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Volume of Components of Lelong Upper-Level Sets

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Abstract
We prove an upper bound for the volume of maximal analytic sets on which the generic Lelong number of a closed positive current is positive. As a particular case, we give a uniform upper bound on the volume of the singular locus of an analytic set in terms of its volume on a compact Kähler manifold.

Keywords Density current · Lelong number · Lelong filtration · Dinh-Sibony product

Mathematics Subject Classification 32U15 · 32Q15

1 Introduction

Let T be a closed positive current on a compact Kähler manifold X of dimension n. For every irreducible analytic subset V in X, we denote by $\nu(T, V)$ the generic Lelong number of T along V. For basics of Lelong numbers, we refer to [8]. Let W be an irreducible analytic subset of dimension m on X. Let $\mathscr{V}_{T,W}$ be the set of (proper) irreducible analytic subsets V of W such that $\nu(T, V) > 0$, and V is maximal with respect to the inclusion of sets, *i.e,* if V' is another proper irreducible analytic set in W so that $\nu(T, V') > 0$ and $V \subset V'$, then $V' = V$. Such a V is called *a component of Lelong upper-level set* of T on W.

Let ω be a fixed Kähler form on X. We denote by $\|T\|$ the mass norm on T given by $\|T\| := \int_X T \wedge \omega^{n-p}$, where T is of bi-degree (p, p). Similarly we define $\mathrm{vol}(V) := \int_V \omega^{\dim V}$ for every irreducible analytic subset V in X. Usually the volume

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of an analytic set involves a positive multiplicative dimensional constant. Since the dimensions of analytic sets in consideration are bounded by $n = \dim X$, we opt to define $\mathrm{vol}(V)$ as above so that the volume of V is simply equal to the mass norm of the current of integration along V. This is only for a practical purpose and is not essential.

Theorem 1.1 *There exists a constant $c > 0$ independent of T and W such that*

$$\sum_{V \in \mathscr{V}_{T,W}} \big(\nu(T, V) - \nu(T, W)\big)^{m-\dim V} \mathrm{vol}(V) \leq c\, \mathrm{vol}(W) \|T\| (1 + \|T\|)^{m-1}. \quad (1.1)$$

The constant c in Theorem 1.1 depends only on X and ω but it is non-explicit: it comes from upper bounds for density currents (see Theorem 2.1 below).

In Theorem 1.1, it is necessary to consider the "relative" generic Lelong number $\nu(T, V) - \nu(T, W)$ of T along V in W in place of $\nu(T, V)$, see Example 2.8 in the end of the paper.

The irreducibility of W is not absolutely necessary. Nevertheless if W is not irreducible, the inequality (1.1) must be modified a bit to take into account the generic Lelong number of T along irreducible components of W. Theorem 1.1 gives in particular an upper bound for the volume of a maximal irreducible analytic subset of W on which T has a strictly positive "relative" generic Lelong number.

Let T be of bi-degree (p, p). If $p = 0$, then T is a constant function, and in this case, Theorem 1.1 is clear because $\mathscr{V}_{T,W}$ is empty. If $p = n$, then T is a measure on X and the only maximal Lelong level sets for T are points where T has positive mass. In this case, the desired inequality (1.1) follows from the fact that $\mathrm{vol}(W)$ is bounded below by a constant $c_0 > 0$ independent of W (see Lemma 2.5 below) and the elementary inequality

$$\sum_{j=1}^{\infty} a_j^m \leq \Big(\sum_{j=1}^{\infty} a_j\Big)\Big(1 + \sum_{j=1}^{\infty} a_j\Big)^{m-1}$$

for positive numbers a_j. Thus the non-trivial case is when $1 \leq p \leq n - 1$. Since a current of bi-degree (p, p) with $p \geq 1$ on X has zero generic Lelong number along X, by taking $W := X$ in Theorem 1.1, one obtains

Corollary 1.2 *Let T be a closed positive current of bi-degree (p, p) with $p \geq 1$ on X. Let $\mathscr{V}_T$ be the set of maximal irreducible analytic subsets V in X along which the generic Lelong number of T is strictly positive. Then we have*

$$\sum_{V \in \mathscr{V}_T} \big(\nu(T, V)\big)^{n-\dim V} \mathrm{vol}(V) \leq c \|T\| (1 + \|T\|)^{n-1},$$

for some constant $c > 0$ independent of T.

