The Journal of Geometric Analysis (2023) 33:331
https://doi.org/10.1007/s12220-023-01390-y

Reflexivity of the Space of Transversal Distributions

J. Kališnik[1,2]

Received: 17 February 2023 / Accepted: 15 July 2023 / Published online: 1 August 2023

Abstract
For any smooth, Hausdorff and second-countable manifold N one can define the Fréchet space $\mathcal{C}^\infty(N)$ of smooth functions on N and its strong dual $\mathcal{E}'(N)$ of compactly supported distributions on N. It is a standard result that the strong dual of $\mathcal{E}'(N)$ is naturally isomorphic to $\mathcal{C}^\infty(N)$, which implies that both $\mathcal{C}^\infty(N)$ and $\mathcal{E}'(N)$ are reflexive locally convex spaces. In this paper we generalise that result to the setting of transversal distributions on the total space of a surjective submersion $\pi : P \to M$. We show that the strong $\mathcal{C}_c^\infty(M)$-dual of the space $\mathcal{E}'_\pi(P)$ of π-transversal distributions is naturally isomorphic to the $\mathcal{C}_c^\infty(M)$-module $\mathcal{C}^\infty(P)$.

Keywords Distributions with compact support · Fréchet spaces · Transversal distributions · Homomorphisms of modules · Reflexive modules

Mathematics Subject Classification 46A04 · 46A08 · 46A13 · 46A25 · 46F05 · 46G05 · 46H25

1 Introduction

Spaces of distributions on a smooth manifold play an important role in the theory of linear partial differential equations. The space $\mathcal{D}'(P)$ of distributions on a manifold P is defined as the continuous dual of the LF-space $\mathcal{C}_c^\infty(P)$ of smooth, compactly supported functions on P. In practical applications the space $\mathcal{D}'(P)$ is usually too large, so we often restrict ourselves to some subspace of it, such as for example Sobolev

This research was supported by Research Grants J1-1690, N1-0137, N1-0237 and research program Analysis and Geometry P1-0291 from the Slovenian Research Agency ARRS.

✉ J. Kališnik
jure.kalisnik@fmf.uni-lj.si

1 Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

2 Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

space $W^{k,p}(P)$ or some $L^p(P)$ space. In our concrete case, we are primarily interested in the convolution of distributions, so it makes sense to restrict ourselves to the space of compactly supported distributions on the manifold P. It is denoted by $\mathcal{E}'(P)$ and can be equivalently described as the continuous dual of the Fréchet space $\mathcal{C}^\infty(P)$ of smooth functions on P. In general, the space $\mathcal{E}'(P)$ is a complete, locally convex vector space. However, in the case of a Lie group G (and in particular if $G = \mathbb{R}^n$ is the euclidean space) we can use the multiplication on G to equip the space $\mathcal{E}'(G)$ with a convolution product, which extends the well-known convolution of functions on the subspace $\mathcal{C}_c^\infty(G)$ of $\mathcal{E}'(G)$. In this way $\mathcal{E}'(G)$ becomes a locally convex algebra. If one wants to construct a similar convolution algebra for any Lie groupoid G, one first needs to replace the space $\mathcal{E}'(G)$ with the space of transversal distributions $\mathcal{E}_t'(G)$, which we will describe next.

The notion of a distribution on a single manifold naturally extends to that of a π-transversal distribution on the surjective submersion $\pi : P \to M$, which is by definition a continuous $\mathcal{C}^\infty(M)$-linear map from $\mathcal{C}_c^\infty(P)$ to $\mathcal{C}^\infty(M)$. The notion of distributions transverse to a submersion appeared first in [1], where they were used to define a pseudodifferential calculus on a singular foliation. A basic example of such a submersion is the trivial bundle $t : M \times M \to M$ over a closed manifold M, where t is the projection on the first factor. In this case we can identify the space $\mathcal{E}_t'(M \times M)$ of t-transversal distributions with the space of Schwartz kernels of continuous linear operators from $\mathcal{C}^\infty(M)$ to $\mathcal{C}^\infty(M)$ in the following way. If $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a continuous linear operator, we denote by K_D its Schwartz kernel, an element of $\mathcal{D}'(M \times M)$. The kernel K_D is semiregular with respect to t (see [14]) and connected to D by the formula

$$D(f)(x) = \int_M K_D(x, y) f(y) \mathrm{d}y,$$

where the integral is meant in the distributional sense and $f \in \mathcal{C}^\infty(M)$. The kernel K_D induces a continuous $\mathcal{C}^\infty(M)$-linear map $T_{K_D} : \mathcal{C}^\infty(M \times M) \to \mathcal{C}^\infty(M)$, given by $T_{K_D}(F) = t_*(F K_D)$ for $F \in \mathcal{C}^\infty(M \times M)$. From the Schwartz kernels theorem it then follows that we have an isomorphism

$$\mathcal{E}_t'(M \times M) = \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\mathcal{C}^\infty(M \times M), \mathcal{C}^\infty(M)) \cong \mathrm{End}(\mathcal{C}^\infty(M))$$

of $\mathcal{C}^\infty(M)$-modules. This point of view gives us an algebra structure on the space $\mathcal{E}_t'(M \times M)$, which corresponds to convolution of Schwartz kernels. However, there is also the third point of view, which is connected to the isomorphism

$$\mathcal{E}_t'(M \times M) \cong \mathcal{C}^\infty(M, \mathcal{E}'(M))$$

of $\mathcal{C}^\infty(M)$-modules. Here we view a t-transversal distribution $T \in \mathcal{E}_t'(M \times M)$ as a smooth family $(T_x)_{x \in M}$ of compactly supported distributions along the fibres of t.

The algebra structure on the space $\mathcal{E}_t'(M \times M)$ is related to the fact that $M \times M$ is the pair Lie groupoid with the target map t. In [11] the authors have constructed a convolution product on the space $\mathcal{E}'_t(G)$ of t-transversal compactly supported distributions on any Lie groupoid G. This construction extends the well known case of $G = M \times M$ and provides a natural representation of $\mathcal{E}'_t(G)$ into $\mathrm{End}(\mathcal{C}^\infty(M))$. The convolution algebra $\mathcal{E}'_t(G)$ plays an important role in [16], where the authors, based on the ideas from [4], used Lie groupoids to characterize classical pseudodifferential operators on manifolds. In [9] the authors constructed for any Lie groupoid G associated bialgebroid $\mathcal{C}_c^\infty(\mathscr{G}^\#, \mathfrak{g})$ with a natural representation into $\mathcal{E}'_t(G)$, while in [2] transversal distributions were used to define the convolution algebra of Schwartz kernels on a singular foliation. Transversal distributions were also used very recently in [3] in the proof of the Helffer-Nourrigat conjecture.

We now turn to the case of an arbitrary surjective submersion $\pi : P \to M$. In this generality we do not have any natural product on the space of compactly supported π-transversal distributions, defined by

$$\mathcal{E}'_\pi(P) = \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{C}^\infty(P), \mathcal{C}_c^\infty(M)).$$

However, we can still consider π-transversal distributions as smooth families of distributions along the fibres. In this paper we focus on the functional analytic aspects of the space $\mathcal{E}'_\pi(P)$. It is a standard result that the space $\mathcal{C}^\infty(P)$ is a reflexive Fréchet space for any manifold P, with an explicit isomorphism $\hat{}: \mathcal{C}^\infty(P) \to \mathcal{E}'(P)'$ given by $\hat{F}(v) = v(F)$ for $F \in \mathcal{C}^\infty(P)$ and $v \in \mathcal{E}'(P)$. Our main theorem extends this result to the setting of transversal distributions.

Theorem Let $\pi : P \to M$ be a surjective submersion. The map

$$\hat{}: \mathcal{C}^\infty(P) \to \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$$

is an isomorphism of locally convex vector spaces. In particular, this implies that $\mathcal{C}^\infty(P)$ and $\mathcal{E}'_\pi(P)$ are reflexive $\mathcal{C}_c^\infty(M)$-modules.

Although the above result is interesting in its own right, let us quickly describe the motivation behind it and some of its applications. Our motivation comes from the classical Gelfand–Naimark correspondence between locally compact Hausdorff topological spaces and C^*-algebras, which gives us a link between algebra and topology. In our case, we are interested in the class of geometric spaces, which can be described by Lie groupoids, such as for example orbifolds, spaces of orbits of Lie groups actions and spaces of leaves of foliations. Our goal is to assign to each Lie groupoid a suitable algebraic object, from which one could reconstruct the groupoid. In the case of étale Lie groupoids this was done in [12], while the case of semidirect products of étale Lie groupoids and bundles of Lie groups was studied in [8]. Both of these constructions assign to a Lie groupoid a certain Hopf algebroid, which is in both cases a purely algebraic object. It seems that in the case of more general Lie groupoids one needs additional structure to recover the groupoid. Our main idea is then to equip the Hopf algebroid with a structure of a locally convex vector space. In [7] this idea was used to assign to each action Lie groupoid a certain locally convex bialgebroid. The main theorem of the present paper plays a crucial role in showing how the action Lie groupoid can be reconstructed. We expect that a similar idea will work also in the general case.

