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Parametrization of the p-Weil–Petersson Curves: Holomorphic Dependence

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Abstract

Similar to the Bers simultaneous uniformization, the product of the p-Weil–Petersson Teichmüller spaces for $p \geq 1$ provides the coordinates for the space of p-Weil–Petersson embeddings γ of the real line $\mathbb{R}$ into the complex plane $\mathbb{C}$. We prove the biholomorphic correspondence from this space to the p-Besov space of $u = \log \gamma'$ on $\mathbb{R}$ for $p > 1$. From this fundamental result, several consequences follow immediately which clarify the analytic structures concerning parameter spaces of p-Weil–Petersson curves. Specifically, it implies that the correspondence of the Riemann mapping parameters to the arc-length parameters preserving the images of curves is a homeomorphism with bi-real-analytic dependence of the change of parameters. This is analogous to the classical theorem of Coifman and Meyer for chord-arc curves.

Keywords Weil–Petersson Teichmüller space · Integrable Beltrami coefficients · Bers simultaneous uniformization · Pre-Schwarzian derivative model · Analytic Besov space · Weil–Petersson curves · Arc-length parametrization

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1 Introduction

The Weil–Petersson metric was originally introduced in the study of Teichmüller spaces of Riemann surfaces. To parametrize the deformation of closed curves on the plane, the universal Teichmüller space can be utilized, and the Weil–Petersson metric was also provided for this space. In particular, the subspace T_2 of the universal Teichmüller space T that admits the Weil–Petersson metric was introduced by Cui [8], and subsequently the Hilbert manifold structure of T and the curvatures of the Weil–Petersson metric were further investigated by Takhtajan and Teo [43]. Building upon these fundamental works, Shen and his coauthors have developed complex-analytic theories of the subspace T_2, which is now referred to as the Weil–Petersson Teichmüller space. As Beltrami coefficients representing elements of T_2 are square integrable with respect to the hyperbolic metric, this is also called the integrable Teichmüller space.

The main focus of this paper lies in exploring the complex analytic aspects of the Weil–Petersson Teichmüller space. In this context, Shen [35] first characterized the Weil–Petersson class W_2, which consists of quasisymmetric homeomorphisms representing elements of T_2, without employing quasiconformal extension. It was given in terms of the fractional dimensional Sobolev space $H_{\mathbb{R}}^{1/2}$ of real-valued functions. Then, Shen and Tang [36] regarded $H_{\mathbb{R}}^{1/2}$ as a new parameter space for T_2 which is real-analytically equivalent to the original complex Hilbert structure. In this work, they considered the Weil–Petersson class W_2 on the real line $\mathbb{R}$ and applied the arguments of chord-arc curves induced by BMO functions in Semmes [34]. Furthermore, Shen and Wu [38] examined the Weil–Petersson curves in the complex plane $\mathbb{C}$, which represent the complex generalization of the elements in W_2, and proved that the Riemann mappings onto the domains defined by Weil–Petersson curves move continuously.

Recently, Bishop [2, 3] accomplished a comprehensive study on Weil–Petersson curves by showing approximately twenty different characterizations from various viewpoints of analysis and geometry including the complex analytic methods mentioned above. For instance, a planar geometric characterization of a bounded Weil–Petersson curve Γ is given by the rate at which the perimeter of the inscribed 2^n-polygon for Γ converges to that of Γ. Other characterizations involve certain measurement of coarse smoothness for closed rectifiable curves γ, the Möbius energy defined on γ similar to knots, hyperbolic geometry of convex cores spanned by γ, the curvatures of minimal surfaces with the boundary γ, and more. One can find other related work in the references therein including Shanon and Mumford [32] on 2D-shape mapping, and Wang [46] arising from SLE theory.

In this present paper, we lay the foundation of the parametrization of the space of Weil–Petersson curves in the framework of quasiconformal Teichmüller theory. We represent this space as the product of the Weil–Petersson Teichmüller spaces in three ways and establish analytic and topological correspondences among these factors. In this approach, we can understand the structure of the space of Weil–Petersson curves clearly and easily and extend several known results that have important applications as immediate consequences from our fundamental theorems. We further develop those arguments in the generalization to p-Weil–Petersson curves for $p > 1$.

For $p \geq 1$, let $\mathcal{M}_p(\mathbb{U})$ be the set of Beltrami coefficients that are *p*-integrable with respect to the hyperbolic metric on the upper half-plane $\mathbb{U} \subset \mathbb{C}$. The *p*-Weil–Petersson Teichmüller space $T_p(\mathbb{U})$ is the quotient space of $\mathcal{M}_p(\mathbb{U})$ by the Teichmüller equivalence. The precise definitions are given in Sect. 2 . It has been proved that $T_p(\mathbb{U})$ possesses a complex Banach manifold structure via the Bers embedding. On the lower half-plane $\mathbb{L} \subset \mathbb{C}$, the corresponding spaces $\mathcal{M}_p(\mathbb{L})$ and $T_p(\mathbb{L})$ are defined similarly. The *p*-Weil–Petersson class W_p is the set of quasisymmetric homeomorphisms $f :$ $\mathbb{R} \to \mathbb{R}$ that extend quasiconformally to $\mathbb{U}$ (and to $\mathbb{L}$) with their complex dilatations in $\mathcal{M}_p(\mathbb{U})$ (and in $\mathcal{M}_p(\mathbb{L})$). Then, the element f in W_p for $p > 1$ can be characterized by the property that f is locally absolutely continuous and $\log f'$ belongs to the *p*-Besov space $B_p^{\mathbb{R}}(\mathbb{R})$ of real-valued functions, which coincides with $H_{\mathbb{R}}^{1/2}(\mathbb{R})$ for $p = 2$. In fact, in our paper [47] generalizing [36], it was proved that if a quasisymmetric homeomorphism f of $\mathbb{R}$ is locally absolutely continuous and $\log f'$ is in $B_p^{\mathbb{R}}(\mathbb{R})$, then the variant of the Beurling–Ahlfors extension by the heat kernel introduced by Fefferman, Kenig and Pipher [10] yields a quasiconformal homeomorphism of $\mathbb{U}$ whose complex dilatation is in $\mathcal{M}_p(\mathbb{U})$. We will elaborate these concepts in Sect. 3 as well as those introduced next.

A *p*-Weil–Petersson embedding $\gamma : \mathbb{R} \to \mathbb{C}$ is defined as the restriction of a quasiconformal homeomorphism of $\mathbb{C}$ whose complex dilatations on $\mathbb{U}$ and $\mathbb{L}$ belong to $\mathcal{M}_p(\mathbb{U})$ and $\mathcal{M}_p(\mathbb{L})$, respectively. (In this paper, we always assume that $\gamma(\infty) = \infty$ and associated quasiconformal homeomorphisms fix ∞. Instead of explicity stating this condition, we use $\mathbb{R}$ and $\mathbb{C}$ to exclude ∞.) With this definition of a curve as a continuous mapping, the space WPC_p of all normalized *p*-Weil–Petersson embeddings for $p \geq 1$ can be parametrized in the same spirit as the Bers simultaneous uniformization. Specifically, WPC_p is identified with the product of the *p*-Weil–Petersson Teichmüller spaces $T_p(\mathbb{U}) \times T_p(\mathbb{L})$. Although this natural viewpoint has been overlooked in the literature, we emphasize in this paper that this can significantly clarify the theory of WPC_p. In our recent paper [49], we employed similar arguments for the space of chord-arc curves and obtained several interesting consequences.

In Theorems 3.6 and 5.1, we establish the following basic result regarding this parametrization of WPC_p. The *p*-Besov space $B_p(\mathbb{R})$ is a complex Banach space of complex-valued functions that contains $B_p^{\mathbb{R}}(\mathbb{R})$ as the real subspace. The crucial point is not only to characterize the elements in WPC_p but also to demonstrate that this correspondence is biholomorphic. This is a novel result even in the case of $p = 2$.

Theorem 1.1 *For any p-Weil–Petersson embedding $\gamma : \mathbb{R} \to \mathbb{C}$, the logarithm of its derivative $\log \gamma'$ belongs to the p-Besov space $B_p(\mathbb{R})$ for $p > 1$. Moreover, this correspondence*

$$L : \mathrm{WPC}_p \cong T_p(\mathbb{U}) \times T_p(\mathbb{L}) \to B_p(\mathbb{R})$$

is a biholomorphic homeomorphism onto the image.

The *p*-Weil–Petersson class W_p consisting of all normalized quasisymmetric homeomorphisms of $\mathbb{R}$ onto itself forms a real-analytic submanifold of WPC_p corresponding to the diagonal axis of the Bers coordinates $T_p(\mathbb{U}) \times T_p(\mathbb{L})$. All normalized

p-Weil–Petersson embeddings that are induced by Riemann mappings on $\mathbb{U}$ consti-
tute a complex-analytic submanifold RM_p of WPC_p corresponding to $\{[0]\} \times T_p(\mathbb{L})$.
Every $\gamma \in WPC_p$ is represented uniquely as the reparametrization of $h \in RM_p$
by $f \in W_p$. We denote these correspondences as $f = \Pi(\gamma)$ and $h = \Phi(\gamma)$. In
the Bers coordinates, these maps are defined as $\Pi([\mu_1], [\mu_2]) = ([\mu_1], [\mu_1])$ and
$\Phi([\mu_1], [\mu_2]) = ([0], [\mu_2] * [\mu_1]^{-1})$. Here, we see that Φ is a continuous surjection
by the topological group property of T_p (Theorem 6.1). Thus, we have a homeomor-
phism

$$(\Pi, \Phi) : WPC_p \to W_p \times RM_p.$$

This is the second product structure of WPC_p.

Moreover, let IW_p be the space of all p-Weil–Petersson embeddings that have
arc-length parametrizations. Every $\gamma \in WPC_p$ is represented uniquely as the
reparametrization of its arc-length parametrization $\gamma_0 \in IW_p$ by $f \in W_p$. In
this way, $WPC_p \cong W_p \times IW_p$. See Sect. 4 for more details. Since the condition
$\gamma \in IW_p$ is equivalent to the condition that $L(\gamma)$ is purely imaginary, we have
$IW_p = L^{-1}(iB_p^{\mathbb{R}}(\mathbb{R})^\circ)$ and this is a real-analytic submanifold of WPC_p. Here,
$iB_p^{\mathbb{R}}(\mathbb{R})^\circ$ is the intersection of the open subset $L(WPC_p) \subset B_p(\mathbb{R})$ with the real sub-
space $iB_p^{\mathbb{R}}(\mathbb{R})$ consisting of purely imaginary functions. Clearly, $L(W_p) = B_p^{\mathbb{R}}(\mathbb{R})$.
Then, there is a bijection J from $B_p^{\mathbb{R}}(\mathbb{R}) \times iB_p^{\mathbb{R}}(\mathbb{R})^\circ$ to $L(WPC_p)$ such that this product
structure is compatible with $W_p \times IW_p$ on WPC_p under $J^{-1} \circ L$ (Lemma 4.2). Again
by the topological group property of T_p, J is a homeomorphism (Theorem 6.3) and
thus we have a homeomorphism

$$J^{-1} \circ L : WPC_p \to B_p^{\mathbb{R}}(\mathbb{R}) \times iB_p^{\mathbb{R}}(\mathbb{R})^\circ.$$

This is the third product structure of WPC_p.

By investigating these product structures, we establish a topological correspondence
between the factor subspaces of the products. This reveals that these subspaces admit
different analytic structures that are topologically equivalent to each other. We develop
those arguments in Sect. 6 and the results are summarized as follows (Fig. 1).

Theorem 1.2 *The space* WPC_p *for* $p > 1$ *admits two other product structures* $W_p \times RM_p$ *and* $W_p \times IW_p$ *that are homeomorphic to* WPC_p. *The fiber structures for the projections to the second factors in both products are the same and each fiber consists of the family of all normalized* p-*Weil–Petersson embeddings of the same image.*

More detailed real-analytic correspondence between the subspaces of WPC_p are
also established. In particular, the dependence of the Riemann mappings on the arc-
length parameters of p-Weil–Petersson curves is an interesting problem, which is
addressed in Corollary 5.2 and Theorem 5.4. The latter theorem contains the following
new achievement on the real-analytic dependence of the reparametrization map.
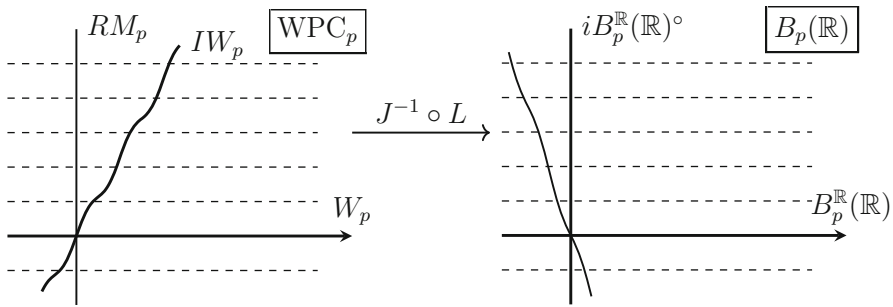
Fig. 1 The fiber structure of WPC$_p$

Theorem 1.3 *The reparametrization map from the arc-length parametrization to the Riemann mapping parametrization defined by*

$$\lambda = L \circ \Pi \circ L^{-1}|_{iB_p^{\mathbb{R}}(\mathbb{R})^\circ} : iB_p^{\mathbb{R}}(\mathbb{R})^\circ \to B_p^{\mathbb{R}}(\mathbb{R})$$

is a real-analytic homeomorphism onto an open contractible domain of $B_p^{\mathbb{R}}(\mathbb{R})$ for $p > 1$ whose inverse is also real-analytic.

This is the Weil–Petersson curve version of the original result for chord-arc curves by Coifman and Meyer [7], which has been a significant contribution to the field. See Semmes [34, Section 6] and Wu [51]. Our formulation of the space of *p*-Weil–Petersson curves can make these arguments transparent, and in particular, the real-analyticity of λ follows from our arguments immediately. The essential step in the original work is the investigation of the inverse correspondence, but once we know that λ is real-analytic, we can apply their result to conclude that λ^{-1} is also real-analytic. These are demonstrated in Sect. 5.

2 The *p*-Weil–Petersson Teichmüller Space and the *p*-Besov Space

A measurable function μ on $\mathbb{U}$ is called a *Beltrami coefficient* if $\|\mu\|_\infty < 1$. By the solution of the Beltrami equation, there exists a quasiconformal homeomorphism F of $\mathbb{U}$ onto itself whose complex dilatation $\mu_F = F_{\bar{z}}/F_z$ coincides with μ uniquely up to the post-composition of affine transformations of $\mathbb{U}$. The definition on the lower half-plane $\mathbb{L}$ can be similarly done for this and all other concepts appearing hereafter.

