Y)$ is a *calibration* if $\mathrm{d}\psi = 0$ and

$$\psi|_V \leq \mathrm{vol}_{h|_V}$$

for any $y \in Y$ and m-dimensional oriented subspace $V \subset T_yY$. Here, $h|_V$ is the induced metric on V and $\mathrm{vol}_{h|_V}$ is its volume form whose orientation is compatible with the one equipped with V. Moreover, an oriented submanifold $X \subset Y$ is a *calibrated submanifold* if

$$\psi|_{T_xX} = \mathrm{vol}|_{h|_{T_xX}}$$

for any $x \in X$.

Now, if X is an oriented manifold with a volume form $\mathrm{vol} \in \Omega^m(X)$, then for every linear map, $A : T_xX \to T_yY$ can be regarded as an $n \times m$-matrix by taking a basis $e_1, \ldots, e_m$ of T_xX and an orthonormal basis of T_yY with $\mathrm{vol}_x(e_1, \ldots, e_m) = 1$. Then $\sqrt{\det({}^tA \cdot A)}$ does not depend on the choice of these basis. Therefore, for $f \in C^\infty(X, Y)$, we can define the energy density $\tau_m(f)(x) := \sqrt{\det({}^t\mathrm{d}f_x \cdot \mathrm{d}f_x)}$ and the energy $\mathcal{E}_{\tau_m}(f) := \int_X \tau_m(f)\mathrm{vol}$.

Proposition 3.4 *Let* (X, vol) *be an oriented manifold equipped with a volume form and* $\psi \in \Omega^m(Y)$ *be closed. Assume that* $\dim_{\mathbb{R}} X = m < n$ *and denote by* $\pi_Y\colon X \times Y \to Y$ *the natural projection. Then* ψ *is a calibration iff* $\pi_Y^* \psi \in \Omega^m(X \times Y)$ *is a* τ_m*-calibration. Moreover, for any embedding* $f\colon X \to Y$*, the following conditions are equivalent.*

(i) $f(X)$ *is a calibrated submanifold, where the orientation of* $f(X)$ *is determined such that* f *preserves the orientation.*
(ii) f *is a* $(\tau_m, \pi_Y^* \psi)$*-calibrated map.*

Proof Note that $(1_X, f)^*(\pi_Y^* \psi) = f^* \psi$ and $\tau_m(f)\mathrm{vol}_g = \mathrm{vol}_{f^* h}$. Hence ψ is a calibration iff $\pi_Y^* \psi \in \Omega^m(X \times Y)$ is a τ_m-calibration. Moreover, suppose that f is an embedding. Then f is a $(\tau_m, \pi_Y^* \psi)$-calibrated map iff $f(X)$ is a calibrated submanifold. □

3.3 Fibrations

Let (X^m, g) be an oriented Riemannian manifold and Y^n be a smooth manifold equipped with a volume form $\mathrm{vol}_Y \in \Omega^n(Y)$. Here, we suppose $n < m$ and let $\varphi \in \Omega^{m-n}$ be a calibration in the sense of Definition 3.3. Fix an orthonormal basis of $T_x X$ and a basis $e'_1, \dots, e'_n \in T_y Y$ with $\mathrm{vol}_Y(e'_1, \dots, e'_n) = 1$, we can regard a linear map $A\colon T_x X \to T_y Y$ as an $m \times n$-matrix. Then the value of $\sqrt{\det(A \cdot {}^t A)}$ does not depend on the choice of above basis. For a smooth map $f\colon X \to Y$, put $\tilde{\tau}_{m,n}(f)|_x := \sqrt{\det(\mathrm{d} f_x \cdot {}^t \mathrm{d} f_x)}$ and $\Phi := \mathrm{vol}_Y \wedge \varphi$.

Put

$$X_{\mathrm{reg}} := \{x \in X \,|\, x \text{ is a regular point of } f\}.$$

Note that X_{reg} is open in X. If $x \in X_{\mathrm{reg}}$, we have the orthogonal decomposition $T_x X = \mathrm{Ker}(\mathrm{d} f_x) \oplus H$ and $\mathrm{d} f_x|_H\colon H \to T_{f(x)} Y$ is a linear isomorphism. Put $y = f(x)$ and suppose that $f^{-1}(y)$ is a calibrated submanifold with respect to the suitable orientation. We say that $\mathrm{d} f_x$ is *orientation preserving* if there is a basis $v_1, \dots, v_m$ of $T_x X$ such that

$$\begin{aligned}
&v_1, \dots, v_n \in H, \quad \mathrm{vol}_Y(\mathrm{d} f_x(v_1), \dots, \mathrm{d} f_x(v_n)) > 0,\\
&v_{n+1}, \dots, v_m \in \mathrm{Ker}(\mathrm{d} f_x), \quad \varphi_x(v_{n+1}, \dots, v_m) > 0,\\
&\mathrm{vol}_g(v_1, \dots, v_m) > 0.
\end{aligned}$$

Proposition 3.5 Φ *is a* $\tilde{\tau}_{m,n}$*-calibration. Moreover, a smooth map* $f\colon X \to Y$ *is a* $(\tilde{\tau}_{m,n}, \Phi)$*-calibrated map iff*

(i) $f^{-1}(y) \cap X_{\mathrm{reg}}$ *is a calibrated submanifold with respect to* φ *and the suitable orientation for any* $y \in Y$*,*
(ii) $\mathrm{d} f_x$ *is orientation preserving for any* $x \in X_{\mathrm{reg}}$.

Proof If $x \in X$ is a critical point of f, then we can see

$$f^*\mathrm{vol}_Y \wedge \varphi|_x = \tilde{\tau}_{m,n}(f)\mathrm{vol}_g|_x = 0.$$

Fix a regular point x and an oriented orthonormal basis $e_1, \ldots, e_m \in T_x X$ such that $e_{m-n+1}, \ldots, e_m \in \mathrm{Ker}(\mathrm{d}f_x)$. Then we have

$$f^*\mathrm{vol}_Y \wedge \varphi(e_1, \ldots, e_m) = \mathrm{vol}_Y(\mathrm{d}f_x(e_1), \ldots, \mathrm{d}f_x(e_n))\varphi(e_{n+1}, \ldots, e_m),$$
$$\tilde{\tau}_{m,n}(f)|_x = |\mathrm{vol}_Y(\mathrm{d}f_x(e_1), \ldots, \mathrm{d}f_x(e_n))|\,.$$

Since φ is a calibration, we have $\varphi(\pm e_{n+1}, e_{n+2}, \ldots, e_m) \leq 1$, hence $|\varphi(e_{n+1}, \ldots, e_m)| \leq 1$. Therefore,

$$f^*\mathrm{vol}_Y \wedge \varphi(e_1, \ldots, e_m) \leq |\mathrm{vol}_Y(\mathrm{d}f_x(e_1), \ldots, \mathrm{d}f_x(e_{m-n}))| = \tilde{\tau}_{m,n}(f)|_x,$$

which implies that Φ is a $\tilde{\tau}_{m,n}$-calibration.