More precisely, in the current paper we have not used the fact that $\mathcal{C}^\infty(P)$ is an algebra over $\mathcal{C}_c^\infty(M)$. The whole $\mathcal{C}_c^\infty(M)$-dual $\mathcal{E}'_\pi(P)$ of $\mathcal{C}^\infty(P)$ is too large to be a coalgebra in the algebraic sense, so we need to restrict ourselves to a suitable subspace of $\mathcal{E}'_\pi(P)$. We then hope to find a functorial construction of a coalgebra, assigned to any smooth surjective submersion $\pi : P \to M$, which generalizes the construction in [13] and contains enough information to reconstruct the submersion $\pi : P \to M$.

2 Preliminaries

For the convenience of the reader and to fix the notations, we will first quickly review some basic definitions concerning locally convex vector spaces, smooth manifolds, spaces of functions and spaces of distributions. See [5, 14] for more details. All our locally convex vector spaces will be complex and Hausdorff and all manifolds will be assumed to be smooth, Hausdorff and second countable. For us a closed manifold M will mean a compact manifold without boundary, while a closed submanifold N of P will be an embedded submanifold without boundary which is closed as a subspace of P.

A subset B of a locally convex space E is bounded if and only if the set $p(B)$ is a bounded subset of $\mathbb{R}$ for any continuous seminorm p on E. We will denote by $E' = \mathrm{Hom}(E, \mathbb{C})$ the space of all continuous linear functionals on E. If F is another locally convex space, we similarly define by $\mathrm{Hom}(E, F)$ the space of all continuous linear maps from E to F. We will equip all these spaces of maps with the strong topology of uniform convergence on bounded subsets. The basis of neighbourhoods of zero in E' consists of sets of the form

$$K(B, \epsilon) = \{v \in E' \,|\, |v(F)| < \epsilon \text{ for all } F \in B\},$$

where B is a bounded subset of E and $\epsilon > 0$. Similarly, the basis of neighbourhoods of zero in $\mathrm{Hom}(E, F)$ consists of sets of the form

$$K(B, V) = \{T \in \mathrm{Hom}(E, F) \,|\, T(B) \subset V\},$$

where B is a bounded subset of E and V is a neighbourhood of zero in F. If E and F are modules over a complex algebra A, we will denote the corresponding space of continuous module homomorphisms by $\mathrm{Hom}_A(E, F)$. Since it is a subspace of $\mathrm{Hom}(E, F)$, it carries the induced topology.

We now recall the definition of the Fréchet topology on $\mathcal{C}^\infty(M)$ for any smooth manifold M. Choose local coordinates $\phi = (x_1, \ldots, x_l) : U \to \mathbb{R}^l$ on M. For any $m \in \mathbb{N}_0$ and any compact subset $K \subset U$ we define the seminorm $p_{K,m}$ on $\mathcal{C}^\infty(M)$ by

$$p_{K,m}(F) = \sup_{x \in K, |\alpha| \leq m} |D^\alpha(F)(x)|$$

for $F \in \mathcal{C}^\infty(M)$. Here, we denoted $D^\alpha(F) = \frac{\partial^{|\alpha|}F}{\partial x_1^{\alpha_1}\cdots\partial x_l^{\alpha_l}}$, where $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}_0^l$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. To keep notation simple, we avoided denoting the dependence of seminorm on the coordinate chart. The sets of the form

$$V_{K,m,\epsilon} = \{F \in \mathcal{C}^\infty(M) \mid p_{K,m}(F) < \epsilon\}$$

for K compact subset in some chart of M, $m \in \mathbb{N}_0$ and $\epsilon > 0$ then form a subbasis of neighbourhoods of zero for the Fréchet topology on $\mathcal{C}^\infty(M)$. With this topology $\mathcal{C}^\infty(M)$ becomes a complete, metrizable locally convex vector space. This topology coincides with the topology of uniform convergence of all derivatives on compact subsets of M.

The subspace $\mathcal{C}_c^\infty(M)$ of $\mathcal{C}^\infty(M)$, consisting of compactly supported functions, is not complete in the Fréchet topology, which makes us consider a finer LF-topology on $\mathcal{C}_c^\infty(M)$. We first denote for any compact subset K of M by $\mathcal{C}_c^\infty(K)$ the space of functions with support contained in K. The space $\mathcal{C}_c^\infty(K)$ is a closed subset of $\mathcal{C}^\infty(M)$ and hence a Fréchet space itself. The LF-topology on $\mathcal{C}_c^\infty(M)$ is then defined as the inductive limit topology with respect to the family of all subspaces of the form $\mathcal{C}_c^\infty(K)$ for $K \subset M$ compact. In particular, an absolutely convex subset V of $\mathcal{C}_c^\infty(M)$ is a neighbourhood of zero in $\mathcal{C}_c^\infty(M)$ if and only if $V \cap \mathcal{C}_c^\infty(K)$ is a neighbourhood of zero in $\mathcal{C}_c^\infty(K)$ for every compact subset K of M. The space $\mathcal{C}_c^\infty(M)$ with LF-topology is a complete locally convex space, which is not metrizable, if M is not compact.

The space of distributions on M is defined as $\mathcal{D}'(M) = \mathcal{C}_c^\infty(M)'$, while the space of compactly supported distributions on M is defined by $\mathcal{E}'(M) = \mathcal{C}^\infty(M)'$. We have a natural inclusion of the space $\mathcal{E}'(M)$ into $\mathcal{D}'(M)$, whose image consists of distributions with compact support. The algebra $\mathcal{C}^\infty(M)$ acts on $\mathcal{D}'(M)$ by $(F \cdot v)(G) = v(FG)$ for $F \in \mathcal{C}^\infty(M)$, $G \in \mathcal{C}_c^\infty(M)$ and $v \in \mathcal{D}'(M)$. It is known that the space $\mathcal{C}^\infty(M)$ is reflexive, which means that $\mathcal{C}^\infty(M) \cong \mathcal{E}'(M)'$.

If M is a smooth manifold and E is a locally convex vector space, a vector valued function $\alpha : M \to E$ is smooth if in local coordinates all partial derivatives exist and are continuous. A map $\alpha : M \to E$ is scalarly smooth if the map $\phi \circ \alpha : M \to \mathbb{C}$ is smooth for every $\phi \in E'$. If α is smooth it is also scalarly smooth, but the implication in the reverse direction does not always hold. However, if the space E is complete, then every scalarly smooth function into E is smooth [10]. We will denote by $\mathcal{C}^\infty(M, E)$ the space of smooth functions on M with values in E and by $\mathcal{C}_c^\infty(M, E)$ its subspace, consisting of compactly supported functions. To make a distinction between scalar functions and vector valued functions, we will denote by $f(x) \in \mathbb{C}$ the value of a function $f \in \mathcal{C}^\infty(M)$ at x and by $u_x \in E$ the value of a function $u \in \mathcal{C}^\infty(M, E)$ at x.

3 Correspondence Between Transversal Distributions and Families Of Distributions

Let P and M be smooth manifolds and let $\pi : P \to M$ be a surjective submersion. We can view P as a smooth family of closed submanifolds $P_x = \pi^{-1}(x)$, parametrized by $x \in M$. Furthermore, smooth families of distributions on these fibres can be equivalently described by π-transversal distributions. In this section we will recall the definition of π-transversal distributions on P and its connection with smooth families of distributions along the fibres of π. This correspondence has been already used in [1, 2, 11, 16]. However, since it plays a very important role in the proof of our main theorem and to make the presentation complete, we will review the main definitions and constructions.

Note that the algebra $\mathcal{C}^\infty(M)$ acts on $\mathcal{C}^\infty(P)$ by $f \cdot F = (f, \pi)F$ for $f \in \mathcal{C}^\infty(M)$ and $F \in \mathcal{C}^\infty(P)$. So we can define the following vector spaces.

Definition 3.1 Let P and M be smooth manifolds and suppose $\pi : P \to M$ is a surjective submersion.