By [22], for every closed positive current T of bi-degree (p, p) with $1 \leq p \leq n-1$, there exists a closed positive current of bi-degree $(1, 1)$ whose Lelong numbers are equal to those of T everywhere. Hence it makes no difference in the above results if we assume that T is of bi-degree $(1,1)$. Thus Corollary 1.2 follows indeed from [25, Theorem 4.1], see Remark 2.6 below.

Corollary 1.2 is related to Demailly's self-intersection inequality ([9, Theorem 1.7]) and other higher bi-degree versions of it in [18, 19, 22]. If one applies directly [9, Theorem 1.7] or results in [18, 19, 22], then one only gets an upper bound for the volume of maximal Lelong sets of maximal dimension. Precisely, let $k := \max_{V\in\mathscr{V}_T} \dim V$ and let $\mathscr{V}'_T$ be the subset of $\mathscr{V}_T$ containing V of dimension k. By these last references, there holds

$$\sum_{V\in\mathscr{V}'_T} \big(\nu(T,V)\big)^{n-k}\,\mathrm{vol}(V) \le c\|T\|^{n-k},$$

for some constant $c > 0$ independent of T. However [9, Theorem 1.7] is not sufficient to treat maximal Lelong level sets which are not of maximal dimension k because these sets are not read in the process of considering jumping values of Lelong level sets if their Lelong numbers are less than or equal to $\max_{V\in\mathscr{V}'_T}\nu(T,V)$, see Remark 2.7 at the end of the paper for an example illustrating this issue. Hence the novelty in Corollary 1.2 (and Theorem 1.1) is that it bounds the volume of every maximal Lelong level set, see also Corollary 1.3 below.

Let W be analytic set in X. We don't require W to be irreducible. We define a sequence of analytic subsets as follows. Let $W_0 := W$ and W_j to be the singular locus of W_{j-1} for $j \ge 1$. The process ends at a finite time when W_j is smooth because of a dimension reason. Hence we obtain a finite sequence $(W_j)_{1\le j\le k_W}$. Note $k_W \le \dim W$. We call $(W_j)_{1\le j\le k_W}$ the *singularity filtration* of W.

Let $T := [W]$ be the current of integration along W. Recall that $\nu(T,x)$ is the multiplicity of W at x for every $x \in X$, and $\nu(T,x) = 1$ if $x \in W_0\backslash W_1$ and $\nu(T,x) \ge 2$ if $x \in W_1$ (see [2, Page 120]). The analytic set W_j might not be irreducible and hence can contain several irreducible components of different dimensions. Write $W_j = \cup_{k=1}^{l} W_{jk}$ which is the decomposition of W_j into irreducible components. We define

$$\mathrm{vol}(W_j) := \sum_{k=1}^{l} \mathrm{vol}(W_{jk}) = \sum_{k=1}^{l} \int_{W_{jk}} \omega^{\dim W_{jk}}.$$

As an application of Theorem 1.1, we obtain

Corollary 1.3 *There exists a constant $c > 0$ independent of W such that*

$$\mathrm{vol}(W_1) \le c\,\mathrm{vol}(W)(1+\mathrm{vol}(W))^{m-1} \tag{1.2}$$

and

$$\sum_{j=1}^{k_W} \mathrm{vol}(W_j) \le c\,\mathrm{vol}(W)(1+\mathrm{vol}(W))^{(m+2)^m}, \tag{1.3}$$

where $m := \dim W$.

If one uses [9, Theorem 1.7], then one only obtains a similar upper bound for the sum of volumes of irreducible components of W_1 of maximal dimension. The reason as explained above is that some maximal Lelong level sets are not taken into account in the sequence of jumping numbers for the Lelong level sets E_c in [9, Theorem 1.7].

We can refine Corollary 1.2 a bit more as follows. *A Lelong filtration associated to* T is a finite sequence of (non-empty) irreducible analytic subsets $\mathbf{V} := (V_j)_{0\le j\le l}$ in X such that $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l$, and for $1 \le j \le l+1$, V_{j-1} is a maximal irreducible analytic subset of T on V_j such that $\nu(T, V_{j-1}) > \nu(T, V_j)$, where $V_{l+1} := X$. The number l is called *the length of the filtration* $\mathbf{V}$. For each Lelong filtration $\mathbf{V}$, we put

$$\mathrm{vol}(\mathbf{V}) := \mathrm{vol}(V_0) \prod_{s=0}^{l} \big(\nu(T, V_s) - \nu(T, V_{s+1})\big)^{\dim V_{s+1} - \dim V_s}.$$

A Lelong filtration $\mathbf{V} = (V_j)_{1\le j\le m}$ is *maximal* if $\nu(T, V_0) = \nu(T, x)$ for every $x \in V_0$, in other words, one can not add one more analytic subset to $\mathbf{V}$ to make it longer. Let $\mathscr{F}_T$ be the set of Lelong filtrations of T.