Next we consider the condition

$$f^*\mathrm{vol}_Y \wedge \varphi|_x = \tilde{\tau}_{m,n}(f)\mathrm{vol}_g|_x,$$

where x is a regular value of f. In this case, we have the orthogonal decomposition $T_x X = \mathrm{Ker}(\mathrm{d}f_x) \oplus H$, where H is an n-dimensional subspace. We can take an orthonormal basis $e_1, \ldots, e_m \in T_x X$ such that

$$e_1, \ldots, e_n \in H,$$
$$e_{n+1}, \ldots, e_m \in \mathrm{Ker}(\mathrm{d}f_x),$$
$$a := \mathrm{vol}_Y(\mathrm{d}f_x(e_1), \ldots, \mathrm{d}f_x(e_{m-n})) > 0,$$
$$\mathrm{vol}_g(e_1, \ldots, e_m) > 0.$$

Then we have

$$f^*\mathrm{vol}_Y \wedge \varphi(e_1, \ldots, e_m) = a\varphi(e_{n+1}, \ldots, e_m),$$
$$\tilde{\tau}_{m,n}(f)\mathrm{vol}_g(e_1, \ldots, e_m) = |a| = a.$$

Therefore, we have

$$f^*\mathrm{vol}_Y \wedge \varphi|_x = \tilde{\tau}_{m,n}(f)\mathrm{vol}_g|_x$$

iff $\varphi(e_{n+1}, \ldots, e_m) = 1$. Now we have taken $x \in X_{\mathrm{reg}}$ arbitrarily, hence we have

$$f^*\mathrm{vol}_Y \wedge \varphi = \tilde{\tau}_{m,n}(f)\mathrm{vol}_g$$

iff $f^{-1}(y) \cap X_{\mathrm{reg}}$ is a calibrated submanifold for any $y \in Y$ and $\mathrm{d}f_x$ is orientation preserving for all $x \in X_{\mathrm{reg}}$. □

3.4 Totally Geodesic Maps Between Tori

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus and we consider smooth maps from $\mathbb{T}^m$ to $\mathbb{T}^n$. Let $G = (g_{ij}) \in M_m(\mathbb{R})$ and $H = (h_{ij}) \in M_n(\mathbb{R})$ be positive symmetric matrices. Denote by $x = (x^1, \ldots, x^m)$ and $y = (y^1, \ldots, y^n)$ the Cartesian coordinate on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, then we have closed 1-forms $\mathrm{d}x^i \in \Omega^1(\mathbb{T}^m)$ and $\mathrm{d}y^i \in \Omega^1(\mathbb{T}^n)$. We define the flat Riemannian metrics $g = \sum_{i,j} g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ on $\mathbb{T}^m$ and $h = \sum_{i,j} h_{ij}\mathrm{d}y^i \otimes \mathrm{d}y^j$ on $\mathbb{T}^n$.

For a smooth map $f : \mathbb{T}^m \to \mathbb{T}^n$, we have the pullback $f^* : H^1(\mathbb{T}^n, \mathbb{R}) \to H^1(\mathbb{T}^m, \mathbb{R})$. Here, since

$$
\begin{aligned}
H^1(\mathbb{T}^m, \mathbb{Z}) &= \mathrm{span}_{\mathbb{Z}}\{[\mathrm{d}x^1], \ldots, [\mathrm{d}x^m]\},\\
H^1(\mathbb{T}^n, \mathbb{Z}) &= \mathrm{span}_{\mathbb{Z}}\{[\mathrm{d}y^1], \ldots, [\mathrm{d}y^n]\},
\end{aligned}
$$

there is $P = (P_i^j) \in M_{m,n}(\mathbb{Z})$ such that $f^*[\mathrm{d}y^j] = \sum_i P_i^j[\mathrm{d}x^i]$. The matrix P is determined by the homotopy class of f. Now, let $*_g$ be the Hodge star operator of g and put

$$
\Phi := \sum_{i,j,k} h_{jk} P_i^j *_g \mathrm{d}x^i \wedge \mathrm{d}y^k \in \Omega^m(\mathbb{T}^m \times \mathbb{T}^n). \tag{1}
$$

Then we can check that

$$
\begin{aligned}
\int_{\mathbb{T}^m} (1_{\mathbb{T}^m}, f)^*\Phi &= \sum_{i,j,k,l} h_{jk} P_i^j P_l^k \int_{\mathbb{T}^m} *_g \mathrm{d}x^i \wedge \mathrm{d}x^l\\
&= \sum_{i,j,k,l} h_{jk} P_i^j P_l^k g^{il} \mathrm{vol}_g(\mathbb{T}^m)\\
&= \mathrm{tr}({}^t P G^{-1} P H)\mathrm{vol}_g(\mathbb{T}^m) =: \|P\|^2 \mathrm{vol}_g(\mathbb{T}^m) \geq 0.
\end{aligned}
$$

Consequently, by the positivity of G^{-1} and H, $\int_{\mathbb{T}^m} (1_{\mathbb{T}^m}, f)^*\Phi = 0$ iff $P = 0$.