(1) *The space of π-transversal distributions* on P is the vector space

$$\mathcal{D}'_\pi(P) = \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\mathcal{C}^\infty_c(P), \mathcal{C}^\infty(M)).$$

(2) *The space of π-transversal distributions with compact support* is defined similarly by

$$\mathcal{E}'_\pi(P) = \mathrm{Hom}_{\mathcal{C}^\infty_c(M)}(\mathcal{C}^\infty(P), \mathcal{C}^\infty_c(M)).$$

(3) *A smooth family of distributions with compact supports along fibres of π* is an element $u = (u_x)_{x\in M} \in \prod_{x\in M} \mathcal{E}'(P_x)$, such that for every $F \in \mathcal{C}^\infty(P)$ the function $u(F)$, defined by $u(F)(x) = u_x(F|_{P_x})$, is smooth on M. The vector space of all such smooth families will be denoted by

$$\mathcal{C}^\infty\left(\prod_{x\in M} \mathcal{E}'(P_x)\right).$$

Let us take a look at some basic examples.

Example 3.2 (1) If P is any smooth manifold and M is a point, then the constant map $\pi : P \to M$ is a surjective submersion and in this case the space of π-transversal distributions $\mathcal{E}'_\pi(P)$ coincides with the space $\mathcal{E}'(P)$ of ordinary compactly supported distributions on P.

(2) Let $\pi : P \to M$ be a surjective submersion and let E be an image of a local section of π (a submanifold of P that maps by π diffeomorphically to an open subset $U = \pi(E)$ of M). Denote by $\sigma_E : U \to E$ the smooth inverse of the map $\pi|_E$. For any $f \in \mathcal{C}^\infty_c(U)$ we define a π-transversal distribution $[\![E, f]\!]$ by

$$[\![E, f]\!](F)(x) = \begin{cases} f(x)F(\sigma_E(x)) \ ; & x \in U, \\ 0 \ ; & x \notin U, \end{cases}$$

for $F \in \mathcal{C}^\infty(P)$. We think of $[\![E, f]\!]$ as a smooth family of Dirac distributions, supported on the section E and weighted by the function f. In particular, we have

$$[\![E, f]\!]_x = f(x)\delta_{\sigma_E(x)}.$$

As an important special case of this construction let us consider the case when M is closed, $P = M \times M$ and $t : M \times M \to M$ is the projection on the first factor. In this case we can, by the Schwartz kernels theorem, identify $\mathcal{E}'_t(M \times M)$ with the space $\mathrm{Hom}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$. If $E \subset M \times M$ is the graph of a diffeomorphism $\Phi : M \to M$ and $f \equiv 1$ is the unit of $\mathcal{C}^\infty(M)$, then $[\![E, f]\!]$ corresponds to the Schwartz kernel of the pullback operator $\Phi^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

(3) Suppose again that M is a closed manifold, $P = M \times M$ and t is the projection on the first factor. For any linear partial differential operator D on M we can define a t-transversal distribution $[\![M, D]\!]$ by

$$[\![M, D]\!](F)(x) = D(F|_{\{x\}\times M})(x).$$

Here, we have considered $F|_{\{x\}\times M}$ as a smooth function on M, so $D(F|_{\{x\}\times M})$ is well defined. This distribution is supported on the diagonal $M \subset M \times M$ and computes the vertical D-derivative of F. It corresponds to the Schwartz kernel of the operator D.

More generally, let $\pi : P \to M$ be a surjective submersion and let E be an image of a local section of π, as in the previous example. Choose a linear partial differential operator D on P which acts along the fibres of π. For any $f \in \mathcal{C}^\infty_c(M)$ we then define $[\![E, fD]\!] \in \mathcal{E}'_\pi(P)$ by

$$[\![E, fD]\!](F)(x) = \begin{cases} f(x)D(F)(\sigma_E(x)) \ ; & x \in U, \\ 0 \ ; & x \notin U. \end{cases}$$

(4) Let $M = \mathbb{R}^n$, $P = \mathbb{R}^n \times \mathbb{R}^n$ and let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection on the first factor. For $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{R}^n)$ we define a π-transversal distribution T_ϕ by

$$T_\phi(F)(x) = \int_{\mathbb{R}^n} \phi(x, y)F(x, y)\mathrm{d}y.$$

This π-transversal distribution corresponds to the family of smooth densities on $\mathbb{R}^n$, parametrized by $\mathbb{R}^n$. Explicitly,

$$(T_\phi)_x = \phi(x, -)\mathrm{d}V,$$

where $\mathrm{d}V$ is Lebesgue measure on $\mathbb{R}^n$. The distribution T_ϕ corresponds to the Schwartz kernel of the integral operator, associated to the function ϕ.

The vector spaces $\mathcal{D}'_\pi(P)$ and $\mathcal{C}^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$ are modules over $\mathcal{C}^\infty(P)$ and $\mathcal{C}^\infty(M)$, with the actions given by:

$$\begin{aligned}(F \cdot T)(G) &= T(FG),\\ (f \cdot T)(G) &= T(f \cdot G),\\ (F \cdot u)_x &= F|_{P_x} \cdot u_x,\\ (f \cdot u)_x &= f(x)u_x,\end{aligned}$$

for $T \in \mathcal{D}'_\pi(P)$, $u \in \mathcal{C}^\infty(\prod_{x \in M} \mathcal{E}'(P_x))$, $f \in \mathcal{C}^\infty(M)$, $G \in \mathcal{C}_c^\infty(P)$ and $F \in \mathcal{C}^\infty(P)$.

We will now recall the definition of the support of a transversal distribution. This will enable us to identify $\mathcal{E}'_\pi(P)$ with the subspace of compactly supported distributions in $\mathcal{D}'_\pi(P)$.

Definition 3.3 *Support* of a π-transversal distribution $T \in \mathcal{D}'_\pi(P)$ is the subset $\mathrm{supp}(T)$ of P, consisting of all points $p \in P$, which satisfy the condition that for every open neighbourhood U of p there exists $F \in \mathcal{C}_c^\infty(U) \subset \mathcal{C}_c^\infty(P)$ with $T(F) \neq 0$.

In the above definition we have identified the space $\mathcal{C}_c^\infty(U)$ with the subspace of $\mathcal{C}_c^\infty(P)$, consisting of all functions with support contained in U, so that $T(F)$ is a well defined element of $\mathcal{C}^\infty(M)$ for $T \in \mathcal{D}'_\pi(P)$ and $F \in \mathcal{C}_c^\infty(U)$. Note that we can equivalently define that $p \in \mathrm{supp}(T)^c$ if and only if there exists an open neighbourhood U of p such that $T(F) = 0$ for every $F \in \mathcal{C}_c^\infty(U)$. From this characterization it easily follows that $\mathrm{supp}(T)$ is a closed subset of P.

The inclusions $\mathcal{C}_c^\infty(M) \hookrightarrow \mathcal{C}^\infty(M)$ and $\mathcal{C}_c^\infty(P) \hookrightarrow \mathcal{C}^\infty(P)$ are continuous, so we can define for any $T \in \mathcal{E}'_\pi(P)$ the composition

$$\mathcal{C}_c^\infty(P) \hookrightarrow \mathcal{C}^\infty(P) \xrightarrow{T} \mathcal{C}_c^\infty(M) \hookrightarrow \mathcal{C}^\infty(M),$$

which is continuous and $\mathcal{C}^\infty(M)$-linear. In particular, this composition defines an element of $\mathcal{D}'_\pi(P)$, which we will again denote by T. As a result we obtain an injective linear map $\mathcal{E}'_\pi(P) \hookrightarrow \mathcal{D}'_\pi(P)$, which enables us to view $\mathcal{E}'_\pi(P)$ as a vector subspace of $\mathcal{D}'_\pi(P)$. Similarly as in the case of distributions on a single manifold, one can prove the following basic result.

Proposition 3.4 *A π-transversal distribution $T \in \mathcal{D}'_\pi(P)$ belongs to $\mathcal{E}'_\pi(P)$ if and only if $\mathrm{supp}(T)$ is compact.*

In the case of smooth families of distributions we have several different notions of support. If we consider a family of distributions as some kind of a vector valued map, it makes sense to define its support as a subset of the base manifold. On the other hand, in view of the correspondence with transversal distributions, we will also consider its support as a subspace of the total space of the submersion.

Definition 3.5 Let $\pi : P \to M$ be a surjective submersion and let u be a smooth family of distributions with compact supports along the fibres of π.

(1) The *support* of u is the subset $\mathrm{supp}(u)$ of M, defined by

$$\mathrm{supp}(u) = \overline{\{x \in M \mid u_x \neq 0\}}.$$

(2) The *total support* of u is the subset $\mathrm{supp}_P(u)$ of P, consisting of all points $p \in P$, which satisfy the condition that for every open neighbourhood U of p there exists $F \in \mathcal{C}_c^\infty(U)$ such that $u(F) \neq 0$.

The relationship between these two types of support is described in the following proposition.

