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Norm Estimates for the $\bar{\partial}$-Equation on a Non-reduced Space

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Abstract
We study norm-estimates for the $\bar{\partial}$-equation on non-reduced analytic spaces. Our main result is that on a non-reduced analytic space, which is Cohen–Macaulay and whose underlying reduced space is smooth, the $\bar{\partial}$-equation for (0, 1)-forms can be solved with L^p-estimates.

Keywords $\bar{\partial}$-Equation · L^p-Estimates · Non-reduced analytic spaces

Mathematics Subject Classification 32A26 · 32A27 · 32B15 · 32W05

1 Introduction

Various estimates for solutions of the $\bar{\partial}$-equation on a smooth complex manifold are known since long ago. The paramount methods are the L^2-methods, going back to Hörmander, Kohn, and others, and integral representation formulas, first used by Henkin and by Skoda. Starting with [28, 29],there has been an increasing interest for L^2- and L^p-estimates for $\bar{\partial}$ on non-smooth reduced analytic spaces in later years, see, e.g., [13, 19, 23, 27, 31]. In [6, 15, 16], there are results about L^2-estimates of extensions from non-reduced subvarieties. In this paper, we try to initiate the study of L^p-estimates for the $\bar{\partial}$-equation on a non-reduced analytic space.

Let X be an analytic space of pure dimension n with structure sheaf $\mathscr{O}_X$. Locally then we have an embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ and a coherent ideal sheaf $\mathcal{J} \subset \mathscr{O}_{\mathcal{U}}$ of pure dimension n such that $\mathscr{O}_X = \mathscr{O}_\Omega/\mathcal{J}$ in $X \cap \mathcal{U}$. In [7], we introduced a notion

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of smooth $(0,*)$-forms on X and proved that if the underlying reduced space X_{red} is smooth and in addition $\mathscr{O}_X$ is Cohen–Macaulay, then there is a smooth solution to $\bar\partial u = \phi$ if ϕ is smooth and $\bar\partial\phi = 0$. More generally, we defined sheaves $\mathscr{A}_X^q$ of $(0,q)$-forms on X that are closed under multiplication by smooth $(0,*)$-forms and coincides with $\mathscr{E}_X^{0,q}$ where X_{red} is smooth and $\mathscr{O}_X$ is Cohen–Macaulay, such that

$$0 \to \mathscr{O}_X \to \mathscr{A}_X^0 \to \cdots \to \mathscr{A}_X^n \to 0$$

is a fine resolution of $\mathscr{O}_X$. The solutions to the $\bar\partial$-equation are obtained by intrinsic integral formulas on X. Variants of the $\bar\partial$-equation on non-reduced spaces have also been studied by Henkin–Polyakov [20, 21].

In [5], it was introduced a pointwise norm $|\cdot|$ on forms $\phi \in \mathscr{E}_X^{0,*}$. That is, $|\phi(x)|_X$ is non-negative function on X_{red} which vanishes in a neighborhood of a point x_0 if and only if ϕ vanishes there. It was proved that $\mathscr{O}_X$ is complete with respect to the topology of uniform convergence on compacts induced by this norm. In this paper, we will only discuss spaces where X_{red} is smooth. In [5], it is defined an intrinsic coherent left $\mathscr{O}_{X_{red}}$-module $\mathscr{N}_X$ of differential operators $\mathscr{O}_X \to \mathscr{O}_{X_{red}}$, and the pointwise norm is defined as

$$|\phi(x)|_X^2 = \sum_k |(\mathcal{L}_k\phi)(x)|^2,$$

where $\mathcal{L}_j$ is a finite set of local generators for $\mathscr{N}_X$. Clearly another set of generators will give rise to an equivalent norm. In particular, it becomes meaningful to say that ϕ vanishes at a point $x \in X$. The norm extends to smooth $(0,*)$-forms. By a partition of unity, we patch together and define a fixed global $|\cdot|_X$. Let us also choose a volume element dV on X_{red}. We define $L^p_{0,*;X}$ as the (local) completion of $\mathscr{E}_X^{0,*}$ with respect to the L^p-norm. In the same way, we define $C_{0,*;X}$ as the completion with respect to the uniform norm. Our main result is