Proposition 3.6 *Assume that $f^* : H^1(\mathbb{T}^n, \mathbb{R}) \to H^1(\mathbb{T}^m, \mathbb{R})$ is not the zero map. Then*

(i) $\|P\|^{-1}\Phi$ *is a σ_1-calibration,*
(ii) f *is a $(\sigma_1, \|P\|^{-1}\Phi)$-calibrated map if $f(x) = Px + a$ for some $a \in \mathbb{T}^m$,*
(iii) f *minimizes $\mathcal{E}_2$ in its homotopy class iff $f(x) = Px + a$ for some $a \in \mathbb{T}^m$.*

Proof We fix $x \in \mathbb{T}^m$ and put $\mathrm{d}f_x := A = (A_i^j) \in M_{n,m}(\mathbb{R})$, and show $(1_{\mathbb{T}^m}, f)^*\Phi \leq \sigma_1(f)\mathrm{vol}_g$ at x. Since

$$\begin{aligned}(1_{\mathbb{T}^m}, f)^*\Phi|_x &= \sum_{i,j,k,l} h_{jk} P_i^j A_l^k *_g \mathrm{d}x^i \wedge \mathrm{d}x^l|_x \\ &= \left(\sum_{i,j,k,l} h_{jk} P_i^j A_l^k g^{il}\right) \mathrm{vol}_g|_x \\ &= \left(\mathrm{tr}({}^t P G^{-1} A H)\right) \mathrm{vol}_g|_x.\end{aligned}$$

Here, by the Cauchy–Schwarz inequality, we have

$$\mathrm{tr}({}^t P G^{-1} A H) \leq \sqrt{\|P\|\|A\|},$$

and the equality holds iff $A = \lambda P$ for a constant $\lambda \geq 0$. Therefore, we have

$$(1_{\mathbb{T}^m}, f)^*\Phi \leq \|P\|\sigma_1(f)\mathrm{vol}_g,$$

which implies that $\|P\|^{-1}\Phi$ is a σ_1-calibration. Moreover, the equality holds iff $\mathrm{d}f_x = \lambda_x \cdot {}^t P$ for some $\lambda_x \geq 0$. Therefore, $f(x) = {}^t Px + a$ for some $a \in \mathbb{T}^m$ is a $(\sigma_1, \|P\|^{-1}\Phi)$-calibrated map.

For any $f \in C^\infty(\mathbb{T}^m, \mathbb{T}^n)$, we have

$$\int_X (1_{\mathbb{T}^m}, f)^*\Phi \leq \|P\| \int_X \sigma_1(f)\mathrm{vol}_g \leq \|P\|\sqrt{\mathrm{vol}_g(\mathbb{T}^m)\mathcal{E}_2(f)}$$

by the Cauchy–Schwartz inequality. Moreover, we have the following equality

$$\int_X (1_{\mathbb{T}^m}, f)^*\Phi = \|P\|\sqrt{\mathrm{vol}_g(\mathbb{T}^m)\mathcal{E}_2(f)}$$

iff $\mathrm{d}f_x = \lambda_x \cdot {}^t P$ for some $\lambda_x \geq 0$ and $\sigma_1(f)$ is a constant function on $\mathbb{T}^m$. Since $\sigma_1(f)(x) = \lambda_x\|P\|$, if $\sigma_1(f)$ is constant, then $\lambda_x = \lambda$ is independent of x. Hence we may write $f(x) = \lambda \cdot {}^t Px + a$ for some $a \in \mathbb{T}^m$. Moreover, since $f^* = P$ on $H^1(\mathbb{T}^n)$, we have $\lambda = 1$. □

In the above proposition, we cannot show that every $(\sigma_1, \|P\|^{-1}\Phi)$-calibrated map is given by $f(x) = {}^t Px + a$ for some a. We can give a counterexample as follows.

Suppose $m = n = 1$ and let $P = 1$. If we put $f(x) = x + \frac{1}{2\pi}\sin(2\pi x)$, then it gives a smooth map $\mathbb{T}^1 \to \mathbb{T}^1$ homotopic to the identity map. Then one can check that f is a $(\sigma_1, \|P\|^{-1}\Phi)$-calibrated map since $f'(x) \geq 0$.

4 The Lower Bound of *p*-Energy

In this section, we give the lower bound of p-energy in the general situation. Let (X, g) and (Y, h) be Riemannian manifolds and assume X is compact and oriented. Now we have the decomposition

$$\Lambda^k T^*_{(x,y)}(X \times Y) \cong \bigoplus_{l=0}^{k} \Lambda^l T^*_x X \otimes \Lambda^{k-l} T^*_y Y,$$

then denote by $\Omega^{l,k-l}(X \times Y) \subset \Omega^k(X \times Y)$ the set consisting of smooth sections of $\Lambda^l T^*_x X \otimes \Lambda^{k-l} T^*_y Y$. For $\Phi \in \Omega^k(X \times Y)$, let $|\Phi_{(x,y)}|$ be the norm with respect to the metric $g \oplus h$ on $X \times Y$.

Lemma 4.1 *Let $\Phi \in \Omega^{m-k,k}(X \times Y)$ be closed and $\sup_{x,y} |\Phi_{(x,y)}| < \infty$. Then there is a constant $C > 0$ depending only on Φ, m, n, k such that $C\Phi$ is a σ_k-calibration.*

Proof Fix $x \in X$ and let $\{e_1, \ldots, e_m\}$ and $\{e'_1, \ldots, e'_n\}$ be an orthonormal basis of $T_x X$ and $T_{f(x)} Y$, respectively. Put

$$\mathcal{I}^m_k := \left\{ I = (i_1, \ldots, i_k) \in \mathbb{Z}^k \mid 0 \le i_1 < \cdots < i_k \le m \right\}.$$

For $I = (i_1, \ldots, i_k) \in \mathcal{I}^m_k$, $J = (j_1, \ldots, j_k) \in \mathcal{I}^n_k$, we write

$$e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad e'_J := e'_{j_1} \wedge \cdots \wedge e'_{j_k}.$$

Then we have

$$\Phi_{(x,f(x))} = \sum_{I \in \mathcal{I}^m_k, J \in \mathcal{I}^n_k} \Phi_{IJ} (*_g e_I) \wedge e'_J$$

for some $\Phi_{IJ} \in \mathbb{R}$ and

$$\left\{(1_X, f)^* \Phi\right\}_x = \sum_{I,J} \Phi_{IJ} (*_g e_I) \wedge \mathrm{d} f^*_x e'_J.$$

If we denote by $(\mathrm{d} f_x)_{IJ}$ the $k \times k$ matrix whose (p, q)-component is given by $g(\mathrm{d} f_x(e_{i_q}), e'_{j_p})$, then we have

$$(*_g e_I) \wedge \mathrm{d} f^*_x e'_J = \det((\mathrm{d} f_x)_{IJ}) \mathrm{vol}_g|_x \le k! |\mathrm{d} f_x|^k \mathrm{vol}_g|_x,$$

therefore, we can see

$$\left\{(1_X, f)^* \Phi\right\}_x \le \left(\sum_{I,J} |\Phi_{IJ}| \right) k! |\mathrm{d} f_x|^k \mathrm{vol}_g|_x.$$

