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Facets of High-Dimensional Gaussian Polytopes

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Abstract
We study the number of facets of the convex hull of n independent standard Gaussian points in $\mathbb{R}^d$. In particular, we are interested in the expected number of facets when the dimension is allowed to grow with the sample size. We establish an explicit asymptotic formula that is valid whenever $d/n \to 0$. We also obtain the asymptotic value when d is close to n.

Keywords Gaussian polytope · Expected number of facets

Mathematics Subject Classification 52A22 · 60D05

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1 Introduction

The convex hull $[X_1, \dots, X_n]$ of n independent standard Gaussian samples $X_1, \dots, X_n$ from $\mathbb{R}^d$ is the Gaussian polytope $P_n^{(d)}$. For fixed dimension d, the face numbers and intrinsic volumes of $P_n^{(d)}$ as n tends to infinity are well understood by now. For $i = 0 \dots, d$ and polytope Q, let $f_i(Q)$ denote the number of i-faces of Q and let $V_i(Q)$ denote the ith intrinsic volume of Q. The asymptotic behavior of the expected value of the number of facets $f_{d-1}(P_n^{(d)})$ as $n \to \infty$ was provided by Rényi and Sulanke [22] if $d = 2$, and by Raynaud [21] if $d \geq 3$. Namely, they proved that, for any fixed d,

$$\mathbb{E} f_{d-1}(P_n^{(d)}) = 2^d \pi^{\frac{d-1}{2}} d^{-\frac{1}{2}} (\ln n)^{\frac{d-1}{2}} (1 + o(1)) \tag{1}$$

as $n \to \infty$. For $i = 0, \dots, d$, expected value of $V_i(P_n^{(d)})$ as $n \to \infty$ was computed by Affentranger [1], and that of $f_i(P_n^{(d)})$ was determined Affentranger and Schneider [2] and Baryshnikov and Vitale [3], see Hug et al. [15] and Fleury [12] for a different approach. More recently, Kabluchko and Zaporozhets [18, 19] proved explicit expressions for the expected value of $V_d(P_n^{(d)})$ and the number of k-faces $f_k(P_n^{(d)})$. Yet these formulas are complicated and it is not immediate how to deduce asymptotic results for large n high dimensions d.

After various partial results, including the variance estimates of Calka and Yukich [6] and Hug and Reitzner [16], central limit theorems were proved for $f_i(P_n^{(d)})$ and $V_d(P_n^{(d)})$ by Bárány and Vu [5], and for $V_i(P_n^{(d)})$ by Bárány and Thäle [4]. These results have been strengthened considerably by Grote and Thäle [14]. The interesting question whether $\mathbb{E} f_{d-1}(P_n^{(d)})$ is an increasing function in n was answered in the positive by Kabluchko and Thäle [17]. It would be interesting to investigate the monotonicity behavior of the facet number if n and d increases simultaneously.

The "high-dimensional" regime, that is, when d is allowed to grow with n, is of interest in numerous applications in statistics, signal processing, and information theory. The combinatorial structure of $P_n^{(d)}$, when d tends to infinity and n grows proportionally with d, was first investigated by Vershik and Sporyshev [23], and later Donoho and Tanner [11] provided a satisfactory description. For any $t > 1$, Donoho and Tanner [11] determined the optimal $\varrho(t) \in (0, 1)$ such that if n/d tends to t, then $P_n^{(d)}$ is essentially $\varrho(t)d$-neighbourly (if $0 < \eta < \varrho(t)$ and $0 \leq k \leq \eta d$, then $f_k(P_n^{(d)})$ is asymptotically $\binom{n}{k+1}$). See Donoho [10], Candés et al. [7], Candés and Tao [8, 9], Mendoza-Smith et al. [20].

In this note, we consider $f_{d-1}(P_n^{(d)})$, the number of facets, when both d and n tend to infinity. Our main result is the following estimate for the expected number of facets of the Gaussian polytope. The implied constant in $O(\cdot)$ is always some absolute constant. We write $\operatorname{lln} x$ for $\ln(\ln x)$.

Theorem 1.1 *Assume $P_n^{(d)}$ is a Gaussian polytope. Then for $d \geq 78$ and $n \geq e^e d$, we have*

$$\mathbb{E} f_{d-1}(P_n^{(d)}) = 2^d \pi^{\frac{d-1}{2}} d^{-\frac{1}{2}} e^{\frac{d-1}{2} \operatorname{lln} \frac{n}{d} - \frac{d-1}{4} \frac{\operatorname{lln} \frac{n}{d}}{\ln \frac{n}{d}} + (d-1) \frac{\theta}{\ln \frac{n}{d}} + O\left(\sqrt{d} e^{-\frac{1}{10} d}\right)}$$

with $\theta = \theta(n, d) \in [-34, 2]$.

When n/d tends to infinity as $d \to \infty$, Theorem 1.1 provides the asymptotic formula

$$\mathbb{E} f_{d-1}(P_n^{(d)}) = \left((4\pi + o(1)) \ln \frac{n}{d}\right)^{\frac{d-1}{2}} .$$

If $n/(de^d) \to \infty$, then we have $\frac{d}{\ln \frac{n}{d}} \to 0$ and hence

$$\mathbb{E} f_{d-1}(P_n^{(d)}) = 2^d \pi^{\frac{d-1}{2}} d^{-\frac{1}{2}} e^{\frac{d-1}{2} \ln\ln \frac{n}{d} - \frac{d-1}{4} \frac{\ln\ln \frac{n}{d}}{\ln \frac{n}{d}} + o(1)}$$

as $d \to \infty$. In the case when n grows even faster such that $(\ln n)/(d \ln d) \to \infty$, the asymptotic formula simplifies to the result (1) of Rényi and Sulanke [22] and Raynaud [21] for fixed dimension.