Theorem 1.1 *Let X be an analytic space such that $\mathscr{O}_X$ is Cohen–Macaulay and X_{red} is smooth. Assume that $1 \le p < \infty$. Given a point x, there are neighborhoods $\mathcal{V}' \subset\subset \mathcal{V} \subset X$ and a constant C_p such that if $\phi \in L^p_{0,1}(\mathcal{V})$ and $\bar\partial\phi = 0$, then there is $\psi \in L^p_{0,0}(\mathcal{V}')$ such that $\bar\partial\psi = \phi$ and*

$$\int_{\mathcal{V}'_{red}} |\psi|_X^p dV \le C_p^p \int_{\mathcal{V}_{red}} |\phi|_X^p dV.$$

Moreover, there is a constant C_∞ such that if $\phi \in C_{0,1}(\mathcal{V})$ and $\bar\partial\phi = 0$, then there is a solution $\psi \in C_{0,0}(\mathcal{V}')$ such that

$$\sup_{\mathcal{V}'_{red}} |\psi|_X \le C_\infty \sup_{\mathcal{V}_{red}} |\phi|_X.$$

By standard sheaf theory, and the fact that $\psi \in L^p_{0,0;X}$ and $\bar\partial\psi = 0$ implies that $\psi \in \mathscr{O}_X$, see Lemma 4.8 and (5.1), we get the following corollaries.

Corollary 1.2 *Assume that X is a compact analytic space such that $\mathscr{O}_X$ is Cohen–Macaulay, X_{red} is smooth. If $\phi \in L^p_{0,1}(X)$, $\bar{\partial}\phi = 0$ and the cohomology class of ϕ in $H^1(X, \mathscr{O}_X)$ vanishes, then there is $\psi \in L^p_{0,0}(X)$ such that $\bar{\partial}\psi = \phi$. If $\phi \in C_{0,1}(X)$, then there is a solution $\psi \in C_{0,0}(X)$.*

Notice that ϕ defines a Čech cohomology class in $H^1(X, \mathscr{O}_X)$ through $(\psi_j - \psi_k)_{j,k}$, where (ψ_j) are local $\bar{\partial}$-solutions on a covering $\mathcal{U}_j$ of X.

Corollary 1.3 *Assume that X is an Stein space such that $\mathscr{O}_X$ is Cohen–Macaulay and X_{red} is smooth. If $\phi \in L^p_{0,1,loc}(X)$ is $\bar{\partial}$-closed, then there is $\psi \in L^p_{0,0,loc}(X)$ such that $\bar{\partial}\psi = \phi$. If $\phi \in C_{0,1}(X)$, then there is a solution $\psi \in C_{0,0}(X)$.*

The proof of Theorem 1.1 relies on the integral formulas in [7] in combination with a new notion of sheaves of $\mathcal{C}^{0,*}_X$ of $(0,*)$-currents on X which provide a fine resolution of $\mathscr{O}_X$ (see Sect. 4). These sheaves should have an independent interest. In Remark 5.4, we give a heuristic argument for Theorem 1.1 which relies on these sheaves but with no reference to integral formulas.

We first consider a certain kind of "simple" non-reduced space for which we prove these L^p-estimates for all $(0, q)$-forms, see Sect. 9. We then prove the general case by means of a local embedding $X \to \hat{X}$, where $\hat{X}$ is simple. To carry out the proof, we need comparison results between the constituents in the integral formulas for the two spaces. One of them is provided by [22], whereas another one, for the so-called Hefer mappings, is new, see Sect. 8. The proof of Theorem 1.1 is in Sect. 10. Technical difficulties restrict us, for the moment, to the case with $(0, 1)$-forms.

We have no idea of whether one could prove Theorem 1.1, e.g., in case $p = 2$, by L^2-methods.

The assumption that X_{red} be smooth and X be Cohen–Macaulay is crucial in this paper. In the reduced case, considerable difficulties appear already with the presence of an isolated singularity, besides the references already mentioned above, see, e.g., [17, 18, 25, 26, 32]. In the non-reduced case even when X_{red} is smooth, an isolated non-Cohen–Macaulay point offers new difficulties. We discuss such an example in Sect. 11.

Throughout this paper, X is a non-reduced space of pure dimension n and the underlying reduced space $Z = X_{red}$ is smooth, if nothing else is explicitly stated.