Since $|\Phi_{x,f(x)}|^2 = \sum_{I,J} |\Phi_{I,J}|^2$, we have

$$(1_X, f)^*\Phi \le k!(\#\mathcal{I}_k^m)(\#\mathcal{I}_k^n) \sup_{x,y} |\Phi_{(x,y)}|\sigma_k(f)\mathrm{vol}_g,$$

which implies the assertion. □

For $f \in C^\infty(X, Y)$, denote by $[f^*]^k$ the pullback $H^k(Y, \mathbb{R}) \to H^k(X, \mathbb{R})$ of f. For a closed form $\alpha \in \Omega^k(Y)$, denote by $[\alpha] \in H^k(Y, \mathbb{R})$ its cohomology class. Put

$$H^k_{\mathrm{bdd}}(Y, \mathbb{R}) := \left\{ [\alpha] \in H^k(Y, \mathbb{R}) | \, \alpha \in \Omega^k(Y), \ \mathrm{d}\alpha = 0, \ \sup_{y \in Y} h(\alpha_y, \alpha_y) < \infty \right\}.$$

This is a subspace of $H^k(Y, \mathbb{R})$, and we have $H^k_{\mathrm{bdd}}(Y, \mathbb{R}) = H^k(Y, \mathbb{R})$ if Y is compact. Denote by $[f^*]^k_{\mathrm{bdd}}$ the restriction of $[f^*]^k$ to $H^k_{\mathrm{bdd}}(Y, \mathbb{R})$. Fixing a basis of $H^k(X, \mathbb{R})$ and $H^k_{\mathrm{bdd}}(Y, \mathbb{R})$, we obtain the matrix $P = P([f^*]^k_{\mathrm{bdd}}) \in M_{N,d}(\mathbb{R})$ of $[f^*]^k_{\mathrm{bdd}}$, where $d = \dim H^k_{\mathrm{bdd}}(Y, \mathbb{R})$ and $N = \dim H^k(X, \mathbb{R})$. Put $|P| := \sqrt{\mathrm{tr}({}^tPP)}$, which may depends on the choice of basis. Here, since d may become infinity, we may have $|P| = \infty$.

Theorem 4.2 *Let (X^m, g) and (Y^n, h) be Riemannian manifolds and X be compact and oriented. For any $1 \le k \le m$, there is a constant $C > 0$ depending only on k, (X, g), and (Y, h) and the basis of $H^k(X, \mathbb{R})$, $H^k_{\mathrm{bdd}}(Y, \mathbb{R})$ such that for any $f \in C^\infty(X, Y)$, we have*

$$\mathcal{E}_k(f) \ge C|P([f^*]^k_{\mathrm{bdd}})|.$$

In particular, if $[f^*]^k_{\mathrm{bdd}}$ is a nonzero map, then the infimum of $\mathcal{E}_k|_{[f]}$ is positive.

Proof Take bounded closed k-forms $\beta_1, \dots, \beta_d \in \Omega^k(Y)$ such that $\{[\beta_l]\}_l$ is a basis of $H^k_{\mathrm{bdd}}(Y, \mathbb{R})$.

By the Hodge Theory, $H^k(X)$ is isomorphic to the space of harmonic k-forms as vector spaces. Therefore, for any basis of $H^k(X, \mathbb{R})$, there is a corresponding basis $\alpha_1, \dots, \alpha_N \in \Omega^k(X)$ of the space of harmonic k-forms. Let $G_{ij} := \int_X \alpha_i \wedge *_g\alpha_j$, which is symmetric positive definite.

Define $P = (P_{ij}) \in M_{N,d}(\mathbb{R})$ by $[f^*]^k_{\mathrm{bdd}}([\beta_j]) = \sum_i P_{ij}[\alpha_i]$. If we put

$$\Phi := \sum_{i,j} P_{ij}\beta_j \wedge (*_g\alpha_i),$$

then every $\beta_j \wedge (*_g\alpha_i)$ is closed and satisfies the assumption of Lemma 4.1, since X is compact and β_j is bounded. Take the constant $C_{ij} > 0$ as in Lemma 4.1. Here, C_{ij}

is depending only on m, n, k, and α_i, β_j. Then for any $f \in C^\infty(X, Y)$, we have

$$(1_X, f)^* \left\{\beta_j \wedge (*_g \alpha_i)\right\} \leq C_{ij} \sigma_k(f) \mathrm{vol}_g,$$

$$\begin{aligned}(1_X, f)^* \Phi &\leq \sum_{i,j} C_{ij} |P_{ij}| \sigma_k(f) \mathrm{vol}_g \\ &\leq \sqrt{\sum_{i,j} C_{ij}^2} |P| \sigma_k(f) \mathrm{vol}_g,\end{aligned}$$

hence

$$\mathcal{E}_k(f) \geq \left(\sum_{i,j} C_{ij}^2\right)^{-1/2} |P|^{-1} \int_X (1_X, f)^* \Phi.$$

Moreover, we have

$$\begin{aligned}\int_X (1_X, f)^* \Phi &= \sum_{i,j} \int_X P_{ij} f^* \beta_j \wedge (*_g \alpha_i) \\ &= \sum_{i,j} \int_X P_{ij} \sum_k P_{kj} \alpha_k \wedge (*_g \alpha_i) \\ &= \sum_{i,j,k} P_{ij} P_{kj} G_{ki}.\end{aligned}$$

If we denote by $\lambda > 0$ the minimum eigenvalue of $(G_{ij})_{i,j}$, then we have $\sum_{i,j,k} P_{ij} P_{kj} G_{ki} \geq \lambda |P|^2$. Hence we obtain

$$\mathcal{E}_k(f) \geq \lambda \left(\sum_{i,j} C_{ij}^2\right)^{-1/2} |P|.$$

□

Remark 4.3 Combining the above theorem with Proposition 3.1, we also have the lower bound of $\mathcal{E}_p$ for any $p \geq k$.

5 Energy of the Identity Maps

In this section, we consider when the identity map on compact oriented Riemannian manifold X minimizes the energy. Here, we consider the family of energies. For Riemannian manifolds (X^m, g), (Y^n, h) and points $x \in X$, $y \in Y$, take a linear map $A\colon T_x X \to T_y Y$. Fixing orthonormal basis of $T_x X$ and $T_y Y$, we can regard A as an

$n \times m$-matirx. Denote by $a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}$ the eigenvalues of ${}^t A \cdot A$, then put

$$|A|_p := \left(\sum_{i=1}^{m} a_i^{p/2}\right)^{1/p}$$

for $p > 0$. Then $|A|_p$ is independent of the choice of the orthonormal basis of $T_x X$. For a smooth map $f : X \to Y$, let

$$\sigma_{p,q}(f)|_x := |\mathrm{d} f_x|_p^q,$$
$$\mathcal{E}_{p,q}(f) := \int_X \sigma_{p,q}(f)\mathrm{vol}_g.$$

Note that $\sigma_{2,p} = \sigma_p$ and $\mathcal{E}_{2,p} = \mathcal{E}_p$.