Proposition 3.6 *Let $\pi : P \to M$ be a surjective submersion.*

(1) *For every smooth family $u \in \mathcal{C}^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$ we have*

$$supp_P(u) = \overline{\bigcup_{x\in M} supp(u_x)}.$$

(2) *Let $u \in \mathcal{C}^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$ be a smooth family of distributions. Then $supp(u)$ is compact if and only if $supp_P(u)$ is compact. Moreover, if $supp(u)$ and $supp_P(u)$ are compact, we have $\pi(supp_P(u)) = supp(u)$.*

Proof (1) Let us denote $A = \overline{\bigcup_{x\in M} \mathrm{supp}(u_x)}$. To show that $A \subset \mathrm{supp}_P(u)$, pick any $p \in A$ and arbitrary neighbourhood U of p. We can then find $x' \in \pi(U)$ and $p' \in U \cap \mathrm{supp}(u_{x'})$. From the definition it follows that there exists $F_{x'} \in \mathcal{C}^\infty(P_{x'})$ with compact support in $U \cap P_{x'}$ such that $u_{x'}(F_{x'}) \neq 0$. If $F \in \mathcal{C}_c^\infty(P)$ is an extension of $F_{x'}$ with $\mathrm{supp}(F) \subset U$, it follows that $u(F)(x') \neq 0$, which shows that $p \in \mathrm{supp}_P(u)$.

To prove the inverse inclusion, pick $p \in \mathrm{supp}_P(u)$ and any neighbourhood U of p. Then there exists $F \in \mathcal{C}_c^\infty(U)$ such that $u_x(F|_{P_x}) \neq 0$ for some $x \in \pi(U)$. This implies that $U \cap \mathrm{supp}(u_x) \neq \emptyset$ and hence $p \in A$.

(2) Using contradiction we first prove that compactness of $\mathrm{supp}(u)$ implies compactness of $\mathrm{supp}_P(u)$. Suppose therefore that $\mathrm{supp}(u)$ is compact and that $\mathrm{supp}_P(u)$ is not compact. Then we can find a sequence of points (p_n) in $\mathrm{supp}_P(u)$, together with pairwise disjoint open neighbourhoods U_n of p_n. Since $p_n \in \mathrm{supp}_P(u)$, there exists a function $G_n \in \mathcal{C}_c^\infty(U_n)$ such that $u(G_n) \neq 0$. The function $u(G_n)$ lies in $\mathcal{C}_c^\infty(M)$, so $\overline{u(G_n)}$ is in $\mathcal{C}_c^\infty(M)$ as well and therefore by $\mathcal{C}_c^\infty(M)$-linearity of u we have $u(\overline{u(G_n)} \cdot G_n) = \overline{u(G_n)} u(G_n) \geq 0$. If we now define $F_n \in \mathcal{C}_c^\infty(U_n)$ by

$$F_n = \frac{n}{\max\limits_{x\in M} \overline{u(G_n)}(x) u(G_n)(x)} \overline{u(G_n)} \cdot G_n,$$

we have that $u(F_n)$ is nonnegative and that $\max(u(F_n)) = n$. Since the sets U_n are pairwise disjoint, the function $F : P \to \mathbb{C}$, defined by $F = \sum_{n=1}^\infty F_n$, is well defined and smooth, hence $u(F) \in \mathcal{C}_c^\infty(M)$ by our assumption on compactness of $\mathrm{supp}(u)$. Now note that for any $x \in M$ the continuity of $u_x : \mathcal{C}^\infty(P_x) \to \mathbb{C}$ implies

$$u(F)(x) = u_x(F|_{P_x}) = u_x\left(\sum_{n=1}^\infty F_n|_{P_x}\right) = \sum_{n=1}^\infty u_x(F_n|_{P_x}) = \sum_{n=1}^\infty u(F_n)(x),$$

which shows that the sum $\sum_{n=1}^\infty u(F_n)$ converges pointwise on M to $u(F)$. Since this sum is unbounded on M, this leads us into contradiction.

Suppose now that $\mathrm{supp}_P(u)$ is compact. We then have

$$\pi(\mathrm{supp}_P(u)) = \pi\left(\overline{\bigcup_{x\in M} \mathrm{supp}(u_x)}\right) = \overline{\pi\left(\bigcup_{x\in M} \mathrm{supp}(u_x)\right)} = \mathrm{supp}(u)$$

hence $\mathrm{supp}(u)$ is compact as well. □

We are thus led to define the space of compactly supported smooth families as follows.

Definition 3.7 Let $\pi : P \to M$ be a surjective submersion. A smooth family $u \in \mathcal{C}^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$ has *compact support* if $\mathrm{supp}(u)$ or equivalently $\mathrm{supp}_P(u)$ is a compact set. The vector space of compactly supported smooth families will be denoted by $\mathcal{C}_c^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$.

In the remainder of this section we will show that we have a natural isomorphism between vector spaces $\mathcal{E}'_\pi(P)$ and $\mathcal{C}_c^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$.

For any closed submanifold N of P we denote by $I_N = \{F \in \mathcal{C}^\infty(P) \mid F|_N = 0\}$ the ideal of functions that vanish on N. Furthermore, for every $x \in M$ we denote by $\mathrm{ev}_x : \mathcal{C}^\infty(M) \to \mathbb{C}$ the evaluation at the point x. The kernel of ev_x is the maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{C}^\infty(M) \mid f(x) = 0\}$ of functions that vanish at x. Using Taylor expansion it was observed in [11] that any $F \in I_{P_x}$ can be written in the form

$$F = f_1 \cdot F_1 + \cdots + f_k \cdot F_k$$

for some functions $F_1, \ldots, F_k \in \mathcal{C}^\infty(P)$ and $f_1, \ldots, f_k \in \mathfrak{m}_x$. This gives us for every $x \in M$ the equality

$$\mathfrak{m}_x \cdot \mathcal{C}^\infty(P) = I_{P_x}.$$

Now choose any $T \in \mathcal{E}'_\pi(P)$ and any $F \in I_{P_x}$. From $\mathcal{C}^\infty(M)$-linearity of T it follows

$$\begin{aligned}(\mathrm{ev}_x \circ T)(F) &= T(f_1 \cdot F_1 + \cdots + f_k \cdot F_k)(x)\\ &= f_1(x)T(F_1)(x) + \cdots + f_k(x)T(F_k)(x) = 0,\end{aligned}$$

hence we obtain the induced continuous map $\mathrm{ev}_x \circ T : \mathcal{C}^\infty(P)/I_{P_x} \to \mathbb{C}$. By composing this induced map with the natural isomorphism of locally convex spaces $\mathcal{C}^\infty(P)/I_{P_x} \cong \mathcal{C}^\infty(P_x)$ we get for every $x \in M$ a continuous linear map

$$T_x : \mathcal{C}^\infty(P_x) \to \mathbb{C}.$$

Since P_x is a closed submanifold of P, every function in $\mathcal{C}^\infty(P_x)$ is a restriction of a function $F \in \mathcal{C}^\infty(P)$ and T_x is then defined by $T_x(F|_{P_x}) = T(F)(x)$, independent of the choice of the extension F. It follows directly from the definition that $(T_x)_{x\in M} \in \prod_{x\in M} \mathcal{E}'(P_x)$ is a smooth compactly supported family and that there is a $\mathcal{C}_c^\infty(M)$-linear map

$$\Phi : \mathcal{E}'_\pi(P) \to \mathcal{C}_c^\infty\left(\prod_{x\in M} \mathcal{E}'(P_x)\right),$$

given by

$$\Phi(T) = (T_x)_{x \in M}.$$

To see that Φ is an isomorphism, we first consider a special case when $P = M \times N$ is a trivial bundle over M with fiber N and π is the projection onto M. In this case we can also define the vector space

$$\mathcal{C}_c^\infty(M, \mathcal{E}'(N))$$

of all smooth compactly supported functions on M with values in the locally convex space $\mathcal{E}'(N)$. As shown in [11] (see also [6]) we have the following basic result.

Proposition 3.8 *Let $P = M \times N$ and let $\pi : M \times N \to M$ be the projection. Then, we have natural isomorphisms of vector spaces*

$$\mathcal{E}'_\pi(P) \cong \mathcal{C}_c^\infty\left(\prod_{x \in M} \mathcal{E}'(P_x)\right) \cong \mathcal{C}_c^\infty(M, \mathcal{E}'(N)).$$

The first isomorphism is given by the map Φ, as described above, while for the second one we have to identify $\mathcal{E}'(P_x) = \mathcal{E}'(\{x\} \times N)$ with $\mathcal{E}'(N)$ for each $x \in M$ and then interpret a smooth family as a smooth $\mathcal{E}'(N)$-valued map on M.

