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Normal Covering Spaces with Maximal Bottom of Spectrum

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Abstract

We study the property of spectral-tightness of Riemannian manifolds, which means that the bottom of the spectrum of the Laplacian separates the universal covering space from any other normal covering space of a Riemannian manifold. We prove that spectral-tightness of a closed Riemannian manifold is a topological property characterized by its fundamental group. As an application, we show that a non-positively curved, closed Riemannian manifold is spectrally-tight if and only if the dimension of its Euclidean local de Rham factor is zero. In their general form, our results extend the state of the art results on the bottom of the spectrum under Riemannian coverings.

Keywords Bottom of spectrum · Schrödinger operator · Riemannian covering · Amenable covering · Spectral-tightness

Mathematics Subject Classification 58J50 · 35P15 · 53C99

1 Introduction

The spectrum of the Laplacian of a Riemannian manifold is an interesting isometric invariant, which attracted much attention over the last years. Aiming to a better comprehension of its relations with the geometry of the underlying manifold, its behavior under maps between Riemannian manifolds that respect the geometry of the manifolds to some extent has been studied. In particular, there are various results on the behavior of the spectrum under Riemannian coverings and open questions arising from them.

To be more precise, let $p\colon M_2 \to M_1$ be a Riemannian covering. Then the bottoms of the spectra of the Laplacians satisfy $\lambda_0(M_1) \leq \lambda_0(M_2)$. Brooks was the first one to investigate when the equality holds, and this is closely related to the notion of

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amenability. The covering p is called amenable if the monodromy action of $\pi_1(M_1)$ on the fiber of p is amenable. It is noteworthy that a normal covering is amenable if and only if its deck transformation group is amenable. Brooks proved in [10] that the universal covering of a closed (that is, compact and without boundary) manifold preserves the bottom of the spectrum if and only if it is amenable. This result has been generalized in various ways over the last years (cf. for instance, the survey [4]). In [3], we showed that if p is amenable, then $\lambda_0(M_1) = \lambda_0(M_2)$, without imposing any assumptions on the geometry or the topology of the manifolds. The converse implication is not true in general, but holds under a natural condition involving the bottom $\lambda_0^{\mathrm{ess}}(M_1)$ of the essential spectrum of the Laplacian. More specifically, according to [20, Theorem 1.2], if $\lambda_0(M_2) = \lambda_0(M_1) < \lambda_0^{\mathrm{ess}}(M_1)$, then p is amenable. The situation is very unclear in the case where $\lambda_0(M_1) = \lambda_0^{\mathrm{ess}}(M_1)$, as pointed out in [4, Question 1.5].

Given a normal Riemannian covering $p \colon M_2 \to M_1$, besides the aforementioned inequality, we have that $\lambda_0(M_2) \leq \lambda_0(\tilde{M})$, where $\tilde{M}$ is the universal covering space of M_1. One of the purposes of this paper is to examine the validity of the equality. This fits into the study of manifolds with maximal bottom of spectrum under some constraint. Examples of remarkable works on this problem are [19] and [18], which focus on complete manifolds with Ricci curvature bounded from below and quotients of symmetric spaces of non-compact type, respectively. This setting, where M_2 is a normal covering space of M_1, may seem more restrictive, but our goal is actually different. In addition to obtaining information about the maximizer M_2, we want to characterize the existence of a maximizer (different from the universal covering space) in terms of properties of M_1.

Our motivation is to investigate to what extent the bottom of the spectrum of the Laplacian separates the universal covering space from the rest of the normal covering spaces of a Riemannian manifold. The corresponding question about the exponential volume growth has been addressed in [23]. To be more precise, if M_2 is a normal covering space of M_1, then the exponential volume growths satisfy $\mu(M_1) \leq \mu(M_2)$. Hence, the universal covering space $\tilde{M}$ of M_1 has maximal bottom of spectrum and maximal exponential volume growth among all the normal covering spaces of M_1. A Riemannian manifold M_1 is called *spectrally-tight* or *growth tight* if the universal covering space of M_1 is the unique normal covering space of M_1 with maximal bottom of spectrum or maximal exponential volume growth, respectively. Sambusetti proved in [23] that negatively curved, closed manifolds are growth tight, which yields that negatively curved, closed, locally symmetric spaces are spectrally-tight. One of the aims of this paper is to study the notion of spectral-tightness, establish that negatively curved, closed manifolds enjoy this property, and more generally, characterize this property for non-positively curved, closed Riemannian manifolds.

To set the stage, we consider Riemannian coverings $p \colon M_2 \to M_1, q \colon M_1 \to M_0$, with q normal, a Schrödinger operator S_0 on M_0 and its lifts S_1, S_2 on M_1, M_2, respectively. It turns out that the validity of $\lambda_0(S_2) = \lambda_0(S_1)$ is intertwined with a property similar, but weaker than the amenability of the covering.

The covering p is called relatively amenable with respect to q, or for short, q-amenable if the monodromy action of $q_*\pi_1(M_1)$ on the fiber of $q \circ p$ is amenable. It is evident that a covering is amenable if and only if it is relatively amenable with respect to the identity. The notion of relative amenability is naturally related to the amenability of the composition. More specifically, the composition $q \circ p$ is amenable if and only if q is amenable and p is q-amenable. In general, amenable coverings are relatively amenable, but there exist relatively amenable coverings that are not amenable, as we will show by an example. It is worth to point out that if q is finite sheeted or more importantly, if $q \circ p$ is normal, then p is q-amenable if and only if it is amenable. Our first result illustrates the role of this notion in the behavior of the bottom of the spectrum, extending [3, Theorem 1.2] and [20, Theorem 1.2].

Theorem 1.1 *Let $p\colon M_2 \to M_1$ and $q\colon M_1 \to M_0$ be Riemannian coverings, with q normal. Consider a Schrödinger operator S_0 on M_0 and denote by S_1, S_2 its lift on M_1, M_2, respectively. Then:*

(i) If p is q-amenable, then $\lambda_0(S_2) = \lambda_0(S_1)$.
(ii) Conversely, if $\lambda_0(S_2) = \lambda_0(S_1) < \lambda_0^{\mathrm{ess}}(S_0)$, then p is q-amenable.

It should be noticed that if q is infinite sheeted, then $\lambda_0(S_1) = \lambda_0^{\mathrm{ess}}(S_1)$ (cf. for example [21, Corollary 1.3]). Therefore, the setting of Theorem 1.1 fits into the context of [4, Question 1.5]. Conceptually, it seems interesting that even in this special case, a property different from amenability has such an effect on the behavior of the bottom of the spectrum.

In view of Theorem 1.1, the study of spectral-tightness becomes quite easier, bearing in mind that in this theorem, if M_2 is simply connected, then p is q-amenable if and only if it is amenable. More precisely, it follows that spectral-tightness of a closed manifold is a topological property determined by its fundamental group as follows:

Theorem 1.2 *A closed Riemannian manifold is spectrally-tight if and only if the unique normal, amenable subgroup of its fundamental group is the trivial one.*

Drawing heavily from the work of Eberlein [16], as an application of Theorem 1.2, we characterize the property of spectral-tightness for non-positively curved, closed Riemannian manifolds.