Theorem 1.4 *Let T be a closed positive current of bi-degree (p, p) with $p \ge 1$ on X. Then we have*

$$\sum_{\mathbf{V}\in\mathscr{F}_T} \mathrm{vol}(\mathbf{V}) \le c\|T\|(1 + \|T\|)^{n^2},$$

for some constant $c > 0$ independent of T.

Since the self-intersection of Demailly mentioned above has found applications in the study of equidistribution of pre-images of subvarieties in complex dynamics (see [13, 21]), we expect that Theorem 1.1 could be also useful for this question. Corollary 1.3 is by the way of independent interest in its own right.

The difficulty in proving Theorem 1.1 is that the intersection of copies of T with the current of integration along W is not well-defined in the classical sense. To solve this problem, we need to use both the analytic regularisation of psh functions by Demailly ([9]) and the theory of density currents by Dinh–Sibony ([14]). The fact that the notion of density currents generalizes the classical intersection of currents of bi-degree $(1, 1)$ (see [16]) is also crucial in our proof.

We suspect that the inequality in Corollary 1.2 is more or less the best possible relation between $\nu(T, V)$ for maximal V if we don't take into account the cohomology class of T. More precisely we would like to pose the following question.

Problem 1.5 Let $(V_j)_{j\in\mathbb{N}}$ be a countable family of irreducible analytic subsets in X together with a sequence of strictly positive real numbers $(\lambda_j)_{j\in\mathbb{N}}$ such that $V_j \not\subset V_{j'}$ for every $j \ne j'$, and

$$\sum_{j=1}^{\infty} \lambda_j^{n-\dim V_j}\, \mathrm{vol}(V_j) < \infty.$$

Does there exist a closed positive current T of bi-degree $(1, 1)$ such that $\nu(T, V_j) \geq \lambda_j$ and V_j is a maximal analytic subset along which T has strictly positive generic Lelong number for every j? Furthermore, if α is a pseudoeffective $(1, 1)$-class, under which additional assumptions on (V_j, λ_j) there exists a closed positive current T in α so that T satisfies the above property?

The problem was solved in [10, Corollary 6.8] in the case where α is big and nef, and $(V_j)_{j\in\mathbb{N}}$ is a countable family of points in X. The necessary and sufficient condition in this case is

$$\sum_{j=1}^{\infty} \lambda_j^n \leq \int_X \alpha^n.$$

Problem 1.5 can be put in a broader context as follows. Consider a big cohomology $(1, 1)$-class α and an analytic subset V in X. One wants to study the space of currents in α having controlled singularities along V (in some reasonable ways). Such a question is relevant in the theory of complex Monge-Ampère equations big cohomology classes where this equation was solved with given prescribed singularities (see [4–6, 15]). We refer also to [3, 7] for some results around this topic in the case where the class is integral.

2 Proofs of Theorem 1.1 and Its Consequences

We first recall some basic properties of density currents. The last notion was introduced in [14].

Let X be a complex Kähler manifold of dimension n and V a smooth complex submanifold of X of dimension l. Let T be a closed positive (p, p)-current on X, where $0 \leq p \leq n$. Denote by $\pi : E \to V$ the normal bundle of V in X and $\overline{E} := \mathbb{P}(E \oplus \mathbb{C})$ the projective compactification of E. By abuse of notation, we also use π to denote the natural projection from $\overline{E}$ to V.

Let U be an open subset of X with $U \cap V \neq \varnothing$. Let τ be a smooth diffeomorphism from U to an open neighborhood of $V \cap U$ in E which is identity on $V \cap U$ such that the restriction of its differential $d\tau$ to $E|_{V\cap U}$ is identity. Such a map is called *an almost-admissible map*. When U is a small enough tubular neighborhood of V, there always exists an almost-admissible map τ by [14, Lemma 4.2]. In general, τ is not holomorphic. When U is a small enough local chart, we can choose a holomorphic almost-admissible map by using suitable holomorphic coordinates on U. For $\lambda \in \mathbb{C}^*$, let $A_\lambda : E \to E$ be the multiplication by λ on fibers of E.