Corollary 1.2 *Assume $P_n^{(d)}$ is a Gaussian polytope. If $(\ln n)/(d \ln d) \to \infty$, we have*

$$\mathbb{E} f_{d-1}(P_n^{(d)}) = 2^d \pi^{\frac{d-1}{2}} d^{-\frac{1}{2}} (\ln n)^{\frac{d-1}{2}} (1 + o(1)) .$$

There is a (simpler) counterpart of our main results stating the asymptotic behavior of the expected number of facets of $P_n^{(d)}$, if $n - d$ is *small* compared to d, that is, if n/d tends to one.

Theorem 1.3 *Assume $P_n^{(d)}$ is a Gaussian polytope. Then for $n - d = o(d)$, we have*

$$\mathbb{E} f_{d-1}(P_n^{(d)}) = \binom{n}{d} 2^{-(n-d)+1} e^{\frac{1}{\pi} \frac{(n-d)^2}{d} + O\left(\frac{(n-d)^3}{d^2}\right) + o(1)}$$

as $d \to \infty$.

This complements a result of Affentranger and Schneider [2] stating the number of k-dimensional faces for $k \le n - d$ and $n - d$ fixed,

$$\mathbb{E} f_k(P_n^{(d)}) = \binom{n}{k+1} (1 + o(1)) ,$$

as $d \to \infty$.

In the next section we sketch the basic idea of our approach, leaving the technical details to later sections. In Sect. 3 we provide asymptotic approximations for the tail of the normal distribution. In Sect. 4 concentration inequalities are derived for the β-distribution. Finally, in Sects. 5 and 6, Corollary 1.2 and Theorem 1.3 are proven.

2 Outline of the Argument

For $z \in \mathbb{R}$, let

$$\Phi(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{y} e^{-s^2}\, ds, \text{ and } \phi(y) = \Phi'(y) = \frac{1}{\sqrt{\pi}} e^{-y^2} .$$

Our proof is based on the approach of Hug, Munsonius, and Reitzner [15]. In particular, [15, Theorem 3.2] states that if $n \geq d+1$ and $X_1, \ldots, X_n$ are independent standard Gaussian points in $\mathbb{R}^d$, then

$$\mathbb{E} f_{d-1}([X_1, \ldots, X_n]) = \binom{n}{d} \mathbb{P}(Y \notin [Y_1, \ldots, Y_{n-d}]) ,$$

where $Y, Y_1, \ldots, Y_{n-d}$ are independent real-valued random variables with $Y \stackrel{d}{=} N\left(0, \frac{1}{2d}\right)$ and $Y_i \stackrel{d}{=} N\left(0, \frac{1}{2}\right)$ for $i = 1, \ldots, n-d$. This gives

$$\mathbb{E} f_{d-1}([X_1, \ldots, X_n]) = 2\binom{n}{d} \frac{\sqrt{d}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Phi(y)^{n-d} e^{-dy^2}\, dy \tag{2}$$

$$= 2\binom{n}{d} \sqrt{d}\, \pi^{\frac{d-1}{2}} \int_{-\infty}^{\infty} \Phi(y)^{n-d} \phi(y)^d\, dy . \tag{3}$$

Note that similar integrals appear in the analysis of the expected number of k-faces for values of k in the entire range $k = 0, \ldots, d-1$. In our case, the analysis boils down to understanding the integral of $\Phi(y)^{n-d}\phi(y)^d$ over the real line. By substituting $(1-u) = \Phi(y)$, we obtain

$$\int_{-\infty}^{\infty} \Phi(y)^{n-d} \phi(y)^d\, dy = \int_0^1 (1-u)^{n-d} \phi(\Phi^{-1}(1-u))^{d-1}\, du .$$

Clearly, $n \geq d+2$ is the nontrivial range. When $n/d \to \infty$, $(1-u)^{n-d}$ is dominating, and we need to investigate the asymptotic behavior of $\phi(\Phi^{-1}(1-u))$ as $u \to 0$. We show that the essential term is precisely $2u$. Hence, it makes sense to rewrite the integral as

$$2^{d-1} \int_0^1 (1-u)^{n-d} u^{d-1} \underbrace{\left((2u)^{-1} \phi(\Phi^{-1}(1-u))\right)^{d-1}}_{=: g_d(u)} du .$$

For $x, y > 0$, the Beta-function is given by $\boldsymbol{B}(x, y) = \int_0^1 (1-u)^{x-1} u^{y-1} du$. It is well known that for $k, l \in \mathbb{N}$ we have $\boldsymbol{B}(k, l) = \frac{(k-1)!(l-1)!}{(k+l-1)!}$. A random variable U is $\boldsymbol{B}_{(x,y)}$ distributed if its density is given by $\boldsymbol{B}(x, y)^{-1}(1-u)^{x-1}u^{y-1}$. With this, we have established the following identity:

Proposition 2.1

$$\mathbb{E} f_{d-1}([X_1, \ldots, X_n]) = 2^d \pi^{\frac{d-1}{2}} d^{-\frac{1}{2}} \mathbb{E} g_d(U) \tag{4}$$

where

$$g_d(u) = \left((2u)^{-1}\phi(\Phi^{-1}(1-u))\right)^{d-1}$$

and U is a $\boldsymbol{B}(n-d+1, d)$ random variable.

In Lemma 3.3 below we show that

$$g_d(u) = (\ln u^{-1})^{-\frac{d-1}{2}} e^{-\frac{d-1}{4}\frac{\ln\ln u^{-1}}{\ln u^{-1}} - (d-1)\frac{O(1)}{\ln u^{-1}}}$$

as $u \to 0$. Because the Beta function is concentrated around $\frac{d}{n}$, see Lemma 4.1 and Lemma 4.2, this yields

$$\mathbb{E} g_d(U) \approx \left(\ln \frac{n}{d}\right)^{\frac{d-1}{2}} e^{-\frac{d-1}{4}\frac{\ln\ln\frac{n}{d}}{\ln\frac{n}{d}} - (d-1)\frac{O(1)}{\ln\frac{n}{d}}}$$

which implies our main result.