Theorem 1.3 *A non-positively curved, closed Riemannian manifold is not spectrally-tight if and only if its universal covering space splits as the Riemannian product of a Euclidean space with another manifold.*

The notion of spectral-tightness seems more complicated on non-compact Riemannian manifolds. However, it is worth to mention that we prove the characterization of Theorem 1.2 for any Riemannian manifold M with $\lambda_0^{\mathrm{ess}}(M) > \lambda_0(\tilde{M})$, where $\tilde{M}$ is the universal covering space of M. Exploiting this, we present examples demonstrating that spectral-tightness of a non-compact Riemannian manifold depends on its Riemannian metric.

As another application of this characterization, we establish the spectral-tightness of certain geometrically finite manifolds, in the sense of Bowditch [9]. Let M be a complete Riemannian manifold of bounded sectional curvature $-b^2 \leq K \leq -a^2 < 0$. Then its universal covering space $\tilde{M}$ is a Hadamard manifold, and M is written as a quotient $\tilde{M}/\Gamma$. Denoting by $\tilde{M}_c = \tilde{M} \cup \tilde{M}_i$ the geometric compactification of $\tilde{M}$, let

$\Lambda \subset \tilde{M}_i$ be the limit set of Γ. Then Γ acts properly discontinuously on $\tilde{M}_c \setminus \Lambda$. The manifold M is called geometrically finite if the topological manifold $(\tilde{M}_c \setminus \Lambda)/\Gamma$ with possibly empty boundary has finitely many ends and each of them is parabolic. It is worth to mention that if M is geometrically finite, then $\lambda_0^{\text{ess}}(M) \geq \lambda_0(\tilde{M})$, according to [6, Theorem B].

Corollary 1.4 *Let M be a non-simply connected, geometrically finite manifold, and denote by $\tilde{M}$ its universal covering space. If $\lambda_0^{\text{ess}}(M) > \lambda_0(\tilde{M})$, then M is not spectrally-tight if and only if $\pi_1(M)$ is elementary.*

The paper is organized as follows: In Sect. 2, we give some preliminaries on Schrödinger operators, amenable actions and coverings. In Sect. 3, we introduce the notion of relatively amenable coverings and discuss some basic properties. Section 4 is devoted to Theorem 1.1 and some applications. In Sect. 5, we focus on the notion of spectral-tightness and establish Theorems 1.2, 1.3, and Corollary 1.4.

2 Preliminaries

Throughout this paper, manifolds are assumed to be connected and without boundary, unless otherwise stated. Moreover, non-connected manifolds are assumed to have at most countably many connected components.

A *Schrödinger operator* S on a possibly non-connected Riemannian manifold M is an operator of the form $S = \Delta + V$, where Δ is the Laplacian and $V \in C^\infty(M)$, such that there exists $c \in \mathbb{R}$ satisfying

$$\langle Sf, f \rangle_{L^2(M)} \geq c \| f \|^2_{L^2(M)}$$

for any $f \in C_c^\infty(M)$. Then the linear operator

$$S \colon C_c^\infty(M) \subset L^2(M) \to L^2(M)$$

is densely defined, symmetric and bounded from below. Hence, it admits Friedrichs extension. Denote by $\bar{S}$ this extension and by $\mathcal{D}(\bar{S})$ its domain of definition. Recall that the *spectrum* of S is defined as

$$\sigma(S) = \{\lambda \in \mathbb{R} : \bar{S} - \lambda \colon \mathcal{D}(\bar{S}) \to L^2(M) \text{ is not bijective}\},$$

and is decomposed into the *essential spectrum* of S, which is given as

$$\sigma_{\text{ess}}(S) = \{\lambda \in \mathbb{R} : \bar{S} - \lambda \colon \mathcal{D}(\bar{S}) \to L^2(M) \text{ is not Fredholm}\},$$

and into the *discrete spectrum* $\sigma_d(S) = \sigma(S) \setminus \sigma_{\text{ess}}(S)$ of S. It is well-known that the discrete spectrum of S consists of isolated points of the spectrum which are eigenvalues of $\bar{S}$ of finite multiplicity.

The bottoms of (that is, the infimums) of the spectrum and the essential spectrum of S are denoted by $\lambda_0(S)$ and $\lambda_0^{\text{ess}}(S)$, respectively. In the case of the Laplacian (that is,

$V = 0$), these sets and quantities are denoted by $\sigma(M)$, $\sigma_{\mathrm{ess}}(M)$ and $\lambda_0(M)$, $\lambda_0^{\mathrm{ess}}(M)$, respectively. We have by definition that $\lambda_0^{\mathrm{ess}}(S) = +\infty$ if $\sigma_{\mathrm{ess}}(S)$ is empty, and we then say that S *has discrete spectrum*.

The *Rayleigh quotient* of a non-zero $f \in \mathrm{Lip}_c(M)$ with respect to S is defined by

$$\mathcal{R}_S(f) = \frac{\int_M (\| \operatorname{grad} f \|^2 + V f^2)}{\int_M f^2}. \tag{1}$$

The Rayleigh quotient of f with respect to the Laplacian is denoted by $\mathcal{R}(f)$. According to the following proposition, the bottom of the spectrum of S is expressed as an infimum of Rayleigh quotients. This is a straightforward consequence of Rayleigh's theorem and standard approximations, and may be found for instance in [20, Proposition 3.2].

Proposition 2.1 *Let S be a Schrödinger operator on a possibly non-connected Riemannian manifold M. Then the bottom of the spectrum of S is given by*

$$\lambda_0(S) = \inf_f \mathcal{R}_S(f),$$

where the infimum is taken over all $f \in C_c^\infty(M) \smallsetminus \{0\}$, or over all $f \in \mathrm{Lip}_c(M) \smallsetminus \{0\}$.

In the case where M is connected, the bottom of the spectrum of S is characterized as the maximum of the positive spectrum of S (cf. for instance [4, Theorem 3.1] and the references therein).

Proposition 2.2 *Let S be a Schrödinger operator on a Riemannian manifold M. Then $\lambda_0(S)$ is the maximum of all $\lambda \in \mathbb{R}$ such that there exists a positive $\varphi \in C^\infty(M)$ satisfying $S\varphi = \lambda\varphi$.*

It should be emphasized that the positive, smooth functions involved in this proposition are not required to be square-integrable.

We now focus on the essential spectrum of a Schrödinger operator S on a (connected) Riemannian manifold M. The decomposition principle asserts that

$$\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(S, M \smallsetminus K)$$

for any smoothly bounded, compact domain K of M. This is well known in the case where M is complete (compare with [15, Proposition 2.1]), but also holds if M is non-complete, as explained for example in [4, Theorem A.17]. This yields the following expression for the bottom of the essential spectrum of S (cf. for instance [8, Proposition 3.2] for complete Riemannian manifolds. The proof for non-complete manifolds is identical, since the decomposition principle holds).