Theorem 2.1 *([14, Theorem 4.6]) Let τ be an almost-admissible map defined on a tubular neighborhood of V. Then, the family $(A_\lambda)_*\tau_*T$ is of mass bounded uniformly in λ on compact sets, and if S is a limit current of the last family as $\lambda \to \infty$, then S is a current on E which can be extended trivially through $\overline{E}\backslash E$ to a closed positive current on $\overline{E}$ such that the cohomology class $\{S\}$ of S in $\overline{E}$ is independent of the choice of S, and $\{S\}|_V = \{T\}|_V$, and $\|S\| \leq C\|T\|$ for some constant C independent of S and T.*

We call S *a tangent current to T along V*. By [14, Theorem 4.6] again, if

$$S = \lim_{k\to\infty} (A_{\lambda_k})_* \tau_* T$$

for some sequence $(\lambda_k)_k$ converging to ∞, then for every open subset U of X and every almost-admissible map $\tau' : U' \to E$, we also have

$$S = \lim_{k\to\infty} (A_{\lambda_k})_* \tau'_* T.$$

This is equivalent to saying that tangent currents are independent of the choice of almost-admissible maps. By this reason, the sequence $(\lambda_k)_k$ is called *a defining sequence* of T_∞. In practice, to study tangent currents, we usually choose τ' to be a holomorphic change of coordinates.

Let $m \geq 2$ be an integer. Let T_j be a closed positive current of bi-degree (p_j, p_j) for $1 \leq j \leq m$ on X and let $T_1 \otimes \cdots \otimes T_m$ be the tensor current of $T_1, \ldots, T_m$ which is a current on X^m. A *density current* associated to $T_1, \ldots, T_m$ is a tangent current to $\otimes_{j=1}^m T_j$ along the diagonal Δ_m of X^m. Let $\pi_m : E_m \to \Delta$ be the normal bundle of Δ_m in X^m. When $m = 2$ and $T_2 = [V]$, the density currents of T_1 and T_2 are naturally identified with the tangent currents to T_1 along V (see [14] or [23, Lemma 2.3]).

We say that the *Dinh–Sibony product* $T_1 \curlywedge \cdots \curlywedge T_m$ of $T_1, \ldots, T_m$ is well-defined if $\sum_{j=1}^m p_j \leq n$ and there is only one density current associated to $T_1, \ldots, T_m$ and this current is the pull-back by π_m of a current S on Δ_m. We define $T_1 \curlywedge \cdots \curlywedge T_m$ to be S.

We note that one can still define density currents on general complex manifolds; see [16, 17, 24] for details. We recall the following from [16].

Let R be a positive closed current and v be a psh function on an open subset Ω in $\mathbb{C}^n$. If v is locally integrable with respect to (the trace measure) of R, we define

$$dd^c v \wedge R := dd^c(vR), \tag{2.1}$$

see [1].

For a collection $v_1, \ldots, v_s$ of psh functions, we can apply the above definition recursively, as long as the integrability conditions are satisfied.

Definition 2.2 We say that the intersection of $dd^c v_1, \ldots, dd^c v_s, R$ is *classically well-defined* if for every non-empty subset $J = \{j_1, \ldots, j_k\}$ of $\{1, \ldots, s\}$, we have that v_{j_k} is locally integrable with respect to the trace measure of R and inductively, v_{j_r} is locally integrable with respect to the trace measure of $dd^c v_{j_{r+1}} \wedge \cdots \wedge dd^c v_{j_k} \wedge R$ for $r = k-1, \ldots, 1$, and the product $dd^c v_{j_1} \wedge \cdots \wedge dd^c v_{j_k} \wedge R$ is continuous under decreasing sequences of psh functions.

The above definition is generalized in an obvious way to the case where Ω is replaced by a complex manifold. If X is a complex manifold and $T_1, \ldots, T_m$ are closed positive currents of bi-degree $(1, 1)$ on X and R is closed positive current on X, then we say that the intersection $T_1 \wedge \cdots T_m \wedge R$ of $T_1, \ldots, T_m, R$ is *classically well-defined* if they are so locally on X.

The reason for Definition 2.2 is that in all of well-known standard situations warranting the well-definedness of the intersection of currents (see, e.g, [8]), the intersection satisfies the continuity under decreasing sequences. In particular if T_j has locally bounded potentials outside some analytic subset V_j in X and $R := [W]$ the current of integration along an analytic set W, and

$$\dim(V_{j_1} \cap \cdots \cap V_{j_s} \cap W) \leq \dim W - s,$$

for every $1 \leq j_1 < \cdots < j_s \leq m$, then $T_1 \wedge \cdots \wedge T_m \wedge R$ is classically well-defined.