From now onward, we consider $(Y, h) = (X, g)$ and a map $f : X \to X$. Let 1_X be the identity map of X.

Proposition 5.1 *If 1_X minimizes $\mathcal{E}_{p,q}|_{[1_X]}$, then it also minimizes $\mathcal{E}_{p',q'}|_{[1_X]}$ for any $p' \geq p$ and $q' \geq q$.*

Proof First of all, for any smooth map f, we have

$$|\mathrm{d} f_x|_p \leq m^{1/p-1/p'}|\mathrm{d} f_x|_{p'},$$
$$\mathcal{E}_{p,q}(f) \leq m^{q/p-q/p'}\mathrm{vol}_g(X)^{1-q/q'}\left(\int_X |\mathrm{d} f|_{p'}^{q'}\mathrm{vol}_g\right)^{q/q'},$$

by the Hölder's inequality, which gives $\mathcal{E}_{p',q'}(f) \geq C\mathcal{E}_{p,q}(f)^{q'/q}$ for some constant $C > 0$. Moreover, we have the equality for $f = 1_X$. Therefore, we can see

$$\inf \mathcal{E}_{p',q'}|_{[1_X]} \geq \inf C\mathcal{E}_{p,q}^{q'/q}|_{[1_X]} = C\mathcal{E}_{p,q}(1_X)^{q'/q} = \mathcal{E}_{p',q'}(1_X) \geq \inf \mathcal{E}_{p',q'}|_{[1_X]}.$$

□

Proposition 5.2 (cf. [7, Lemma 2.2]) *Let (X, g) be a compact oriented Riemannian manifold of dimension m. Then 1_X minimizes $\mathcal{E}_{1,m}$ in its homotopy class.*

Proof The proof is essentially given by [7, Lemma 2.2]. For any map $f : X \to X$, we can see

$$f^*\mathrm{vol}_g = \det(\mathrm{d} f)\mathrm{vol}_g \leq m^{-m}\sigma_{1,m}(f)\mathrm{vol}_g.$$

Here, the second inequality follows from the inequality

$$\frac{\sum_{i=1}^m a_i}{m} \geq \left(\prod_{i=1}^m a_i\right)^{1/m}$$

for any $a_i \geq 0$. Therefore, we can see

$$\mathcal{E}_{1,m}(f) \geq m^m \int_X f^* \mathrm{vol}_g.$$

Moreover, the equality holds if $f = 1_X$. □

Next we consider the analogy of the above proposition. We assume that X has a nontrivial parallel k-form.

Denote by g_0 the standard metric on $\mathbb{R}^m$, which also induces the metric on $\Lambda^k(\mathbb{R}^m)^*$. Let $\varphi_0 \in \Lambda^k(\mathbb{R}^m)^*$ and fix an orientation of $\mathbb{R}^m$. For a k-form φ and a Riemannian metric g on an oriented manifold X, we say that (g_0, φ_0) *is a local model of* (g, φ) if for any $x \in X$ there is an orientation preserving isometry $I : \mathbb{R}^m \to T_x X$ such that $I^*(\varphi|_x) = \varphi_0$.

Denote by $*_{g_0} : \Lambda^k(\mathbb{R}^m)^* \to \Lambda^{m-k}(\mathbb{R}^m)^*$ the Hodge star operator induced by the standard metric and let $\mathrm{vol}_{g_0} \in \Lambda^m(\mathbb{R}^m)^*$ be the volume form. First of all, we show the following proposition for the local model (g_0, φ_0).

Proposition 5.3 *Let (g_0, φ_0) be as above. Assume that $|\iota_u \varphi_0|_{g_0}$ is independent of $u \in \mathbb{R}^m$ if $|u|_{g_0} = 1$. We have*

$$A^*\varphi_0 \wedge *_{g_0}\varphi_0 \leq \frac{|\varphi_0|^2_{g_0}}{m} |A|^k_k \mathrm{vol}_{g_0}$$

for any $A \in M_m(\mathbb{R})$. Moreover, if $A = \lambda T$ for $\lambda \in \mathbb{R}$, $T \in O(m)$ and $A^*\varphi_0 = \lambda'\varphi_0$ for some $\lambda' \geq 0$, then we have the equality.

Proof For any A, we can take oriented orthonormal basis $\{e_1, \ldots, e_m\}$ and $e'_1, \ldots, e'_m$ of $(\mathbb{R}^m)^*$ such that $A^*e'_i = a_i e_i$ for some $a_i \in \mathbb{R}$. We put

$$\varphi_0 = \sum_{I \in \mathcal{I}^m_k} F_I e_I = \sum_{I \in \mathcal{I}^m_k} F'_I e'_I$$

for some $F_I, F'_I \in \mathbb{R}$. Now, put $a_I := a_{i_1} \cdots a_{i_k}$ for $I = (i_1, \ldots, i_k) \in \mathcal{I}^m_k$. The we have $A^*\varphi_0 = \sum_I F'_I a_I e_I$ and

$$\begin{aligned} A^*\varphi_0 \wedge *_{g_0}\varphi_0 = g_0(A^*\varphi_0, \varphi_0)\mathrm{vol}_{g_0} &= \sum_I F_I F'_I a_I \mathrm{vol}_{g_0} \\ &\leq \sum_I |F_I F'_I||a_I| \mathrm{vol}_{g_0}. \end{aligned}$$

If we put $\{I\} := \{i_1, \ldots, i_k\}$, then

$$|a_I| = \left(|a_{i_1}|^k \cdots |a_{i_k}|^k\right)^{1/k} \leq \frac{1}{k} \sum_{j \in \{I\}} |a_j|^k,$$

therefore, we obtain

$$\begin{aligned}\sum_I |F_I F_I'||a_I| &\le \frac{1}{k}\sum_I |F_I F_I'| \sum_{j\in\{I\}} |a_j|^k\\ &= \frac{1}{k}\sum_{j=1}^m |a_j|^k \sum_{I\in\mathcal{I}_k^m,\, j\in\{I\}} |F_I F_I'|.\end{aligned}$$