For $p \geq 1$, let $\mathcal{M}_p(\mathbb{U})$ be the set of Beltrami coefficients μ satisfying

$$\|\mu\|_p^p = \int_{\mathbb{U}} \frac{|\mu(z)|^p}{y^2} dx dy < \infty.$$

By the norm $\|\cdot\|_\infty + \|\cdot\|_p$, $\mathcal{M}_p(\mathbb{U})$ is a domain of the corresponding Banach space.

The *p-Weil–Petersson Teichmüller space* $T_p(\mathbb{U})$ on the upper half-plane $\mathbb{U}$ is defined to be the set of all Teichmüller equivalence classes $[\mu]$ for $\mu \in \mathcal{M}_p(\mathbb{U})$. Here, μ and μ' are equivalent if the quasiconformal homeomorphisms F_μ and $F_{\mu'}$ of $\mathbb{U}$ onto itself

determined by μ and μ' have the same boundary extension to $\mathbb{R}$ up to the post-composition of an affine transformation of $\mathbb{R}$. The quotient map $\pi : \mathcal{M}_p(\mathbb{U}) \to T_p(\mathbb{U})$ taking the equivalence class is called the *Teichmüller projection*. The canonical complex Banach manifold structure of $T_p(\mathbb{U})$ for $p \geq 1$ is introduced via the Bers embedding into certain complex Banach space (see [52, Theorem 4.4], [42, Theorem 2.1], [50, Theorem 4.1], and the Appendix).

The boundary extension to $\mathbb{R}$ of a quasiconformal homeomorphism $F : \mathbb{U} \to \mathbb{U}$ is called a *quasisymmetric homeomorphism*. If such an $f : \mathbb{R} \to \mathbb{R}$ is the extension of F whose complex dilatation is in $\mathcal{M}_p(\mathbb{U})$, we say that f is a *p-Weil–Petersson class homeomorphism*. Then, the above definition of $T_p(\mathbb{U})$ is equivalent to saying that $T_p(\mathbb{U})$ is the set of all p-Weil–Petersson class homeomorphisms modulo affine transformations of $\mathbb{R}$. Later, this set is defined as W_p for $p \geq 1$.

A p-Weil–Petersson class homeomorphism for $p > 1$ can be intrinsically defined as an increasing homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that f is locally absolutely continuous and $\log f'$ belongs to $B_p^{\mathbb{R}}(\mathbb{R})$ defined below (see [35, Theorem 1.1], [42, Theorem 1.2], [37, Theorems 1.2 and 1.3] and [50, Theorem 5.5]). Partially, this will be proved in Lemma 3.3.

The *p-Besov space* $B_p(\mathbb{R})$ for $p > 1$ is the set of all locally integrable complex-valued functions u on $\mathbb{R}$ satisfying

$$\|u\|_{B_p}^p = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|u(t) - u(s)|^p}{|t - s|^2} ds dt < \infty.$$

Clearly, if $u \in B_p(\mathbb{R})$ then $|u|$, $\operatorname{Re} u$, $\operatorname{Im} u \in B_p(\mathbb{R})$. We can regard $B_p(\mathbb{R})$ as a complex Banach space with norm $\| \cdot \|_{B_p}$ by taking the quotient modulo constant functions. Namely, we regard $B_p(\mathbb{R})$ as the homogeneous Besov space, which is often denoted by $\dot{B}_p(\mathbb{R})$ in the literature. The real Banach subspace of $B_p(\mathbb{R})$ consisting of all elements represented by real-valued functions is denoted by $B_p^{\mathbb{R}}(\mathbb{R})$.

Remark When $p = 1$, $B_p(\mathbb{R})$ degenerates into the space of constant functions (see [19, Exercise 17.14]). In this case, we change $B_1(\mathbb{R})$ into the space of all bounded complex-valued functions u on $\mathbb{R}$ that satisfy

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|u(t) + u(s) - 2u(\frac{t+s}{2})|}{|t - s|^2} ds dt < \infty.$$

Correspondingly, the analytic Besov space on $\mathbb{U}$ defined below should be defined for $p = 1$ as the space $\mathcal{B}_1(\mathbb{U})$ of all bounded holomorphic functions φ on $\mathbb{U}$ satisfying

$$\frac{1}{\pi} \iint_{\mathbb{U}} |\varphi''(z)| dx dy < \infty.$$

See [28, Sections 2, 9]. Then, the arguments below can also be developed under these replacements even in the case of $p = 1$. However, we do not pursue this in the present paper and put the restriction $p > 1$ at the places where the Besov spaces are involved.

Concerning the composition operator on this space, the following result was shown in [5, Theorem 12 and Remark 5] (see also [45, Theorem 2.2] and [4, Theorem 1.3 and Section 3.4]).

Proposition 2.1 *Let $p > 1$. An increasing homeomorphism h from $\mathbb{R}$ onto itself is quasisymmetric if and only if the composition operator $P_h : u \mapsto u \circ h$ gives an isomorphism of the p-Besov space $B_p(\mathbb{R})$ onto itself, that is, P_h and $(P_h)^{-1}$ are bounded linear operators.*

Remark In the case of $p = 2$, it is known that the operator norm $\|P_h\|$ of the composition operator $P_h : B_p(\mathbb{R}) \to B_p(\mathbb{R})$ depends only on the doubling constant of h (or equivalently, the quasisymmetric constant of h or the Teichmüller distance $d_\infty(h, \mathrm{id})$). See [26, Theorem 3.1]. In contrast, in the case of $p \neq 2$, the estimate of the operator norm becomes more difficult. See [5, Remark 4]. In a special case where $\log h'$ belongs to $B_p(\mathbb{R})$, a certain dependence of $\|P_h\|$ on h will be shown later in Proposition 6.9.

Next, we consider analytic function spaces. Let $\mathcal{B}(\mathbb{U})$ denote the *Bloch space* of functions φ holomorphic on $\mathbb{U}$ with semi-norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{z \in \mathbb{U}} |\varphi'(z)| y.$$

Let $\mathcal{B}_p(\mathbb{U})$ denote the *analytic p-Besov space* for $p > 1$ (or *p*-Dirichlet space) of holomorphic functions φ on $\mathbb{U}$ with semi-norm

$$\|\varphi\|_{\mathcal{B}_p} = \left(\frac{1}{\pi} \iint_{\mathbb{U}} |\varphi'(z)|^p y^{p-2} dx dy \right)^{\frac{1}{p}}.$$

Then, $\mathcal{B}_p(\mathbb{U}) \subset \mathcal{B}_q(\mathbb{U}) \subset \mathcal{B}(\mathbb{U})$ for $1 < p \leq q$, and the inclusion maps are continuous. By considering functions in these spaces modulo additive constants, which we always do hereafter, the semi-norms become norms and the spaces become complex Banach spaces.

These spaces can be defined on the unit disk $\mathbb{D}$ in the same way as on $\mathbb{U}$, and any conformal homeomorphism $\mathbb{U} \to \mathbb{D}$ induces an isometric isomorphism between the corresponding spaces. For example, we take the Cayley transformation $\Theta(z) = (z-i)/(z+i)$, which maps $\overline{\mathbb{U}}$ onto $\overline{\mathbb{D}}$, and define the push-forward $\Theta_* : \varphi \mapsto \varphi \circ \Theta^{-1}$ for functions φ on $\mathbb{U}$. Then,

$$\frac{1}{\pi} \iint_{\mathbb{D}} |(\Theta_*\varphi)'(w)|^p \left(\frac{1 - |w|^2}{2} \right)^{p-2} du dv = \frac{1}{\pi} \iint_{\mathbb{U}} |\varphi'(z)|^p y^{p-2} dx dy.$$

By defining $\mathcal{B}_p(\mathbb{D})$ as the analytic *p*-Besov space of holomorphic functions on $\mathbb{D}$ with the finite norm in the left side integral of the above equation, we have that $\mathcal{B}_p(\mathbb{D})$ and $\mathcal{B}_p(\mathbb{U})$ are isometric.

We also see that the Cayley transformation Θ induces an isometric isomorphism between $B_p(\mathbb{S})$ and $B_p(\mathbb{R})$, where $B_p(\mathbb{S})$ is defined similarly. Indeed,

$$
\frac{1}{4\pi^2} \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{|(\Theta_* u)(w_1) - (\Theta_* u)(w_2)|^p}{|w_1 - w_2|^2} |dw_1||dw_2|
$$
$$
= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(z_1) - u(z_2)|^p}{|\Theta(z_1) - \Theta(z_2)|^2} |\Theta'(z_1)||\Theta'(z_2)||dz_1||dz_2|
$$
$$
= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(z_1) - u(z_2)|^p}{|z_1 - z_2|^2} |dz_1||dz_2|,
$$

where we used the identity

$$
\frac{|\Theta(z_1) - \Theta(z_2)|^2}{|z_1 - z_2|^2} = |\Theta'(z_1)||\Theta'(z_2)|
$$

for a Möbius transformation Θ.

We use the fact that each function $\varphi \in \mathcal{B}_p(\mathbb{U})$ has a boundary value almost everywhere on $\mathbb{R}$, and this boundary function $b(\varphi)$ belongs to the p-Besov space $B_p(\mathbb{R})$. As we have seen above, the results on the pairs $(\mathcal{B}_p(\mathbb{U}), B_p(\mathbb{R}))$ and $(\mathcal{B}_p(\mathbb{D}), B_p(\mathbb{S}))$ correspond under Θ_*; we can consider this problem for $(\mathcal{B}_p(\mathbb{D}), B_p(\mathbb{S}))$.

The boundary function $b(\phi)$ is given by non-tangential limits of $\phi \in \mathcal{B}_p(\mathbb{D})$. The existence of the non-tangential limit, and moreover, the reproduction of ϕ from $b(\phi)$ by the Poisson integral have been proved (see [53, Lemma 10.13]).

For $p = 2$, the fact that $b(\phi) \in B_2(\mathbb{S})$ for $\phi \in \mathcal{B}_2(\mathbb{D})$ is well known as the Douglas formula for the Dirichlet integral:

$$
\|\phi\|_{\mathcal{B}_2}^2 = \int_{\mathbb{D}} |\phi'(z)|^2 \frac{dxdy}{\pi} = \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{|b(\phi)(z) - b(\phi)(w)|^2}{|z - w|^2} \frac{|dz|}{2\pi} \frac{|dw|}{2\pi} = \|b(\phi)\|_{B_2}^2.
$$

See [1, Theorem 2-5] for example. The statement for the general case is as follows (see [53, pp.131, 301]).

Lemma 2.2 *The boundary function $b(\phi)$ of $\phi \in \mathcal{B}_p(\mathbb{D})$ belongs to $B_p(\mathbb{S})$ for $p > 1$. The boundary extension operator $b : \mathcal{B}_p(\mathbb{D}) \to B_p(\mathbb{S})$ is a bounded linear isomorphism onto the image.*

A proof for the inhomogeneous Besov space on $\mathbb{R}$ (and on $\mathbb{R}^n$) can be found in [39, Section V.5]. A more explicit proof for the homogeneous case on $\mathbb{S}$ is in [27, Theorems 2.1 and 5.1]. These are referred to in [31, p.505].

The inverse map of the boundary extension b can be extended to $B_p(\mathbb{S})$ after composing the Riesz–Szegö projection $\mathcal{P} : B_p(\mathbb{S}) \to b(\mathcal{B}_p(\mathbb{D}))$. The boundedness of $\mathcal{P}$ is also known (see [28, Section 2.3]). This implies the boundedness of the conjugate operator $\mathcal{H}$ because $\mathcal{P} = (I + i\mathcal{H})/2$. On the real line $\mathbb{R}$, $\mathcal{H}$ is represented as the *Hilbert transformation*, which is defined by

$$
(\mathcal{H}u)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t| > \varepsilon} \frac{u(t)}{x - t} dt
$$

(see [13, Section III.1]).

Lemma 2.3 *The Hilbert transformation $\mathcal{H}$ gives a bounded linear surjective isomorphism $\mathcal{H} : B_p(\mathbb{R}) \to B_p(\mathbb{R})$ such that $\mathcal{H}^2 = -I$ for $p > 1$.*

This also follows from the results on more general operators, for example, in [18, Théorème A] and [14, Proposition 4.7].

The *p*-Besov space $B_p(\mathbb{R})$ is closely related to BMO functions defined as follows. A locally integrable complex-valued function u on $\mathbb{R}$ is of *bounded mean oscillation* (BMO) if

$$\|u\|_* = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |u(x) - u_I| dx < \infty,$$

where the supremum is taken over all bounded intervals I on $\mathbb{R}$ and u_I denotes the integral mean of u over I. The set of all BMO functions on $\mathbb{R}$ is denoted by $\mathrm{BMO}(\mathbb{R})$. This is regarded as a Banach space with the BMO-norm $\|\cdot\|_*$ by ignoring the difference of constant functions. It is said that $u \in \mathrm{BMO}(\mathbb{R})$ is of *vanishing mean oscillation* (VMO) if

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_I |u(x) - u_I| dx = 0,$$

and the subspace of all such functions is denoted by $\mathrm{VMO}(\mathbb{R})$.

The following relation between $B_p(\mathbb{R})$ and $\mathrm{VMO}(\mathbb{R})$ is known. See [38, Section 3] and [47, Propositions 2.2 and 2.3].

Proposition 2.4 *(1) If $u \in B_p(\mathbb{R})$ then $u \in \mathrm{VMO}(\mathbb{R})$. Moreover, $\|u\|_* \leq \|u\|_{B_p}$.*
(2) If $u \in B_p^{\mathbb{R}}(\mathbb{R})$ then e^u is an A_∞-weight.

Here, we say that a non-negative locally integrable function $\omega \geq 0$ is an A_∞-*weight* if there exists a constant $C_\infty(\omega) \geq 1$ such that

$$\frac{1}{|I|} \int_I \omega(x) dx \leq C_\infty(\omega) \exp\left(\frac{1}{|I|} \int_I \log \omega(x) dx\right)$$

for every bounded interval $I \subset \mathbb{R}$. If ω is an A_∞-weight, then $\log \omega$ is a BMO function (see [11, Corollary IV.2.19]).