Proposition 2.3 *Let S be a Schrödinger operator on a Riemannian manifold M, and $(K_n)_{n \in \mathbb{N}}$ an exhausting sequence of M consisting of compact domains of M. Then the bottom of the essential spectrum of S is given by*

$$\lambda_0^{\mathrm{ess}}(S) = \lim_n \lambda_0(S, M \smallsetminus K_n).$$

Consider a positive $\varphi \in C^\infty(M)$ with $S\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{R}$. Denote by $L^2_\varphi(M)$ the L^2-space of M with respect to the measure φ^2 dv, where dv stands for the volume element of M induced from its Riemannian metric. It is immediate to verify that the map $m_\varphi\colon L^2_\varphi(M) \to L^2(M)$ defined by $m_\varphi(f) = f\varphi$, is an isometric isomorphism. It is easily checked that m_φ intertwines $S - \lambda$ with the diffusion operator

$$L = m_\varphi^{-1} \circ (S - \lambda) \circ m_\varphi = \Delta - 2\operatorname{grad} \ln \varphi.$$

The operator L is called the *renormalization* of S with respect to φ. The *Rayleigh quotient* of a non-zero $f \in C_c^\infty(M)$ with respect to L is defined as

$$\mathcal{R}_L(f) = \frac{\langle Lf, f\rangle_{L^2_\varphi(M)}}{\|f\|^2_{L^2_\varphi(M)}} = \frac{\int_M \|\operatorname{grad} f\|^2\varphi^2}{\int_M f^2\varphi^2}.$$

Proposition 2.4 *The Rayleigh quotients of any non-zero $f \in C_c^\infty(M)$ are related by $\mathcal{R}_L(f) = \mathcal{R}_S(f\varphi) - \lambda$. In particular, we have that*

$$\lambda_0(S) - \lambda = \inf_f \mathcal{R}_L(f),$$

where the infimum is taken over all non-zero $f \in C_c^\infty(M)$.

Proof The first equality follows from a straightforward computation, using the definition of L and that m_φ is an isometric isomorphism. This, together with Proposition 2.1, implies the second statement. □

Even though our main results involve manifolds without boundary, it is quite important to consider manifolds with boundary in intermediate steps. Let M be a possibly non-connected Riemannian manifold with smooth boundary, and denote by ν the outward pointing, unit normal to the boundary. Then the Laplacian on M regarded as

$$\Delta\colon \{f \in C_c^\infty(M) : \nu(f) = 0 \text{ on } \partial M\} \subset L^2(M) \to L^2(M)$$

admits Friedrichs extension, being densely defined, symmetric and bounded from below. The spectrum of the Friedrichs extension of this operator is called the *Neumann spectrum* of M, and its bottom is denoted by $\lambda_0^N(M)$. We recall the following expression for the bottom of the Neumann spectrum, where $\mathcal{R}(f)$ is defined as in (1) with $V = 0$. This may be found for instance in [20, Proposition 3.2].

Proposition 2.5 *Let M be a possibly non-connected Riemannian manifold with smooth boundary. Then the bottom of the Neumann spectrum of M is given by*

$$\lambda_0^N(M) = \inf_f \mathcal{R}(f),$$

where the infimum is taken over all non-zero $f \in C_c^\infty(M)$.

It should be noticed that in this proposition, the test functions $f \in C_c^\infty(M)$ do not have to satisfy any boundary condition.

2.1 Amenable Actions and Coverings

Let X be a countable set and consider a right action of a discrete, countable group Γ on X. This action is called *amenable* if there exists an *invariant mean* on $\ell^\infty(X)$; that is, a linear functional $\mu\colon \ell^\infty(X) \to \mathbb{R}$ such that

$$\inf f \leq \mu(f) \leq \sup f \text{ and } \mu(g^* f) = \mu(f)$$

for any $f \in \ell^\infty(X)$ and $g \in \Gamma$, where $g^* f(x) := f(xg)$ for any $x \in X$. It should be observed that if the action of Γ on the orbit of some $x \in X$ is amenable, then the action of Γ on X is amenable.

A group Γ is called *amenable* if the right action of Γ on itself is amenable. Standard examples of amenable groups are solvable groups and finitely generated groups of subexponential growth. It is worth to mention that the free group in two generators, as well as any group containing it, is non-amenable. It is not difficult to see that if Γ is an amenable group, then any action of Γ is amenable.

The following characterization of amenability is due to Følner in the case of groups [17, Main Theorem and Remark], and extended to actions by Rosenblatt [22, Theorems 4.4 and 4.9].

Proposition 2.6 *The right action of Γ on X is amenable if and only if for any $\varepsilon > 0$ and any finite subset G of Γ there exists a finite subset F of X such that $|Fg \smallsetminus F| < \varepsilon|F|$ for any $g \in G$.*

In particular, it follows that the right action of Γ on X is amenable if and only if the right action of any finitely generated subgroup of Γ on X is amenable. Moreover, if the action of Γ on X has finitely many orbits X_i, $1 \leq i \leq n$, then the action of Γ on X is amenable if and only if the action of Γ on X_i is amenable for some $1 \leq i \leq n$.

Let $p\colon M_2 \to M_1$ be a smooth covering, where M_1 has possibly empty, smooth boundary and M_2 is possibly non-connected. Fix a point $x \in M_1$ and consider the fundamental group $\pi_1(M_1)$ with base point x. For $g \in \pi_1(M_1)$, let γ_g be a representative loop of g based at x. Given $y \in p^{-1}(x)$, let $\tilde{\gamma}_g$ be the lift of γ_g starting at y and denote its endpoint by yg. In this way, we obtain a right action of $\pi_1(M_1)$ on $p^{-1}(x)$, which is called the *monodromy action* of the covering. The covering p is called *amenable* if its monodromy action is amenable. It is easy to see that if M_2 is connected and p is normal, then p is amenable if and only if its deck transformation group is amenable.

Example 2.7 For any smooth covering $p\colon M_2 \to M_1$, the covering

$$p \sqcup \mathrm{Id}\colon M_2 \sqcup M_1 \to M_1$$

is amenable.

Recall that Følner's condition characterizes the amenability of an action in terms of the action of finitely generated subgroups. In the context of coverings, this yields the following characterization in terms of smoothly bounded, compact domains.

Proposition 2.8 *([4, Proposition 2.14]) Let $p\colon M_2 \to M_1$ be a smooth covering, where M_2 is possibly non-connected, and $(K_n)_{n\in\mathbb{N}}$ an exhausting sequence of M_1 consisting of smoothly bounded, compact domains. Then p is amenable if and only the restriction $p\colon p^{-1}(K_n) \to K_n$ is amenable for any $n \in \mathbb{N}$.*

This demonstrates the importance of considering non-connected covering spaces, since $p^{-1}(K)$ does not have to be connected even if M_2 is connected.