Now choose an open subset U of P. We will denote by

$$\mathcal{E}'_\pi(U) = \{T \in \mathcal{E}'_\pi(P) \mid \operatorname{supp}(T) \subset U\}$$

the subspace of $\mathcal{E}'_\pi(P)$, consisting of distributions with support contained in U. We can also consider the restriction $\pi|_U : U \to \pi(U)$, which is again a surjective submersion. We then have a continuous injective map

$$\mathcal{E}'_{\pi|_U}(U) \hookrightarrow \mathcal{E}'_\pi(P)$$

with image $\mathcal{E}'_\pi(U)$. In particular, the vector spaces $\mathcal{E}'_{\pi|_U}(U)$ and $\mathcal{E}'_\pi(U)$ are isomorphic, but in general the topology on $\mathcal{E}'_{\pi|_U}(U)$ is finer than the topology on $\mathcal{E}'_\pi(U)$ induced from $\mathcal{E}'_\pi(P)$.

Theorem 3.9 *For every surjective submersion $\pi : P \to M$, the map*

$$\Phi : \mathcal{E}'_\pi(P) \to \mathcal{C}_c^\infty\left(\prod_{x \in M} \mathcal{E}'(P_x)\right),$$

defined by $\Phi(T) = (T_x)_{x \in M}$, is an isomorphism of $\mathcal{C}_c^\infty(M)$-modules.

Proof It follows directly from the definition that Φ is injective and $\mathcal{C}_c^\infty(M)$-linear, so we have to show that it is surjective. Take any $u \in \mathcal{C}_c^\infty(\prod_{x \in M} \mathcal{E}'(P_x))$. Since π is a submersion and $\mathrm{supp}_P(u) \subset P$ is compact, we can cover $\mathrm{supp}_P(u)$ with open subsets $U_1, U_2, \ldots, U_k$ of P, such that π restricted to each U_i is a trivial fiber bundle. By using a suitable partition of unity and the $\mathcal{C}^\infty(P)$-module structure on $\mathcal{C}_c^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$, we can decompose

$$u = u_1 + u_2 + \cdots + u_k,$$

such that $\mathrm{supp}_P(u_i) \subset U_i$ for each i. If we choose trivializations $U_i \approx \pi(U_i) \times N_i$ for restrictions $\pi|_{U_i}$, we can view u_i as an element of the space $\mathcal{C}_c^\infty(\pi(U_i), \mathcal{E}'(N_i))$. Proposition 3.8 gives us a series of isomorphisms

$$\mathcal{C}_c^\infty(\pi(U_i), \mathcal{E}'(N_i)) \cong \mathcal{E}'_{\pi|_{U_i}}(U_i) \cong \mathcal{E}'_\pi(U_i).$$

We then have $u = \Phi(T_1 + \cdots + T_k)$, where $T_i \in \mathcal{E}'_\pi(U_i) \subset \mathcal{E}'_\pi(P)$ is the element, corresponding to u_i under the above isomorphism. □

Let us mention that one can define a locally convex topology on the space $\mathcal{C}_c^\infty(\prod_{x\in M} \mathcal{E}'(P_x))$, so that Φ becomes an isomorphism of locally convex spaces (see [11] for more details). However, in our case we will be interested in a particular description of topology on the space $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$.

We start by describing a basis of neighbourhoods of zero in $\mathcal{C}_c^\infty(\mathbb{R}^l)$ (see [5] for more details). Denote by $K_0 = \emptyset \subset K_1 \subset K_2 \subset \ldots$ an exhaustion of $\mathbb{R}^l$ by balls $K_n = \{x \in \mathbb{R}^l \mid |x| \leq n\}$ with centres at zero and radius $n \in \mathbb{N}$. Furthermore, denote by $\mathbf{m} = (m_1, m_2, \ldots)$ an increasing sequence of natural numbers and by $\mathbf{e} = (\epsilon_1, \epsilon_2, \ldots)$ a decreasing sequence of positive real numbers. The neighbourhood basis of zero for the LF-topology on $\mathcal{C}_c^\infty(\mathbb{R}^l)$ then consists of the sets of the form

$$V_{\mathbf{m},\mathbf{e}} = \{f \in \mathcal{C}_c^\infty(\mathbb{R}^l) \mid |D^\alpha f(x)| < \epsilon_n \text{ for } x \in K_{n-1}^c \text{ and } |\alpha| \leq m_n\},$$

as $\mathbf{m}$ and $\mathbf{e}$ vary over all sequences as above. In this spirit we also define a basis of neighbourhoods of zero in $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$. For any bounded subset $B \subset \mathcal{C}^\infty(N)$ we define a seminorm p_B on $\mathcal{E}'(N)$ by

$$p_B(v) = \sup_{F\in B} |v(F)|$$

for $v \in \mathcal{E}'(N)$. For any increasing sequence $\mathbf{B} = (B_1, B_2, \ldots)$ of bounded subsets of $\mathcal{C}^\infty(N)$ and $\mathbf{m}$ and $\mathbf{e}$ as above we define

$$V_{\mathbf{B},\mathbf{m},\mathbf{e}} = \{u \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N)) \mid p_{B_n}((D^\alpha u)_x) < \epsilon_n \text{ for } x \in K_{n-1}^c \text{ and } |\alpha| \leq m_n\}.$$

All such sets form a basis of neighbourhoods of zero for a locally convex topology on $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$ and we have the following proposition.

Proposition 3.10 *Let $\pi : \mathbb{R}^l \times N \to \mathbb{R}^l$ be the projection onto $\mathbb{R}^l$. The map*

$$\Phi : \mathcal{E}'_\pi(\mathbb{R}^l \times N) \to \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$$

is then an isomorphism of locally convex spaces.

Proof We will use the isomorphism $\mathcal{C}^\infty(\mathbb{R}^l \times N) \cong \mathcal{C}^\infty(\mathbb{R}^l, \mathcal{C}^\infty(N))$ under which a function $F \in \mathcal{C}^\infty(\mathbb{R}^l \times N)$ is identified with the smooth $\mathcal{C}^\infty(N)$-valued function on $\mathbb{R}^l$, given by $x \mapsto F_x$, where $F_x = F|\{x\} \times N$. This will enable us to compute partial derivatives of such functions in the directions on the base manifold.

First we show that Φ is continuous. Take any basic neighbourhood $V_{\mathbf{B},\mathbf{m},\mathbf{e}}$ of zero in $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$. We can then find a sequence (μ_n) of positive real numbers, such that $\bigcup_{n=1}^\infty \mu_n B_n$ is a bounded subset of $\mathcal{C}^\infty(N)$. If we denote by $\pi_N : \mathbb{R}^l \times N \to N$ the projection onto N, the set $B = \pi_N^*\left(\bigcup_{n=1}^\infty \mu_n B_n\right)$ is then a bounded subset of $\mathcal{C}^\infty(\mathbb{R}^l \times N)$. Any function in B is of the form F, π_N for some $F \in \bigcup_{n=1}^\infty \mu_n B_n$ and hence corresponds to a constant map $x \mapsto F$ in $\mathcal{C}^\infty(\mathbb{R}^l, \mathcal{C}^\infty(N))$. For any $u \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$ and any multi-index α we thus have

$$D^\alpha(u(F, \pi_N))(x) = D^\alpha(x \mapsto u_x(F))(x) = (D^\alpha u)_x(F).$$

Now define a sequence $\mathbf{e}'$ by $\epsilon_n' = \frac{\mu_n \epsilon_n}{2}$ and take any $T \in K(B, V_{\mathbf{m},\mathbf{e}'})$. For any $x \in K_{n-1}^c$ and any multi-index α with $|\alpha| \leq m_n$ we have

$$\begin{aligned} p_{B_n}((D^\alpha \Phi(T))_x) &= \sup_{F \in B_n} |(D^\alpha \Phi(T))_x(F)| = \sup_{F \in B_n} |D^\alpha(T(F, \pi_N))(x)|, \\ &= \frac{1}{\mu_n} \sup_{F \in \mu_n B_n} |D^\alpha(T(F, \pi_N))(x)|, \\ &\leq \frac{1}{\mu_n} \sup_{G \in B} |D^\alpha(T(G))(x)|. \end{aligned}$$

Since $G \in B$ and $T \in K(B, V_{\mathbf{m},\mathbf{e}'})$, it follows that $T(G) \in V_{\mathbf{m},\mathbf{e}'}$. So for $x \in K_{n-1}^c$ and any multi-index α with $|\alpha| \leq m_n$ we have $|D^\alpha(T(G))(x)| < \epsilon_n'$, which shows that

$$p_{B_n}((D^\alpha \Phi(T))_x) \leq \frac{1}{\mu_n} \sup_{G \in B} |D^\alpha(T(G))(x)| < \epsilon_n$$

and hence $\Phi(K(B, V_{\mathbf{m},\mathbf{e}'})) \subset V_{\mathbf{B},\mathbf{m},\mathbf{e}}$.