Finally, we introduce certain classes of Beltrami coefficients on $\mathbb{U}$ including $\mathcal{M}_p(\mathbb{U})$. Let λ be a positive Borel measure on the upper half-plane $\mathbb{U}$. We say that λ is a *Carleson measure* if

$$\|\lambda\|_c = \sup_{I \subset \mathbb{R}} \frac{\lambda(I \times (0, |I|])}{|I|} < \infty,$$

where the supremum is taken over all bounded closed intervals $I \subset \mathbb{R}$ and $I \times (0, |I|] \subset \mathbb{U}$ is the Carleson box over I. The set of all Carleson measures on $\mathbb{U}$ is denoted by

CM($\mathbb{U}$). A Carleson measure $\lambda \in \text{CM}(\mathbb{U})$ is called *vanishing* if

$$\lim_{|I| \to 0} \frac{\lambda(I \times (0, |I|))}{|I|} = 0.$$

The set of all vanishing Carleson measures on $\mathbb{U}$ is denoted by $\text{CM}_0(\mathbb{U})$.

For a Beltrami coefficient μ on $\mathbb{U}$, we define a positive Borel measure λ_μ so that it is absolutely continuous with respect to the Lebesgue measure and satisfies

$$d\lambda_\mu(z) = |\mu(z)|^2 y^{-1} dxdy.$$

Using this, a norm of μ is defined by $\|\mu\|_c = \|\lambda_\mu\|_c^{1/2}$. Let $\mathcal{M}_c(\mathbb{U})$ be the set of all Beltrami coefficients on $\mathbb{U}$ with $\lambda_\mu \in \text{CM}(\mathbb{U})$, which is a domain of the Banach space with norm $\|\cdot\|_c + \|\cdot\|_\infty$. The following claim implies the inclusion $\mathcal{M}_p(\mathbb{U}) \subset \mathcal{M}_c(\mathbb{U})$.

Proposition 2.5 *If $\mu \in \mathcal{M}_p(\mathbb{U})$ for $p \geq 1$, then $\lambda_\mu \in \text{CM}_0(\mathbb{U})$. Moreover, $\|\mu\|_c \leq C_p \|\mu\|_p$ for some constant $C_p > 0$ depending only on p.*

Proof We may assume that $p \geq 2$ because $\mathcal{M}_p(\mathbb{U}) \subset \mathcal{M}_2(\mathbb{U})$ and $\|\mu\|_2 \leq \|\mu\|_p$ if $p \leq 2$. Let $p' = p/2 \geq 1$ and take $q' > 1$ satisfying $1/p' + 1/q' = 1$. When $p' = 1$, the inequality below can be modified suitably. For any bounded interval $I \subset \mathbb{R}$, we have

$$\frac{1}{|I|} \int_0^{|I|} \int_I \frac{|\mu(z)|^2}{y} dxdy = \frac{1}{|I|} \int_0^{|I|} \int_I \frac{|\mu(z)|^2}{y^{2/p'}} \cdot y^{\frac{2}{p'}-1} dxdy$$

$$\leq \left(\int_0^{|I|} \int_I \frac{|\mu(z)|^p}{y^2} dxdy \right)^{2/p} \left(|I|^{1-q'} \int_0^{|I|} y^{-\left(1 - \frac{1}{p'-1}\right)} dy \right)^{1/q'}.$$

The first factor in the last line of the above inequality is bounded by $\|\mu\|_p^2$ and tends to 0 uniformly as $|I| \to 0$. The second factor is equal to $(p' - 1)^{-1/q'} (= 1$ when $p' = 1)$, which we define as C_p^2. Taking the square root shows the statement. $\qquad \square$

3 The Bers Coordinates of the Space of *p*-Weil–Petersson Curves

In this section, we introduce the Bers coordinates for the space of p-Weil–Petersson embeddings $\mathbb{R} \to \mathbb{C}$ and show the holomorphic correspondence to the p-Besov space for $p > 1$.

Definition A continuous embedding $\gamma : \mathbb{R} \to \mathbb{C}$ passing through ∞ is called a *p-Weil–Petersson embedding* for $p \geq 1$ if there is a quasiconformal homeomorphism $G : \mathbb{C} \to \mathbb{C}$ such that $G|_{\mathbb{R}} = \gamma$ and its complex dilatation $\mu_G = G_{\bar{z}}/G_z$ on $\mathbb{U}$ belongs to $\mathcal{M}_p(\mathbb{U})$ and μ_G on $\mathbb{L}$ belongs to $\mathcal{M}_p(\mathbb{L})$. We call such G a *p-Weil–Petersson quasiconformal homeomorphism* associated with γ.

The image of $\gamma : \mathbb{R} \to \mathbb{C}$ as above is called a *p-Weil–Petersson curve*. To consider the space of *p*-Weil–Petersson curves, we incorporate their parametrizations and treat them as *p*-Weil–Petersson embeddings. Special types of *p*-Weil–Petersson embeddings γ are as follows. If such an embedding γ maps $\mathbb{R}$ onto itself, this is nothing but a *p*-Weil–Petersson class homeomorphism. If γ extends conformally to $\mathbb{U}$, we call it the *Riemann mapping parametrization* of a *p*-Weil–Petersson curve.

We can define a *BMO embedding* $\gamma : \mathbb{R} \to \mathbb{C}$ by replacing the above $\mathcal{M}_p(\mathbb{U})$ and $\mathcal{M}_p(\mathbb{L})$ with $\mathcal{M}_c(\mathbb{U})$ and $\mathcal{M}_c(\mathbb{L})$. By Proposition 2.5, we see that any *p*-Weil–Petersson embedding γ for $p \geq 1$ is a BMO embedding. Hence, we can utilize the following known properties of BMO embeddings (see [49, Proposition 3.3, Theorem 3.6]) for *p*-Weil–Petersson embeddings.

Proposition 3.1 *A BMO-embedding* $\gamma : \mathbb{R} \to \mathbb{C}$ *has its derivative* γ' *almost everywhere on* $\mathbb{R}$ *and* $\log \gamma' \in \mathrm{BMO}(\mathbb{R})$. *Moreover, if* $|\gamma'|$ *is an* A_∞-*weight on* $\mathbb{R}$*, then* γ *is locally absolutely continuous and the image* $\gamma(\mathbb{R})$ *is a chord-arc curve.*

A locally rectifiable Jordan curve Γ passing through ∞ is called a *chord-arc curve* if there is a constant $K \geq 1$ such that the length of the arc between $a, b \in \Gamma$ is bounded by $K|a - b|$. Any *p*-Weil–Petersson curve is a chord-arc curve as we see in the proof of the next claim.

Lemma 3.2 *If* $h : \mathbb{R} \to \mathbb{C}$ *is a Riemann mapping parametrization of a p-Weil–Petersson curve for* $p > 1$*, then* h *is locally absolutely continuous with* $\log h' \in B_p(\mathbb{R})$.

Proof By definition, h extends to a conformal homeomorphism H on $\mathbb{U}$. By Theorem 7.1 in the Appendix, we have $\log H' \in \mathcal{B}_p(\mathbb{U})$. Then, the non-tangential limit $b(\log H')$ belongs to $B_p(\mathbb{R})$ by Lemma 2.2. Moreover, $\log h'$ coincides with $b(\log H')$ (a.e.) since H has the quasiconformal extension to $\mathbb{C}$ (see [30, Theorem 5.5]). Thus, we have $\log h' \in B_p(\mathbb{R})$. In particular, $|h'| \in A_\infty$ by Proposition 2.4. Then, we conclude by Proposition 3.1 that h is locally absolutely continuous on $\mathbb{R}$. □

The following result is given in [35, p.1056] for $p = 2$, and in [42, p.669] for $p \geq 2$. The generalization to $p > 1$ is also possible as we do it in [50, Theorem 5.5].

Lemma 3.3 *If* $f : \mathbb{R} \to \mathbb{R}$ *is a p-Weil–Petersson class homeomorphism for* $p > 1$*, then* f *is locally absolutely continuous and* $\log f' \in B_p^{\mathbb{R}}(\mathbb{R})$.

Proof By the well-known conformal welding principle (see [17, Section III.1.4] and [43, p.11]), there exists a pair of quasiconformal homeomorphisms H and H_* on the whole plane $\mathbb{C}$ such that H is conformal on $\mathbb{U}$, H^* is conformal on $\mathbb{L}$, and $f = h_*^{-1} \circ h$ on $\mathbb{R}$ for $h = H|_\mathbb{R}$ and $h_* = H_*|_\mathbb{R}$. Moreover, we can choose these H and H_* so that the complex dilatation of $H|_\mathbb{L}$ is in $\mathcal{M}_p(\mathbb{L})$ and the complex dilatation of $H_*|_\mathbb{U}$ is in $\mathcal{M}_p(\mathbb{U})$. This is a crucial step and its argument is given in [50]. We note that to obtain the appropriate mapping H_*, we have to show that the inverse f^{-1} has a quasiconformal extension whose complex dilatation belongs to $\mathcal{M}_p(\mathbb{U})$. This is a part of the property that T_p has the group structure, which will be explained in Sect. 6.

Lemma 3.2 implies that both h and h_* are locally absolutely continuous with $\log h' \in B_p(\mathbb{R})$ and $\log h_*' \in B_p(\mathbb{R})$. From $h_* \circ f = h$ on $\mathbb{R}$, we see that the

increasing homeomorphism f maps a set of null measure to a set of null measure because h is locally absolutely continuous and $|h'_*(x)| > 0$ almost everywhere on $\mathbb{R}$. Hence, f is locally absolutely continuous. Taking the derivatives of both sides of the above equality, we have

$$P_f(\log h'_*) + \log f' = \log h'.$$

Moreover, Proposition 2.1 shows that $P_f(\log h'_*) \in B_p(\mathbb{R})$. Hence, $\log f' \in B_p(\mathbb{R})$. $\qquad\square$

The combination of the above two lemmas proves the characteristic property of p-Weil–Petersson embeddings.

Theorem 3.4 *A p-Weil–Petersson embedding $\gamma : \mathbb{R} \to \mathbb{C}$ is locally absolutely continuous and $\log \gamma'$ belongs to $B_p(\mathbb{R})$ for $p > 1$.*

Proof Let $G : \mathbb{C} \to \mathbb{C}$ be a p-Weil–Petersson quasiconformal homeomorphism associated with γ such that $\mu_1 = \mu_G|_{\mathbb{U}} \in \mathcal{M}_p(\mathbb{U})$ and $\mu_2 = \mu_G|_{\mathbb{L}} \in \mathcal{M}_p(\mathbb{L})$. We take a quasiconformal homeomorphism $F : \mathbb{C} \to \mathbb{C}$ whose complex dilatation is $\mu_1(z)$ for $z \in \mathbb{U}$ and $\overline{\mu_1(\bar{z})}$ for $z \in \mathbb{L}$, which maps $\mathbb{R}$ onto itself. By Lemma 3.3, $f = F|_{\mathbb{R}}$ is locally absolutely continuous and $\log f'$ belongs to $B_p^{\mathbb{R}}(\mathbb{R})$. Next, we take a quasiconformal homeomorphism $H : \mathbb{C} \to \mathbb{C}$ that is conformal on $\mathbb{U}$ and whose complex dilatation on $\mathbb{L}$ is the push-forward $F_*\mu_2$ of μ_2 by F. Namely, the complex dilatation of $H \circ F$ is μ_2. Then, $H \circ F$ coincides with G up to an affine transformation of $\mathbb{C}$, and hence, we may assume that $H \circ F = G$.

We may replace $F|_{\mathbb{L}}$ with a bi-Lipschitz diffeomorphism under the hyperbolic metric whose complex dilatation $\tilde{\mu}_1$ belongs to $\mathcal{M}_p(\mathbb{L})$ (see [8, Theorem 6], [52, Theorem 2.4], and [50, Lemma 3.4]). The complex dilatation of $H|_{\mathbb{L}}$ is explicitly given by

$$F_*\mu_2(\zeta) = \frac{\mu_2(z) - \tilde{\mu}_1(z)}{1 - \overline{\tilde{\mu}_1(z)}\mu_2(z)} \cdot \frac{F_z}{\overline{F_z}}$$

for $\zeta = F(z) \in \mathbb{L}$. Using the property that F is a bi-Lipschitz diffeomorphism, we see from this formula that $F_*\mu_2$ also belongs to $\mathcal{M}_p(\mathbb{L})$. Then by Lemma 3.2, $h = H|_{\mathbb{R}}$ is locally absolutely continuous with $\log h' \in B_p(\mathbb{R})$.

By $h \circ f = \gamma$, we see that γ is also locally absolutely continuous, and taking the derivative, we have

$$\log h' \circ f + \log f' = \log \gamma'.$$

By $\log f' \in B_p(\mathbb{R})$ and $\log h' \in B_p(\mathbb{R})$ combined with Proposition 2.1, we obtain that $\log \gamma' \in B_p(\mathbb{R})$. $\qquad\square$

We impose the normalization $\gamma(0) = 0$ and $\gamma(1) = 1$ (and $\gamma(\infty) = \infty$) on a p-Weil–Petersson embedding γ. Let WPC_p be the set of all such normalized p-Weil–Petersson embeddings for $p \geq 1$. We also denote the subset of WPC_p consisting

of all normalized p-Weil–Petersson class homeomorphisms by W_p, and the subset consisting of all normalized Riemann mapping parametrizations of p-Weil–Petersson curves by RM_p. For $\mu_1 \in \mathcal{M}_p(\mathbb{U})$ and $\mu_2 \in \mathcal{M}_p(\mathbb{L})$, we denote by $G = G(\mu_1, \mu_2)$ the normalized p-Weil–Petersson quasiconformal homeomorphism of $\mathbb{C}$ ($G(0) = 0$, $G(1) = 1$, and $G(\infty) = \infty$) with $\mu_G|_{\mathbb{U}} = \mu_1$ and $\mu_G|_{\mathbb{L}} = \mu_2$. We define a map

$$\widetilde{\iota} : \mathcal{M}_p(\mathbb{U}) \times \mathcal{M}_p(\mathbb{L}) \to \mathrm{WPC}_p$$

by $\widetilde{\iota}(\mu_1, \mu_2) = G(\mu_1, \mu_2)|_{\mathbb{R}}$. Then, by the famous argument of simultaneous uniformization due to Bers, we see the following fact. The proof is essentially the same as that for BMO embeddings, which is in [49, Proposition 4.1].

Proposition 3.5 *The space* WPC_p *of all normalized p-Weil–Petersson embeddings is identified with* $T_p(\mathbb{U}) \times T_p(\mathbb{L})$ *for* $p \geq 1$. *More precisely,* $\widetilde{\iota}$ *splits into a well-defined bijection*

$$\iota : T_p(\mathbb{U}) \times T_p(\mathbb{L}) \to \mathrm{WPC}_p$$

by the product of the Teichmüller projections $\widetilde{\pi} : \mathcal{M}_p(\mathbb{U}) \times \mathcal{M}_p(\mathbb{L}) \to T_p(\mathbb{U}) \times T_p(\mathbb{L})$ *such that* $\widetilde{\iota} = \iota \circ \widetilde{\pi}$.