Theorem 2.3 *([16, Corollary 3.11]) Let X be a complex manifold. Let $T_1, \ldots, T_m$ be closed positive currents of bi-degree $(1, 1)$ on X and let R be a closed positive current on X such that $T_1 \wedge \cdots \wedge T_m \wedge R$ is classically well-defined. Then the Dinh-Sibony product of $T_1, \ldots, T_m$, R is well-defined and equal to $T_1 \wedge \cdots \wedge T_m \wedge R$.*

Note that the last result was proved in [14] when local potentials of T_j's are continuous. We recall the following.

Lemma 2.4 *The set of Lelong numbers of an arbitrary closed positive current on an open subset in $\mathbb{C}^n$ is countable.*

Proof For readers' convenience, we present below a proof communicated to us by Tien–Cuong Dinh. It is more or less a direct consequence of Siu's analyticity of Lelong upper level sets ([20]). Let T be a closed positive current on X. We show that for every analytic set V in X, the set $\{x \in V : \nu(T, x) > 0\}$ is at most countable. The desired assertion is the case where $V = X$. We prove this claim by induction on the dimension of V. If $\dim V = 0$, there is nothing to prove. Assume that the claim holds for every analytic set of dimension $< m$. Let V be now of dimension m. For $k > 0$ integer, consider

$$E_{1/k} := \{x \in V : \nu(T, x) \geq \nu(T, V) + 1/k\}$$

which is a proper analytic subset of V by Siu's analyticity theorem [20]. Applying the induction hypothesis to $E_{1/k}$, one obtains that the set of Lelong numbers of T on $E_{1/k}$ is at most countable. Since there is a countable number of $E_{1/k}$, we infer the claim for V. This finishes the proof. □

The following result is also standard.

Lemma 2.5 *Let X be a compact complex manifold of dimension n and ω be a Hermitian form on X. Then there exists a constant $C > 0$ such that for every $0 \leq p \leq n$ and every closed positive (p, p)-current T and every $x \in X$, we have*

$$\int_X T \wedge \omega^{n-p} \geq C\nu(T, x). \tag{2.2}$$

In particular there exists a constant $C > 0$ such that if W is an irreducible analytic subset of dimension m on X, then the $\mathrm{vol}(W) := \int_W \omega^m \geq C$.

Proof Cover X be a finitely many local charts $U_1, \dots, U_m$. We can assume furthermore that $\overline{U}_j$ is contained in a bigger local chart U'_j for $1 \leq j \leq m$. Let $x \in X$. Then $x \in U_j$ for some j. We work locally on $U_j \subset \mathbb{C}^n$. Let ω_0 be the standard Kähler form on $\mathbb{C}^n$. Note that there is a constant $\delta > 0$ independent of x such that $\mathbb{B}(x, 2\delta) \subset U'_j$. By the definition of Lelong numbers, we have

$$\int_{\mathbb{B}(x,\delta)} T \wedge \omega_0^{n-p} \geq C_n \delta^{2(n-p)} \nu(T, x),$$

where C_n is an explicit dimensional constant. Since ω is Hermitian, we obtain a constant $C_1 > 0$ such that $C_1 \omega \geq \omega_0$. Thus

$$\int_{\mathbb{B}(x,\delta)} T \wedge \omega_0^{n-p} \leq C_1^{n-p} \int_{\mathbb{B}(x,\delta)} T \wedge \omega^{n-p} \leq C_1^{n-p} \int_X T \wedge \omega^{n-p}.$$

Hence (2.2) follows. As for the second desired assertion, we recall that $\nu([W], x)$ is equal to the multiplicity of x in W which is a positive integer. Hence $\nu([W], x) \geq 1$ for every $x \in W$. The uniform lower bound for the volume of W thus follows. □

Proof of Theorem 1.1 By [22, Theorem 3.1], we can assume that T is of bi-degree $(1, 1)$.

Step 1. We assume first that T has analytic singularities in X (i.e., potentials of T are locally the sum of a bounded function and the logarithmic of a sum of holomorphic functions) and $\nu(T, W) = 0$. We will explain how to remove this assumption at the end of the proof.