Finally, we briefly recall some results on the bottom of the spectrum under Riemannian coverings, which will be used in the sequel. Let $p\colon M_2 \to M_1$ be a Riemannian covering, with M_2 possibly non-connected, S_1 a Schrödinger operator on M_1 and S_2 its lift on M_2. From Proposition 2.2 it is not hard to see that the bottoms of the spectra satisfy

$$\lambda_0(S_1) \leq \lambda_0(S_2). \tag{2}$$

The validity of the equality is closely related to the amenability of the covering.

Theorem 2.9 *Let $p\colon M_2 \to M_1$ be an amenable Riemannian covering, with M_2 possibly non-connected. Consider a Schrödinger operator S_1 on M_1 and its lift S_2 on M_2. Then $\lambda_0(S_2) = \lambda_0(S_1)$.*

In the case where M_2 is connected, this coincides with [3, Theorem 1.2]. With very slight modifications, its proof extends [3, Theorem 1.2] to the case where M_2 is possibly non-connected. This may be found also in [5, Theorem A], which involves Riemannian coverings of orbifolds, where the covering space may be non-connected.

Amenability of a Riemannian covering of a compact manifold with smooth boundary is characterized in terms of the Neumann spectrum, according to the following analogue of Brooks' result [10].

Theorem 2.10 *Let $p\colon M_2 \to M_1$ be a Riemannian covering, where M_1 is compact with smooth boundary and M_2 is possibly non-connected. Then p is amenable if and only if $\lambda_0^N(M_2) = 0$.*

Proof The converse implication is known by [20, Theorem 4.1]. For the other direction, consider a Riemannian metric on M_1 such that its boundary has a neighborhood isometric to a cylinder $\partial M_1 \times [0, \varepsilon)$, and endow M_2 with the lifted metric. Since this metric is uniformly equivalent to the original, it suffices to show that $\lambda_0^N(M_2) = 0$ with respect to this metric. Denote by $2M_i$ the Riemannian manifold obtained by gluing two copies of M_i along their boundaries, $i = 1, 2$. Then $p\colon M_2 \to M_1$ extends to a Riemannian covering $2p\colon 2M_2 \to 2M_1$. Choose $x \in \partial M_1 \subset 2M_1$ as base point for $\pi_1(2M_1)$, and observe that any loop c based at x is written as $c = c_{2n}\star \dots \star c_1$ for some paths (not necessarily loops, since ∂M_1 may be non-connected) c_i, with the image of c_{2i-1} contained in M_1, and the image of c_{2i} in $2M_1 \smallsetminus M_1^\circ$, $1 \leq i \leq n$. Denote by c_i' the reflection of c_i along ∂M_1, and observe that the lifts of c_i and c_i' starting from the same point also have the same endpoint. It is now apparent that given $y \in (2p)^{-1}(x) = p^{-1}(x)$, the lifts of c and $c_{2n}'\star c_{2n-1}\star \dots \star c_2'\star c_1$ starting at y have the same endpoint. Since the image of the latter loop is contained in M_1, it is now easy to verify that $2p$ is amenable, p being amenable. Since $2M_1$ is closed, we derive from Theorem 2.9 that $\lambda_0(2M_2) = 0$. In view of Proposition 2.1, this means that for any $\varepsilon > 0$ there exists $f \in C_c^\infty(2M_2) \smallsetminus \{0\}$ with $\mathcal{R}(f) < \varepsilon$. Without loss of generality, we may assume that f is not identically zero neither on M_2 nor on $2M_2 \smallsetminus M_2$. Indeed, otherwise one may extend f beyond ∂M_2 to obtain a function invariant under reflection along ∂M_2, with the same Rayleigh quotient and the aforementioned property. Then we readily see from Proposition 2.5 that

$$\varepsilon > \mathcal{R}(f) \geq \min\{\mathcal{R}(f|_{M_2}), \mathcal{R}(f|_{2M_2 \smallsetminus M_2^\circ})\} \geq \min\{\lambda_0^N(M_2), \lambda_0^N(2M_2 \smallsetminus M_2^\circ)\}.$$

Because M_2 and $2M_2 \smallsetminus M_2^\circ$ are isometric, we conclude that $\lambda_0^N(M_2) = 0$. □

By virtue of the preceding theorem, we may reformulate Proposition 2.8 as follows.

Corollary 2.11 *Let $p\colon M_2 \to M_1$ be Riemannian covering, with M_2 possibly non-connected, and $(K_n)_{n\in\mathbb{N}}$ an exhausting sequence of M_1 consisting of smoothly bounded, compact domains. Then p is amenable if and only if $\lambda_0^N(p^{-1}(K_n)) = 0$ for any $n \in \mathbb{N}$.*

3 Relatively Amenable Coverings

In this section, we introduce the notion of relatively amenable coverings and present some of their properties.

Definition 3.1 Let $p\colon M_2 \to M_1$ and $q\colon M_1 \to M_0$ be smooth coverings, where q is normal. Fix a base point $x \in M_1$ and set $x_0 = q(x)$. The covering p is called *relatively amenable with respect to* q, or for short, q*-amenable* if the monodromy action of $q_*\pi_1(M_1)$ on $(q \circ p)^{-1}(x_0)$ is amenable.

In the setting of this definition, to provide another description of this action, denote by Γ the deck transformation group of q. Let $s \in \pi_1(M_1)$ and γ_s a representative loop of s based at x. It is clear that for any $z \in (q \circ p)^{-1}(x_0)$, there exists a unique $g \in \Gamma$, such that $z \in p^{-1}(gx)$. Then xs is the endpoint of the lift of $g \circ \gamma_s$ starting at z.

For $g \in \Gamma$, consider the covering $p_g\colon M_2 \to M_1$ defined by $p_g = g^{-1} \circ p$, and denote by $\hat{p}\colon \hat{M} \to M_1$ the induced covering

$$\sqcup_{g\in\Gamma} p_g\colon \sqcup_{g\in\Gamma} M_2 \to M_1.$$

From the above discussion, we arrive at the following characterization of relatively amenable coverings.

Lemma 3.2 *The covering p is q-amenable if and only if the induced covering $\hat{p}$ is amenable.*

Proof It is easily checked that the monodromy action of $\hat{p}$ coincides with the monodromy action of $q_*\pi_1(M_1)$ on $(q \circ p)^{-1}(x_0)$. □

In view of the preceding lemma, it is evident that amenable coverings are relatively amenable. Furthermore, we readily see that if q is finite sheeted, then p is q-amenable if and only if it is amenable. We now discuss an analogue of Corollary 2.11.

Proposition 3.3 *Let $p\colon M_2 \to M_1$ and $q\colon M_1 \to M_0$ be Riemannian coverings, where q is normal with deck transformation group Γ, and fix an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of M_1 consisting of smoothly bounded, compact domains. Then p is q-amenable if and only if*

$$\inf_{g\in\Gamma} \lambda_0^N(p^{-1}(gK_n)) = 0$$

for any $n \in \mathbb{N}$.