We call $T_p(\mathbb{U}) \times T_p(\mathbb{L})$ the *Bers coordinates* of WPC_p. Any normalized p-Weil–Petersson embedding γ is represented by a pair $([\mu_1], [\mu_2])$ of the Teichmüller equivalence classes of $\mu_1 \in \mathcal{M}(\mathbb{U})$ and $\mu_2 \in \mathcal{M}(\mathbb{L})$ via $G(\mu_1, \mu_2)$.

We may provide complex Banach manifold structures for $T_p(\mathbb{U})$ and $T_p(\mathbb{L})$ by using the pre-Schwarzian derivative models as in Theorem 7.2 of the Appendix. Namely, $T_p(\mathbb{U})$ is identified with the domain $\mathcal{T}_p(\mathbb{L})$ of the analytic p-Besov space $\mathcal{B}_p(\mathbb{L})$, and $T_p(\mathbb{L})$ is identified with the domain $\mathcal{T}_p(\mathbb{U})$ of $\mathcal{B}_p(\mathbb{U})$ for $p > 1$:

$$T_p(\mathbb{U}) \cong \mathcal{T}_p(\mathbb{L}) = \{\mathcal{L}_{G(\mu,0)} \in \mathcal{B}_p(\mathbb{L}) \mid \mu \in \mathcal{M}_p(\mathbb{U})\};$$
$$T_p(\mathbb{L}) \cong \mathcal{T}_p(\mathbb{U}) = \{\mathcal{L}_{G(0,\mu)} \in \mathcal{B}_p(\mathbb{U}) \mid \mu \in \mathcal{M}_p(\mathbb{L})\}.$$

Then, by Proposition 3.5, we may also regard WPC_p as a domain of $\mathcal{B}_p(\mathbb{L}) \times \mathcal{B}_p(\mathbb{U})$ for $p > 1$.

By Theorem 3.4, we can consider an injective map $L : \mathrm{WPC}_p \to B_p(\mathbb{R})$ defined by $L(\gamma) = \log \gamma'$. Then, with respect to the complex structure of WPC_p given as above, we see the following.

Theorem 3.6 *The map* $L : \mathrm{WPC}_p \to B_p(\mathbb{R})$ *is a holomorphic injection for* $p > 1$.

Proof We will prove that L is holomorphic at any point $\gamma = G(\mu_1, \mu_2)|_{\mathbb{R}}$ in WPC_p. Since WPC_p can be regarded as a domain of the product $\mathcal{B}_p(\mathbb{L}) \times \mathcal{B}_p(\mathbb{U})$ of the Banach spaces, the Hartogs theorem for Banach spaces (see [6, Theorem 14.27] and [25, Theorem 36.8]) implies that we have only to prove that L is separately holomorphic. Thus, by fixing $[\mu_1] \in T_p(\mathbb{U})$, we will show that $\log(G(\mu_1, \mu)|_{\mathbb{R}})' \in B_p(\mathbb{R})$ depends holomorphically on $[\mu] \in T_p(\mathbb{L})$. The other case is similarly verified.

By the proof of Theorem 3.4, we have

$$\log(G(\mu_1, \mu)|_{\mathbb{R}})' = \log \gamma' = \log h' \circ f + \log f'.$$

As before, we may choose a bi-Lipschitz diffeomorphism $F : \mathbb{L} \to \mathbb{L}$ that is the extension of $f : \mathbb{R} \to \mathbb{R}$ and whose complex dilatation still belongs to $\mathcal{M}_p(\mathbb{L})$. Let $h : \mathbb{R} \to \mathbb{C}$ be the restriction of the quasiconformal homeomorphism $H_{F_*\mu}$ of $\mathbb{C}$ that is conformal on $\mathbb{U}$ and has the complex dilatation $F_*\mu$ on $\mathbb{L}$. Since F is a bi-Lipschitz diffeomorphism, we see that F_* acts on $\mathcal{M}_p(\mathbb{L})$ as a biholomorphic automorphism, and its action projects down to $T_p(\mathbb{L})$ also as a biholomorphic automorphism. For $p \geq 2$, this is shown in [43, Chap.1, Corollary 2.12] and [52, Proposition 5.3], and the same proof is valid for $p \geq 1$ once we know that $F_*\mu \in \mathcal{M}_p(\mathbb{L})$ for every $\mu \in \mathcal{M}_p(\mathbb{L})$ (see [50, Lemma 3.1]). The continuity or local boundedness of F_* is enough to show the holomorphy of F_*, which is also explained in [48, Proposition 3.1] in a similar setting. Hence, $\mathcal{L}_{H_{F_*\mu}|_{\mathbb{U}}} = \log(H_{F_*\mu}|_{\mathbb{U}})' \in \mathcal{B}_p(\mathbb{U})$ depends on $[\mu] \in T_p(\mathbb{L})$ holomorphically as we see that $\alpha^{-1} : \mathscr{T}_p \cong T_p(\mathbb{L}) \to \mathcal{B}_p(\mathbb{U})$ is holomorphic in the proof of Theorem 7.2 in the Appendix.

By Lemma 2.2, the boundary extension $b : \mathcal{B}_p(\mathbb{U}) \to B_p(\mathbb{R})$ is a bounded linear operator for $p > 1$. Moreover, by Proposition 2.1, the composition operator $P_f : B_p(\mathbb{R}) \to B_p(\mathbb{R})$ induced by f is also a bounded linear operator. Therefore,

$$\log h' \circ f = P_f \circ b(\mathcal{L}_{H_{F_*\mu}|_{\mathbb{U}}}) \in B_p(\mathbb{R})$$

depends on $[\mu] \in T_p(\mathbb{L})$ holomorphically, and so does $\log(G(\mu_1, \mu)|_{\mathbb{R}})'$. □

4 Conformal Welding and Curve Theoretical Coordinates

We introduce other coordinates of WPC_p and $L(\mathrm{WPC}_p)$ and investigate their relationship. To this end, we utilize the canonical automorphisms of WPC_p.

For $\nu \in \mathcal{M}_p(\mathbb{U})$, the same symbol ν still denotes the complex dilatation $\overline{\nu(\bar{z})}$ for $z \in \mathbb{L}$ in $\mathcal{M}_p(\mathbb{L})$. This also gives the identification of $T_p(\mathbb{U})$ and $T_p(\mathbb{L})$, which is often denoted by T_p hereafter. For any $[\nu] \in T_p$, we define the *right translation* of WPC_p for $p \geq 1$ by

$$\widetilde{R}_{[\nu]} : ([\mu_1], [\mu_2]) \mapsto ([\mu_1] * [\nu], [\mu_2] * [\nu]),$$

where $R_{[\nu]}([\mu]) = [\mu] * [\nu]$ is the composition of elements in T_p that is given by the Teichmüller class of the complex dilatation of $F^\mu \circ F^\nu$ for the normalized p-Weil–Petersson class homeomorphisms F^μ and F^ν of $\mathbb{U}$ (or $\mathbb{L}$) onto itself with the given complex dilatations. The right translation defined by $R_{[\nu]}$ is a biholomorphic automorphism of T_p for $p \geq 1$ as is explained in the proof of Theorem 3.6. Hence, $\widetilde{R}_{[\nu]}$ gives a biholomorphic automorphism of WPC_p.

First, we consider the *conformal welding coordinates* of WPC_p for $p \geq 1$. Under the Bers coordinates $\mathrm{WPC}_p \cong T_p(\mathbb{U}) \times T_p(\mathbb{L})$, the subspace $W_p \subset \mathrm{WPC}_p$ is identified

with the diagonal locus

$$\{([\mu], [\mu]) \in T_p(\mathbb{U}) \times T_p(\mathbb{L}) \mid [\mu] \in T_p\},$$

where the second coordinate $[\mu] \in T_p(\mathbb{L})$ stands for the Teichmüller class of the Beltrami coefficient $\overline{\mu(\bar{z})} \in \mathcal{M}_p(\mathbb{L})$ for $\mu(z) \in \mathcal{M}_p(\mathbb{U})$. Since this is the fixed point locus of the anti-holomorphic involution $([\mu_1], [\mu_2]) \mapsto ([\mu_2], [\mu_1])$, we see that W_p is a real-analytic submanifold of WPC$_p$ by the implicit function theorem (see [6, Theorem 7.18]). A more explicit claim in the finite dimensional case can be found in [16, Proposition 1.2], which is also applicable to our case. See also [9, p.38].

Moreover, the subspace $RM_p \subset$ WPC$_p$ is identified with the second coordinate axis

$$\{([0], [\mu]) \in T_p(\mathbb{U}) \times T_p(\mathbb{L}) \mid [\mu] \in T_p\},$$

which is a complex-analytic submanifold of WPC$_p$.

We define the projections to these submanifolds

$$\Pi : \text{WPC}_p \to W_p, \qquad \Phi : \text{WPC}_p \to RM_p$$

by $\Pi([\mu_1], [\mu_2]) = ([\mu_1], [\mu_1])$ and $\Phi([\mu_1], [\mu_2]) = ([0], [\mu_2] * [\mu_1]^{-1})$ in the Bers coordinates, where $[\mu]^{-1}$ is the inverse of an element in T_p that is given by the Teichmüller class of the complex dilatation μ^{-1} of $(F^\mu)^{-1}$. Then, every $\gamma \in$ WPC$_p$ is decomposed uniquely into $\gamma = \Phi(\gamma) \circ \Pi(\gamma)$. This corresponds to the decomposition $\gamma = h \circ f$ in the proof of Theorem 3.4. Clearly, Π is real-analytic. We see that Φ is continuous later by Theorem 6.1. The biholomorphic automorphism $\widetilde{R}_{[\nu]}$ of WPC$_p$ for $[\nu] \in T_p$ satisfies that $\Phi \circ \widetilde{R}_{[\nu]} = \Phi$.

The projections Π and Φ define another product structure $W_p \times RM_p$ on WPC$_p$ for $p \geq 1$. Namely, we have a bijection

$$(\Pi, \Phi) : \text{WPC}_p \to W_p \times RM_p.$$

Once we see that Φ is continuous, (Π, Φ) is a homeomorphism. This is the coordinate change of WPC$_p$ from the Bers coordinates to the one we may call the conformal welding coordinates. Since W_p and RM_p are both identified with T_p, by marking T_p with $W_p \cong T_p^W$ and $RM_p \cong T_p^{RM}$, the coordinate change is expressed as

$$T_p(\mathbb{U}) \times T_p(\mathbb{L}) \to T_p^W \times T_p^{RM} : \quad ([\mu_1], [\mu_2]) \mapsto ([\mu_1], [\mu_2] * [\mu_1]^{-1}).$$

Next, we consider the *curve theoretical coordinates* of the space of *p*-Weil–Petersson embeddings γ by using the image $L(\text{WPC}_p)$ in $B_p(\mathbb{R})$ for $p > 1$. We see that L is injective because γ can be reproduced from $w = \log \gamma' \in L(\text{WPC}_p)$ by

$$\gamma(x) = \int_0^x e^{w(t)} dt.$$

We have assumed that $B_p(\mathbb{R})$ is the Banach space of all equivalence classes of complex-valued functions modulo additive constants, and can also regard it as the set of representatives w satisfying the normalization condition $\int_0^1 e^{w(t)} dt = 1$. For $w \in L(\mathrm{WPC}_p)$, this is always possible by adding some complex constant to w.

Let $u \in B_p^{\mathbb{R}}(\mathbb{R})$ and let $\gamma_u : \mathbb{R} \to \mathbb{R}$ be the p-Weil–Petersson class homeomorphism in W_p defined by $\gamma_u(x) = \int_0^x e^{u(t)} dt$. Then, the composition operator $P_{\gamma_u} : B_p(\mathbb{R}) \to B_p(\mathbb{R})$ is given by $w \mapsto w \circ \gamma_u$ for $w \in B_p(\mathbb{R})$, which is a bounded linear isomorphism of the Banach space $B_p(\mathbb{R})$ onto itself by Proposition 2.1. Moreover, we define $Q_u(w) = P_{\gamma_u}(w) + u$ for $w \in B_p(\mathbb{R})$, which is an affine isomorphism of $B_p(\mathbb{R})$ onto itself. Since P_{γ_u} preserves $B_p^{\mathbb{R}}(\mathbb{R})$, we see that Q_u maps $u_0 + i B_p^{\mathbb{R}}(\mathbb{R})$ onto $u_1 + i B_p^{\mathbb{R}}(\mathbb{R})$ for some $u_1 \in B_p^{\mathbb{R}}(\mathbb{R})$ depending on $u_0 \in B_p^{\mathbb{R}}(\mathbb{R})$, where $i B_p^{\mathbb{R}}(\mathbb{R})$ denotes the real subspace of $B_p(\mathbb{R})$ consisting of all purely imaginary functions modulo complex-valued constant functions.

We see that the affine isomorphism Q_u of $B_p(\mathbb{R})$ keeps the subset $L(\mathrm{WPC}_p)$ invariant as the following claim asserts.

Proposition 4.1 *The right translation $\widetilde{R}_{[v]}$ satisfies*

$$L \circ \widetilde{R}_{[v]} = Q_{L([v],[v])} \circ L$$

on WPC_p *for every* $[v] \in T_p$. *Hence, the affine isomorphism* Q_u *of* $B_p(\mathbb{R})$ *for any* $u \in B_p^{\mathbb{R}}(\mathbb{R})$ *maps* $L(\mathrm{WPC}_p)$ *onto itself.*

The proof is the same as that of [49, Proposition 5.1] for BMO embeddings. Indeed, by the correspondence between $u \in B_p^{\mathbb{R}}(\mathbb{R})$ and $[v] \in T_p$ through $\gamma_u \in W_p \cong T_p$, we have

$$Q_{L([v],[v])} \circ L = Q_u \circ L = P_{\gamma_u} \circ L + u = L \circ \widetilde{R}_{[v]}.$$

We define the following subset of $L(\mathrm{WPC}_p)$ corresponding to the *arc-length parametrization*:

$$i B_p^{\mathbb{R}}(\mathbb{R})^\circ = i B_p^{\mathbb{R}}(\mathbb{R}) \cap L(\mathrm{WPC}_p).$$

Let $iv \in i B_p^{\mathbb{R}}(\mathbb{R})^\circ$. Then, $\gamma_{iv}(x) = \int_0^x e^{iv(t)} dt$ is a p-Weil–Petersson embedding of arc-length parametrization. Precisely speaking, due to the normalization, γ_{iv} is parametrized by the multiple of its arc-length by a positive constant. We can regard $i B_p^{\mathbb{R}}(\mathbb{R})^\circ$ as a parameter space of such p-Weil–Petersson embeddings for $p > 1$. All p-Weil–Petersson embeddings are obtained by the reparametrization of their arc-length parametrizations as follows.