3 Asymptotics of the Φ-Function

To estimate $\Phi(z)$, we need a version of Gordon's inequality [13] for the Mill's ratio:

Lemma 3.1 *For any $z > 1$ there exists $\theta \in (0, 1)$, such that*

$$\Phi(z) = 1 - \frac{e^{-z^2}}{2\sqrt{\pi}z}\left(1 - \frac{\theta}{2z^2}\right)$$

Proof It follows by partial integration that

$$\int_z^\infty e^{-t^2}\,dt = \int_z^\infty 2te^{-t^2}\frac{1}{2t}\,dt = \frac{e^{-z^2}}{2z} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\,dt = \frac{e^{-z^2}}{2z} - \frac{\theta e^{-z^2}}{4z^3}$$

which yields the lemma. □

Lemma 3.2 *For any $u \in (0, e^{-1}]$ there is a $\delta = \delta(u) \in (0, 16)$ such that*

$$\Phi^{-1}(1-u) = \sqrt{\ln u^{-1} - \frac{1}{2}\ln\ln u^{-1} - \ln(2\sqrt{\pi}) + \frac{1}{4}\frac{\ln\ln u^{-1}}{\ln u^{-1}} + \frac{\delta}{\ln u^{-1}}}. \tag{5}$$

Proof It is useful to prove (5) for the transformed variable $u = e^{-t}$. We define

$$z(t) = \sqrt{t - \frac{1}{2}\ln t - \ln(2\sqrt{\pi}) + \frac{1}{4}\frac{\ln t}{t} + \frac{\delta(t)}{t}} \tag{6}$$

which exists for $t > 0$. In a first step we prove that this is the asymptotic expansion of $z = \Phi^{-1}(1-e^{-t})$ as $z, t \to \infty$ with a suitable function $\delta = \delta(t) = O(1)$. In a second step we show the bound on δ. Observe that $z \geq 1$ implies $t \geq \ln \Phi(-1)) = -2, 54\ldots$. By Lemma 3.1, for $z \geq 1$

$$e^{-t} = 1 - \Phi(z) = \frac{1}{2\sqrt{\pi}\, z} e^{-z^2} \left(1 - \frac{\theta(z)}{2z^2}\right) \tag{7}$$

as $z \to \infty$ with some $\theta(z) \in (0, 1)$, which immediately implies that $z = z(t) \to \infty$ as $t \to \infty$. Equation (7) shows that $e^t \geq 2\sqrt{\pi} z e^{z^2}$ and thus

$$t \geq \ln(2\sqrt{\pi}) + \ln z(t) + z(t)^2 \geq z(t)^2$$

for $z \geq 1$. The function $z = z(t)$ is the inverse function we are looking for, if it satisfies

$$4\pi z(t)^2 e^{-2t} = e^{-2z(t)^2} \left(1 - \frac{\theta(z)}{2z^2}\right)^2. \tag{8}$$

We plug (6) into this equation. This leads to

$$\begin{aligned} t - \frac{1}{2}\ln t - \ln(2\sqrt{\pi}) + \frac{1}{4}\frac{\ln t}{t} + \frac{\delta(t)}{t} &= t e^{-\frac{1}{2}\frac{\ln t}{t} - 2\frac{\delta(t)}{t}} \left(1 - O(t^{-1})\right) \\ &= t - \frac{1}{2}\ln t - 2\delta(t) - O(1) \end{aligned}$$

and shows $-\ln(2\sqrt{\pi}) + o(1) = -2\delta(t) - O(1)$. Thus the function $z(t)$ given by (6) in fact satisfies (7) and therefore it is the asymptotic expansion of the inverse function.

The desired estimate for δ follows from some more elaborate but elementary calculations. First we prove that $\delta \geq 0$. By (8) and because $e^x \geq 1 + x$,

$$\begin{aligned} t - \frac{1}{2}\ln t - \ln(2\sqrt{\pi}) + \frac{1}{4}\frac{\ln t}{t} + \frac{\delta(t)}{t} &\geq t\left(1 - \frac{1}{2}\frac{\ln t}{t} - 2\frac{\delta(t)}{t}\right)\left(1 - \frac{\theta}{2t}\right)^2 \\ &\geq (t - \frac{1}{2}\ln t - 2\delta(t))\left(1 - \frac{\theta}{t}\right) \end{aligned}$$

which is equivalent to

$$\delta(t) \geq \frac{\ln(2\sqrt{\pi}) - \theta - \frac{1-2\theta\ln t}{4t}}{(2 + \frac{1-2\theta}{t})} > 0$$

for $t \geq 1$. On the other hand, again by (8),

$$t \geq \left(t - \frac{1}{2}\ln t - \ln(2\sqrt{\pi}) + \frac{1}{4}\frac{\ln t}{t} + \frac{\delta(t)}{t}\right) e^{\frac{1}{2}\frac{\ln t}{t} + 2\frac{\delta(t)}{t}}$$

and using $e^x \geq 1 + x$ implies

$$\delta(t) \leq \frac{\ln(2\sqrt{\pi}) + \frac{2\ln(2\sqrt{\pi})-1}{4}\frac{\ln t}{t} + \frac{1}{4}\frac{(\ln t)^2}{t} + \frac{1}{8}\frac{(\ln t)^2}{t^2}}{2 - (2\ln(2\sqrt{\pi}) - 1)\frac{1}{t} - \frac{\ln t}{t}} \leq 16.$$