Proof It is obvious that $\hat{p}^{-1}(K)$ is isometric to the disjoint union of $p^{-1}(gK)$ with $g \in \Gamma$, for any smoothly bounded, compact domain K of M_1. In particular, the bottoms of the Neumann spectra are related by

$$\lambda_0^N(\hat{p}^{-1}(K)) = \inf_{g\in\Gamma} \lambda_0^N(p^{-1}(gK)).$$

The proof is completed by Corollary 2.11 and Lemma 3.2. □

Corollary 3.4 *Let $p\colon M_2 \to M_1$ and $q\colon M_1 \to M_0$ be smooth coverings, with q and $q \circ p$ normal. Then p is q-amenable if and only if it is amenable.*

Proof It is clear that if p is amenable, then it is q-amenable. For the converse implication, endow M_0 with a Riemannian metric and M_1, M_2 with the lifted metrics. Denote by Γ the deck transformation group of q and let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of M_1 consisting of smoothly bounded, compact domains. Then Proposition 3.3 states that

$$\inf_{g\in\Gamma} \lambda_0^N(p^{-1}(gK_n)) = 0$$

for any $n \in \mathbb{N}$. Since $q \circ p$ is normal, we deduce that any $g \in \Gamma$ can be lifted to an isometry of M_2. Indeed, given $x \in M_2$, write $x_0 = (q \circ p)(x)$ and observe that $(q_* \circ g_* \circ p_*)(\pi_1(M_2, x)) = (q_* \circ p_*)(\pi_1(M_2, x))$. Therefore, we derive that $(q_* \circ g_* \circ p_*)(\pi_1(M_2, x)) = (q_* \circ p_*)(\pi_1(M_2, y))$ for any $y \in p^{-1}(gp(x))$, $q \circ p$ being normal. Keeping in mind that $q_*\colon \pi_1(M_1, gp(x)) \to \pi_1(M_0, x_0)$ is injective, this implies that $(g_* \circ p_*)(\pi_1(M_2, x)) = p_*(\pi_1(M_2, y))$. It now follows from the lifting theorem that g can be lifted to a local isometry of M_2 mapping x to y. Since $x \in M_2$, $g \in \Gamma$ and $y \in p^{-1}(gp(x))$ are arbitrary, we readily see that the lift is invertible, and hence, an isometry. In particular, we derive that $p^{-1}(gK_n)$ is isometric to $p^{-1}(K_n)$ for any $g \in \Gamma$ and $n \in \mathbb{N}$, which yields that

$$\lambda_0^N(p^{-1}(K_n)) = \inf_{g\in\Gamma} \lambda_0^N(p^{-1}(gK_n)) = 0$$

for any $n \in \mathbb{N}$. We conclude from Corollary 2.11 that p is amenable. □

4 Spectrum Under Relatively Amenable Coverings

In this section we study the behavior of the bottom of the spectrum under relatively amenable coverings, and give some applications and examples.

Proof of Theorem 1.1 Suppose first that p is q-amenable. Then the induced covering $\hat{p}$ is amenable, from Lemma 3.2. Denoting by $\hat{S}$ the lift of S_1 on $\hat{M}$, we derive from Theorem 2.9 that $\lambda_0(\hat{S}) = \lambda_0(S_1)$. Moreover, bearing in mind that S_1 is the lift of S_0, we readily see that S_1 is invariant under deck transformations of q. Therefore, any connected component of $\hat{M}$ is isometric to M_2 via an isometry that identifies $\hat{S}$ with S_2. This yields that $\lambda_0(\hat{S}) = \lambda_0(S_2)$, which establishes the asserted equality.

To prove the second assertion, notice that the assumption that $\lambda_0^{\mathrm{ess}}(S_0) > \lambda_0(S_1)$, together with Proposition 2.3, implies that there exists a compact domain D_0 of M_0 such that

$$\lambda_0(S_0, M_0 \smallsetminus D_0) > \lambda_0(S_1).$$

Let $(K_m)_{m\in\mathbb{N}}$ be an exhausting sequence of M_1 consisting of smoothly bounded, compact domains, such that D_0 is contained in the interior of $q(K_1)$.

Assume to the contrary that p is not q-amenable, and denote by Γ the deck transformation group of q. By virtue of Proposition 3.3, there exists $m \in \mathbb{N}$ and $c > 0$, such that

$$\lambda_0^N(p^{-1}(gK_m)) \geq c$$

for any $g \in \Gamma$. Set $K = K_m$ and $D = q(K_m)$.

We know from Proposition 2.2 that there exists a positive $\varphi_1 \in C^\infty(M_1)$ satisfying $S_1\varphi_1 = \lambda_0(S_1)\varphi_1$. Denote by φ_2 the lift of φ_1 on M_2 and by L the renormalization of S_2 with respect to φ_2. Since $\lambda_0(S_2) = \lambda_0(S_1)$, we derive from Proposition 2.4 that there exists $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(M_2)$ with $\|f_n\|_{L^2_{\varphi_2}(M_2)} = 1$ and $\mathcal{R}_L(f_n) \to 0$.

According to the gradient estimate [14, Theorem 6], there exists $C > 0$ such that

$$\max_K \psi \leq C \min_K \psi$$

for any positive $\psi \in C^\infty(M_1)$ with $S_1\psi = \lambda_0(S_1)\psi$. For $\psi = \varphi_1 \circ g$, this means that

$$\max_{p^{-1}(gK)} \varphi_2 = \max_{gK} \varphi_1 \leq C \min_{gK} \varphi_1 = C \min_{p^{-1}(gK)} \varphi_2$$

for any $g \in \Gamma$. Using this and Proposition 2.5, we compute

$$\begin{aligned}\int_{p^{-1}(gK)} \|\operatorname{grad} f_n\|^2 \varphi_2^2 &\geq \Big(\min_{p^{-1}(gK)} \varphi_2^2\Big) \int_{p^{-1}(gK)} \|\operatorname{grad} f_n\|^2 \\ &\geq \frac{1}{C^2}\Big(\max_{p^{-1}(gK)} \varphi_2^2\Big) \lambda_0^N(p^{-1}(gK)) \int_{p^{-1}(gK)} f_n^2 \\ &\geq \frac{c}{C^2} \int_{p^{-1}(gK)} f_n^2 \varphi_2^2 \end{aligned} \tag{3}$$

for any $n \in \mathbb{N}$ and $g \in \Gamma$.