2 Some Preliminaries

Let Y and Y' be complex manifolds and $f : Y' \to Y$ a proper mapping. If τ is a current on Y, then the pushforward, or direct image, $f_*\tau$ is defined by the relation $f_*\tau.\xi = \tau.f^*\xi$ for test forms ξ. If α is a smooth form on Y, then we have the simple but useful relation

$$\alpha \wedge f_*\tau = f_*(f^*\alpha \wedge \tau). \tag{2.1}$$

In [9, 11], was introduced the sheaf of *pseudomeromorphic currents* on Y. Roughly speaking, a pseudomeromorphic current is the direct image under a holomorphic mapping of a smooth form times a tensor product of one-variable principal value current

$1/z_j^m$ and $\bar\partial(1/z_k^{m'})$. This sheaf is closed under $\bar\partial$ and under multiplication by smooth forms. If a pseudomeromorphic current τ has support on a subvariety V and the holomorphic function h vanishes on V, then $\bar h\tau = 0$ and $d\bar h \wedge \tau = 0$. This leads to the crucial *dimension principle*

Proposition 2.1 *Let τ be a pseudomeromorphic current of bidegree $(*, q)$, and assume that the support of μ is contained in a subvariety of codimension $> q$. Then $\tau = 0$.*

We say that a current a is *almost semi-meromorphic* in Y if there is a modification $\pi\colon Y' \to Y$, such that a is the direct image of a form α/f, where α is smooth and f is a holomorphic section of some line bundle on Y'. Assume that μ is pseudomeromorphic, a is an almost semi-meromorphic current, $\chi_\epsilon = \chi(|F|^2/\epsilon)$, where χ is a smooth cutoff function, and F is a tuple of holomorphic functions such that $\{F = 0\}$ contains the set where a is not smooth. Then the limit

$$\lim_{\epsilon\to 0} \chi_\epsilon a \wedge \mu \tag{2.2}$$

exists and defines a pseudomeromorphic current $a \wedge \mu$ that is independent of the choice of χ. We define $\bar\partial a \wedge \mu := \bar\partial(a \wedge \mu) - (-1)^{\deg a} a \wedge \bar\partial\mu$. It is readily verified that $\bar\partial a \wedge \mu = \lim_{\epsilon\to 0} \bar\partial\chi_\epsilon \wedge a \wedge \mu$. By Hironaka's theorem, any almost semi-meromorphic current is pseudomeromorphic.

Let $\mathcal{U} \subset \mathbb{C}^N$ be an open set, let Z be a submanifold of dimension $n < N$, and let $\kappa = N - n$. The $\mathscr{O}_{\mathcal{U}}$-sheaf of Coleff–Herrera currents, $\mathcal{CH}_{\mathcal{U}}^Z$, see [14], consists of all $\bar\partial$-closed (N, κ)-currents in $\mathcal{U}$ with support on Z that are annihilated by $\bar{\mathcal{J}}_Z$, i.e., by all $\bar h$ where h is in $\mathcal{J}_Z$. If $\mathcal{J} \subset \mathscr{O}_{\mathcal{U}}$ is an ideal sheaf with zero set Z, then $\mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ is the subsheaf of μ in $\mathcal{CH}_{\mathcal{U}}^Z$ that are annihilated by $\mathcal{J}$. It is well known that $\mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ is coherent, cf. e.g., [4, Theorem 1.5]

Remark 2.2 If Z is not smooth, then $\mathcal{CH}_{\mathcal{U}}^Z$ is defined in the same way, but one needs an additional regularity condition, the so-called standard extension property, SEP, see, e.g., [7, Section 2.1]. When Z is smooth, the currents in $\mathcal{CH}_{\mathcal{U}}^Z$ (with the definition given here) admit an expansion as in [3, (3.4)], and so the SEP follows.

Let us recall some properties of residue currents associated to a locally free resolution

$$0 \to \mathscr{O}(E_{N_0}) \stackrel{f_{N_0}}{\to} \mathscr{O}(E_{N_0-1}) \cdots \stackrel{f_1}{\to} \mathscr{O}(E_0) \to 0 \tag{2.3}$$

of a coherent (ideal) sheaf $\mathscr{O}_{\mathcal{U}}/\mathcal{J}$. The precise definitions and claimed results can all be found in [10]. Let us denote the complex (2.3) by (E, f). Assume that the vector bundles E_k are equipped with Hermitian metrics. The corresponding complex of vector bundles is pointwise exact on $\mathcal{U} \setminus Z$, where $Z = Z(\mathcal{J})$. There are associated currents U and R. The current U is almost semi-meromorphic on $\mathcal{U}$ and smooth on $\mathcal{U} \setminus Z$, and takes values in $\operatorname{Hom}(E, E)$. The current R is a pseudomeromorphic current on $\mathcal{U}$ that takes values in $\operatorname{Hom}(E_0, E)$ and has support on Z. One may write $R = \sum_k R_k$, where R_k is a $(0, k)$-current that takes values in $\operatorname{Hom}(E_0, E_k)$. They satisfy the relation