For an irreducible analytic set V, we denote by $[V]$ the current of integration along V. Denote by $L(T)$ the unbounded locus of T (for a definition, see [8, Page 150]). In our present setting, the last set is an analytic subset of X. Since $\nu(T, W) = 0$, we have $W \not\subset L(T)$. Let $0 \leq l \leq m - 1$ be an integer. Let $\mathscr{V}^l_{T,W}$ be the set of $V \in \mathscr{V}_{T,W}$ such that $\dim V = l$. The last set is finite because T has analytic singularities. Note that

$$\mathscr{V}_{T,W} = \cup_{l=0}^{m-1} \mathscr{V}^l_{T,W}.$$

Let $V \in \mathscr{V}^l_{T,W}$. By the maximality of V, we see that V is an irreducible component of $L(T) \cap W$. Thus, since $\dim V = l$, we deduce that the intersection $T^{m-l} \wedge [W]$ is classically well-defined on some open neighborhood U_V of $V \backslash \mathrm{Sing}\big(L(T) \cap W\big)$, where $\mathrm{Sing}\big(L(T) \cap W\big)$ denotes the singular locus of the analytic set $L(T) \cap W$.

Let $Q := T^{m-l} \wedge [W]$ on U_V. Now, using a comparison result on Lelong numbers ([8, Page 169]) gives that

$$Q \geq \big(\nu(T, V)\big)^{m-l} [V]$$

on U. Letting V run over every element of $\mathscr{V}^l_{T,W}$, we infer that Q is well-defined on some open neighborhood U of

$$\bigcup_{V\in\mathscr{V}^l_{T,W}} V\backslash\mathrm{Sing}\big(L(T)\cap W\big),$$

and

$$Q \geq \sum_{V\in\mathscr{V}^l_{T,W}} \big(\nu(T,V)\big)^{m-l}[V] \tag{2.3}$$

on U. We now estimate the mass of Q. Let Δ_{m-l+1} be the diagonal of X^{m-l+1} and $\pi : E \to \Delta_{m-l+1}$ the normal bundle of Δ_{m-l+1} in X^{m-l+1}. Let $\tilde{Q}$ be a density current associated to $T,\dots,T$ ($m-l$ times), and $[W]$. Recall that $\tilde{Q}$ is a closed positive current on $\overline{E} := \mathbb{P}(E\oplus\mathbb{C})$. By Theorem 2.1, we have

$$\|\tilde{Q}\| \lesssim \|T\|^{m-l}\,\mathrm{vol}(W). \tag{2.4}$$

Since Q is well-defined on U, using Theorem 2.3, we obtain that $\tilde{Q} = \pi^*Q$ on $\pi^{-1}(U)$, where we have identified Δ_{m-l+1} with X. This combined with (2.4) yields that

$$\|Q\| \lesssim \|T\|^{m-l}\,\mathrm{vol}(W).$$

Using this and (2.3) gives

$$\sum_{V\in\mathscr{V}^l_{T,W}} \big(\nu(T,V)\big)^{m-l}\,\mathrm{vol}(V) \lesssim \|T\|^{m-l}\,\mathrm{vol}(W)$$

for every l. Summing the last inequality over l gives the desired inequality.

Step 2. We now get rid of the assumption that T has analytic singularities (but we still require $\nu(T,W)=0$).

By Lemma 2.4 there are at most countably many maximal Lelong level sets of T on W. Hence we can write

$$\mathscr{V}_{T,W} = \{V_1,\dots,V_k,\dots\}.$$

Let $r>0$ be a small constant, and let A_r be the finite subset of $\mathscr{V}_{T,W}$ consisting of V such that $\nu(T,V)\geq r$. Therefore A_r increases to $\mathscr{V}_{T,W}$ as $r\to 0$.

Let $(T_\epsilon)_\epsilon$ be the sequence of closed positive $(1,1)$-current regularising T given by Demailly's analytic approximation of psh functions ([12, Corollary 14.13]). These currents satisfy the following properties. One has $T_\epsilon \to T$ weakly as $\epsilon\to 0$, and $\nu(T_\epsilon,\cdot)\leq\nu(T,\cdot)$ for every ϵ, and $\nu(T_\epsilon,\cdot)\to\nu(T,\cdot)$ uniformly on X as $\epsilon\to 0$. It follows that for every $r > 0$ there exists an $\epsilon > 0$ small enough such that $A_r \subset \mathscr{V}_{T_\epsilon, W}$. Applying the first part of the proof to T_ϵ yields that

$$\sum_{V \in A_r} \big(\nu(T_\epsilon, V)\big)^{m-\dim V} \operatorname{vol}(V) \le c \operatorname{vol}(W) \|T_\epsilon\| (1 + \|T_\epsilon\|)^{m-1},$$

for some constant $c > 0$ independent of T, W. Letting $\epsilon \to 0$, and then $r \to 0$ gives the desired inequality.