□

An asymptotic expansion for $\phi(\Phi^{-1}(1-u))$ follows immediately:

Lemma 3.3 *For any* $u \in (0, e^{-1}]$ *there is a* $\delta = \delta(u) \in (0, 16)$ *such that*

$$g_d(u) = \left((2u)^{-1}\phi(\Phi^{-1}(1-u))\right)^{d-1} = (\ln u^{-1})^{\frac{d-1}{2}} e^{-\frac{d-1}{4}\frac{\operatorname{lln} u^{-1}}{\ln u^{-1}} - (d-1)\frac{\delta}{\ln u^{-1}}} .$$

4 Concentration of the β-Distribution

A basic integral for us is the Beta-integral

$$\boldsymbol{B}(\alpha, \beta) = \int_0^1 (1-x)^{\alpha-1} x^{\beta-1}\, dx = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}. \tag{9}$$

Let $U \sim \boldsymbol{B}(\alpha, \beta)$ distributed. Then $\mathbb{E}U = \frac{\beta}{\alpha+\beta}$ and $\operatorname{var}(U) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ Next we establish concentration inequalities for a Beta-distributed random variable around its mean. Observe that if $U \sim \mathbf{B}(\alpha, \beta)$, then $1-U \sim \mathbf{B}(\beta, \alpha)$. Hence we may concentrate on the case $\alpha \geq \beta$.

Lemma 4.1 *Let* $U \sim \boldsymbol{B}(a+1, b+1)$ *distributed with* $a \geq b$ *and set* $n = a + b$. *Then*

$$\mathbb{P}\left(U \leq \frac{b}{n} - s\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{n^{\frac{3}{2}}}\right) \leq \frac{3e^3}{\pi}\frac{1}{s}\left(e^{-\frac{1}{6}s^2} - e^{-\frac{1}{6}\frac{nb}{a}}\right)_+ .$$

Proof We have to estimate the integral

$$\frac{1}{\boldsymbol{B}(a+1, b+1)} \int_0^{\frac{b-s\sqrt{\frac{ab}{n}}}{n}} (1-x)^a x^b\, dx$$

For an estimate from above we substitute $x = \frac{b}{n} - \frac{y}{n}\sqrt{\frac{ab}{n}}$.

$$J_- = \int_0^{\frac{b-s\sqrt{\frac{ab}{n}}}{n}} (1-x)^a x^b\, dx$$

$$= \frac{a^{a+\frac{1}{2}} b^{b+\frac{1}{2}}}{n^{n+\frac{3}{2}}} \int_s^{\sqrt{\frac{nb}{a}}} \left(1 + y\sqrt{\frac{b}{an}}\right)^a \left(1 - y\sqrt{\frac{a}{bn}}\right)^b dy$$

It is well known that

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \le x - \frac{x^2}{6}, \tag{10}$$

for $x \in (-1, 1]$. Since $a \ge b$, we have

$$\left(1 + y\sqrt{\frac{b}{an}}\right)^a \left(1 - y\sqrt{\frac{a}{bn}}\right)^b \le e^{-\frac{1}{6}y^2},$$

which implies

$$\begin{aligned} J_- &\le \frac{a^{a+\frac{1}{2}} b^{b+\frac{1}{2}}}{n^{n+\frac{3}{2}}} \int_s^{\sqrt{\frac{nb}{a}}} e^{-\frac{1}{6}y^2} dy \\ &\le \frac{3a^{a+\frac{1}{2}} b^{b+\frac{1}{2}}}{n^{n+\frac{3}{2}}} \frac{1}{s} \left(e^{-\frac{1}{6}s^2} - e^{-\frac{1}{6}\frac{nb}{a}}\right). \end{aligned}$$

In the last step we use Stirling's formula,

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \le n! \le e\, n^{n+\frac{1}{2}} e^{-n},$$

to see that

$$\frac{a^{a+\frac{1}{2}} b^{b+\frac{1}{2}}}{n^{n+\frac{3}{2}}} \le \frac{e^3}{\pi} \boldsymbol{B}(a+1, b+1). \tag{11}$$

□

Lemma 4.2 *Let $U \sim \boldsymbol{B}(a+1, b+1)$ distributed with $a \ge b$ and set $n = a + b$. Then for $\lambda \ge 2$,*

$$\mathbb{P}\left(U \ge \lambda \frac{b}{n}\right) \le \frac{e^3}{\pi} \lambda^b b^{\frac{1}{2}} e^{b+\frac{3}{2}} e^{-\lambda \frac{ab}{n}}.$$

Proof We assume that $a \ge b$ and thus $a \ge \frac{n}{2}$. We have to estimate the probability

$$\mathbb{P}\left(U \ge \lambda \frac{b}{n}\right) \le \frac{1}{\boldsymbol{B}(a+1, b+1)} \int_{\lambda \frac{b}{n}}^1 (1-x)^a x^b \, dx$$

We substitute $x \to \frac{1}{a}x + \lambda\frac{b}{n}$ and obtain

$$\int_{\lambda\frac{b}{n}}^{1} (1-x)^a x^b\,dx \le \int_0^\infty e^{-x-\lambda\frac{ab}{n}} \left(\frac{1}{a}x + \lambda\frac{b}{n}\right)^b \frac{1}{a}\,dx$$

$$\le a^{-(b+1)} e^{-\lambda\frac{ab}{n}} \int_0^\infty e^{-x}\left(x + \lambda\frac{ab}{n}\right)^b dx.$$