$$(f - \bar\partial) \circ U + U \circ (f - \bar\partial) = I_E - R. \tag{2.4}$$

Here we use the compact notation $E = \oplus E_k$, $f = \sum f_k$. By the dimension principle $R_k = 0$ if $k < \kappa = \operatorname{codim} \mathcal{J}$. In particular, since $(f - \bar\partial)^2 = 0$, it follows by (2.4) that $(f - \bar\partial)R = 0$, so

$$f_\kappa R_\kappa = \bar\partial R_{\kappa-1} = 0. \tag{2.5}$$

Moreover, R is annihilated by $\bar{\mathcal{J}}_Z$, and it satisfies the duality principle

$$R\Phi = 0 \text{ if and only if } \Phi \in \mathcal{J}. \tag{2.6}$$

We will typically assume that the resolution is chosen to be minimal at level 0, i.e., such that $E_0 \cong \mathscr{O}$. Thus, $\operatorname{Hom}(E_0, E_k) \cong E_k$, so we may consider R_k as an E_k-valued current. If $\mathscr{O}_\mathcal{U}/\mathcal{J}$ is Cohen–Macaulay, then we can choose the resolution so that $N_0 = \kappa$. Then it follows that R consists of the only term R_κ that takes values in E_κ, and from (2.4) that $\bar\partial R_\kappa = 0$. We conclude that the components $\mu_1, \ldots, \mu_\rho$ of R_κ, $\rho = \operatorname{rank} E_\kappa$, are in the sheaf $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$. It is proved in [4, Example 1] that these components μ_j actually generate this sheaf. It follows from (2.6) that

$$\mu\Phi = 0,\ \mu \in \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U}), \text{ if and only if } \Phi \in \mathcal{J}. \tag{2.7}$$

By continuity (2.7) holds everywhere if $\mathcal{J}$ is has pure dimension.

3 Pointwise Norm on a Non-reduced Space *X*

Recall that X is a non-reduced space of pure dimension n with smooth underlying manifold $X_{red} = Z$.

Consider a local embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ and assume that $\pi : \mathcal{U} \to Z \cap \mathcal{U}$ is a submersion. Possibly after shrinking $\mathcal{U}$, we can assume that we have coordinates $(\zeta, \tau) = (\zeta_1, \ldots, \zeta_n, \tau_1, \ldots, \tau_\kappa)$ in $\mathcal{U}$ so that $Z \cap \mathcal{U} = \{\tau = 0\}$ and π is the projection $(\zeta, \tau) \mapsto \zeta$. Let $d\zeta = d\zeta_1 \wedge \ldots, d\zeta_n$.

If μ is a section of $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$ in $\mathcal{U}$, then

$$\pi_*(\phi\mu) =: \mathcal{L}\phi\, d\zeta \tag{3.1}$$

defines a holomorphic differential operator $\mathcal{L}\colon \mathscr{O}(X \cap \mathcal{U}) \to \mathscr{O}(Z \cap \mathcal{U})$. Following [5, Section 1] we define $\mathcal{N}_X$ as the set of all such local operators $\mathcal{L}$ obtained from some μ in $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$ and a local submersion. It follows from (2.1) and (3.1) that if ξ is in $\mathscr{O}_Z$, then $\xi\mathcal{L}\phi = \mathcal{L}(\pi^*\xi\phi)$. Thus $\mathcal{N}_X$ is a left $\mathscr{O}_Z$-module. It is coherent, in particular locally finitely generated, and if $\mathcal{L}_j$ is a set of local generators, then $\phi = 0$ if and only if $\mathcal{L}_j\phi = 0$ for all j, see [5, Theorem 1.3]. If $\mathcal{L}_j$ is a finite set of local generators, therefore

$$|\phi|^2_X = \sum_j |\mathcal{L}_j\phi|^2 \tag{3.2}$$

defines a local norm, and any other finite set of local generators gives rise to an equivalent local norm.