Since K is compact, it is clear that there exists $k \in \mathbb{N}$ such that any point of M_1 belongs to at most k different translates gK of K, with $g \in \Gamma$. Thus, any point of M_2 belongs to at most k different $p^{-1}(gK)$, with $g \in \Gamma$. This, together with (3), gives the estimate

$$\int_{(q\circ p)^{-1}(D)} \|\operatorname{grad} f_n\|^2 \varphi_2^2 \geq \frac{1}{k} \sum_{g\in\Gamma} \int_{p^{-1}(gK)} \|\operatorname{grad} f_n\|^2 \varphi_2^2 \geq \frac{c}{kC^2} \int_{(q\circ p)^{-1}(D)} f_n^2 \varphi_2^2,$$

where we used that $(q \circ p)^{-1}(D)$ is the union of $p^{-1}(gK)$ with $g \in \Gamma$. Bearing in mind that $\| f_n \|_{L^2_{\varphi_2}(M_2)} = 1$ and $\mathcal{R}_L(f_n) \to 0$, this yields that

$$\int_{(q\circ p)^{-1}(D)} f_n^2 \varphi_2^2 \to 0 \text{ and } \int_{M_2 \smallsetminus (q\circ p)^{-1}(D)} f_n^2 \varphi_2^2 \to 1. \tag{4}$$

Let $\chi_0 \in C_c^\infty(M_0)$ with $\chi_0 = 1$ in a neighborhood of D_0 and supp χ_0 contained in the interior of D. Set $\chi_2 = \chi_0 \circ q \circ p$ and set $h_n = (1 - \chi_2) f_n \in C_c^\infty(M_2)$. In view of (4), it is immediate to verify that

$$\int_{M_2} h_n^2 \varphi_2^2 \to 1 \text{ and } \int_{M_2} \|\operatorname{grad} h_n\|^2 \varphi_2^2 \to 0,$$

which shows that that $\mathcal{R}_L(h_n) \to 0$, and thus, $\mathcal{R}_{S_2}(h_n \varphi_2) \to \lambda_0(S_1)$, by Proposition 2.4. Since $h_n \varphi_2$ is compactly supported in $M_2 \smallsetminus (q \circ p)(D_0)$, Proposition 2.1 implies that

$$\mathcal{R}_{S_2}(h_n \varphi_2) \geq \lambda_0(S_2, M_2 \smallsetminus (q \circ p)^{-1}(D_0)) \geq \lambda_0(S_0, M_0 \smallsetminus D_0) > \lambda_0(S_1),$$

where the intermediate inequality follows from (2) applied to the restriction of $q \circ p$ over any connected component of $M_0 \smallsetminus D_0$. This is a contradiction, which completes the proof. □

The next example illustrates that in Theorem 1.1(ii), the covering p is q-amenable, but not necessarily amenable, and in particular, that relative amenability is indeed a weaker property than amenability. This example also demonstrates that in Corollary 3.4, the assumption that $q \circ p$ is normal cannot be replaced with p being normal.

Example 4.1 Let N be a closed Riemannian manifold of dimension $n \geq 3$ with non-amenable fundamental group. Fix two sufficiently small open balls B_i with disjoint closures and ∂B_i diffeomorphic to the sphere S^{n-1}, $i = 1, 2$. Denote by M_0 the closed manifold obtained by gluing a cylinder $S^{n-1} \times [0, 1]$ along the boundary of $N \smallsetminus (B_1 \cup B_2)$, so that $S^{n-1} \times \{0\}$ gets identified with ∂B_1, and $S^{n-1} \times \{1\}$ with ∂B_2. Endow M_0 with a Riemannian metric.

Consider now the disjoint union of copies N_k of $N \smallsetminus (B_1 \cup B_2)$, with $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, glue a cylinder $S^{n-1} \times [0, 1]$ along the boundary of this disjoint union, so that $S^{n-1} \times \{0\}$ gets identified with ∂B_1 in N_k, and $S^{n-1} \times \{1\}$ with ∂B_2 in N_{k+1}. In this way, we obtain a manifold M_1 on which $\mathbb{Z}$ acts via diffeomorphsms and the quotient is diffeomorphic to M_0. That is, we have a covering $q : M_1 \to M_0$ with deck transformation group $\mathbb{Z}$. We endow M_1 with the lift of the Riemannian metric of M_0.

Let F be a fundamental domain of q which is diffeomorphic to N_0 with two cylinders $S^{n-1} \times [0, 1/2]$ attached along its boundary. Then $\pi_1(F)$ is non-amenable and ∂F has two connected components C_1 and C_2, which are diffeomorphic to S^{n-1}. Hence, the universal covering $p : \tilde{F} \to F$ of F is non-amenable. It should be noticed that the restriction of p on any connected component of $\partial \tilde{F}$ is a covering over a connected component of ∂F. Since the connected components of ∂F are diffeomorphic to S^{n-1}, where $n \geq 3$, we derive that p restricted to any connected component of $\partial \tilde{F}$ is an isometry. This means that $p^{-1}(C_i)$ is a disjoint union of copies of C_i, $i = 1, 2$. Write $M_1 \smallsetminus F^{\circ} = D_1 \sqcup D_2$, with C_i contained in D_i, $i = 1, 2$. Denote by M_2 the manifold obtained by gluing a copy of D_i to $\tilde{F}$ along any connected component of $p^{-1}(C_i)$, $i = 1, 2$. We readily see that the covering $p : \tilde{F} \to F$ is extended to a normal covering $p : M_2 \to M_1$ with the same deck transformation group. Therefore, $p : M_2 \to M_1$ is non-amenable.

It remains to establish that $p : M_2 \to M_1$ is q-amenable. To this end, we endow M_2 with the lift of the Riemannian metric of M_1, and by virtue of Theorem 1.1, it suffices to prove that $\lambda_0(M_2) = 0$. Since q is amenable, Theorem 2.9 states that $\lambda_0(M_1) = 0$. From Proposition 2.1, for any $\varepsilon > 0$ there exists a non-zero $f \in C_c^{\infty}(M_1)$ with $\mathcal{R}(f) < \varepsilon$. Then there exists a deck transformation g of q, such that $\mathrm{supp}(f \circ g)$ is contained in the interior of D_1, $\mathrm{supp}\, f$ being compact. It is evident that $\mathcal{R}(f \circ g) < \varepsilon$, and since D_1 is isometric to a domain of M_2, the corresponding $f_2 \in C_c^{\infty}(M_2)$ satisfies $\mathcal{R}(f_2) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude from Proposition 2.1 that $\lambda_0(M_2) = 0$, as we wished.

Based on the preceding example, we now present examples of amenable smooth coverings $p : M_2 \to M_1$ with M_2 non-connected, such that the restriction of p on any connected component of M_2 is non-amenable.

Example 4.2 Let $p : M_2 \to M_1$ and $q : M_1 \to M_0$ be smooth coverings, where q is normal with deck transformation group Γ. Suppose that p is q-amenable, but not amenable. Then the induced covering $\hat{p} : \hat{M} \to M_1$ is amenable, from Lemma 3.2. It is evident that the restriction of $\hat{p}$ on any connected component of $\hat{M}$ is of the form $p_g = g^{-1} \circ p : M_2 \to M_1$ for some $g \in \Gamma$. It follows from Corollary 2.11 that p_g is non-amenable for any $g \in \Gamma$, p being non-amenable.