The use of the binomial formula and the Gamma functions yields

$$\int_0^\infty e^{-x}\left(x+\lambda\frac{ab}{n}\right)^b dx = \sum_{k=0}^{b}\binom{b}{k}\int_0^\infty e^{-x}x^{b-k}\left(\lambda\frac{ab}{n}\right)^k dx$$

$$= \sum_{k=0}^{b}\binom{b}{k}(b-k)!\left(\lambda\frac{ab}{n}\right)^k$$

$$\le b\left(\lambda\frac{ab}{n}\right)^b$$

because $b \le \lambda\frac{ab}{n}$ for $a \ge \frac{n}{2} \ge b$ and $\lambda \ge 2$, and $\frac{1}{k!}\left(\lambda\frac{ab}{n}\right)^k$ is increasing for $k \le \left(\lambda\frac{ab}{n}\right)$. Using (11) this gives

$$\mathbb{P}\left(U \ge \lambda\frac{b}{n}\right) \le \frac{e^3}{\pi}\left(1+\frac{b}{a}\right)^{a+\frac{3}{2}} b^{\frac{1}{2}}\lambda^b e^{-\lambda\frac{ab}{n}}$$

and with $(1+x) \le e^x$ the lemma. □

5 The Case *n* − *d* Large

In this section we combine Lemma 3.3 which gives the asymptotic behavior of $g_d(u)$ as $u \to 0$, with the concentration properties of the Beta function just obtained. We split our proof in two Lemmata.

Lemma 5.1 *For $d \ge d_0 = 78$ and $n \ge e^e d$ we have*

$$\mathbb{E}g_d(U) \le e^{\frac{d-1}{2}\,\mathrm{lln}(\frac{n}{d}) - \frac{d-1}{4}\frac{\mathrm{lln}(\frac{n}{d})}{\ln(\frac{n}{d})} + (d-1)\frac{2}{\ln(\frac{n}{d})}} e^{\frac{e^6}{\pi}\sqrt{d}e^{-\frac{1}{10}d}}.$$

Lemma 5.2 *For $d \ge d_0 = 78$ and $n \ge e^e d$ we have*

$$\mathbb{E}g_d(U) \ge e^{\frac{d-1}{2}\,\mathrm{lln}(\frac{n}{d}) - \frac{d-1}{4}\frac{\mathrm{lln}\frac{n}{d}}{\ln\frac{n}{d}} - (d-1)\frac{34}{\ln\frac{n}{d}}} e^{-\frac{2e^6}{\pi}\sqrt{d}e^{-\frac{1}{10}d}}.$$

These two bounds prove Theorem 1.1. The idea is to split the expectation into the main term close to $\frac{d}{n}$ and two error terms,

$$\begin{aligned}\mathbb{E}g_d(U) = \mathbb{E}g_d(U)\,\mathbb{1}\left(U \le e^{-2}\frac{d}{n}\right)\\ +\mathbb{E}g_d(U)\,\mathbb{1}\left(U \in \left[e^{-2}\frac{d}{n}, 2\frac{d}{n}\right]\right)\\ +\mathbb{E}g_d(U)\,\mathbb{1}\left(U \ge 2\frac{d}{n}\right).\end{aligned}$$

Proof of Lemma 5.2 Recall that U is $\boldsymbol{B}(n-d+1, d)$-distributed. Lemma 4.2 with $a = n-d$ and $b = d-1$ shows that

$$\mathbb{P}\left(U \ge \lambda\frac{d}{n}\right) \le \mathbb{P}\left(U \ge \lambda\frac{d-1}{n-1}\right) \le \frac{e^3}{\pi}\lambda^{d-1}(d-1)^{\frac{1}{2}}e^{(d-1)+\frac{3}{2}}e^{-\lambda\frac{(n-d)(d-1)}{n-1}}$$

because $\frac{d-1}{n-1} < \frac{d}{n}$. For $\lambda = 2$ this gives

$$\mathbb{P}\left(U \ge 2\frac{d}{n}\right) \le \frac{e^6}{2\pi}\sqrt{d}e^{(\ln 2-1+2\frac{d}{n})d} \le \frac{e^6}{2\pi}\sqrt{d}e^{-\frac{1}{10}d} \tag{12}$$

for $n \ge 10d$. The probability that U is small is estimated by Lemma 4.1 with $s = (1-e^{-2})\sqrt{\frac{(d-1)(n-1)}{n-d}}$,

$$\begin{aligned}\mathbb{P}\left(U \le e^{-2}\frac{d-1}{n-1}\right) &\le \frac{3e^3}{\pi}(1-e^{-2})^{-1}\sqrt{\frac{n-d}{(d-1)(n-1)}}e^{-\frac{1}{6}(1-e^{-2})^2\frac{(d-1)(n-1)}{n-d}}\\ &\le \frac{e^6}{2\pi}e^{-\frac{1}{10}d}\end{aligned}$$

for $d \ge 6$. Combining both estimates and using

$$\ln(1-x) \ge -2x \tag{13}$$

for $x \in [0, \frac{1}{2}]$, we have

$$\mathbb{P}\left(U \in \left[\frac{1}{2}\frac{d}{n}, 2\frac{d}{n}\right]\right) \ge 1 - \frac{e^6}{2\pi}\sqrt{d}e^{-\frac{1}{10}d} - \frac{e^6}{2\pi}e^{-\frac{1}{10}d} \ge e^{-\frac{2e^6}{\pi}\sqrt{d}e^{-\frac{1}{10}d}} \tag{14}$$

for $d \ge d_0 = 78$. (Observe that $\frac{2e^6}{\pi}\sqrt{d_0}e^{-\frac{1}{10}d_0} \le \frac{1}{2}$.) In the last step we compute

$$\min_{u\in\left[e^{-2}\frac{d}{n}, 2\frac{d}{n}\right]} g_d(u) = \min_{u\in\left[e^{-2}\frac{d}{n}, 2\frac{d}{n}\right]} e^{\frac{d-1}{2}\ln u^{-1} - \frac{d-1}{4}\frac{\ln\ln u^{-1}}{\ln u^{-1}} - (d-1)\frac{\delta}{\ln u^{-1}}}$$