Example 3.1 Assume we have a local embedding and local coordinates (ζ, τ) as above in $\mathcal{U}$. Let $M = (M_1, \ldots, M_\kappa)$ be a tuple of non-negative integers and consider the ideal sheaf

$$\mathcal{I} = \left\langle \tau_1^{M_1+1}, \ldots, \tau_\kappa^{M_\kappa+1} \right\rangle.$$

Let $\hat{X}$ be the analytic space with structure sheaf $\mathscr{O}_{\hat{X}} = \mathscr{O}_{\mathcal{U}}/\mathcal{I}$. Consider the tensor product of currents

$$\hat{\mu} = \bar\partial \frac{d\tau_1}{\tau_1^{M_1+1}} \wedge \cdots \wedge \bar\partial \frac{d\tau_\kappa}{\tau_\kappa^{M_\kappa+1}}, \tag{3.3}$$

where $d\tau_j/\tau_j^{M_j+1}$ is the principal value current. We recall that if $\varphi = \varphi_0(\zeta,\tau) d\zeta \wedge d\bar\zeta$ is a test form, then

$$\hat{\mu}.\varphi = \bar\partial \frac{d\tau_1}{\tau_1^{M_1+1}} \wedge \ldots \wedge \bar\partial \frac{d\tau_\kappa}{\tau_\kappa^{M_\kappa+1}}.\varphi = \frac{(2\pi i)^\kappa}{M!} \int_\zeta \frac{\partial \varphi_0}{\partial \tau^M}(\zeta, 0) d\zeta \wedge d\bar\zeta, \tag{3.4}$$

where $M! = M_1! \cdots M_\kappa!$. It follows, e.g., by [3, Theorem 4.1] that $\hat{\mu} \wedge d\zeta$ is a generator for the $\mathscr{O}_{\mathcal{U}}$-module (and $\mathscr{O}_{\hat{X}}$-module) $\mathcal{H}om(\mathscr{O}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z)$. For a multiindex m, we will use the shorthand notation

$$\bar\partial \frac{d\tau}{\tau^m} = \bar\partial \frac{d\tau_1}{\tau_1^{m_1}} \wedge \ldots \wedge \bar\partial \frac{d\tau_\kappa}{\tau_\kappa^{m_\kappa}}. \tag{3.5}$$

Moreover, $m \le M$ means that $m_j \le M_j$ for $j = 1, \ldots, \kappa$. It is readily verified that

$$\tau^\beta \bar\partial \frac{d\tau}{\tau^\alpha} = \bar\partial \frac{d\tau}{\tau^{\alpha-\beta}} \tag{3.6}$$

if $\beta \le \alpha$. Any ψ in $\mathscr{O}_{\hat{X}}$ has a unique representative in $\mathcal{U}$ of the form

$$\psi = \sum_{m \le M} \hat{\psi}_m(\zeta) \tau^m. \tag{3.7}$$

By [5, Proposition 3.1],

$$\mathcal{L}_{m,\beta} := \left. \frac{\partial^{|m|+|\beta|}}{\partial \tau^m \partial \zeta^\beta} \right|_{\tau=0}, \quad m \le M, \; |\beta| \le |M-m|,$$

is a generating set for $\mathcal{N}_{\hat{X}}$. If $\Psi(\zeta, \tau)$ is any representative in $\mathcal{U}$ for ψ, thus, cf. (3.2),

$$|\psi|_{\hat{X}} \sim \sum_{m \le M, \; |\beta| \le |M-m|} \left| \frac{\partial}{\partial \tau^m \partial \zeta^\beta} \Psi(\zeta, 0) \right| \sim \sum_{m \le M, \; |\beta| \le |M-m|} \left| \frac{\partial \hat{\psi}_m}{\partial \zeta^\beta} \right|. \tag{3.8}$$