We end this section with a characterization of the amenability of a composition of coverings.

Corollary 4.3 *Let $p: M_2 \to M_1$ and $q: M_1 \to M_0$ be smooth coverings, with q normal. Then $q \circ p$ is amenable if and only if q is amenable and p is q-amenable.*

Proof Endow M_0 with a Riemannian metric and M_1, M_2 with the lifted metrics. Let S_0 be a Schrödinger operator on M_0 with discrete spectrum (for instance, $S_0 = \Delta + V$, where V has compact sublevels), and denote by S_1, S_2 its lift on M_1, M_2, respectively.

If $q \circ p$ is amenable, then $\lambda_0(S_0) = \lambda_0(S_2)$, from Theorem 2.9. Furthermore, we know from (2) that $\lambda_0(S_0) \leq \lambda_0(S_1) \leq \lambda_0(S_2)$. As a consequence, we obtain that $\lambda_0(S_1) = \lambda_0(S_0) < \lambda_0^{\text{ess}}(S_0)$, which yields that q is amenable, from [20, Theorem 1.2]. Since $\lambda_0(S_2) = \lambda_0(S_1) < \lambda_0^{\text{ess}}(S_0)$, Theorem 1.1 implies that p is q-amenable.

Conversely, if q is amenable and p is q-amenable, then $\lambda_0(S_2) = \lambda_0(S_1) = \lambda_0(S_0)$, where we used Theorem 1.1 for the first equality, and Theorem 2.9 for the second one. Since S_0 has discrete spectrum, [20, Theorem 1.2] states that $q \circ p$ is amenable. □

5 Spectral-Tightness of Riemannian Manifolds

In this section we study the notion of spectral-tightness of Riemannian manifolds. Recall that a Riemannian manifold M is called *spectrally-tight* if $\lambda_0(M') < \lambda_0(\tilde{M})$ for any non-simply connected, normal covering space M' of M, where $\tilde{M}$ is the universal covering space of M. We begin with a straightforward consequence of Theorem 1.1.

Corollary 5.1 *Let M be a Riemannian manifold with $\lambda_0(\tilde{M}) < \lambda_0^{\text{ess}}(M)$, where $\tilde{M}$ is the universal covering space of M. Then M is spectrally-tight if and only if the unique normal, amenable subgroup of $\pi_1(M)$ is the trivial one.*

Proof If M is not spectrally-tight, then there exists a normal covering $q: M' \to M$ with $\lambda_0(M') = \lambda_0(\tilde{M})$ and M' non-simply connected. In view of Theorem 1.1, the universal covering $p: \tilde{M} \to M'$ is q-amenable. Since $q \circ p$ is normal, Corollary 3.4 asserts that p is amenable, or equivalently, that $\pi_1(M')$ is amenable. It is clear that $\pi_1(M')$ is a non-trivial, normal subgroup of $\pi_1(M)$.

Conversely, if $\pi_1(M)$ has a non-trivial, amenable, normal subgroup Γ, then the action of Γ on $\tilde{M}$ gives rise to an amenable Riemannian covering $p: \tilde{M} \to \tilde{M}/\Gamma$. We deduce from Theorem 2.9 that $\lambda_0(\tilde{M}) = \lambda_0(\tilde{M}/\Gamma)$, while $\tilde{M}/\Gamma$ is a non-simply connected, normal covering space of M. □

Proof of Theorem 1.2 It is an immediate consequence of Corollary 5.1, since the spectrum of the Laplacian on a closed Riemannian manifold is discrete. □

Before proceeding to the proof of Theorem 1.3, we recall some terminology from [16]. Let M be a non-positively curved, closed Riemannian manifold. The Euclidean local de Rham factor of M is the maximum of all $n \in \mathbb{N} \cup \{0\}$ such that the universal covering space $\tilde{M}$ of M splits as the Riemannian product $\mathbb{R}^n \times N$ for some Riemannian manifold N (under the convention that if $\tilde{M}$ cannot be written as $\mathbb{R} \times N$, then this number is zero). According to [16, Theorem], the Euclidean local de Rham factor of M coincides with the rank of the unique maximal normal abelian subgroup of $\pi_1(M)$.

Finally, we recall the well-known result of Bieberbach. An isometry of the Euclidean space $\mathbb{R}^n$ is of the form $\varphi(x) = Ax + v$, where A is an orthogonal transformation and $v \in \mathbb{R}^n$. An isometry is called translation if it is of the form $\varphi(x) = x + v$. A *Bieberbach group* Γ is a discrete group of isometries of $\mathbb{R}^n$ such that $\mathbb{R}^n / \Gamma$ is compact. According to Bieberbach's result, the subgroup G consisting of the translations in Γ is the unique maximal normal abelian subgroup of Γ (cf. for instance [1, § 9]).

Proof of Theorem 1.3 Let M be a non-positively curved, closed Riemannian manifold. Suppose first that the dimension of the Euclidean local de Rham factor of M is non-zero. We derive from [16, Theorem] that there exists a non-trivial, normal abelian subgroup of $\pi_1(M)$. By virtue of Theorem 1.2, this shows that M is not spectrally-tight.

Conversely, if M is not spectrally-tight, then there exists a non-trivial, normal amenable subgroup Γ of $\pi_1(M)$, according to Theorem 1.2. We obtain from [12, Corollary 2] that Γ is a Bieberbach group. Denote by G the subgroup of translations in Γ. It should be noticed that G is non-trivial, since otherwise, Γ is finite and in particular, there exist non-trivial elements of finite order in $\pi_1(M)$, which is a contradiction. Furthermore, it is known that G is the unique maximal normal abelian subgroup of Γ, and thus, a characteristic subgroup of Γ. Since Γ is a normal subgroup of $\pi_1(M)$, we readily see that so is G. Since G is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$, we conclude that the rank of the unique normal maximal abelian subgroup of $\pi_1(M)$ is non-zero and the proof is completed by [16, Theorem]. □

We now recall some notions on manifolds of non-positive curvature, which may be found in [1]. Let g be an isometry of a Hadamard manifold M. The *displacement function* $d_g \colon M \to \mathbb{R}$ of g is defined as $d_g(x) = d(x, gx)$. Then g is called:

- *elliptic* if $\min_M d_g = 0$,
- *axial* if $\min_M d_g > 0$,
- *parabolic* if d_g does not achieve its infimum.

It is evident that g is elliptic if and only if it fixes a point of M. If g is axial, then it acts as translation by $\min_M d_g$ along a geodesic of M (cf. [1, Lemma 6.5]). Finally, if g is parabolic, then it fixes a point of the ideal boundary M_i and all horospheres (as sets) centered at that point (see [1, Lemma 6.6.]).