$$\geq e^{\frac{d-1}{2}\,\mathrm{lln}\left(\frac{1}{2}\frac{n}{d}\right)-\frac{d-1}{4}\frac{\mathrm{lln}\left(\frac{1}{2}\frac{n}{d}\right)}{\ln\left(\frac{1}{2}\frac{n}{d}\right)}-(d-1)\frac{\max\delta}{\ln\left(\frac{1}{2}\frac{n}{d}\right)}}$$

for $n \geq e^e d$. Here, note that $\frac{\mathrm{lln}\,x}{\ln x}$ is decreasing for $x \geq e^e$. Now using

$$\mathrm{lln}\left(\frac{n}{d}\right) \geq \mathrm{lln}\left(\frac{1}{2}\frac{n}{d}\right) = \mathrm{lln}\left(\frac{n}{d}\right) + \ln\left(1 - \frac{\ln 2}{\ln\left(\frac{n}{d}\right)}\right) \geq \mathrm{lln}\left(\frac{n}{d}\right) - \frac{2\ln 2}{\ln\left(\frac{n}{d}\right)},$$

and

$$\frac{1}{\ln\left(\frac{1}{2}\frac{n}{d}\right)} = \frac{1}{\ln\left(\frac{n}{d}\right) - \ln 2} \leq \frac{1}{\ln\left(\frac{n}{d}\right)}\left(1 + 2\frac{\ln 2}{\ln\left(\frac{n}{d}\right)}\right) \leq 2\frac{1}{\ln\left(\frac{n}{d}\right)}$$

for $n \geq e^e d$, we have

$$\min_{u\in\left[e^{-2}\frac{d}{n},2\frac{d}{n}\right]} g_d(u) \geq e^{\frac{d-1}{2}\,\mathrm{lln}\,\frac{n}{d}-\frac{d-1}{4}\frac{\mathrm{lln}\,\frac{n}{d}}{\ln\frac{n}{d}}-(d-1)\frac{\delta'}{\ln\frac{n}{d}}}$$

with $\delta' = \frac{3\ln 2}{2} + 2\max\delta \in [0, 34]$. Combining this estimate with (14) we obtain

$$\begin{aligned}\mathbb{E}g_d(U) &\geq \min_{u\in\left[e^{-2}\frac{d}{n},2\frac{d}{n}\right]} g_d(u)\,\mathbb{E}\mathbb{1}\left(U \in \left[e^{-2}\frac{d}{n}, 2\frac{d}{n}\right]\right)\\ &\geq e^{\frac{d-1}{2}\,\mathrm{lln}\,\frac{n}{d}-\frac{d-1}{4}\frac{\mathrm{lln}\,\frac{n}{d}}{\ln\frac{n}{d}}-(d-1)\frac{\delta'}{\ln\frac{n}{d}}}\; e^{-\frac{2e^6}{\pi}\sqrt{d}e^{-\frac{1}{10}d}}\end{aligned}$$

for $d \geq d_0$ and $n \geq e^e d$.

Proof of Lemma 5.1 As an upper bound we have

$$\begin{aligned}\mathbb{E}g_d(U) &\leq \mathbb{E}g_d(U)\mathbb{1}\left(U \leq e^{-2}\frac{d}{n}\right)\\ &\quad+ \max_{u\in[e^{-2}\frac{d}{n},2\frac{d}{n}]} g_d(u)\,\mathbb{P}\left(U \in \left[e^{-2}\frac{d}{n}, 2\frac{d}{n}\right]\right)\\ &\quad+ \underbrace{\max_{u\in\left[2\frac{d}{n},1\right]} g_d(u)}_{\leq \max_{u\in\left[\frac{d}{n},1\right]} g_d(u)}\,\mathbb{P}\left(U \geq 2\frac{d}{n}\right)\\ &\leq \mathbb{E}g_d(U)\mathbb{1}\left(U \leq e^{-2}\frac{d}{n}\right)\\ &\quad+ e^{\frac{d-1}{2}\,\mathrm{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\mathrm{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\end{aligned}$$

$$+e^{\frac{d-1}{2}\operatorname{lln}\left(\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(\frac{n}{d}\right)}{\ln\left(\frac{n}{d}\right)}}\frac{e^6}{2\pi}\sqrt{d}e^{-\frac{1}{10}d}$$

since $\delta \geq 0$, and where the last term follows from (12). For the first term we use that $\phi(\Phi^{-1}(\cdot))$ is a symmetric and concave function and thus increasing on $[0, e^{-2}\frac{d}{n}]$, and that $\delta \geq 0$.

$$\begin{aligned}
&\mathbb{E}g_d(U)\mathbb{1}\left(U \leq e^{-2}\frac{d}{n}\right)\\
&\leq \frac{1}{\boldsymbol{B}(n-d+1,d)}\int_0^{e^{-2}\frac{d}{n}} e^{\frac{d-1}{2}\operatorname{lln} x^{-1}-\frac{d-1}{4}\frac{\operatorname{lln} x^{-1}}{\ln x^{-1}}}(1-x)^{n-d}x^{d-1}dx\\
&\leq \frac{1}{\boldsymbol{B}(n-d+1,d)}e^{\frac{d-1}{2}\operatorname{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\left(e^{-2}\frac{d}{n}\right)^{d-1}\int_0^{\infty} e^{-(n-d)x}dx
\end{aligned}$$