Let us now return to the setting of a local embedding $i : X \to \mathcal{U} \subset \mathbb{C}^N$ as above. Notice that if M is large enough in the example, then $\mathcal{I} \subset \mathcal{J}$. Let $\mu_1, \dots, \mu_\rho$ be local generators for the coherent $\mathscr{O}_{\mathcal{U}}$-module $\mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}^Z_{\mathcal{U}})$. Then we have a natural mapping $\mathscr{O}_{\mathcal{U}}/\mathcal{I} \to \mathscr{O}_{\mathcal{U}}/\mathcal{J}$, that is, a mapping $\iota^* \colon \mathscr{O}_{\hat{X}} \to \mathscr{O}_X$. It is natural to say that we have an embedding

$$\iota\colon X \to \hat{X}.$$

It is well known, see, e.g., [4, Theorem 1.5] that there are holomorphic functions $\gamma_1, \dots, \gamma_\rho$ (possibly after shrinking $\mathcal{U}$) such that

$$\mu_j = \gamma_j \hat{\mu}, \quad j = 1, \dots, \rho. \tag{3.9}$$

From [5, Theorem 1.4] we have that

$$|\phi|_X \sim \sum_{j=1}^{\rho} |\gamma_j \phi|_{\hat{X}}. \tag{3.10}$$

In this way, the norm $|\cdot|_X$ is thus expressed in terms of the simpler norm $|\cdot|_{\hat{X}}$.

3.1 The Norm When *X* is Cohen–Macaulay

So far we have only used the assumption the Z is smooth. Let us now assume in addition that $\mathscr{O}_X$ is Cohen–Macaulay. Then one can find monomials $1, \tau^{\alpha_1}, \dots, \tau^{\alpha_{\nu-1}}$ such that each ϕ in $\mathscr{O}_X$ has a unique representative

$$\hat{\phi} = \hat{\phi}_0(z) \otimes 1 + \cdots + \hat{\phi}_{\nu-1}(z) \otimes \tau^{\alpha_{\nu-1}}, \tag{3.11}$$

where $\hat{\phi}_j$ are in $\mathscr{O}_Z$, see, e.g., [7, Corollary 3.3]. In this way, $\mathscr{O}_X$ becomes a free $\mathscr{O}_Z$-module (in a non-canonical way). Let $|\cdot|_{X,\pi}$ be the norm obtained from the subsheaf $\mathcal{N}_{X,\pi}$ of $\mathcal{N}_X$, consisting of operators $\mathcal{L}$ obtained, cf. (3.1), from the submersion π such that $(\zeta, \tau) \mapsto \zeta$ in $\mathcal{U}$. It turns out that

$$|\phi|^2_{X,\pi} \sim |\hat{\phi}_0(z)|^2 + \cdots + |\hat{\phi}_{\nu-1}(z)|^2, \tag{3.12}$$

cf. [5, Theorem 1.5]. By [5, Proposition 3.4], the whole sheaf $\mathcal{N}_X$ is generated by $\mathcal{N}_{X,\pi}$ for a finite number of generic submersions π^ι. It follows that

$$|\phi|_X \sim \sum_\iota |\phi|_{X,\pi^\iota}. \tag{3.13}$$

3.2 The Sheaf $\mathscr{E}^{0,*}_X$ of Smooth Forms on *X*

Assume that we have a local embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$. If Φ is in $\mathscr{E}^{0,*}_{\mathcal{U}}$, we say that $i^*\Phi = 0$, or equivalently Φ is in $\mathcal{K}er\, i^*$, if Φ is in $\mathscr{E}^{0,*}_{\mathcal{U}}\mathcal{J} + \mathscr{E}^{0,*}_{\Omega}\bar{\mathcal{J}}_Z + \mathscr{E}^{0,*}_{\mathcal{U}} d\bar{\mathcal{J}}_Z$

on X_{reg}, where $\mathcal{J}_Z$ is the radical sheaf of Z and we by X_{reg} denote the set of points of X where Z is smooth and $\mathscr{O}_X$ is Cohen–Macaulay.

Remark 3.2 If the underlying reduced space X_{red} is not smooth, or $\mathscr{O}_X$ is not Cohen–Macaulay, then this definition of $\mathcal{K}er\, i^*$ is not valid. Instead is used as definition that $\Phi \wedge \mu = 0$ for all μ in $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$. However, it is true that $i^*\Phi = 0$ if $i^*\Phi = 0$ where X_{red} is smooth and X is Cohen–Macaulay. See [7, Lemma 2.2].

We define $\mathscr{E}^{0,*}_X = \mathscr{E}^{0,*}_\mathcal{U}/\mathcal{K}er\, i^*$ and have the natural mapping $i^*\colon \mathscr{E}^{0,*}_\mathcal{U} \to \mathscr{E}^{0,*}_X$. By standard arguments, one can check that the $\mathscr{O}_X$-module $\mathscr{E}^{0,*}_X$ so defined does not depend on the choice of local embedding.