Lemma 4.2 *Let* $u \in B_p^{\mathbb{R}}(\mathbb{R})$ *and* $iv \in i B_p^{\mathbb{R}}(\mathbb{R})^\circ$. *Then,* $\gamma_{Q_u(iv)}(x)$ *is obtained from the p-Weil–Petersson embedding* $\gamma_{iv}(x')$ *of arc-length parametrization by the change of parameter* $x' = \gamma_u(x)$, *which is also a p-Weil–Petersson embedding. Conversely, every p-Weil–Petersson embedding is obtained in this way. Hence, the map*

$$J : B_p^{\mathbb{R}}(\mathbb{R}) \times i B_p^{\mathbb{R}}(\mathbb{R})^\circ \to L(\mathrm{WPC}_p) \subset B_p(\mathbb{R})$$

defined by $J(u, iv) = Q_u(iv) = u + i P_{\gamma_u}(v)$ *is bijective.*

Proof Since $Q_u(iv) = u + i P_{\gamma_u}(v) = u + iv \circ \gamma_u$, we have

$$\gamma_{Q_u(iv)}(x) = \int_0^x e^{u(t)} e^{iv \circ \gamma_u(t)} dt = \int_0^x \gamma_u'(t) e^{iv \circ \gamma_u(t)} dt$$
$$= \int_0^{\gamma_u(x)} e^{iv(s)} ds = \gamma_{iv}(\gamma_u(x))$$

by $s = \gamma_u(t)$. Proposition 4.1 implies that the reparametrization of a *p*-Weil–Petersson embedding is also a *p*-Weil–Petersson embedding. Conversely, let $\gamma_{u+iv'}$ be any *p*-Weil–Petersson embedding for $u + iv' \in L(\text{WPC}_p)$. Then, by choosing $v \in B_p^{\mathbb{R}}(\mathbb{R})$ satisfying $P_{\gamma_u}(v) = v'$, we see that $\gamma_{u+iv'}$ is obtained from γ_{iv} by changing the parameter. Here, γ_{iv} is a *p*-Weil–Petersson embedding, and hence $iv \in i B_p^{\mathbb{R}}(\mathbb{R})^\circ$. □

We will see later in Theorem 6.3 that the above bijection J is in fact a homeomorphism.

Now, we have two product structures $W_p \times RM_p$ on WPC_p and $B_p^{\mathbb{R}}(\mathbb{R}) \times i B_p^{\mathbb{R}}(\mathbb{R})^\circ$ on $L(\text{WPC}_p)$ for $p > 1$. There is a close relation between these structures through L and J. Each fiber of the projection Φ consists of a family of embeddings with the same image, and hence their arc-length parametrizations are the same. See Fig. 1 in Sect. 1. This observation leads the following.

Proposition 4.3 *For any* $iv \in i B_p^{\mathbb{R}}(\mathbb{R})^\circ$, *let* $\gamma = \Phi \circ L^{-1} \circ J(0, iv) \in RM_p$. *Then, the fiber* $\Phi^{-1}(\gamma)$ *coincides with* $L^{-1} \circ J(B_p^{\mathbb{R}}(\mathbb{R}) \times \{iv\})$, *which is the family of all normalized p-Weil–Petersson embeddings with the same image* $\gamma(\mathbb{R})$.

5 Biholomorphic Correspondence

All the results in this section are stated under the assumption $p > 1$. We show the main theorem in this section as follows.

Theorem 5.1 *The holomorphic map* $L : \text{WPC}_p \to B_p(\mathbb{R})$ *is a biholomorphic homeomorphism onto its image. In particular,* $L(\text{WPC}_p)$ *is an open contractible domain of* $B_p(\mathbb{R})$ *which contains* $B_p^{\mathbb{R}}(\mathbb{R})$.

Proof By virtue of Theorem 3.6, to prove that L is biholomorphic, it suffices to show that L has a local holomorphic inverse at any point $w \in L(\text{WPC}_p) \subset B_p(\mathbb{R})$. This in particular shows that $L(\text{WPC}_p)$ is open.

It is proved in [38, Theorem 6.1] based on the arguments in [34] that if $w = iv \in i B_p^{\mathbb{R}}(\mathbb{R})^\circ$ for $p = 2$, then there is a neighborhood $V_{iv} \subset L(\text{WPC}_p)$ of iv and a holomorphic map $\lambda_{iv} : V_{iv} \to \mathcal{M}_p(\mathbb{U}) \times \mathcal{M}_p(\mathbb{L})$ such that $L \circ \tilde{\pi} \circ \lambda_{iv} = \text{id}|_{V_{iv}}$. For general $p > 1$, the proof is essentially the same.

By [34, Lemma 4.11] and [38, Proposition 5.3], the quasiconformal homeomorphism G of $\mathbb{C}$ onto itself defined by the complex dilatation $\lambda_{iv}(iv) \in \mathcal{M}_p(\mathbb{U}) \times \mathcal{M}_p(\mathbb{L})$ is bi-Lipschitz in the Euclidean metric on $\mathbb{C}$. This implies that both $G|_{\mathbb{U}}$ and $G|_{\mathbb{L}}$

are bi-Lipschitz with respect to the hyperbolic metrics on $\mathbb{U}$, $\mathbb{L}$, and their images (see [23, Proposition 11]). In this case, the product of the Teichmüller projections $\widetilde{\pi} : \mathcal{M}_p(\mathbb{U}) \times \mathcal{M}_p(\mathbb{L}) \to T(\mathbb{U}) \times T(\mathbb{L})$ is continuous at $\lambda_{iv}(iv)$ and in fact holomorphic. This follows from [50, Lemma 3.2] with [48, Lemma 6.1] for $p \geq 1$. We note that the Teichmüller projection π is known to be holomorphic for $p \geq 2$ in [41, Theorem 3.1]. Then, $\Psi_{iv} = \widetilde{\pi} \circ \lambda_{iv}$ is a local holomorphic inverse of L defined on V_{iv} by choosing a smaller neighborhood V_{iv} if necessary.

If $w = u + iv'$ is an arbitrary point in $L(\mathrm{WPC}_p)$, then we utilize Q_u, and find $iv \in iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$ with $Q_u(iv) = u + iv'$ by Lemma 4.2. Since Q_u is a biholomorphic automorphism of $L(\mathrm{WPC}_p)$ by Proposition 4.1, we see that $\widetilde{R}_{[v]} \circ \Psi_{iv} \circ Q_u^{-1}$ is holomorphic on $Q_u(V_{iv})$ for $[v] \in T_p$ corresponding to $\gamma_u \in W_p$. Then, Proposition 4.1 implies that

$$L \circ \widetilde{R}_{[v]} \circ \Psi_{iv} \circ Q_u^{-1} = Q_{L([v],[v])} \circ L \circ \Psi_{iv} \circ Q_u^{-1} = Q_{L([v],[v])} \circ Q_u^{-1} = \mathrm{id}$$

on $Q_u(V_{iv})$. Hence, $\widetilde{R}_{[v]} \circ \Psi_{iv} \circ Q_u^{-1}$ is a local holomorphic inverse of L on $Q_u(V_{iv})$.

We know that the p-Weil–Petersson Teichmüller space T_p is contractible by [8, Theorem 6] for $p = 2$, by [52, Proposition 3.5] for $p \geq 2$, and by [47, Corollary 1.4] and [50, Proposition 5.6] for general $p > 1$. (In fact, the above argument giving the identification $T_p \cong W_p \cong B_p^{\mathbb{R}}(\mathbb{R})$ for $p > 1$ produces Corollary 5.2 (1) below, which implies that T_p is contractible.) Hence, the product $T_p(\mathbb{U}) \times T_p(\mathbb{L})$ is contractible, and so is $L(\mathrm{WPC}_p)$. □

Remark The space $L(\mathrm{WPC}_p)$ is denoted by $\widehat{\mathcal{T}_e}$ in [38, Theorem 2.5] in the case of $p = 2$ and proved that it is a contractible open domain in $H^{1/2}(\mathbb{R}) = B_2(\mathbb{R})$. It is also shown in [38, Theorem 2.2] that $iB_2^{\mathbb{R}}(\mathbb{R})^{\circ}$, the parameter space for Weil–Petersson curves with arc-length parametrization, coincides with an open subset of $iB_2^{\mathbb{R}}(\mathbb{R})$ consisting of all elements corresponding to chord-arc curves with arc-length parametrization. This result can be generalized as follows: *For $p > 1$, $iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$ is the subset consisting of all $iv \in iB_p^{\mathbb{R}}(\mathbb{R})$ such that $\gamma_{iv}(x) = \int_0^x e^{iv(t)}dt$ is a quasisymmetric embedding of $\mathbb{R}$.* This claim follows from Proposition 5.5 at the end of this section.

Let $IW_p \subset \mathrm{WPC}_p$ denote the subset of all arc-length parametrizations of normalized p-Weil–Petersson curves. Namely,

$$IW_p = L^{-1}(iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}).$$

As $iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$ is a real-analytic submanifold of the domain $L(\mathrm{WPC}_p)$ in the complex Banach space $B_p(\mathbb{R})$, IW_p is a real-analytic submanifold of the complex manifold WPC_p. By $W_p = L^{-1}(B_p^{\mathbb{R}}(\mathbb{R}))$, we see again that W_p is a real-analytic submanifold of WPC_p.

Corollary 5.2 (1) W_p *is a real-analytic submanifold of* WPC_p, *and* $L|_{W_p}$ *is a real-analytic homeomorphism onto* $B_p^{\mathbb{R}}(\mathbb{R})$ *whose inverse is also real-analytic.* (2) IW_p *is a real-analytic submanifold of* WPC_p, *and* $L|_{IW_p}$ *is a real-analytic homeomorphism onto* $iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$ *whose inverse is also real-analytic.*

The real-analytic property of $L|_{W_p}$ has been shown in [36, Theorem 2.3] in the case of $p = 2$ by a different method. Part (1) of the above corollary shows that the real-analytic structure of W_p is equivalent to that of $B_p^{\mathbb{R}}(\mathbb{R})$. This is subordinate to the complex-analytic structure of T_p. Part (2) shows that the real-analytic structure of IW_p is equivalent to that of $iB_p^{\mathbb{R}}(\mathbb{R})^\circ$. Later in Theorem 6.6, we will see that IW_p is topologically equivalent to T_p.

In [47, Theorem 4.4], we construct a holomorphic map $\Lambda : U(B_p^{\mathbb{R}}(\mathbb{R})) \to \mathcal{M}_p(\mathbb{U})$ on some neighborhood $U(B_p^{\mathbb{R}}(\mathbb{R}))$ of the real-valued subspace $B_p^{\mathbb{R}}(\mathbb{R})$ for $p > 1$. In the same way, we have the correspondence to the complex dilatations on $\mathbb{L}$. Thus, we can extend Λ to a holomorphic map

$$\widetilde{\Lambda} : U(B_p^{\mathbb{R}}(\mathbb{R})) \to \mathcal{M}_p(\mathbb{U}) \times \mathcal{M}_p(\mathbb{L}).$$

This induces the inverse of the biholomorphic map L on the neighborhood $U(B_p^{\mathbb{R}}(\mathbb{R}))$ as shown in [47, Theorem 4.5].

Theorem 5.3 *The neighborhood $U(B_p^{\mathbb{R}}(\mathbb{R}))$ is contained in $L(\mathrm{WPC}_p)$, and*

$$\widetilde{\pi} \circ \widetilde{\Lambda} : U(B_p^{\mathbb{R}}(\mathbb{R})) \to \mathrm{WPC}_p$$

is the inverse of L on $U(B_p^{\mathbb{R}}(\mathbb{R}))$ which is holomorphic.

We compare the arc-length parametrizations in IW_p with the Riemann mapping parametrizations in RM_p. Both are the sets of all representatives of p-Weil–Petersson curves, which follows from Proposition 4.3. Hence, there is a canonical bijection between IW_p and RM_p giving the change of the representatives, namely, keeping the images of the corresponding embeddings the same. For the projection $\Phi : \mathrm{WPC}_p \to RM_p$, this bijection is nothing but its restriction $\Phi|_{IW_p} : IW_p \to RM_p$. We will see that $\Phi|_{IW_p}$ is a homeomorphism by Proposition 6.5 in the next section.

Here, we consider the other projection Π restricted to IW_p, which has been studied with great interest in the literature. For any $\gamma \in IW_p$, $\Pi(\gamma) \in W_p$ is defined by the p-Weil–Petersson class homeomorphism inducing the reparametrization from γ to $\Phi(\gamma) \in RM_p$. We will prove the bi-real-analytic property of this mapping. For the space of chord-arc curves, this property for the corresponding map was proved in [7, Theorem 1] by operator theoretical arguments. The first part of the following theorem asserting that λ is real-analytic appeared in [38, Theorem 7.1] in the case of $p = 2$.

Theorem 5.4 *The map $\Pi|_{IW_p} : IW_p \to W_p$ is real-analytic. Hence,*

$$\lambda = L \circ \Pi \circ L^{-1}|_{iB_p^{\mathbb{R}}(\mathbb{R})^\circ} : iB_p^{\mathbb{R}}(\mathbb{R})^\circ \to B_p^{\mathbb{R}}(\mathbb{R})$$

is also real-analytic. Moreover, λ is injective and the inverse λ^{-1} is real-analytic. Namely, λ is a real-analytic homeomorphism onto an open subset of $B_p^{\mathbb{R}}(\mathbb{R})$ whose inverse is also real-analytic. This is also the case for $\Pi|_{IW_p}$.

Proof By Corollary 5.2, IW_p is a real-analytic submanifold of WPC_p. Hence, the restriction $\Pi|_{IW_p}$ of the projection $\Pi : \mathrm{WPC}_p \to W_p$ is real-analytic. Since L is biholomorphic by Theorem 5.1, the conjugate map λ is real-analytic.

We will prove the real-analyticity of the inverse of λ. To this end, we use the corresponding result for the space of chord-arc curves as in [49, Theorem 7.3]. From Propositions 2.4 and 3.1, we see that the space CA of all normalized BMO embeddings with chord-arc images contains WPC_p. Then, there are the corresponding subspaces of CA that contain W_p, IW_p, and RM_p. We also have the inclusion relations of subspaces

$$B_p^{\mathbb{R}}(\mathbb{R}) \subset \mathrm{BMO}_{\mathbb{R}}^*(\mathbb{R}), \ iB_p^{\mathbb{R}}(\mathbb{R})^\circ \subset i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^\circ$$

in $B_p(\mathbb{R}) \subset \mathrm{BMO}(\mathbb{R})$, where $\mathrm{BMO}_{\mathbb{R}}^*(\mathbb{R})$ stands for the space of all real-valued BMO functions u with e^u being an A_∞-weight, and $i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^\circ$ is the open subset consisting of imaginary-valued BMO functions iv such that $\int_0^x e^{iv(t)}dt$ is the arc-length parametrization of a chord-arc curve.