Step 3. Finally we treat the general case where $\nu(T, W) > 0$. We use [9, Theorem 1.1] which is a regularisation result of (1, 1)-currents (see also [11, Theorem 6.1]). The last theorem allows one to cut down the Lelong level set $E_c := \{x \in X : \nu(T, x) \ge c\}$ from T, where $c > 0$ is a constant. Choose $c := \nu(T, W)$ which is bounded by a uniform constant times $\|T\|$. By [9, Theorem 1.1], we obtain a closed positive (1, 1)-current T' such that $\|T'\| \le M\|T\|$ (M is a constant independent of T), and $\nu(T', x) = \max\{\nu(T, x) - c, 0\}$, and additionally T' is smooth outside E_c (but we don't need this property at this point). Hence

$$\mathscr{V}_{T,W} = \mathscr{V}_{T',W}, \quad \nu(T', W) = 0,$$

and for every $V \in \mathscr{V}_{T,W}$ one gets $\nu(T, V) - \nu(T, W) = \nu(T', V)$. Thus the desired inequality follows from Step 2 applied to T' and W. This finishes the proof. □

Proof of Theorem 1.4 Observe that the length of a Lelong filtration is at most $n - 1$, so it is bounded uniformly. Let $\mathscr{F}_{T,l}$ be the set of Lelong filtrations of T of length l for $0 \le l \le n - 1$. We prove by induction on l that

$$\sum_{\mathbf{V} \in \mathscr{F}_{T,l}} \operatorname{vol}(\mathbf{V}) \le c\|T\|(1 + \|T\|)^{n^{l+1}}, \tag{2.5}$$

for some constant $c > 0$ independent of T. If $l = 0$, the desired estimate is Corollary 1.2. We assume that (2.5) holds for $l - 1$. For every irreducible analytic subset W in X, define $\mathscr{F}_{T,l,W}$ to be the subset of $\mathscr{F}_{T,l}$ consisting of $\mathbf{V} = (V_j)_{0 \le j \le l}$ such that $V_1 = W$. We have

$$\mathscr{F}_{T,l} = \cup_W \mathscr{F}_{T,l,W}.$$

For every $\mathbf{V} = (V_j)_{0 \le j \le l} \in \mathscr{F}_{T,l,W}$, we associate to it a Lelong filtration $\mathcal{C}_W(\mathbf{V})$ of length $l - 1$ by putting

$$\mathcal{C}_W(\mathbf{V}) := (V_j)_{1 \le j \le l}.$$

Hence $\mathcal{C}_W$ is a map from $\mathscr{F}_{T,l,W}$ to $\mathscr{F}_{T,l-1}$. The image $\mathscr{F}_{T,l,W}$ under $\mathcal{C}_W$ is the subset of $\mathscr{F}_{T,l-1}$ consisting of Lelong filtrations starting from W. The fiber of $\mathcal{C}_W$ is naturally identified with $\mathscr{V}_{T,W}$. Applying now Theorem 1.1 to T and W yields

$$\sum_{V\in\mathscr{V}_{T,W}} (\nu(T,V)-\nu(T,W))^{\dim W-\dim V}\,\mathrm{vol}(V)\lesssim \mathrm{vol}(W)\|T\|(1+\|T\|)^{m-1}.$$

Thus

$$\begin{aligned}\sum_{\mathbf{V}\in\mathscr{F}_{T,l,W}} \mathrm{vol}(\mathbf{V}) &\lesssim \sum_{\mathbf{V}_1\in\mathcal{C}_W(\mathscr{F}_{T,l,W})} \mathrm{vol}(\mathbf{V}_1)\|T\|(1+\|T\|)^{m-1}\\ &\le \sum_{\mathbf{V}_1\in\mathcal{C}_W(\mathscr{F}_{T,l,W})} \mathrm{vol}(\mathbf{V}_1)(1+\|T\|)^{n}\end{aligned}$$

Summing again over W, one get

$$\sum_W\sum_{\mathbf{V}\in\mathscr{F}_{T,l,W}} \mathrm{vol}(\mathbf{V})\lesssim \sum_{\mathbf{V}_1\in\mathscr{F}_{T,l-1}} \mathrm{vol}(\mathbf{V}_1)(1+\|T\|)^{n}.$$

Since the left-hand side of the last inequality is equal to

$$\sum_{\mathbf{V}\in\mathscr{F}_{T,l}} \mathrm{vol}(\mathbf{V}),$$

the induction hypothesis now implies the desired assertion for l. This finishes the proof. □

Remark 2.6 We explain now how to deduce Theorem 1.1 in the case where $W=X$ from [25, Theorem 4.1]. Firstly by considering $T/\|T\|$ instead of T, we can assume that T of mass 1. Observe now that there is a constant $c>0$ independent of T such that $T+c\omega$ lies in a fixed compact subset of Kähler cone in X. Thus Theorem 1.1 for $W=X$ follows directly from [25, Theorem 4.1].