Each $\mathcal{L} \in \mathcal{N}_X$ extends to a mapping $\mathscr{E}^{0,*}_X \to \mathscr{E}^{0,*}_Z$ so that (3.1) holds, see [5, Lemma 8.1]. If we choose a Hermitian metric on the tangent space TZ, we get an induced norm on $\mathscr{E}^{0,*}_Z$, and so we get a pointwise norm by (3.2) of smooth $(0,*)$-forms on X. In particular, if ϕ and ξ are smooth forms on X, then

$$|\xi \wedge \phi|_X \le C_\xi |\phi|_X, \tag{3.14}$$

cf. the remark after [5, (4.23)]. Choosing an embedding $\iota\colon X \to \hat{X}$ as above, (3.14) follows from (3.8) and (3.10).

If $\mathscr{O}_X$ is Cohen–Macaulay and $i: X \to \mathcal{U}$ is a local embedding with coordinates (ζ, τ) and a monomial basis τ^{α_ℓ}, then we have a unique local representation (3.11) of each ϕ in $\mathscr{E}^{0,*}_X$ with $\hat{\phi}_\ell$ in $\mathscr{E}^{0,*}_Z$, and the other statements in Sect. 3.2 hold verbatim, with the same proofs, for smooth $(0,*)$-forms.

4 Intrinsic Currents on *X*

In the reduced case, one can define currents just as dual elements of smooth forms. In the non-reduced case, one has to be cautious because there are two natural kinds of currents:suitable limits of smooth forms and dual elements of smooth forms. We have to deal with both kinds. In this paper, the former type appears as $(0,*)$-currents, while the latter appears as $(n,*)$-currents. In [8], we study the $\bar\partial$-equation on a non-reduced space for general (p,q)-forms, and then both type of currents appear in arbitrary bidegrees.

4.1 The Sheaf of Currents $\mathcal{C}^Z_\mathcal{U}$

Let $\mathcal{U} \subset \mathbb{C}^N$ be an open subset and Z a submanifold as before. Let $\mathcal{C}^Z_\mathcal{U}$ be the $\mathscr{O}_\mathcal{U}$-sheaf of all $(N,*)$-currents in $\mathcal{U}$ that are annihilated by $\bar{\mathcal{J}}_Z$ and $d\bar{\mathcal{J}}_Z$. Clearly these currents have support on Z.

Lemma 4.1 *If (ζ, τ) are local coordinates in $\mathcal{U}$ so that $Z = \{\tau = 0\}$, then each current μ in $\mathcal{C}^Z_\mathcal{U}$ has a unique representation*

$$\mu = \sum_{\alpha \ge 0} a_\alpha(\zeta) \wedge \bar\partial \frac{d\tau}{\tau^{\alpha+\mathbf{1}}} \wedge d\zeta, \tag{4.1}$$

where a_α are in $\mathcal{C}_Z^{0,*}$ and the sum is finite, and we use the shorthand notation (3.5).

Proof Since $\bar\tau_j\mu = 0$ for all j, μ must have support on Z. Since it is a current, there is a tuple M of positive integers such that $\tau^\alpha \wedge \mu$ is non-zero only when $\alpha \le M$. Let π be the projection $(\zeta, \tau) \mapsto \zeta$. We claim that (4.1) holds with

$$a_\alpha(\zeta) \wedge d\zeta = \pm \frac{1}{(2\pi i)^p} \pi_*(\tau^\alpha \mu). \tag{4.2}$$

In fact, given a test form ϕ with Taylor expansion

$$\phi(\zeta,\tau) = \sum_{\alpha \le M} \frac{1}{\alpha!} \frac{\partial^\alpha \phi}{\partial \tau^\alpha}(\zeta, 0)\tau^\alpha + \mathscr{O}(\bar\tau, d\bar\tau) + \mathscr{O}(\tau_1^{M_1+1}, \dots, \tau_\kappa^{M_\kappa+1}),$$

and using (3.4), we see that

$$\mu.\phi = \sum_{\alpha \le M} \mu. \frac{1}{\alpha!} \frac{\partial^\alpha \phi}{\partial \tau^\alpha}(\zeta,0)\tau^\alpha = \sum_{\alpha \le M} \frac{1}{(2\pi i)^p} \pi_*(\tau^\alpha \mu) \wedge \bar\partial \frac{d\tau}{\tau^{\alpha+\mathbf{1}}}.\phi.$$