The corresponding map to λ between these larger spaces is denoted by $\tilde{\lambda} : i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^\circ \to \mathrm{BMO}_{\mathbb{R}}^*(\mathbb{R})$. Then, $\lambda = \tilde{\lambda}|_{iB_p^{\mathbb{R}}(\mathbb{R})^\circ}$. It is known that $\tilde{\lambda}$ is a real-analytic homeomorphism onto an open subset of $\mathrm{BMO}_{\mathbb{R}}^*(\mathbb{R})$ whose inverse is also real-analytic (see [33, Theorem 5]).

First, we prove the injectivity of λ. This is the same as the case of the space of chord-arc curves. Every $\gamma_0 \in IW_p$ is decomposed uniquely into $\gamma_0 = h \circ f$ for $h \in RM_p$ and $f \in W_p$. Taking the logarithm of the derivative of this equation, we have

$$\log \gamma_0' = \log h' \circ f + \log f'.$$

Since $\log \gamma_0' = iv$ is purely imaginary and $\log f'$ is real, the real and imaginary parts of this equation become

$$0 = \mathrm{Re} \log h' \circ f + \log f' \quad \text{and} \quad v = \mathrm{Im} \log h' \circ f. \tag{†}$$

Moreover, since $\log h'$ is the boundary extension of the holomorphic function $\log H'$ for the Riemann mapping H on $\mathbb{U}$, $\mathrm{Re} \log h'$ and $\mathrm{Im} \log h'$ are related by the Hilbert transformation $\mathcal{H}$ on $\mathbb{R}$:

$$\mathrm{Im} \log h' = \mathcal{H}(\mathrm{Re} \log h'). \tag{‡}$$

Then, the combination of these equations yields

$$-P_f \circ \mathcal{H} \circ P_f^{-1}(\log f') = v. \tag{*}$$

This shows that v is determined by f and thus $\lambda : \log \gamma_0' \mapsto \log f'$ is injective.

Claim 1 *Suppose that we have the decomposition $\tilde{\gamma}_0 = \tilde{h} \circ \tilde{f}$ of the arc-length parametrization $\tilde{\gamma}_0$ of a chord-arc curve by the Riemann mapping parametrization*

$\tilde{h}$ and the strongly quasisymmetric homeomorphism $\tilde{f}$ of $\mathbb{R}$. In this situation, if $\tilde{f}$ belongs to W_p, then $\tilde{h} \in RM_p$ and $\tilde{\gamma}_0 \in IW_p$.

Proof The formula corresponding to (*) reads as

$$-P_{\tilde{f}} \circ \mathcal{H} \circ P_{\tilde{f}}^{-1}(\log \tilde{f}') = \tilde{v}. \qquad (**)$$

Here, $\tilde{f} \in W_p$ implies $\log \tilde{f}' \in B_p^{\mathbb{R}}(\mathbb{R})$ by Lemma 3.3. Moreover, since $(P_{\tilde{f}})^{-1} = P_{\tilde{f}^{-1}}$ preserves $B_p^{\mathbb{R}}(\mathbb{R})$ by Proposition 2.1, we have $P_{\tilde{f}}^{-1}(\log \tilde{f}') \in B_p^{\mathbb{R}}(\mathbb{R})$. By Lemma 2.3, the Hilbert transformation $\mathcal{H}$ maps $B_p^{\mathbb{R}}(\mathbb{R})$ onto $B_p^{\mathbb{R}}(\mathbb{R})$. This implies $\mathcal{H} \circ P_{\tilde{f}}^{-1}(\log \tilde{f}') \in B_p^{\mathbb{R}}(\mathbb{R})$. By applying $P_{\tilde{f}}$ again, we see that the left side of (**) is in $B_p^{\mathbb{R}}(\mathbb{R})$, and hence $\tilde{v} \in B_p^{\mathbb{R}}(\mathbb{R})$. Since $\log \tilde{\gamma}_0' = i\tilde{v}$, we have $\tilde{\gamma}_0 \in IW_p$ and thus $\tilde{h} = \tilde{\gamma}_0 \circ \tilde{f}^{-1} \in RM_p$. $\qquad \square$

By the conjugation of L, this claim is equivalent to saying that if $\tilde{\lambda}(w) \in B_p^{\mathbb{R}}(\mathbb{R})$ for $w \in i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^\circ$ then $w \in iB_p^{\mathbb{R}}(\mathbb{R})^\circ$.

We move to the investigation of the derivatives of λ and $\tilde{\lambda}$. We note the following two facts: (1) As λ is real-analytic, the derivative $d_w\lambda : iB_p^{\mathbb{R}}(\mathbb{R}) \to B_p^{\mathbb{R}}(\mathbb{R})$ is a bounded linear operator at every point w of the domain of λ; (2) As $\tilde{\lambda}$ is real-analytic and $\tilde{\lambda}^{-1}$ is also real-analytic, the derivative $d_{\tilde{w}}\tilde{\lambda} : i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R}) \to \mathrm{BMO}_{\mathbb{R}}(\mathbb{R})$ is a surjective bounded linear isomorphism at every point $\tilde{w}$ of the domain of $\tilde{\lambda}$.

Claim 2 $d_w\tilde{\lambda}|_{iB_P^{\mathbb{R}}(\mathbb{R})} = d_w\lambda$ at every point w in the domain $iB_p^{\mathbb{R}}(\mathbb{R})^\circ$ of λ.

Proof Take any $iv \in iB_P^{\mathbb{R}}(\mathbb{R})$, and set $d_w\lambda(iv) = u$ and $d_w\tilde{\lambda}(iv) = \tilde{u}$. Then,

$$\lim_{t \to 0} \left\| \frac{\lambda(w + tiv) - \lambda(w)}{t} - u \right\|_{B_p} = 0; \quad \lim_{t \to 0} \left\| \frac{\tilde{\lambda}(w + tiv) - \tilde{\lambda}(w)}{t} - \tilde{u} \right\|_* = 0.$$

Since $w + tiv \in iB_p^{\mathbb{R}}(\mathbb{R})^\circ$ for all $t \in \mathbb{R}$ sufficiently close to 0, we have $\tilde{\lambda}(w + tiv) = \lambda(w + tiv)$ as well as $\tilde{\lambda}(w) = \lambda(w)$. Combined with the estimate of the norms $\|\cdot\|_* \leq \|\cdot\|_{B_p}$ by Proposition 2.4, these two limits imply $u = \tilde{u}$. Hence, $d_w\tilde{\lambda}(iv) = d_w\lambda(iv)$ for every $iv \in iB_P^{\mathbb{R}}(\mathbb{R})$, that is, $d_w\tilde{\lambda}|_{iB_P^{\mathbb{R}}(\mathbb{R})} = d_w\lambda$. $\qquad \square$

Fact (2) as above implies that $d_{\tilde{w}}\tilde{\lambda}$ is injective at every $\tilde{w}$. Then, $d_w\lambda$ is also injective at every $w \in iB_p^{\mathbb{R}}(\mathbb{R})^\circ$ by Claim 2. Next, we will show that $d_w\lambda$ is surjective. After this, we see from Fact (1) and the open mapping theorem that $d_w\lambda : iB_p^{\mathbb{R}}(\mathbb{R}) \to B_p^{\mathbb{R}}(\mathbb{R})$ is a bounded linear isomorphism. Under this condition, the inverse mapping theorem implies that λ^{-1} is real-analytic in some neighborhood of any point in the image $\lambda(iB_P^{\mathbb{R}}(\mathbb{R})^\circ)$, and thus λ^{-1} is globally real-analytic on $\lambda(iB_P^{\mathbb{R}}(\mathbb{R})^\circ)$ which is an open subset of $B_P^{\mathbb{R}}(\mathbb{R})$.

The remaining task is to show that $d_w\lambda$ is surjective at every $w \in iB_p^{\mathbb{R}}(\mathbb{R})^\circ$. We take any tangent vector $u \in B_p^{\mathbb{R}}(\mathbb{R})$ at $\lambda(w)$, and consider a segment $\{\lambda(w) + tu\} \subset$

$B_p^{\mathbb{R}}(\mathbb{R})$ with t in a sufficiently small interval $[0, \varepsilon]$. Since $\lambda(w) \in \tilde{\lambda}(i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^{\circ})$ and $\tilde{\lambda}(i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^{\circ})$ is open, we may assume that $\{\lambda(w) + tu\} \subset \tilde{\lambda}(i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^{\circ})$. Then, the inverse image $\tilde{\lambda}^{-1}\{\lambda(w) + tu\}$ of the segment is a real-analytic curve $\beta(t)$ in $i\mathrm{BMO}_{\mathbb{R}}(\mathbb{R})^{\circ}$ starting at $w = \beta(0)$. The tangent vector $iv = \frac{d}{dt}\beta(t)|_{t=0}$ of $\beta(t)$ at $t = 0$ satisfies $d_w\tilde{\lambda}(iv) = u$. On the other hand, since $\{\lambda(w) + tu\}$ is contained in $B_p^{\mathbb{R}}(\mathbb{R})$, Claim 1 implies that $\beta(t)$ is contained in $iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$. Hence, $iv \in iB_p^{\mathbb{R}}(\mathbb{R})$. Then by Claim 2, we have $d_w\lambda(iv) = u$. This shows that $d_w\lambda$ is surjective. ☐

In this Weil–Petersson curve version of the Coifman–Meyer theorem, we can also ask a question about the characterization of the domains $iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$ and $\lambda(iB_p^{\mathbb{R}}(\mathbb{R})^{\circ})$ which are contractible and real-analytically equivalent to each other. The contractibility will be seen by Proposition 6.5.

Claim 1 in the above proof can be generalized to some extent. The claim in the remark after Theorem 5.1 is also related to this. A continuous embedding $\gamma : \mathbb{R} \to \mathbb{C}$ is a quasisymmetric embedding if and only if it extends to a quasiconformal homeomorphism of $\mathbb{C}$ (see [44]).

Proposition 5.5 *Let $\gamma_0 : \mathbb{R} \to \mathbb{C}$ be a quasisymmetric embedding that is given as the arc-length parametrization $\gamma_0(x) = \int_0^x e^{iv(t)}dt$ by a measurable function v on $\mathbb{R}$. Let $H : \mathbb{U} \to \Omega$ be the Riemann mapping onto the domain Ω bounded by the image Γ of γ_0, and $h : \mathbb{R} \to \Gamma$ its extension. Then, the reparametrization $f : \mathbb{R} \to \mathbb{R}$ satisfying $\gamma_0 = h \circ f$ is a locally absolutely continuous quasisymmetric homeomorphism. In these circumstances, the following conditions are equivalent: (i) $\log f' \in B_p^{\mathbb{R}}(\mathbb{R})$; (ii) $\log \gamma_0' \in B_p(\mathbb{R})$ $(iv \in iB_p^{\mathbb{R}}(\mathbb{R}))$; (iii) $\log h' \in B_p(\mathbb{R})$; (iv) Γ is a p-Weil–Petersson curve.*

Proof Since Γ is locally rectifiable, the extension h of H to $\mathbb{R}$ is locally absolutely continuous and $h'(x) \neq 0$ almost everywhere on $\mathbb{R}$ (see [30, Theorem 6.8]). Then, for the same reason as in Lemma 3.3, f is locally absolutely continuous from the fact that $\gamma_0 = h \circ f$ is locally absolutely continuous. Since both γ_0 and h extend quasiconformally to $\mathbb{C}$, so does f, and hence f is quasisymmetric.

The equivalence of (i), (ii), and (iii) follows from formulas (†) and (‡) combined with the fact that the composition operator P_f and the Hilbert transformation $\mathcal{H}$ preserve the p-Besov space by Proposition 2.1 and Lemma 2.3. Lemma 3.2 gives the implication (iv) $\Rightarrow$ (iii). Conversely, Theorem 7.1 with Lemma 2.2 gives (iii) $\Rightarrow$ (iv). ☐

The corresponding statements to chord-arc curves are also true for the same reason.

6 The Topological Group Structure and Its Applications

For further investigation, we use the following fact.

Theorem 6.1 *The p-Weil–Petersson Teichmüller space T_p for $p \geq 1$ is a topological group under the operation $*$.*

For $p = 2$, this was proved in [43, Chap.1, Theorem 3.8]. A similar argument to this case using the bi-Lipschitz quasiconformal extension and estimating the integral of the complex dilatation also works for any $p \geq 1$. We show the following basic fact.

Lemma 6.2 *If* $[\mu]$ *and* $[\nu]$ *converge to* $[0]$ *in* T_p *for* $p \geq 1$, *then* $[\mu] * [\nu] \to [0]$ *and* $[\nu]^{-1} \to [0]$.

Proof Let $F : \mathbb{U} \to \mathbb{U}$ be a quasiconformal homeomorphism with its complex dilatation $\mu \in \mathcal{M}_p(\mathbb{U})$ in the equivalence class $[\mu]$. We may choose μ so that $\|\mu\|_p \to 0$ and $\|\mu\|_\infty \to 0$ as $[\mu] \to [0]$. Let $H : \mathbb{U} \to \mathbb{U}$ be a bi-Lipschitz diffeomorphism with its complex dilatation $\nu \in \mathcal{M}_p(\mathbb{U})$ in the equivalence class $[\nu]$. The existence of such an extension is guaranteed by [50, Lemma 3.4]. This also implies that $\|\nu\|_p \to 0$ and $\|\nu\|_\infty \to 0$ as $[\nu] \to [0]$ and that the bi-Lipschitz constant $L \geq 1$ of H is uniformly bounded while $[\nu]$ tends to $[0]$. We use the chain rule for the complex dilatations:

$$\mu * \nu(z) = \frac{(\mu \circ H(z)) \cdot \frac{\overline{H_z}}{H_z} + \nu(z)}{1 + \mu \circ H(z) \cdot \bar{\nu}(z) \frac{\overline{H_z}}{H_z}}; \quad \nu^{-1}(H(z)) = -\nu(z) \frac{H_z}{\overline{H_z}}.$$

For the composition, we estimate the integral as

$$\int_{\mathbb{U}} |\mu * \nu(z)|^p \frac{dxdy}{y^2} \leq \frac{2^{p-1}}{(1 - \|\mu\|_\infty \|\nu\|_\infty)^p} \left(\int_{\mathbb{U}} |\mu \circ H(z)|^p \frac{dxdy}{y^2} + \int_{\mathbb{U}} |\nu(z)|^p \frac{dxdy}{y^2} \right).$$

Since H^{-1} is Lipschitz with the constant L in the hyperbolic metric, we have

$$\int_{\mathbb{U}} |\mu \circ H(z)|^p \frac{dxdy}{y^2} = \int_{\mathbb{U}} |\mu(\zeta)|^p \frac{d\xi d\eta}{J_H(z)y^2} \leq KL^2 \int_{\mathbb{U}} |\mu(\zeta)|^p \frac{d\xi d\eta}{\eta^2}$$

for the Jacobian determinant J_H of H and the maximal dilatation $K \geq 1$ of H. Thus,

$$\|\mu * \nu\|_p \leq 2^{\frac{p-1}{p}} \frac{(KL^2 \|\mu\|_p^p + \|\nu\|_p^p)^{1/p}}{1 - \|\mu\|_\infty \|\nu\|_\infty}; \quad \|\mu * \nu\|_\infty \leq \frac{\|\mu\|_\infty + \|\nu\|_\infty}{1 - \|\mu\|_\infty \|\nu\|_\infty}.$$

This implies that $[\mu] * [\nu] \to [0]$ as $[\mu] \to [0]$ and $[\nu] \to [0]$.