Proof of Corollary 1.3 We prove first (1.2). Assume for the moment that W is irreducible.

Let $T:=[W]$. Note that elements of $\mathscr{V}_{T,W}$ are irreducible components of W_1. Moreover $\nu(T,V)-\nu(T,W)\ge 1$ for $V\in\mathscr{V}_{T,W}$ because it is a strictly positive integer. This combined with Theorem 1.1 applied to $T:=[W]$ and W yields (1.2). The case where W is not necessarily irreducible follows by applying the previous case to each irreducible component of W.

It remains to prove (1.3). Let $(W_j)_{1\le j\le m}$ be the singularity filtration of W. Note that $m\le n$. Note that $\dim W_{s-1}\le \dim W-(s-1)$. Applying (1.2) to W_{s-1} in place of W for $1\le s\le m$, we obtain

$$\mathrm{vol}(W_s)\le c\,\mathrm{vol}(W_{s-1})(1+\mathrm{vol}(W_{s-1}))^{m-(s-1)},$$

for some constant $c>0$ independent of W. An induction argument thus gives

$$\mathrm{vol}(W_s)\lesssim \mathrm{vol}(W_l)(1+\mathrm{vol}(W_l))^{(m-l+2)^{s-l}}$$

for every $0 \leq l \leq s \leq m$. Applying the last inequality to $l = 0$ gives

$$\operatorname{vol}(W_s) \lesssim \operatorname{vol}(W)(1 + \operatorname{vol}(W))^{(m+2)^s}$$

for every $1 \leq s \leq m$. This finishes the proof. □

We end the paper with some examples.

Remark 2.7 Let $X := \mathbb{P}^n$ with $n > 3$ and $x = [x_0 : \cdots : x_n]$ homogeneous coordinates. Let V_1, V_2, V_3 be linear subspaces in $\mathbb{P}^n$ such that $\dim V_1 = n - 1$, $\dim V_2 = n - 2$, and $\dim V_3 = 1$, and $V_3 \not\subset V_2 \cup V_1$, and $V_2 \not\subset V_1$. Let T_1, T_2, T_3 be closed positive currents of bi-degree $(1, 1)$ such that the set of points with positive Lelong number for T_j is equal exactly to V_j, and $\nu(T_j, x) = \nu(T_j, V_j)$ for every $x \in V_j$, and for $j = 1, 2, 3$ (e.g., $V_2 = \{x_0 = x_1 = 0\}$, choose $T_2 := dd^c(|x_0|^2 + |x_1|^2)$). Choosing positive constants λ_j suitably, one see that the current

$$T := \lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3$$

satisfies

$$\nu(T, V_1) = 1/100, \quad \nu(T, V_2) = 1/100, \quad \nu(T, V_3) = 1.$$

Hence V_1, V_2, V_3 are all maximal Lelong upper level sets for T. On the other hand, if $E_c := \{x \in \mathbb{P}^n, \nu(T, x) \geq c\}$, then $E_c = V_1 \cup V_2 \cup V_3$ if $c \in (0, 1/100]$, and $E_c = V_3$ if $c \in (1/100, 1]$, and $E_c = \varnothing$ if $c > 1$. One sees that the maximal set V_2 is not taken into account in the self-intersection inequality in [9, Theorem 1.7] as well as its generalizations in [18, 22] (because V_2 is of codimension 2, but only appears in $E_{1/100}$ which is of codimension 1).

Example 2.8 Let $X = \mathbb{P}^3$, $W \approx \mathbb{P}^2$ a hyperplane in $\mathbb{P}^3$ and D a hypersurface of degree d in $\mathbb{P}^3$ so that D intersects W at a smooth curve V (of degree d). Let $T := \epsilon[D] + [W]$ for some small constant $\epsilon > 0$. We see that $\mathscr{V}_{T,W} = \{V\}$ and $\operatorname{vol}(V) = d$ and

$$\nu(T, V) \operatorname{vol}(V) = (1 + \epsilon)d, \quad c \operatorname{vol}(W)\|T\|(1 + \|T\|) \leq c(2 + \epsilon d)^2 < (1 + \epsilon)d$$

if d is big enough and $\epsilon := 1/d$ (recall c is independent of T, W). Thus the inequality (1.1) fails to be true in general if $\nu(T, V) - \nu(T, W)$ is replaced by $\nu(T, V)$.

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