To see that Φ is an open map, take any basic neighbourhood $K(B, V_{\mathbf{m},\mathbf{e}})$ of zero in $\mathcal{E}_\pi'(\mathbb{R}^l \times N)$. Now define a sequence (B_n) of bounded subsets of $\mathcal{C}^\infty(N)$ by

$$B_n = \{(D^\gamma F)_x \mid x \in K_n, |\gamma| \leq m_n, F \in B\}.$$

Furthermore, define a sequence $\mathbf{e}'$ by $\epsilon_n' = \frac{\epsilon_n}{2^{m_n}}$. For any $u \in V_{\mathbf{B},\mathbf{m},\mathbf{e}'}$, any $x \in K_{n-1}^c$, any $F \in B$ and any multi-index α with $|\alpha| \leq m_n$ we then have

$$|D^\alpha(u(F))(x)| = |D^\alpha(x \mapsto u_x(F_x))(x)| = \left| \sum_{\alpha=\beta+\gamma} (D^\beta u)_x (D^\gamma F)_x \right|,$$
$$\leq \sum_{\alpha=\beta+\gamma} |(D^\beta u)_x (D^\gamma F)_x|.$$

By definition we have $(D^\gamma F)_x \in B_{n'}$ for some $n' \geq n$, so we have a bound

$$|(D^\beta u)_x (D^\gamma F)_x| \leq p_{B_{n'}}((D^\beta u)_x) < \epsilon'_{n'} < \epsilon'_n.$$

Since there are at most 2^{m_n} possibilities of writing $\alpha = \beta + \gamma$, we get that

$$|D^\alpha(u(F))(x)| < \epsilon_n.$$

This shows that $V_{\mathbf{B},\mathbf{m},\mathbf{e}'} \subset \Phi(K(B, V_{\mathbf{m},\mathbf{e}}))$ which concludes the proof. □

4 Reflexivity of the Space of Transversal Distributions

The space of π-transversal distributions $\mathcal{E}'_\pi(P)$ is the strong $\mathcal{C}^\infty_c(M)$-dual of the algebra of smooth functions $\mathcal{C}^\infty(P)$. In particular, $\mathcal{E}'_\pi(P)$ is again a $\mathcal{C}^\infty_c(M)$-module, so we can consider its strong $\mathcal{C}^\infty_c(M)$-dual. We will show in this section that we have a canonical isomorphism between $\mathcal{C}^\infty(P)$ and the strong $\mathcal{C}^\infty_c(M)$-dual of $\mathcal{E}'_\pi(P)$.

Let M be a smooth manifold and let $\mathcal{M}$ be a $\mathcal{C}^\infty_c(M)$-module. For any $x \in M$ we define 'evaluation' of $\mathcal{M}$ at x as the quotient

$$\mathcal{M}(x) = \frac{\mathcal{M}}{\mathfrak{m}_x \cdot \mathcal{M}}.$$

We will first show that for any smooth surjective submersion $\pi : P \to M$ and any $x \in M$ the space $\mathcal{E}'_\pi(P)(x)$ is isomorphic to $\mathcal{E}'(P_x)$ as a locally convex vector space.

Proposition 4.1 *Let $\pi : P \to M$ be a surjective submersion.*

1. *For every $x \in M$ we have*

$$\mathfrak{m}_x \cdot \mathcal{E}'_\pi(P) = \{T \in \mathcal{E}'_\pi(P) \mid T_x = 0\}.$$

2. *The linear map $\Phi_x : \mathcal{E}'_\pi(P) \to \mathcal{E}'(P_x)$, defined by $\Phi_x(T) = T_x$, induces an isomorphism of vector spaces $\Phi_x : \mathcal{E}'_\pi(P)(x) \to \mathcal{E}'(P_x)$ for every $x \in M$.*

Proof (1) By definition it follows that $\mathfrak{m}_x \cdot \mathcal{E}'_\pi(P) \subset \{T \in \mathcal{E}'_\pi(P) \mid T_x = 0\}$, so we only have to show the inclusion in the other direction. Choose any $T \in \mathcal{E}'_\pi(P)$ such that $T_x = 0$. The set $K = \text{supp}(T) \cap P_x$ is then a compact subset of P_x, so we can find an open neighbourhood U of K in P, such that $\pi|_U$ is a trivial fiber bundle. Choose a partition of unity $\{\rho_U, \rho_{K^c}\}$, subordinated to the open cover $\{U, K^c\}$ of P. We can then decompose

$$T = \rho_U T + \rho_{K^c} T.$$

By construction, the support of $\rho_{K^c} T$ is compact and disjoint from P_x. So we can find $f_{K^c} \in \mathfrak{m}_x$ such that $f_{K^c} \equiv 1$ on a neighbourhood of $\pi(\mathrm{supp}(\rho_{K^c} T))$, hence

$$\rho_{K^c} T = f_{K^c} \cdot (\rho_{K^c} T) \in \mathfrak{m}_x \cdot \mathcal{E}'_\pi(P).$$

On the other hand, let us choose a trivialization $U \approx \pi(U) \times N$ for $\pi|_U$. We can then consider $\rho_U T$ as a smooth $\mathcal{E}'(N)$-valued function on $\pi(U)$ with zero at x. Taylor expansion (see [10]) now gives us $T_1, \ldots, T_k \in \mathcal{C}^\infty_c(\pi(U), \mathcal{E}'(N)) \cong \mathcal{E}'_\pi(U)$ and functions $f_1, \ldots, f_k \in \mathfrak{m}_x$, such that

$$\rho_U T = f_1 \cdot T_1 + \cdots + f_k \cdot T_k \in \mathfrak{m}_x \cdot \mathcal{E}'_\pi(P).$$

(2) First we show that $\Phi_x : \mathcal{E}'_\pi(P) \to \mathcal{E}'(P_x)$ is surjective. Choose any $v \in \mathcal{E}'(P_x)$ with $\mathrm{supp}(v) = K \subset P_x$. Similarly as above, let us choose an open neighbourhood $U \approx \pi(U) \times N$ of K such that $\pi|_U$ is a trivial fiber bundle. Choose $f \in \mathcal{C}^\infty_c(\pi(U))$, such that $f(x) = 1$, and define $\overline{v} \in \mathcal{C}^\infty_c(\pi(U), \mathcal{E}'(N)) \cong \mathcal{E}'_\pi(U) \subset \mathcal{E}'_\pi(P)$ by

$$\overline{v}_y = f(y)v$$

for $y \in \pi(U)$. We then have $\Phi_x(\overline{v}) = v$.

Since $\ker(\Phi_x) = \{T \in \mathcal{E}'_\pi(P) \mid T_x = 0\}$, we get by (1) the induced isomorphism

$$\Phi_x : \mathcal{E}'_\pi(P)(x) \to \mathcal{E}'(P_x).$$

□

To show that the map Φ_x is an isomorphism of locally convex spaces, we need the following lemma. Let N be a smooth manifold and $K \subset N$ a compact subset. We then denote by $\mathcal{E}'(K)$ the vector subspace of $\mathcal{E}'(N)$ consisting of all distributions with support contained in K.

Lemma 4.2 *Let N be a smooth manifold and let $V \subset \mathcal{E}'(N)$ be an absolutely convex set. Then, V is a neighbourhood of zero in $\mathcal{E}'(N)$ if and only if $V \cap \mathcal{E}'(K)$ is a neighbourhood of zero in $\mathcal{E}'(K)$ for each compact subset K of N.*

Proof The space $\mathcal{E}'(N)$ is barrelled and we have $\mathcal{E}'(N) = \bigcup_K \mathcal{E}'(K)$ as K ranges over all compact subsets of N. The proof now follows from Corollary 1.5 in [15]. □

Proposition 4.3 *Let $\pi : P \to M$ be a surjective submersion. The isomorphism*

$$\Phi_x : \mathcal{E}'_\pi(P)(x) \to \mathcal{E}'(P_x)$$

is an isomorphism of locally convex vector spaces for every $x \in M$.

Proof We have already shown that Φ_x is an isomorphism of vector spaces, so it remains to be shown that it is continuous and open. It is actually enough to show that $\Phi_x : \mathcal{E}'_\pi(P) \to \mathcal{E}'(P_x)$ is continuous and open.

To see that Φ_x is continuous, choose any basic open neighbourhood $K(B, \epsilon)$ of zero in $\mathcal{E}'(P_x)$, where $B \subset \mathcal{C}^\infty(P_x)$ is bounded and $\epsilon > 0$. Note that the restriction map $\mathrm{Res}_x : \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P_x)$ is surjective and continuous. Using a tubular neighbourhood of P_x in P we can construct a continuous section $\mathrm{Ext}_x : \mathcal{C}^\infty(P_x) \to \mathcal{C}^\infty(P)$ of Res_x. It follows that $\mathrm{Ext}_x(B)$ is bounded in $\mathcal{C}^\infty(P)$ and

$$\Phi_x(K(\mathrm{Ext}_x(B), V)) \subset K(B, \epsilon),$$

where $V = \{f \in \mathcal{C}_c^\infty(M) \mid |f(x)| < \epsilon\}$ is an open neighbourhood of zero in $\mathcal{C}_c^\infty(M)$.