For the inverse operation, we obtain similarly

$$\int_{\mathbb{U}} |\nu^{-1}(z)|^p \frac{dxdy}{y^2} \leq \int_{\mathbb{U}} |\nu \circ H^{-1}(z)|^p \frac{dxdy}{y^2} \leq KL^2 \int_{\mathbb{U}} |\nu(\zeta)|^p \frac{d\xi d\eta}{\eta^2}$$

since H is Lipschitz with the constant L. Thus,

$$\|\nu^{-1}\|_p \leq (KL^2)^{1/p} \|\nu\|_p; \quad \|\nu^{-1}\|_\infty = \|\nu\|_\infty.$$

This implies that $[\nu]^{-1} \to [0]$ as $[\nu] \to [0]$. □

Proof of Theorem 6.1 Lemma 6.2 implies that T_p is a partial topological group. We have already seen that the right translation $R_{[\nu]}$ for any $[\nu] \in T_p$ is continuous (in fact, biholomorphic) at the beginning of Sect. 4. Hence, to show that T_p is a topological group according to [12, Lemma 1.1], we have only to prove that the adjoint map $T_p \to T_p$ defined by $[\nu] \mapsto [\mu] * [\nu] * [\mu]^{-1}$ for any fixed $[\mu] \in T_p$ is continuous at $[\nu] = [0]$. The arguments for this fact are essentially the same as those in [43, Chap.1, Lemma 3.5]. We omit the details here.   □

There are several consequences from Theorem 6.1. We first consider the curve theoretical product structure for the space WPC_p of normalized p-Weil–Petersson embeddings, and verify the topological equivalence between this real-analytic product structure and the complex-analytic product structure of WPC_p. Then, the continuity of Riemann mappings defined by p-Weil–Petersson curves is proved, and the boundedness of the image of a bounded set under the biholomorphic homeomorphism L on WPC_p is investigated.

Theorem 6.1 in particular implies that $R_{[\nu]}([\mu]) = [\mu] * [\nu]$ and $(R_{[\nu]})^{-1}([\mu]) = [\mu] * [\nu]^{-1}$ are continuous with respect to two variables $([\mu], [\nu]) \in T_p \times T_p$. Since $\widetilde{R}_{[\nu]}$ and $Q_{L([\nu],[\nu])}$ correspond as in Proposition 4.1, this fact yields the following consequence on the bijection J given in Lemma 4.2.

Theorem 6.3 *The bijection* $J : B_p^{\mathbb{R}}(\mathbb{R}) \times i B_p^{\mathbb{R}}(\mathbb{R})^\circ \to L(\text{WPC}_p)$ *is a homeomorphism for* $p > 1$.

Proof We prove that $Q_u(w)$ and $Q_u^{-1}(w)$ are continuous for $(u, w) \in B_p^{\mathbb{R}}(\mathbb{R}) \times L(\text{WPC}_p)$. By Proposition 4.1 with $L([\nu], [\nu]) = u$ for some $[\nu] \in T_p$, we obtain that

$$Q_u(w) = L \circ \widetilde{R}_{[\nu]} \circ L^{-1}(w).$$

Then, Theorems 5.1 and 6.1 imply that this is continuous. Similarly, $Q_u^{-1}(w)$ is also continuous. Since $J(u, iv) = Q_u(iv)$ and $J^{-1}(u + iv) = (u, Q_u^{-1}(u + iv))$, both J and J^{-1} are continuous.   □

Remark The continuity of $Q_u(w)$ also implies that the composition operator $P : B_p(\mathbb{R}) \times W_p \to B_p(\mathbb{R})$ defined by $P_h(w) = w \circ h$ is continuous for both $w \in B_p(\mathbb{R})$ and $h \in W_p$. Indeed, $P_h(w) = Q_{\log h'}(w) - \log h'$. For $p = 2$, a stronger claim than this continuity was asked in [35, Question 4.2].

Corollary 6.4 *The complex Banach manifold structure on* $\text{WPC}_p \cong T_p(\mathbb{U}) \times T_p(\mathbb{L})$ *is topologically equivalent to the real Banach structure of* $B_p^{\mathbb{R}}(\mathbb{R}) \times i B_p^{\mathbb{R}}(\mathbb{R})^\circ$ *under* $J^{-1} \circ L$.

Next, we investigate the continuity of Riemann mappings defined by p-Weil–Petersson curves with respect to the topological structure of WPC_p. By Corollary 6.4, we can also use the product structure $\text{WPC}_p \cong B_p^{\mathbb{R}}(\mathbb{R}) \times i B_p^{\mathbb{R}}(\mathbb{R})^\circ$ in order to consider this continuity.

Proposition 6.5 *The map* $\Phi : \text{WPC}_p \to RM_p \cong T_p^{RM}$ *is a continuous surjection for* $p \geq 1$. *Moreover,* $\Phi|_{IW_p} : IW_p \to RM_p$ *is a homeomorphism for* $p > 1$.

Proof The first statement follows from Theorem 6.1. For $(\Phi|_{IW_p})^{-1} : RM_p \to IW_p$, we consider its conjugate by $J^{-1} \circ L$. This is nothing but the projection $B_p^{\mathbb{R}}(\mathbb{R}) \times iB_p^{\mathbb{R}}(\mathbb{R})^{\circ} \to iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$ to the second factor by Proposition 4.3. Hence, $(\Phi|_{IW_p})^{-1}$ is continuous and $\Phi|_{IW_p}$ is a homeomorphism. $\square$

Remark It is proved in [38, Theorem 2.4] that the map $iB_2^{\mathbb{R}}(\mathbb{R})^{\circ} \to T_2^{RM}$ is a homeomorphism, which corresponds to our $\Phi|_{IW_2}$ under $IW_2 \cong iB_2^{\mathbb{R}}(\mathbb{R})^{\circ}$. In our framework, IW_2 and $iB_2^{\mathbb{R}}(\mathbb{R})^{\circ}$ are real-analytically equivalent under L. Hence, this claim can be rephrased as the homeomorphy of $\Phi|_{IW_2}$ as in Proposition 6.5.

Thus, we conclude that IW_p and RM_p, both of which can be regarded as the space of all *p*-Weil–Petersson curves, are naturally endowed with the two analytic structures in the following sense.

Theorem 6.6 *The real-analytic submanifold IW_p and the complex-analytic submanifold RM_p of* WPC_p *for $p > 1$ are equipped with both the complex-analytic structure of T_p and the real-analytic structure of $iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$, which are topologically equivalent.*

By $J^{-1} \circ L$, we can introduce the product structure $W_p \times IW_p$ to WPC_p from $B_p^{\mathbb{R}}(\mathbb{R}) \times iB_p^{\mathbb{R}}(\mathbb{R})^{\circ}$. Both the products $W_p \times RM_p$ and $W_p \times IW_p$ are homeomorphic to WPC_p for $p > 1$. We summarize the correspondence of these product structures more precisely than Proposition 4.3 by incorporating the homeomorphic property of the sections for the projection Φ. The proof is similar. The fiber structure of the projection to the second factors in both products are preserved since such a fiber consists of *p*-Weil–Petersson curves of the same image.

Proposition 6.7 (1) *For every $\gamma_0 \in IW_p$, the projection $\Phi : \mathrm{WPC}_p \to RM_p$ restricted to $W_p \times \{\gamma_0\} \subset W_p \times IW_p$ is a constant map, and hence $W_p \times \{\gamma_0\}$ is the fiber of Φ over $\Phi(\gamma_0)$. (2) For every $f \in W_p$, Φ restricted to $\{f\} \times IW_p \subset W_p \times IW_p$ is a homeomorphism onto $RM_p \cong T_p^{RM}$, and hence $\{f\} \times IW_p$ is the section of Φ through f.*

We remark that in the comparison of the product structures $W_p \times RM_p$ and $W_p \times IW_p$ on WPC_p, the fibers of the projection to the first factors are not preserved. The reparametrization $\Pi|_{IW_p} : IW_p \to W_p$ considered in Theorem 5.4 measures the difference between the fibers RM_p and IW_p over the origin.

Finally, we consider the correspondence of bounded subsets under the biholomorphic mapping $L : \mathrm{WPC}_p \to B_p(\mathbb{R})$. Here, the boundedness on $\mathrm{WPC}_p \cong T_p(\mathbb{U}) \times T_p(\mathbb{L})$ is defined with respect to the product of the canonical metric structure of T_p. The invariant metric provided for T_p is the *p*-Weil–Petersson metric (see [8, 21, 43]), and let d_p denote the *p*-Weil–Petersson distance in T_p. This has been defined for $p \geq 2$, but it can be extended similarly to $p \geq 1$.

The correspondence $f \mapsto \log f'$ for $f \in W_p$ gives a real-analytic equivalence of the *p*-Weil–Petersson Teichmüller space T_p to $B_p^{\mathbb{R}}(\mathbb{R})$. Translating Lemma 6.2 to $B_p^{\mathbb{R}}(\mathbb{R})$ for $p > 1$, we see that $\|\log(f \circ h)'\|_{B_p} \to 0$ as $\|\log f'\|_{B_p} \to 0$ and $\|\log h'\|_{B_p} \to 0$ for $f, h \in W_p$. Extending this consequence to a claim for the composition operator $P_h : B_p(\mathbb{R}) \to B_p(\mathbb{R})$ defined by $w \mapsto w \circ h$ for every

$w \in B_p(\mathbb{R})$, we obtain the following lemma. We have seen that P_h is a bounded linear operator in Proposition 2.1.

Lemma 6.8 *Let $p > 1$. There exist constants $\tau_0 > 0$ and $C_0 \geq 1$ such that the operator norm of the composition operator P_h on $B_p(\mathbb{R})$ satisfies $\|P_h\| \leq C_0$ for every $h \in W_p$ with $\|\log h'\|_{B_p} \leq \tau_0$.*

For the BMO norm, an analogues result was stated in [7, p.18]. For this case, a proof is given in [49, Proposition 6.3]. The proof in the present case is the same as this.

Proposition 6.9 *Let $h \in W_p$ for $p > 1$. (1) The operator norm $\|P_h\|$ of the composition operator P_h on $B_p(\mathbb{R})$ is bounded by a constant depending only on the p-Weil–Petersson distance $d_p(h, \mathrm{id})$ on $W_p \cong T_p$. (2) The p-Besov norm $\|\log h'\|_{B_p}$ on $B_p(\mathbb{R})$ is bounded by a constant depending only on $d_p(h, \mathrm{id})$.*

Proof For the constant τ_0 in Lemma 6.8, we choose a constant $r_0 > 0$ such that if $h \in W_p$ satisfies $d_p(h, \mathrm{id}) \leq r_0$ then $\|\log h'\|_{B_p} \leq \tau_0$. Any element $h \in W_p$ can be joined to id by a curve in W_p with its length arbitrarily close to $d_p(h, \mathrm{id})$. We choose the minimal number of consecutive points

$$\mathrm{id} = h_0, h_1, \ldots, h_n = h$$

on the curve such that $d_p(h_i, h_{i-1}) < r_0$ for any $i = 1, \ldots, n$. Then, the number n is determined by $d_p(h, \mathrm{id})$, and the invariance of d_p under the right translation implies that the composition $h_i \circ h_{i-1}^{-1}$ satisfies $d_p(h_i \circ h_{i-1}^{-1}, \mathrm{id}) < r_0$, and hence $\|\log(h_i \circ h_{i-1}^{-1})'\|_{B_p} \leq \tau_0$.

By decomposing h into these n mappings, we have

$$P_h = P_{h_1 \circ h_0^{-1}} \circ P_{h_2 \circ h_1^{-1}} \circ \cdots \circ P_{h_n \circ h_{n-1}^{-1}}.$$

Then, Proposition 6.8 shows that $\|P_h\| \leq C_0^n$. This proves statement (1). Moreover,

$$
\begin{aligned}
\|\log h'\|_{B_p} &= \|\log((h_n \circ h_{n-1}^{-1}) \circ (h_{n-1} \circ h_{n-2}^{-1}) \circ \cdots \circ (h_1 \circ h_0^{-1}))'\|_{B_p} \\
&\leq \|P_{h_1 \circ h_0^{-1}} \circ P_{h_2 \circ h_1^{-1}} \circ \cdots \circ P_{h_{n-1} \circ h_{n-2}^{-1}} (\log(h_n \circ h_{n-1}^{-1})')\|_{B_p} \\
&\quad + \cdots + \|\log(h_1 \circ h_0^{-1})'\|_{B_p} \\
&\leq C_0^{n-1}\tau_0 + C_0^{n-2}\tau_0 + \cdots + \tau_0.
\end{aligned}
$$

This proves statement (2).   □

Remark For $p = 2$, statement (1) follows from the stronger result mentioned in the remark after Proposition 2.1. This is because the Teichmüller distance d_∞ is bounded by a certain multiple of the p-Weil–Petersson distance d_p, that is, $d_\infty \lesssim d_p$. See [22, Proposition 6.10].

We show that L maps a bounded set in WPC_p to a bounded set in $B_p(\mathbb{R})$ in a special case. We expect that this is valid in general.