A discrete group G of isometries of M is called:

- *elliptic* if G fixes a point of M,
- *axial* if G fixes a geodesic of M (as a set) but does not fix any point of M,
- *parabolic*, if G fixes a point $x \in M_i$ and horospheres centered at x (as sets), but does not fix any point of $M \cup M_i \smallsetminus \{x\}$,
- *elementary* if G is elliptic, axial or parabolic.

Suppose that G acts freely on M. Then G is elliptic if and only if it is the trivial group. If G is axial, taking into account that any element of G acts by translation along the same geodesic, it is not hard to see that G is abelian. According to [9, Proposition 4.1], parabolic groups are virtually nilponent. It is worth to point out that these groups are amenable, being of subexponential growth.

Finally, we recall another definition (equivalent to the one stated in the Introduction) of geometrically finite manifolds, in the sense of Bowditch [9]. Let M be a complete Riemannian manifold of bounded sectional curvature $-b^2 \le K \le -a^2 < 0$. Then the universal covering space $\tilde{M}$ of M is a Hadamard manifold, and M is the quotient $\tilde{M}/\Gamma$, for some discrete group Γ of isometries of $\tilde{M}$. Let $\tilde{M}_i$ be the ideal boundary of $\tilde{M}$ and consider the compactification $\tilde{M}_c = \tilde{M} \cup \tilde{M}_i$. Denote by $\Lambda \subset \tilde{M}_i$ the limit set of Γ and by H the convex hull of Γ. The set $C = (H \cap \tilde{M})/\Gamma$ is called the convex core of M, and M is called geometrically finite if some/any tubular neighborhood of C has finite volume. The reader may consult the seminal paper of Bowditch [9] for more details on geometrically finite manifolds, or [6, Section 3] for a brief exposition.

Before proceeding to the proof of Corollary 1.4, we need the following observation, which will be also exploited in the examples in the sequel.

Lemma 5.2 *Let M be a geometrically finite manifold. If $\pi_1(M)$ contains a non-trivial, amenable normal subgroup, then $\pi_1(M)$ is elementary.*

Proof Denote by $\tilde{M}$ the universal covering space of M and write $M = \tilde{M}/\Gamma$. Let G be a non-trivial, amenable normal subgroup of Γ. It follows from [7, Corollary 1.2] that G is elementary. Since G is non-trivial and acts freely on $\tilde{M}$, this yields that G is either parabolic or axial.

Suppose first that G is parabolic and contained in the stabilizer of a point x of the ideal boundary of $\tilde{M}$. Given $h \in \Gamma$, using that $hgh^{-1} \in G$, we readily see that G fixes $h^{-1}x$. This shows that $h^{-1}x = x$ for any $h \in \Gamma$, and thus, Γ is parabolic and fixes x.

Assume now that G is axial and contained in the stabilizer of the image of a geodesic $\gamma : \mathbb{R} \to \tilde{M}$ joining two points of the ideal boundary of $\tilde{M}$. Then for any $g \in G$, there exists $t_g \in \mathbb{R}$ such that $g(\gamma(t)) = \gamma(t + t_g)$ for any $t \in \mathbb{R}$. Given $h \in \Gamma$, there exists $t' \in \mathbb{R}$ such that $hgh^{-1}(\gamma(t)) = \gamma(t + t')$ for any $t \in \mathbb{R}$, since $hgh^{-1} \in G$. Then G fixes the image of the geodesic $h^{-1} \circ \gamma$, which implies that the images of $h^{-1} \circ \gamma$ and γ coincide for any $h \in \Gamma$. We conclude that Γ is axial and fixes the image of γ. □

Proof of Corollary 1.4 If M is not spectrally-tight, we derive from Corollary 5.1 that there exists a non-trivial, amenable normal subgroup of $\pi_1(M)$. Then $\pi_1(M)$ is elementary, by Lemma 5.2. Conversely, if $\pi_1(M)$ is elementary, then $\lambda_0(M) = \lambda_0(\tilde{M})$, in view of Theorem 2.9, $\pi_1(M)$ being amenable. Since M is non-simply connected, this means that M is not spectrally-tight. □

Even though spectral-tightness of closed Riemannian manifolds is a topological property, the next examples illustrate that for non-compact manifolds, this property depends on the Riemannian metric.

Example 5.3 Let M be a non-compact surface of finite type with non-cyclic fundamental group, and denote by $\tilde{M}$ its universal covering space. Since M is diffeomorphic to a closed surface with finitely many points removed, it is not hard to see that M carries a complete Riemannian metric which is flat outside a compact domain. We derive from Proposition 2.3 that $\lambda_0^{\mathrm{ess}}(M) = 0$. Furthermore, there exists a compact $K \subset M$ such that the fundamental group of any connected component of $M \smallsetminus K$ is amenable (as a matter of fact, cyclic). We deduce from [21, Corollary 1.6] that $\lambda_0(\tilde{M}) = 0$, and thus, M is not spectrally-tight with respect to this Riemannian metric.

It follows from [2, Proposition 1.5 and (1.3)] that M carries a complete Riemannian metric such that $\lambda_0^{\mathrm{ess}}(M) > \lambda_0(\tilde{M})$. We now show that M is spectrally-tight with respect to this Riemannian metric. Otherwise, we obtain from Theorem 5.1 that there exists a non-trivial, amenable normal subgroup of $\pi_1(M)$. It is known that M admits a complete hyperbolic metric, and with respect to such a metric, M is geometrically finite (cf. for instance [2, Example 1.4]). It now follows from Lemma 5.2 that $\pi_1(M)$ is elementary, and thus, cyclic, M being two-dimensional, which is a contradiction.

Although this gives a quite wide class of examples, it seems reasonable to present a more explicit one.

Example 5.4 Let M be a two-dimensional torus with a cusp attached. Initially, we endow M with a complete Riemannian metric, so that the cusp D is isometric to a domain of a flat cylinder. Then it is clear that $\lambda_0(M) = \lambda_0(D) = \lambda_0^{\text{ess}}(M) = 0$. Since the fundamental group of D is amenable, it follows from [21, Corollary 1.6] that $\lambda_0(M) = \lambda_0(\tilde{M})$, where $\tilde{M}$ stands for the universal covering space of M. Therefore, M endowed with this Riemannian metric is not spectrally-tight.

We now endow M with a complete Riemannian metric, so that the cusp is isometric to the surface of revolution generated by e^{-t^α} with $t \geq 1$, for some $\alpha > 1$. Then the spectrum of M is discrete (cf. [11, Theorem 2]) and Corollary 5.1 characterizes spectral-tightness. It should be observed that $\pi_1(M)$ is the free group F_2 in two generators. Since the fundamental group of any negatively curved, closed manifold contains a subgroup isomorphic to F_2, [13, Theorem 1] shows that any amenable subgroup of F_2 is cyclic. It is not difficult to verify that the unique normal, cyclic subgroup of F_2 is the trivial one. Hence, M endowed with this Riemannian metric is spectrally-tight, from Corollary 5.1.

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