We will now show that Φ_x is an open map. First we consider the case when $P = \mathbb{R}^l \times N$ is a trivial bundle. It is enough to show that for any basic open neighbourhood $V_{\mathbf{B},\mathbf{m},\mathbf{e}}$ of zero in $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$ the set $\Phi_x(V_{\mathbf{B},\mathbf{m},\mathbf{e}})$ is a neighbourhood of zero in $\mathcal{E}'(P_x) \cong \mathcal{E}'(N)$. Take such $n \in \mathbb{N}$ that $x \in \mathrm{Int}(K_n)$ and then choose $\rho \in \mathcal{C}_c^\infty(\mathrm{Int}(K_n))$ with $\rho(x) = 1$ and denote $M = \max_{y \in K_n, |\alpha| \le m_n} |(D^\alpha \rho)(y)|$. Now define $\overline{v} = \rho v \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(N))$ for any $v \in \mathcal{E}'(N)$. If $v \in K(B_n, \frac{\epsilon_n}{M})$, we have for any $y \in K_n$ and any $|\alpha| \le m_n$ the following bound

$$p_{B_n}(D^\alpha(\overline{v})_y) = p_{B_n}((D^\alpha \rho)(y) v) \le M p_{B_n}(v) < \epsilon_n,$$

which shows that $\overline{v} \in V_{\mathbf{B},\mathbf{m},\mathbf{e}}$ and hence $K(B_n, \frac{\epsilon_n}{M}) \subset \Phi_x(V_{\mathbf{B},\mathbf{m},\mathbf{e}})$.

In the case of a general submersion $\pi : P \to M$, choose an absolutely convex neighbourhood V of zero in $\mathcal{E}'_\pi(P)$. We need to show that $\Phi_x(V) \cap \mathcal{E}'(K)$ is a neighbourhood of zero in $\mathcal{E}'(K)$ for every compact subset $K \subset P_x$. First choose an open neighbourhood $U \approx \pi(U) \times N$ of K in P, on which π is trivial and $\pi(U) \approx \mathbb{R}^l$. The set $V \cap \mathcal{E}'_\pi(U)$ is then a neighbourhood of zero in $\mathcal{E}'_\pi(U) \cong \mathcal{C}_c^\infty(\pi(U), \mathcal{E}'(N))$, which by the previous paragraph implies that $\Phi_x(V \cap \mathcal{E}'_\pi(U))$ is a neighbourhood of zero in $\mathcal{E}'(N)$. Since $\mathcal{E}'(K) \subset \mathcal{E}'(N) \subset \mathcal{E}'(P_x)$, this also implies that $\Phi_x(V) \cap \mathcal{E}'(K)$ is a neighbourhood of zero in $\mathcal{E}'(K)$. The proof now follows from Lemma 4.2. □

In the remainder of this section we will compute the strong $\mathcal{C}_c^\infty(M)$-dual of the space $\mathcal{E}'_\pi(P)$. It is the locally convex space

$$\mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M)),$$

equipped with the topology of uniform convergence on bounded subsets of $\mathcal{E}'_\pi(P)$. For every $F \in \mathcal{C}^\infty(P)$ we can define an element $\hat{F} \in \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$ by $\hat{F}(T) = T(F)$ for $T \in \mathcal{E}'_\pi(P)$, to obtain a $\mathcal{C}_c^\infty(M)$-linear map

$$\hat{\ } : \mathcal{C}^\infty(P) \to \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M)).$$

Our goal will be to show that the above map is an isomorphism of locally convex vector spaces.

Choose any $a \in \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$, any $x \in M$ and any $T \in \mathfrak{m}_x \cdot \mathcal{E}'_\pi(P)$, which we write as $T = f_1 \cdot T_1 + \cdots + f_k \cdot T_k$ for some $f_i \in \mathfrak{m}_x$ and $T_i \in \mathcal{E}'_\pi(P)$. We now have

$$\begin{aligned} a(T)(x) &= a(f_1 \cdot T_1 + \cdots + f_k \cdot T_k)(x) \\ &= f_1(x)a(T_1)(x) + \cdots + f_k(x)a(T_k)(x) = 0, \end{aligned}$$

which implies that we have the induced continuous map

$$a_x : \mathcal{E}'_\pi(P)(x) \to \mathbb{C}.$$

Since $\mathcal{E}'_\pi(P)(x) \cong \mathcal{E}'(P_x)$ by Proposition 4.1 and since $\mathcal{E}'(P_x)' \cong \mathcal{C}^\infty(P_x)$, we can identify a_x with some function $\check{a}_x \in \mathcal{C}^\infty(P_x)$ such that $(\check{a}_x)\hat{} = a_x$. We can therefore assign to a a family of smooth functions $(\check{a}_x)_{x\in M} \in \prod_{x\in M} \mathcal{C}^\infty(P_x)$, which can be seen as a function $\check{a} : P \to \mathbb{C}$, which is smooth along the fibres of π.

To show that $\check{a}$ is a smooth function on P, we first recall that $\mathcal{C}^\infty(P)$ acts on $\mathcal{E}'_\pi(P)$ by $(F \cdot T)(G) = T(FG)$ for $F, G \in \mathcal{C}^\infty(P)$ and $T \in \mathcal{E}'_\pi(P)$. Since the multiplication map $F \cdot \_ : \mathcal{E}'_\pi(P) \to \mathcal{E}'_\pi(P)$ is continuous, we get the induced action of $\mathcal{C}^\infty(P)$ on $\mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$, defined by

$$(F \cdot a)(T) = a(F \cdot T),$$

for $F \in \mathcal{C}^\infty(P)$, $a \in \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$ and $T \in \mathcal{E}'_\pi(P)$. We have the basic relation

$$\widetilde{F \cdot a} = F\check{a},$$

which shows that the operation $a \mapsto \check{a}$ is $\mathcal{C}^\infty(P)$-linear.

Proposition 4.4 *Let $\pi : P \to M$ be a surjective submersion. Then for every $a \in \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$ the induced function $\check{a} : P \to \mathbb{C}$ is smooth and we have $(\check{a})\hat{} = a$.*

Proof Assume first that $P = M \times N$ is a trivial bundle over M with fiber N. To show the main idea, we will for simplicity also assume that M is compact. We can then view the family $\check{a} = (\check{a}_x)_{x\in M}$ as a function on M with values in the Fréchet space $\mathcal{C}^\infty(N)$. To show that it is smooth, it is enough to show that it is scalarly smooth since $\mathcal{C}^\infty(N)$ is complete (see Theorem 2.14 in [10]). Denote for any $v \in \mathcal{E}'(N)$ by $\overline{v} \in \mathcal{C}^\infty(M, \mathcal{E}'(N))$ the constant transversal distribution with value v. The function $x \mapsto \check{a}_x$ is then scalarly smooth on M since

$$x \mapsto v(\check{a}_x) = a_x(v) = a(\overline{v})(x)$$

and $a(\overline{v}) \in \mathcal{C}^\infty(M)$. This shows that $\check{a} : M \to \mathcal{C}^\infty(N)$ is smooth and, using the isomorphism $\mathcal{C}^\infty(M, \mathcal{C}^\infty(N)) \cong \mathcal{C}^\infty(M \times N)$ (see Theorem 3.12 in [10]), we can now conclude that $\check{a} \in \mathcal{C}^\infty(M \times N)$. If M is not compact, one can show by using partitions of unity that $\check{a}$ is locally smooth, which however implies smoothness.

Now we consider the case of a general submersion $\pi : P \to M$. Choose any $p \in P$ and an open neighbourhood $U \approx \pi(U) \times N$ of p, on which π is trivial. Furthermore, choose $\chi \in \mathcal{C}_c^\infty(U)$ which is equal to 1 on some neighbourhood of p and suppose $\mathrm{supp}(\chi)$ is contained in $L \times K$, where $L \subset \pi(U)$ and $K \subset N$ are compact subsets. The element $\chi \cdot a \in \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$ then maps into $\mathcal{C}_c^\infty(L)$ and induces a continuous mapping

$$\mathcal{E}'_{\pi|_U}(U) \hookrightarrow \mathcal{E}'_\pi(P) \xrightarrow{\chi \cdot a} \mathcal{C}_c^\infty(L) \hookrightarrow \mathcal{C}_c^\infty(\pi(U)).$$

By the above paragraph $\widecheck{\chi \cdot a}$ is a smooth function on U. But since $\widecheck{\chi \cdot a} = \chi \check{a}$ and since $\chi \equiv 1$ on some neighbourhood of p, this implies that $\check{a}$ is smooth on some neighbourhood of p. The equality $(\check{a})\hat{} = a$ then follows directly from the definition. □

We are now ready to state and prove our main theorem.