Theorem 6.10 *Under the holomorphic mapping $L : \mathrm{WPC}_p \to B_p(\mathbb{R})$ for $p > 2$, the image $L(\widetilde{W})$ of any bounded subset $\widetilde{W} \subset \mathrm{WPC}_p$ is bounded in $B_p(\mathbb{R})$. More precisely, for $\gamma = G(\mu_1, \mu_2)|_{\mathbb{R}}$ with $\mu_1 \in \mathcal{M}_p(\mathbb{U})$ and $\mu_2 \in \mathcal{M}_p(\mathbb{L})$, the norm $\|L(\gamma)\|_{B_p}$ is bounded by a constant depending only on $d_p([\mu_1], [0])$ and $d_p([\mu_2], [0])$.*

Proof By the proof of Theorem 3.4, we have

$$L(\gamma) = \log(G(\mu_1, \mu_2)|_{\mathbb{R}})' = \log h' \circ f + \log f' = P_f(\log h') + \log f',$$

where $f \in W_p$ is the extension of the quasiconformal homeomorphism $F : \mathbb{U} \to \mathbb{U}$ to $\mathbb{R}$ determined by $[\mu_1] \in T_p(\mathbb{U})$, and $h \in \mathrm{WPC}_p$ is the restriction of the quasiconformal homeomorphism $H : \mathbb{C} \to \mathbb{C}$ to $\mathbb{R}$ that is conformal on $\mathbb{U}$ and quasiconformal on $\mathbb{L}$ determined by $R_{[\mu_1]}^{-1}([\mu_2]) \in T_p(\mathbb{L})$. By Proposition 6.9, $\|\log f'\|_{B_p}$ and $\|P_f\|$ are bounded in terms of $d_p([0], [\mu_1])$. Hence, we have only to estimate $\|\log h'\|_{B_p}$.

As $R_{[\mu_1]}$ is a biholomorphic automorphism of T_p and the p-Weil–Petersson distance d_p is invariant under $R_{[\mu_1]}$, we have

$$d_p(R_{[\mu_1]}^{-1}([\mu_2]), [0]) = d_p([\mu_2], [\mu_1]) \leq d_p([0], [\mu_1]) + d_p([0], [\mu_2]).$$

By [21, Proposition 8.4], we see that there is $\nu \in \mathcal{M}_p(\mathbb{L})$ such that $[\nu] = R_{[\mu_1]}^{-1}([\mu_2] \in T_p(\mathbb{L})$ and $\|\nu\|_p$ is bounded by a constant depending only on $d_p(R_{[\mu_1]}^{-1}([\mu_2]), [0])$.

We consider the estimate of $\|\log(H|_{\mathbb{U}})'\|_{\mathcal{B}_p}$ in terms of $\|\nu\|_p$. For the Schwarzian derivative of $H|_{\mathbb{U}}$ in the Banach space $\mathcal{A}_p(\mathbb{U})$ (see the Appendix), this is known. By modifying the arguments for [43, Chap.1, Theorem 2.3, Lemma 2.9] which implies this estimate for the Schwarzian derivative case, we can prove that $\|\log(H|_{\mathbb{U}})'\|_{\mathcal{B}_p}$ is bounded by a constant multiple of $\|\nu\|_p$ under the condition $p > 2$.

Finally, the boundary extension $b : \mathcal{B}_p(\mathbb{U}) \to B_p(\mathbb{R})$, which maps $\log(H|_{\mathbb{U}})'$ to $\log h'$, is a bounded linear operator by Lemma 2.2. Then, the combination of all estimates we have obtained proves the statement. $\qquad\square$

Conversely, we also expect that for any bounded subset $B \subset B_p^{\mathbb{R}}(\mathbb{R})$, the inverse image $L^{-1}(B)$ is bounded in W_p. If we could prove that $\|\mu_u\|_p$ and $\|\mu_u\|_\infty < 1$ are dominated by $\|u\|_{B_p}$ for the complex dilatation μ_u of a certain quasiconformal extension $F_u : \mathbb{U} \to \mathbb{U}$ of $\gamma_u : \mathbb{R} \to \mathbb{R}$, we would obtain this result from [21, Theorem 5.4]. However, we only have the estimates of $\|\mu_u\|_p$ and $\|\mu_u\|_\infty$ in terms of $\|u\|_{B_p}$ in the case where F_u is the variant of the Beurling–Ahlfors extension and $\|u\|_{B_p}$ is sufficiently small (see [47, Proposition 3.5, Lemma 4.2]). To obtain the estimate for general $u \in B_p^{\mathbb{R}}(\mathbb{R})$, we have to decompose u into pieces u_n of small norms and construct a quasiconformal extension F_u out of F_{u_n}.

7 Appendix: The pre-Schwarzian Derivative Model on the Upper Half-Plane

In this appendix, we provide a complex Banach manifold structure for the p-Weil–Petersson Teichmüller space $T_p = T_p(\mathbb{U})$ by using pre-Schwarzian derivatives on $\mathbb{U}$. This is well known in the case of the unit disk $\mathbb{D}$ for $p = 2$. However, since the pre-Schwarzian derivative is not Möbius invariant, we carefully treat the case of $\mathbb{U}$. We will see below that there is a certain advantage of considering the pre-Schwarzian derivative model on $\mathbb{U}$ compared with $\mathbb{D}$. The generalization to any $p > 1$ is also mentioned. We note that if we use the Schwarzian derivative model, there is no difference between $\mathbb{U}$ and $\mathbb{D}$ due to its Möbius invariance, and T_p is equipped with the complex Banach manifold structure for $p \geq 1$.

Let $\mathcal{A}(\mathbb{U})$ denote the Banach space of holomorphic functions φ on $\mathbb{U}$ with norm

$$\|\varphi\|_{\mathcal{A}} = \sup_{z \in \mathbb{U}} |\varphi(z)| y^2.$$

For $p \geq 1$, we also denote by $\mathcal{A}_p(\mathbb{U})$ the Banach space of holomorphic functions φ on $\mathbb{U}$ with norm

$$\|\varphi\|_{\mathcal{A}_p} = \left(\frac{1}{\pi} \iint_{\mathbb{U}} |\varphi(z)|^p y^{2p-2} dx dy \right)^{\frac{1}{p}}.$$

For any locally univalent function H, the derivative of the logarithm $\mathcal{L}_H$ and the Schwarzian derivative $\mathcal{S}_H$ are defined by

$$\mathcal{L}_H = \log H', \quad \mathcal{S}_H = \mathcal{L}_H'' - \frac{1}{2}(\mathcal{L}_H')^2.$$

The derivative of $\mathcal{L}_H$ is called the *pre-Schwarzian derivative* of H. We will show the following result on the upper half-plane $\mathbb{U}$. In the case of the unit disk $\mathbb{D}$, the corresponding theorem for $p \geq 2$ was proved in [15, Theorems 1, 2]. However, since $\mathcal{L}_H$ is not Möbius invariant, this is not straightforward from that on $\mathbb{D}$. As mentioned below, the case of $p = 2$ was proved in [37].

Theorem 7.1 *Let $p > 1$. Let $H : \mathbb{U} \to \mathbb{C}$ be a conformal mapping on $\mathbb{U}$ extending to the whole plane $\mathbb{C}$ quasiconformally such that $\lim_{z \to \infty} H(z) = \infty$. Then, the following conditions are equivalent*:

(a) H extends to a quasiconformal homeomorphism of $\mathbb{C}$ whose complex dilatation μ on $\mathbb{L}$ belongs to $\mathcal{M}_p(\mathbb{L})$;
(b) $\mathcal{L}_H \in \mathcal{B}_p(\mathbb{U})$;
(c) $\mathcal{S}_H \in \mathcal{A}_p(\mathbb{U})$.

Proof The equivalence of (b) and (c) for $p = 2$ was investigated in [37, Theorem 4.4]. Essentially the same argument is valid for general $p > 1$. See [20, Theorem 3.3]. The implication (a) $\Rightarrow$ (c) for $p \geq 1$ is asserted in [50, Lemma 3.2], and (c) $\Rightarrow$ (a) for $p \geq 1$ is contained in [50, Theorem 4.1]. The equivalence of (a) and (c) was proved

formerly by [8, Theorem 2] for $p = 2$, and by [15, Theorems 2] and [36, Theorem 2.1] for $p \geq 2$. □

Under this preparation, we introduce the pre-Schwarzian derivative model of Teichmüller spaces on $\mathbb{U}$. Let $H : \mathbb{U} \to \mathbb{C}$ be a conformal mapping on $\mathbb{U}$ satisfying the condition $H(\infty) = \infty$ (i.e., $\lim_{z \to \infty} H(z) = \infty$) that extends to a quasiconformal homeomorphism of the whole plane $\mathbb{C}$. Then, the set $\mathscr{T}$ of all $\mathcal{S}_H \in \mathcal{A}(\mathbb{U})$ for such H is the Schwarzian derivative model of the universal Teichmüller space T, and the set $\mathcal{T}$ of all $\mathcal{L}_H \in \mathcal{B}(\mathbb{U})$ for such H is the pre-Schwarzian derivative model of T. It is known that $\mathscr{T}$ is a bounded domain in $\mathcal{A}(\mathbb{U})$ identified with T (the Bers embedding), which defines the complex Banach manifold structure for T (see [17, Section III.4]).

However, if we do not impose the condition $H(\infty) = \infty$ on the conformal mapping H and consider all $\mathcal{L}_H$ for those H, then they are classified into uncountably many components and $\mathcal{T}$ is the one containing 0. To see this, we consider the conformal mapping $\widetilde{H} = H \circ \Theta^{-1}$ of $\mathbb{D}$ pushed forward by the Cayley transformation $\Theta : \mathbb{U} \to \mathbb{D}$. Since

$$\log \widetilde{H}' = \Theta_* \mathcal{L}_H + \log(\Theta^{-1})'$$

and $\log(\Theta^{-1})' \in \mathcal{B}(\mathbb{D})$, this defines an affine isometry $\mathcal{B}(\mathbb{U}) \to \mathcal{B}(\mathbb{D})$. Under this isometric isomorphism, the components in $\mathcal{B}(\mathbb{U})$ correspond to those in $\mathcal{B}(\mathbb{D})$ bijectively, which are $\mathcal{T}_\omega(\mathbb{D})$ ($\omega \in \mathbb{S}$) and $\mathcal{T}_{\mathrm{bdd}}(\mathbb{D})$ characterized by the property that a point $\omega \in \mathbb{S}$ or no point of $\mathbb{S}$ is mapped to ∞ by the extension of the conformal mapping $\widetilde{H}$ to $\mathbb{S}$. See [54] and [40, Section 4]. Then, the component $\mathcal{T} \subset \mathcal{B}(\mathbb{U})$ containing 0 corresponds to $\mathcal{T}_{\Theta(\infty)}(\mathbb{D}) = \mathcal{T}_1(\mathbb{D})$, which is biholomorphically equivalent to the universal Teichmüller space $T \cong \mathscr{T}$.

We can also consider the pre-Schwarzian derivative model of the *p*-Weil–Petersson Teichmüller space T_p for $p > 1$ in the same manner. However, unlike the case of the universal Teichmüller space, no unbounded components appear in $\mathcal{B}_p(\mathbb{D})$; namely, $\mathcal{T}_\omega(\mathbb{D}) \cap \mathcal{B}_p(\mathbb{D}) = \emptyset$ for every $\omega \in \mathbb{S}$ (see [41, Theorem 4.1]). Concerning the correspondence between the spaces on $\mathbb{U}$ and on $\mathbb{D}$, there is also a difference, which is due to the fact that $\log(\Theta^{-1})'$ does not belong to $\mathcal{B}_p(\mathbb{D})$. Nevertheless, Theorem 7.1 implies that there is a bijective correspondence between $\mathcal{T}_p = \mathcal{T} \cap \mathcal{B}_p(\mathbb{U})$ and $\mathscr{T}_p = \mathscr{T} \cap \mathcal{A}_p(\mathbb{U})$ under the map $\alpha : \mathcal{B}_p(\mathbb{U}) \to \mathcal{A}_p(\mathbb{U})$ given by $\alpha(\varphi) = \varphi'' - (\varphi')^2/2$ which stems from $\mathcal{S}_H = \mathcal{L}_H'' - \frac{1}{2}(\mathcal{L}_H')^2$. It was proved in [42, Lemma 2.3] that α is holomorphic in the case of $\mathbb{D}$ for $p \geq 2$. This is also true in the case of $\mathbb{U}$ for $p > 1$. Moreover, $\mathscr{T}_p$ is a contractible domain in $\mathcal{A}_p(\mathbb{U})$ identified with the *p*-Weil–Petersson Teichmüller space T_p for $p \geq 1$, which provides the complex Banach manifold structure for T_p (see [8, 15, 50, 52]).

We finish our discussion by stating the following theorem.

Theorem 7.2 *The holomorphic map α restricted to $\mathcal{T}_p$ is a biholomorphic homeomorphism onto $\mathscr{T}_p$ for $p > 1$. Hence, the complex Banach manifold structures on these two models of the p-Weil–Petersson Teichmüller space T_p are biholomorphically equivalent.*

Proof We have mentioned that $\alpha : \mathcal{T}_p \to \mathscr{T}_p$ is a holomorphic bijection. For a Beltrami coefficient μ in $\mathcal{M}_p(\mathbb{L})$, we take a conformal homeomorphism F_μ on $\mathbb{U}$ with $F_\mu(\infty) = \infty$ that is quasiconformally extendable to $\mathbb{L}$ having the complex dilatation μ. Then, the map $\ell : \mathcal{M}_p(\mathbb{L}) \to \mathcal{T}_p$ defined by $\mu \mapsto \mathcal{L}_{F_\mu}$ is continuous at μ if F_μ (or F^μ) is bi-Lipschitz on $\mathbb{L}$ with respect to the hyperbolic metric. To see this, even in the case of $\mathbb{L}$ for $p > 1$, the proof of [42, Theorem 2.4] in the case of $\mathbb{D}^*$ for $p \geq 2$ can be applied once we fill the step of showing that the Bers projection $\sigma : \mathcal{M}_p(\mathbb{L}) \to \mathscr{T}_p$ given by $\mu \mapsto \mathcal{S}_{F_\mu}$ is continuous at μ if F_μ is bi-Lipschitz on $\mathbb{L}$. This is verified in [50, Lemma 3.2]. (In fact, we obtain that σ is holomorphic for $p \geq 1$ by applying [48, Lemma 6.1] though this is not mentioned in [50].)

Moreover, as the composition $\alpha \circ \ell$ coincides with σ, at any point $\psi \in \mathscr{T}_p$, there is a local continuous right inverse s of $\alpha \circ \ell$ such that $s(\psi)$ is an arbitrary $\mu \in \mathcal{M}_p(\mathbb{L})$ with F_μ bi-Lipschitz on $\mathbb{L}$ (see [42, Theorem 2.1], [52, Proposition 4.3], and [50, Theorem 4.1]). It follows that $\ell \circ s$ becomes a local continuous right inverse of α at ψ, from which we see that α^{-1} is continuous. By the standard argument in this situation, the holomorphy of $\alpha^{-1} : \mathscr{T}_p \to \mathcal{T}_p$ follows from its continuity. Thus, α is biholomorphic. □

Declarations

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