Now the remaining integration is trivial. We use Stirling's formula (11) to estimate the Beta-function and obtain

$$\begin{aligned}
&\mathbb{E}g_d(U)\mathbb{1}\left(U \leq e^{-2}\frac{d}{n}\right)\\
&\leq \frac{e^3}{\pi}\frac{(n-1)^{n+\frac{1}{2}}}{(n-d)^{n-d+\frac{3}{2}}(d-1)^{d-\frac{1}{2}}}e^{\frac{d-1}{2}\operatorname{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\left(e^{-2}\frac{d}{n}\right)^{d-1}\\
&\leq e^{\frac{d-1}{2}\operatorname{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\frac{e^5}{\pi}e^{(d-1)+\frac{(d-1)}{(n-d)}\left(\frac{3}{2}\right)+1+\frac{1}{(d-1)}\frac{1}{2}-2d}\\
&\leq e^{\frac{d-1}{2}\operatorname{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\frac{e^5}{\pi}e^{-\frac{1}{10}d}
\end{aligned}$$

e.g. for $n \geq e^e d$ and $d \geq 78$. Combining our results gives

$$\begin{aligned}
\mathbb{E}g_d(U) \leq\ & e^{\frac{d-1}{2}\operatorname{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\frac{e^5}{\pi}e^{-\frac{1}{10}d}\\
&+e^{\frac{d-1}{2}\operatorname{lln}\left(e^2\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(e^2\frac{n}{d}\right)}{\ln\left(e^2\frac{n}{d}\right)}}\\
&+e^{\frac{d-1}{2}\operatorname{lln}\left(\frac{n}{d}\right)-\frac{d-1}{4}\frac{\operatorname{lln}\left(\frac{n}{d}\right)}{\ln\left(\frac{n}{d}\right)}}\frac{e^6}{2\pi}\sqrt{d}e^{-\frac{1}{10}d}
\end{aligned}$$

In a similar way as above, we get rid of the involved constant e^2 by using

$$\operatorname{lln}\left(\frac{n}{d}\right) \leq \operatorname{lln}\left(e^2\frac{n}{d}\right) = \operatorname{lln}\left(\frac{n}{d}\right) + \ln\left(1+\frac{2}{\ln(\frac{n}{d})}\right) \leq \operatorname{lln}\left(\frac{n}{d}\right) + \frac{2}{\ln(\frac{n}{d})},$$

and

$$\frac{1}{\ln\left(e^2\frac{n}{d}\right)} = \frac{1}{\ln(\frac{n}{d})}\left(1 + \frac{2}{\ln(\frac{n}{d})}\right)^{-1} \geq \frac{1}{\ln(\frac{n}{d})}\left(1 - \frac{2}{\ln(\frac{n}{d})}\right).$$

This yields

$$\mathbb{E}g_d(U) \leq e^{\frac{d-1}{2}\operatorname{lln}(\frac{n}{d}) - \frac{d-1}{4}\frac{\operatorname{lln}(\frac{n}{d})}{\ln(\frac{n}{d})} + (d-1)\frac{\frac{3}{2}}{\ln(\frac{n}{d})}}\left(1 + \frac{e^6}{\pi}\sqrt{d}e^{-\frac{1}{10}d}\right) \tag{15}$$

□

6 The Case $n - d$ Small

Finally, it remains to prove Theorem 1.3. The starting point here is again formula (2), together with the substitution $y \to \frac{y}{\sqrt{d}}$.

$$\begin{aligned}\mathbb{E}f_{d-1}([X_1, \ldots, X_n]) &= 2\binom{n}{d}\frac{\sqrt{d}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\Phi(y)^{n-d}e^{-dy^2}\,dy \\ &= 2\binom{n}{d}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\Phi\left(\frac{y}{\sqrt{d}}\right)^{n-d}e^{-y^2}\,dy \end{aligned}\tag{16}$$

The Taylor expansion of Φ at $y = 0$ is given by

$$\Phi(y) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}y + \frac{1}{\sqrt{\pi}}(-\theta_1)e^{-\theta_1^2}\,y^2 = \frac{1}{2} + \frac{1}{\sqrt{\pi}}y(1 - \theta_2 y)$$

with some $\theta_1, \theta_2 \in \mathbb{R}$ depending on y. Since $\Phi(y)$ is above its tangent at 0 for $y > 0$ and below it for $y < 0$, we have $0 \leq 1 - \theta_2 y \leq 1$. Further,

$$|\theta_2| \leq \max_{\theta_1}\theta_1 e^{-\theta_1^2} = \frac{1}{\sqrt{2e}}.$$

Hence an expression for $\ln\Phi$ at $y = 0$ is given by

$$\ln\Phi(y) = -\ln 2 + \ln\left(1 + \frac{2}{\sqrt{\pi}}y(1 - \theta_2 y)\right).$$

We need again estimates for the logarithm, namely $\ln(1 + x) = x - \theta_3 x^2 < x$ with some $\theta_3 = \theta_3(x) \geq 0$. In addition, there exists $c_3 \in \mathbb{R}$ such that $\theta_3 < c_3$ if x is bounded away from -1, for example, for $x \geq 2\Phi(-1) - 1$. This gives

$$\ln\Phi(y) \leq -\ln 2 + \frac{2}{\sqrt{\pi}}y - \frac{2}{\sqrt{\pi}}\theta_2 y^2$$

and

$$\ln \Phi(y) = -\ln 2 + \frac{2}{\sqrt{\pi}} y(1-\theta_2 y) - \theta_3 \frac{4}{\pi} y^2 \underbrace{(1-\theta_2 y)^2}_{\leq 1}$$
$$\geq -\ln 2 + \frac{2}{\sqrt{\pi}} y - \frac{2}{\sqrt{\pi}} \theta_2 y^2 - \theta_3 \frac{4}{\pi} y^2$$

with $\theta_3 < c_3$ for $y \geq -1$. Thus the Taylor expansion of $\ln \Phi$ at $y = 0$ is given by

$$\ln \Phi(y) = -\ln 2 + \frac{2}{\sqrt{\pi}} y - \theta_4 y^2$$

with some $\theta_4 = \theta_4(y) > -\frac{1}{2}$, and there exists a $c_4 \in \mathbb{R}$ with $\theta_4 \leq c_4$ for $y \geq -1$. We plug this into (16) and obtain

$$\int_{-\infty}^{\infty} \Phi\left(\frac{y}{\sqrt{d}}\right)^{n-d} e^{-y^2}\, dy = e^{-(n-d)\ln 2} \int_{-\infty}^{\infty} e^{\frac{2}{\sqrt{\pi}} \frac{n-d}{\sqrt{d}} y - \theta_4 \frac{n-d}{d} y^2 - y^2}\, dy\,.$$