Denote by $\hat{g}_0\colon (\mathbb{R}^m)^* \to \mathbb{R}^m$ the isomorphism induced by the metric g_0. Put

$$\varphi_1 := \sum_{I\in\mathcal{I}_k^m} |F_I|e_I,\quad \varphi_2 := \sum_{I\in\mathcal{I}_k^m} |F_I'|e_I$$

and define an orthogonal matrix $U\colon \mathbb{R}^m \to \mathbb{R}^m$ by $U\circ\hat{g}_0(e_j) = \hat{g}_0(e_j')$. Now we can see

$$\begin{aligned}\sum_{I\in\mathcal{I}_k^m,\, j\in\{I\}} |F_I F_I'| = g_0\left(\iota_{\hat{g}_0(e_j)}\varphi_1, \iota_{\hat{g}_0(e_j)}\varphi_2\right) &\le \left|\iota_{\hat{g}_0(e_j)}\varphi_1\right|_{g_0}\cdot\left|\iota_{\hat{g}_0(e_j)}\varphi_2\right|_{g_0}\\ &= \left|\iota_{\hat{g}_0(e_j)}\varphi_0\right|_{g_0}\cdot\left|\iota_{\hat{g}_0(e_j)}(U^*\varphi_0)\right|_{g_0}\end{aligned}$$

and

$$\left|\iota_{\hat{g}_0(e_j)}(U^*\varphi_0)\right|_{g_0} = \left|U^*(\iota_{U\circ\hat{g}_0(e_j)}\varphi_0)\right|_{g_0} = \left|\iota_{U\circ\hat{g}_0(e_j)}\varphi_0\right|_{g_0}.$$

Then by the assumption, we can see that $C = \left|\iota_{\hat{g}_0(e_j)}\varphi_0\right|_{g_0} = \left|\iota_{U\circ\hat{g}_0(e_j)}\varphi_0\right|_{g_0}$ is independent of j, therefore, we have $\sum_{I\in\mathcal{I}_k^m,\, j\in\{I\}} |F_I F_I'| \le C^2$ and

$$A^*\varphi_0 \wedge *_{g_0}\varphi_0 \le \frac{C^2}{k}\sum_{j=1}^m |a_j|^k \mathrm{vol}_{g_0} = \frac{C^2}{k}|A|_k^k \mathrm{vol}_{g_0}.$$

In the above inequalities, we have the equality if $A = 1_m$, then we can determine the constant C. Moreover, we can also check that the equality holds if $A = \lambda T$, where $\lambda \in \mathbb{R}$, $T \in O(m)$ and $A^*\varphi_0 = \lambda'\varphi_0$ for some $\lambda' \ge 0$. □

Proposition 5.4 *Let (X^m, g) be a compact oriented Riemannian manifold and $\varphi \in \Omega^k(X)$ be a harmonic form. Assume that there is a local model (g_0, φ_0) of (g, φ) and $|\iota_u\varphi_0|_{g_0}$ is independent of $u \in \mathbb{R}^m$ if $|u|_{g_0} = 1$. Denote by $\mathrm{pr}_i\colon X\times X \to X$ the projection to i-th component for $i = 1, 2$. Then $\Phi = m|\varphi_0|_{g_0}^{-2}\mathrm{pr}_2^*\varphi \wedge \mathrm{pr}_1^*(*_g\varphi)$ is an $\sigma_{k,k}$-calibration. Moreover, any isometry $f\colon X \to X$ with $f^*\varphi = \varphi$ is $(\sigma_{k,k}, \Phi)$-calibrated.*

Proof Φ is an $\sigma_{k,k}$-calibration iff

$$f^*\varphi \wedge *_g\varphi \leq \frac{|\varphi_0|_{g_0}^2}{m}|\mathrm{d}f|_k^k \mathrm{vol}_g.$$

By putting $A = \mathrm{d}f_x$ and identifying $\mathbb{R}^m \cong T_xX$, this is equivalent to the inequality in Proposition 5. Moreover, the equality holds if $(\mathrm{d}f_x)^*\varphi|_x = \varphi|_x$ for all $x \in X$ and $\mathrm{d}f_x$ is isometry. □

Next we have to consider when the assumption for (g_0, φ_0) is satisfied. If $G \subset SO(m)$ is a closed subgroup, then the linear action of $SO(m)$ on $\mathbb{R}^m$ induces the action of G on $\mathbb{R}^m$. Similarly, since $SO(m)$ acts on $\Lambda^k(\mathbb{R}^m)^*$ for all k, G also acts on them. Here, $\mathbb{R}^m$ is *irreducible as a G-representation* if any subspace $W \subset \mathbb{R}^m$ which is closed under the G-action is equal to $\mathbb{R}^m$ or $\{0\}$. For $\varphi_0 \in \Lambda^k(\mathbb{R}^m)^*$, denote by $\mathrm{Stab}(\varphi_0) \subset SO(m)$ the stabilizer of φ_0.

Lemma 5.5 *Let G be a closed subgroup of $SO(m)$ and assume that $\mathbb{R}^m$ is irreducible as a G-representation. Moreover, assume that $G \subset \mathrm{Stab}(\varphi_0)$. Then $|\iota_u\varphi_0|_{g_0}$ is independent of $u \in \mathbb{R}^m$ if $|u|_{g_0} = 1$.*

Proof Define a linear map $\Psi\colon \mathbb{R}^m \to \Lambda^{k-1}(\mathbb{R}^m)^*$ by $\Psi(u) := \iota_u\varphi_0$, then we can see Ψ is G-equivariant map since the G-action preserves φ_0. Since the $SO(m)$-action on $\Lambda^{k-1}(\mathbb{R}^m)^*$ preserves the inner product, we can see

$$g_0(A\Psi(u), A\Psi(v)) = g_0(\Psi(u), \Psi(v))$$

for any $A \in G$ and $u, v \in \mathbb{R}^m$. Moreover, the left-hand side is equal to $g_0(\Psi(Au), \Psi(Av))$ since Ψ is G-equivariant.