Theorem 4.5 *Let $\pi : P \to M$ be a surjective submersion. The map*

$$\hat{}: \mathcal{C}^\infty(P) \to Hom_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$$

is an isomorphism of locally convex vector spaces. In particular, this implies that $\mathcal{C}^\infty(P)$ and $\mathcal{E}'_\pi(P)$ are reflexive $\mathcal{C}_c^\infty(M)$-modules.

Proof We first show that the given map is an isomorphism of vector spaces. To show that it is injective, suppose $\hat{F} = \hat{G}$ for some $F, G \in \mathcal{C}^\infty(P)$. This implies that $T(F) = T(G)$ for every $T \in \mathcal{E}'_\pi(P)$. In particular, for every $x \in M$ we have $T_x(F_x) = T_x(G_x)$. Since the elements of $\mathcal{E}'(P_x)$ separate points of $\mathcal{C}^\infty(P_x)$, we have that $F_x = G_x$ for every $x \in M$ and hence $F = G$. The surjectivity follows from the fact that for all $a \in \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$ we have $(\check{a})\hat{} = a$.

Next we show that $\hat{} : \mathcal{C}^\infty(P) \to \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$ is continuous. Choose an arbitrary basic neighbourhood $K(B, V)$ of zero in $\mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(P), \mathcal{C}_c^\infty(M))$. The subset B of $\mathcal{E}'_\pi(P)$ is bounded, hence it is, by the Banach-Steinhaus theorem, equicontinuous. So we can find a neighbourhood V' of zero in $\mathcal{C}^\infty(P)$, such that $B(V') \subset V$. For every $F \in V'$ and every $T \in B$ it then follows $\hat{F}(T) = T(F) \in V$, which shows that $\widehat{V'} \subset K(B, V)$.

Finally, we have to show that the given map is open. We can choose a subbasis of open neighbourhoods of zero in $\mathcal{C}^\infty(P)$ in the following way. Choose an open subset $U \approx \pi(U) \times \mathbb{R}^k$ of P on which π is a trivial bundle with fiber $\mathbb{R}^k$ and assume $\pi(U) \approx \mathbb{R}^l$. Choose compact subsets $L \subset \pi(U)$ and $K \subset \mathbb{R}^k$, $n \in \mathbb{N}_0$, $\epsilon > 0$ and define

$$V_{L \times K, n, \epsilon} = \{F \in \mathcal{C}^\infty(P) \,|\, |D_x^\alpha D_y^\beta F(x, y)| < \epsilon \text{ for } x \in L,\ y \in K,\ |\alpha + \beta| \le n\}.$$

Every neighbourhood of zero in $\mathcal{C}^\infty(P)$ contains a finite intersection of sets as above, so it is enough to show that the image of $V_{L \times K, n, \epsilon}$ is open. To this extent choose

$\eta \in \mathcal{C}_c^\infty(\pi(U))$, such that $\eta \equiv 1$ on some neighbourhood of L. Using the notation from Example 3.2 we define a subset B of $\mathcal{E}'_\pi(P)$ by

$$B = \{[\![E_y, \eta\, D_y^\beta]\!] \mid y \in K, |\beta| \le n\},$$

where E_y is the image of the constant section $\sigma_{E_y} : \pi(U) \to \pi(U) \times \mathbb{R}^k$ with value y. To see that B is bounded in $\mathcal{E}'_\pi(P)$, it is by the Banach–Steinhaus theorem enough to show that $B(F)$ is bounded in $\mathcal{C}_c^\infty(M)$ for every $F \in \mathcal{C}^\infty(P)$. Note that $B(F)$ lies in the Fréchet space $\mathcal{C}_c^\infty(\mathrm{supp}(\eta))$, so we have to show that $p_{\mathrm{supp}(\eta),m}(B(F))$ is bounded in $\mathbb{R}$ for every $m \in \mathbb{N}_0$. This follows from

$$\begin{aligned} p_{\mathrm{supp}(\eta),m}(B(F)) &= \sup_{\substack{T\in B\\ x\in\mathrm{supp}(\eta)\\ |\alpha|\le m}} |D_x^\alpha(T(F))(x)| = \sup_{\substack{(x,y)\in\mathrm{supp}(\eta)\times K\\ |\alpha|\le m\\ |\beta|\le n}} |D_x^\alpha D_y^\beta(\eta\cdot F)(x,y)|,\\ &\le p_{\mathrm{supp}(\eta)\times K, m+n}(\eta\cdot F). \end{aligned}$$

Now denote by $V_{L,n,\frac{\epsilon}{2}}$ the neighbourhood of zero in $\mathcal{C}_c^\infty(M)$, given by

$$V_{L,n,\frac{\epsilon}{2}} = \{f \in \mathcal{C}_c^\infty(M) \mid |D^\alpha f(x)| < \tfrac{\epsilon}{2} \text{ for } x \in L, |\alpha| \le n\}.$$

For any $a \in K(B, V_{L,n,\frac{\epsilon}{2}})$ we then have

$$\sup_{\substack{T\in B\\ x\in L\\ |\alpha|\le n}} |D_x^\alpha(T(\check{a}))(x)| = \sup_{\substack{T\in B\\ x\in L\\ |\alpha|\le n}} |D_x^\alpha(a(T))(x)| < \epsilon$$

and consequently

$$\sup_{\substack{(x,y)\in L\times K\\ |\alpha+\beta|\le n}} |D_x^\alpha D_y^\beta \check{a}(x,y)| \le \sup_{\substack{(x,y)\in L\times K\\ |\alpha|,|\beta|\le n}} |D_x^\alpha D_y^\beta \check{a}(x,y)| = \sup_{\substack{T\in B\\ x\in L\\ |\alpha|\le n}} |D_x^\alpha(T(\check{a}))(x)| < \epsilon.$$

This shows that $\check{a} \in V_{L\times K,n,\epsilon}$ and hence $K(B, V_{L,n,\frac{\epsilon}{2}}) \subset \widehat{V_{L\times K,n,\epsilon}}$. □

Acknowledgements I would like to thank O. Dragičević, F. Forstnerič, A. Kostenko and J. Mrčun for support during research.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Androulidakis, I., Skandalis, G.: Pseudodifferential calculus on a singular foliation. J. Noncommut. Geom. **5**, 125–152 (2011)
2. Androulidakis, I., Mohsen, O., Yuncken, R.: The convolution algebra of Schwarz kernels on a singular foliation. J. Oper. Theory **85**(2), 475–503 (2021)
3. Androulidakis, I., Mohsen, O., Yuncken, R.: A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer–Nourrigat conjecture. arXiv:2201.12060
4. Debord, C., Skandalis, G.: Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and pseudodifferential calculus. Adv. Math. **257**, 66–91 (2014)
5. Horváth, J.: Topological Vector Spaces, vol. I. Addison-Wesley Series in Mathematics, p. XII. Addison-Wesley Publishing Company, London (1966)
6. Kališnik, J.: Automatic continuity of transversal distributions. Proc. Am. Math. Soc. **150**(12), 5243–5251 (2022)
7. Kališnik, J.: Locally convex bialgebroid of an action Lie groupoid. arXiv:2303.11386
8. Kališnik, J., Mrčun, J.: A Cartier–Gabriel–Kostant structure theorem for Hopf algebroids. Adv. Math. **232**, 295–310 (2013)
9. Kališnik, J., Mrčun, J.: Convolution bialgebra of a Lie groupoid and transversal distributions. J. Geom. Phys. **180**, 104642 (2022)
10. Kriegl, A., Michor, P.W.: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence (1997)
11. Lescure, J.-M., Manchon, D., Vassout, S.: About the convolution of distributions on groupoids. J. Noncommut. Geom. **11**, 757–789 (2017)
12. Mrčun, J.: On duality between étale groupoids and Hopf algebroids. J. Pure Appl. Algebra **210**, 267–282 (2007)
13. Mrčun, J.: Sheaf coalgebras and duality. Topol. Appl. **154**, 2795–2812 (2007)
14. Tréves, F.: Topological Vector Spaces. Distributions and Kernels. Academic Press, New York (1967)
15. Valdivia, M.: Absolutely convex sets in barrelled spaces. Ann. Inst. Fourier **21**(2), 3–13 (1971)
16. van Erp, E., Yuncken, R.: A groupoid approach to pseudodifferential operators. J. Reine Angew. Math. **756**, 151–182 (2019)