Since $\frac{n-d}{d} \to 0$ we assume that $1 + \theta_4 \frac{n-d}{d} \geq 1 - \frac{1}{2}\frac{n-d}{d} > 0$. As an estimate from above we have

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{\frac{2}{\sqrt{\pi}} \frac{n-d}{\sqrt{d}} y - (1+\theta_4 \frac{n-d}{d}) y^2}\, dy &\leq \int_{-\infty}^{\infty} e^{\frac{2}{\sqrt{\pi}} \frac{n-d}{\sqrt{d}} y - (1-\frac{1}{2}\frac{n-d}{d}) y^2}\, dy \\
&= e^{\frac{\frac{4}{\pi}\frac{(n-d)^2}{d}}{4(1-\frac{1}{2}\frac{n-d}{d})}} \int_{-\infty}^{\infty} e^{-\left(\frac{\frac{2}{\sqrt{\pi}}\frac{n-d}{\sqrt{d}}}{2\sqrt{(1-\frac{1}{2}\frac{n-d}{d})}} - \sqrt{(1-\frac{1}{2}\frac{n-d}{d})}\,y\right)^2} dy \\
&= e^{\frac{1}{\pi}\frac{(n-d)^2}{d}\left(1+O\left(\frac{n-d}{d}\right)\right)} \frac{\sqrt{\pi}}{\sqrt{\left(1-\frac{1}{2}\frac{n-d}{d}\right)}} \\
&= \sqrt{\pi}\, e^{\frac{1}{\pi}\frac{(n-d)^2}{d} + O\left(\frac{(n-d)^3}{d^2}\right) + O\left(\frac{n-d}{d}\right)}. \qquad (17)
\end{aligned}$$

The estimate from below is slightly more complicated. For $y \geq -\sqrt{d}$ there is an upper bound c_4 for θ_4. Using this we have

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{\frac{2}{\sqrt{\pi}} \frac{n-d}{\sqrt{d}} y - \theta_4 \frac{n-d}{d} y^2 - y^2}\, dy &\geq e^{\frac{1}{\pi}\frac{(n-d)^2}{d}} \int_{\frac{1}{\sqrt{\pi}}\frac{n-d}{\sqrt{d}} - \sqrt{d}}^{\infty} e^{-\left(\frac{1}{\sqrt{\pi}}\frac{n-d}{\sqrt{d}} - y\right)^2 - c_4 \frac{n-d}{d} y^2}\, dy \\
&\geq e^{\frac{1}{\pi}\frac{(n-d)^2}{d}} \int_{-\infty}^{\sqrt{d}} e^{-y^2 - c_4 \frac{n-d}{d}\left(\frac{1}{\sqrt{\pi}}\frac{n-d}{\sqrt{d}} - y\right)^2}\, dy\,.
\end{aligned}$$

Now we use $(a-b)^2 \leq 2a^2 + 2b^2$ which shows that

$$\int_{-\infty}^{\infty} e^{\frac{2}{\sqrt{\pi}}\frac{n-d}{\sqrt{d}}y-\theta_4\frac{n-d}{d}y^2-y^2}\,dy \geq e^{\frac{1}{\pi}\frac{(n-d)^2}{d}+O(\frac{(n-d)^3}{d^2})}\int_{-\infty}^{\sqrt{d}} e^{-(1+2c_4\frac{n-d}{d})y^2}\,dy$$

$$= e^{\frac{1}{\pi}\frac{(n-d)^2}{d}+O\left(\frac{(n-d)^3}{d^2}\right)}\frac{1}{\sqrt{(1+2c_4\frac{n-d}{d})}}\int_{-\infty}^{\sqrt{d(1+2c_4\frac{n-d}{d})}} e^{-y^2}\,dy$$

$$\geq e^{\frac{1}{\pi}\frac{(n-d)^2}{d}+O\left(\frac{(n-d)^3}{d^2}\right)+O\left(\frac{n-d}{d}\right)}\int_{-\infty}^{\sqrt{d}} e^{-y^2}\,dy. \tag{18}$$

Recall the estimate for $\Phi(z)$ from Lemma 3.1,

$$\int_{-\infty}^{\sqrt{d}} e^{-y^2}\,dy = \sqrt{\pi}\,\Phi(\sqrt{d}) \geq \sqrt{\pi}(1-e^{-d}) = \sqrt{\pi}e^{O(e^{-d})}. \tag{19}$$

We combine Eqs. (17), (18) and (19) and obtain

$$\int_{-\infty}^{\infty} e^{\frac{2}{\sqrt{\pi}}\frac{n-d}{\sqrt{d}}y-\theta_4\frac{n-d}{d}y^2-y^2}\,dy = \sqrt{\pi}e^{\frac{1}{\pi}\frac{(n-d)^2}{d}+O\left(\frac{(n-d)^3}{d^2}\right)+O\left(\frac{n-d}{d}\right)+O(e^{-d})}$$

which yields Theorem 1.3.

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