Now, let $e_1, \dots, e_m$ be the standard orthonormal basis of $\mathbb{R}^m$ and define the symmetric matrix $H = (H_{ij})_{i,j}$ by $H_{ij} := g_0(\Psi(e_i), \Psi(e_j))$. Then by the above argument, we have ${}^tAHA = H$. Let $\lambda \in \mathbb{R}$ be any eigenvalue of H and denote by $V(\lambda) \subset \mathbb{R}^m$ the eigenspace associated with λ. Then we can see that $V(\lambda)$ is closed under the G-action, hence we have $V(\lambda) = \mathbb{R}^m$ by the irreducibility, which implies

$$|\Psi(Au)|_{g_0}^2 = \lambda|u|_{g_0}^2$$

for all $u \in \mathbb{R}^m$ and $A \in G$. □

Let (X^m, g) be an oriented Riemannian manifold and denote by $\mathrm{Hol}_g \subset SO(m)$ the holonomy group. We consider $(X, g, \varphi, G, g_0, \varphi_0)$, where $\varphi \in \Omega^k(X)$ is closed, (g_0, φ_0) is a local model of (g, φ), and G is a closed subgroup of $SO(m)$ such that $\mathrm{Hol}_g \subset G \subset \mathrm{Stab}(\varphi_0)$. The followings are examples.

We can apply Proposition 5.4 and Lemma 5.5 to the above cases and obtain the following result.

Theorem 5.6 *Let (X, g, φ) be an oriented compact Riemannian manifold whose geometric structure is one of Table 1 and let Φ be as in Proposition 5.4. Then the identity map 1_X is a $(\sigma_{k,k}, \Phi)$-calibrated map. In particular, 1_X minimizes $\mathcal{E}_{k,k}$ in its homotopy class.*

Table 1 Examples of $(X, g, \varphi, G, g_0, \varphi_0)$

(X, g, φ)	m	G	k
Kähler manifold	$2q$	$U(q)$	2
quaternionic Kähler manifold	$4q \geq 8$	$Sp(q) \cdot Sp(1)$	4
G_2 manifold	7	G_2	3
Spin(7) manifold	8	Spin(7)	4

6 Intersection of Smooth Maps

In [2], Croke and Fathi introduced the homotopy invariant of a smooth map $f: X \to Y$ which gives the lower bound to the 2-energy $\mathcal{E}_2$. In this section, we compare our invariant with the invariant in [2].

First of all, we review the intersection of smooth map introduced in [2]. Let (X, g) and (Y, h) be Riemannian manifolds and suppose X is compact. Here, we do not assume X is oriented, and we use the volume measure μ_g of g instead of the volume form.

Croke and Fathi defined the following quantity

$$i_f(g, h) = \lim_{t\to\infty} \frac{1}{t} \int_{S_g(X)} \phi_t(v) \mathrm{dLiou}_g(v)$$

for a smooth map $f: X \to Y$ and called it *the intersection of* f. Here, Liou_g is the Liouville measure on the unit tangent bundle $S_g(X)$ and $\phi_t^f(v) = \phi_t(v)$ is the minimum length of all paths in Y homotopic with the fixed endpoints to

$$s \mapsto f(\gamma_v(s)), \quad 0 \leq s \leq t,$$

where γ_v is the geodesic from $p \in X$ with $\gamma_v'(0) = v \in S_g(X)$.

Theorem 6.1 [2] *For a smooth map $f: X \to Y$, the intersection $i_f(g, h)$ is homotopy invariant, that is, $i_f(g, h) = i_{f'}(g, h)$ if $[f'] = [f]$. Moreover, for any f, we have*

$$\int_X \sigma_2(f) \mathrm{d}\mu_g \geq \frac{m}{V(S^{m-1})^2 \mu_g(X)} i_f(g, h)^2,$$

where $V(S^{m-1})$ is the volume of the unit sphere S^{m-1} in $\mathbb{R}^m$.

First of all, we introduce the variant of $i_f(g, h)$ and improve the above theorem. We put

$$j_f(g, h) := \lim_{t\to\infty} \frac{1}{t^2} \int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v).$$

Theorem 6.2 *For a smooth map $f\colon X \to Y$, $j_f(g,h)$ is homotopy invariant. Moreover, for any f, we have*

$$\int_X \sigma_2(f)\mathrm{d}\mu_g \geq \frac{m}{V(S^{m-1})} j_f(g,h),$$

where the equality holds iff the image of the geodesic in X by f minimizes the length in its homotopy class with the fixed endpoints.

Proof The proof is parallel to that of Theorem 6.1. The homotopy invariance of $j_f(g,h)$ is same as the case of $i_f(g,h)$. See the proof of [2, Lemma 1.3].

Next we show the inequality. Here we can see

$$\int_X \sigma_2(f)\mathrm{d}\mu_g = \frac{m}{V(S^{m-1})}\int_{S_g(X)} |\mathrm{d}f(v)|_h^2 \mathrm{dLiou}_g(v).$$

For $s \geq 0$, let $g_s\colon S_g(X) \to S_g(X)$ be the geodesic flow. Since g_s preserves the Liouville measure, we can see

$$\begin{aligned}\int_{S_g(X)} |\mathrm{d}f(v)|_h^2 \mathrm{dLiou}_g(v) &= \frac{1}{t}\int_{S_g(X)}\left(\int_0^t |\mathrm{d}f(g_s v)|_h^2 ds\right)\mathrm{dLiou}_g(v)\\ &= \frac{1}{t}\int_{S_g(X)} \mathcal{E}_2(f\circ\gamma_v|_{[0,t]})\mathrm{dLiou}_g(v),\end{aligned}$$

where $\mathcal{E}_2$ is the 2-energy of the curves in (Y,h). If $L(c)$ is the length of c, then we have

$$\begin{aligned}\mathcal{E}_2(c) = \int_a^b |c'(s)|_h^2 ds &\geq \frac{1}{b-a}\left(\int_a^b |c'(s)|_h ds\right)^2\\ &= \frac{1}{b-a}\min_c L(c)^2,\end{aligned}$$

therefore

$$\int_{S_g(X)} |\mathrm{d}f(v)|_h^2 \mathrm{dLiou}_g(v) \geq \frac{1}{t^2}\int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v)$$

for any $t > 0$. Consequently, we have the second assertion by considering $t \to \infty$. Finally, we consider the condition when

$$\int_{S_g(X)} |\mathrm{d}f(v)|_h^2 \mathrm{dLiou}_g(v) = \lim_{t\to\infty}\frac{1}{t^2}\int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v)$$

holds. To consider it, we show

$$\lim_{t\to\infty}\frac{1}{t^2}\int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v) = \inf_{t>0}\frac{1}{t^2}\int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v). \tag{2}$$

By [2, Lemma 1.2], we have

$$\phi_{t+t'}(v) \leq \phi_{t'}(g_t v) + \phi_t(v)$$

for any $t, t' \geq 0$. Then by combining the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{S_g(X)} \phi_{t+t'}(v)^2 \mathrm{dLiou}_g(v) \leq & \int_{S_g(X)} \phi_{t'}(g_t v)^2 \mathrm{dLiou}_g(v) \\ & + 2\sqrt{\int_{S_g(X)} \phi_{t'}(g_t v)^2 \mathrm{dLiou}_g(v) \int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v)} \\ & + \int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v).\end{aligned}$$

Since the Liouville measure is invariant under the geodesic flow, we can see $\int_{S_g(X)} \phi_{t'}(g_t v)^2 \mathrm{dLiou}_g(v) = \int_{S_g(X)} \phi_{t'}(v)^2 \mathrm{dLiou}_g(v)$, hence

$$\int_{S_g(X)} \phi_{t+t'}(v)^2 \mathrm{dLiou}_g(v) \leq \left(\sqrt{\int_{S_g(X)} \phi_{t'}(v)^2 \mathrm{dLiou}_g(v)} + \sqrt{\int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v)} \right)^2.$$

If we put

$$P_t := \sqrt{\int_{S_g(X)} \phi_t(v)^2 \mathrm{dLiou}_g(v)},$$

then we have $P_{t+t'} \leq P_t + P_{t'}$, hence

$$\inf_{t>0} \frac{P(t)}{t} = \lim_{t\to\infty} \frac{P(t)}{t}.$$

Thus, we obtain (2).

Now, suppose

$$\int_X \sigma_2(f) \mathrm{d}\mu_g = \frac{m}{V(S^{m-1})} j_f(g, h).$$

By the above argument, we can see that $f \circ \gamma_v|_{[0,t]}$ is geodesic for any $v \in S_g(X)$ and $t > 0$, and $L(f \circ \gamma_v|_{[0,t]})$ gives the minimum of

$$\left\{ L(c) |\, c \text{ is homotopic with the fixed endpoints to } f \circ \gamma_v|_{[0,t]} \right\}.$$

□

Remark 6.3 By the Cauchy–Schwarz inequality, we have

$$j_f(g,h) \geq \frac{i_f(g,h)^2}{\mu_g(X)V(S^{m-1})},$$

therefore, the inequality in Theorem 6.2 implies the inequality in Theorem 6.1.

Next we compute $j_f(g,h)$ in the case of flat tori and compare with the lower bound obtained by Proposition 3.6. Let $(\mathbb{T}^m, g)$ and $(\mathbb{T}^n, h)$ be as in Sect. 3.4 and take a coordinate x on $\mathbb{T}^m$ and y on $\mathbb{T}^n$ as in Sect. 3.4.

Proposition 6.4 *Let $f\colon \mathbb{T}^m \to \mathbb{T}^n$ be a smooth map such that $f^*([\mathrm{d}y^j]) = \sum_i P_i^j[\mathrm{d}x^i]$ for $P = (P_i^j) \in M_{m,n}(\mathbb{Z})$. If we define Φ by (1) in Sect. 3.4, then we have*

$$j_f(g,h) = \frac{V(S^{m-1})}{m}\int_{\mathbb{T}^m}(1_{\mathbb{T}^m}, f)^*\Phi.$$

Proof First of all, we can see that f is homotopic to the map given by $x \mapsto Px$ for $x \in \mathbb{T}^m$, hence it suffices to show the equality by putting $f(x) = {}^tPx$.

Since the image of the geodesic by f minimizes the length in its homotopy class with the fixed endpoints, then by Theorem 6.2, we have $\mathcal{E}_2(f) = \frac{m}{V(S^{m-1})}j_f(g,h)$. we can compute $\mathcal{E}_2(f)$ directly as

$$\mathcal{E}_2(f) = \int_{\mathbb{T}^m}|\mathrm{d}f|^2\mathrm{vol}_g = \sum_{i,j,k,l} h_{ij}P_k^iP_l^jg^{kl}\mathrm{vol}_g(\mathbb{T}^m) = \|P\|^2\mathrm{vol}_g(\mathbb{T}^m).$$

Moreover, by the computation in Sect. 3.4, we have shown that

$$\int_{\mathbb{T}^m}(1_{\mathbb{T}^m}, f)^*\Phi = \|P\|^2\mathrm{vol}_g(\mathbb{T}^m).$$

Therefore,

$$\int_{\mathbb{T}^m}(1_{\mathbb{T}^m}, f)^*\Phi = \|P\|^2\mathrm{vol}_g(\mathbb{T}^m) = \frac{m}{V(S^{m-1})}j_f(g,h).$$

□

Acknowledgements I would like to thank Professor Frank Morgan for his advice on this paper. This work was supported by JSPS KAKENHI Grant numbers JP19K03474, JP20H01799.

Data Availability The author confirms that the data supporting the findings of this study is available within the article.

References

1. Bott, R., Loring, W.T.: Differential Forms in Algebraic Topology. Graduate Texts in Mathematics, vol. 82. Springer, New York–Berlin (1982)
2. Croke, C., Fathi, A.: An inequality between energy and intersection. Bull. Lond. Math. Soc. **22**(5), 489–494 (1990)
3. Croke, C.B.: Lower bounds on the energy of maps. Duke Math. J. **55**(4), 901–908 (1987)
4. Eells, J., Jr., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)
5. Hartman, P.: On homotopic harmonic maps. Can. J. Math. **19**, 673–687 (1967)
6. Harvey, R., Blaine Lawson, H.: Calibrated geometries. Acta Mathematica **148**(1), 47–157 (1982)
7. Hoisington, J.: Calibrations and energy-minimizing mappings of rank-1 symmetric spaces. (2021). arXiv preprint arXiv:2112.03160
8. Lichnerowicz, A.: Applications harmoniques et variétés Kähleriennes. Rend. Sem. Mat. Fis. Milano **39**, 186–195 (1969)
9. Wei, S.W.: Representing homotopy groups and spaces of maps by p-harmonic maps. Indiana Univ. Math. J. **47**(2), 625–670 (1998)
10. White, B.: Infima of energy functionals in homotopy classes of mappings. J. Differ. Geom. **23**(2), 127–142 (1986)
11. Wirtinger, W.: Eine Determinantenidentität und ihre Anwendung auf analytische Gebilde in euklidischer und Hermitescher Maßbestimmung. Monatsh. Math. Phys. **44**(1), 343–365 (1936)