It follows from (4.2) that if μ has the expansion (4.1), then $\bar\partial\mu$ has an expansion

$$\bar\partial\mu = \sum_{\alpha \ge 0} \bar\partial a_\alpha(\zeta) \wedge \bar\partial \frac{d\tau}{\tau^{\alpha+\mathbf{1}}} \wedge d\zeta.$$

In particular, $\bar\partial\mu = 0$ if and only if each $a_\alpha(\zeta)$ is $\bar\partial$-closed. It is also readily verified that a sequence μ_k tends to 0 if and only if the associated sums (4.1) have uniformly bounded length and their coefficients $a_{k,\alpha}$ tend to 0 for each fixed α.

4.2 The Intrinsic Sheaf $\mathcal{C}_X^{n,*}$

We define the sheaf $\mathcal{C}_X^{n,*}$ of intrinsic $(n,*)$-currents on X as the dual of $\mathscr{E}_X^{0,n-*}$. Assume that $i\colon X \to \mathcal{U}$ is a local embedding. Since $\mathscr{E}_X^{0,n-*} = \mathscr{E}_{\mathcal{U}}^{0,n-*}/\mathcal{K}er\, i^*$ and $\mathcal{K}er\, i^*$ is closed, the elements in $\mathcal{C}_X^{n,*}$ are represented by the currents in $\mathcal{U}$ that vanish when acting on test forms with a factor in $\mathcal{J}, \bar{\mathcal{J}}_Z, d\bar{\mathcal{J}}_Z$, which in turn are the currents in $\mathcal{U}$ that are annihilated by $\mathcal{J}, \bar{\mathcal{J}}_Z, d\bar{\mathcal{J}}_Z$, that is, $\mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{C}_{\mathcal{U}}^Z)$. Therefore,we have the isomorphism

$$i_*\colon \mathcal{C}_X^{n,*} \xrightarrow{\simeq} \mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{C}_{\mathcal{U}}^Z).$$

Let ω_X be the subspace of $\bar\partial$-closed elements in $\mathcal{C}_X^{n,0}$. We then obtain the isomorphism

$$i_*\colon \omega_X \xrightarrow{\simeq} \mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z).$$

In case X is reduced, ω_X is the well-known Barlet sheaf of holomorphic n-forms on X, cf. [7, Section 5] and [12].

4.3 Representations of $\mathscr{O}_X$ and $\mathscr{E}_X^{0,*}$

Assume that we have a local embedding $i : X \to \Omega$. Notice that we have a well-defined mapping

$$\mathscr{O}_X \to \mathcal{H}om\big(\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z), \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)\big), \quad \phi \mapsto (\mu \mapsto \phi\mu). \tag{4.3}$$

It follows from (2.7) that (4.3) is injective.

Remark 4.2 It is in fact an isomorphism where X is Cohen–Macaulay (or more generally where X is S_2), see [7, Theorem 7.3].

Let $\mu_1, \ldots, \mu_\rho$ be generators for $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$, and consider an element Φ in $\mathcal{H}om\big(\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\Omega^Z), \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{C}_\mathcal{U}^Z)\big)$. Moreover, let us choose coordinates (ζ, τ) in $\mathcal{U}$ as before. Since each $\Phi(\mu_j)$ is in $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{C}_\mathcal{U}^Z)$,it has a unique representation (4.1), and if we choose M such that $\tau_1^{M_1+1}, \ldots, \tau_\kappa^{M_\kappa+1} \in \mathcal{J}$, then the sum only runs over $\alpha \le M$. Thus, Φ is represented by the tuple $\tilde{\Phi} \in (\mathcal{C}_Z^{0,*})^r$ consisting of all the $a_\alpha(\zeta)$ for the currents $\Phi(\mu_j)$, $j = 1, \ldots, \rho$.

Now assume that X is Cohen–Macaulay and choose a monomial basis τ^{α_ℓ} as in Sect. 3.1. Each $\phi \in \mathscr{O}_X$ is then, cf. (3.11), represented by a tuple $\hat{\phi} \in (\mathscr{O}_Z)^\nu$. Thus the mapping (4.3) defines a holomorphic sheaf morphism (matrix) $T : (\mathscr{O}_Z)^\nu \to (\mathscr{O}_Z)^r$. It is injective by (2.7), so T is generically pointwise injective. In fact, we have, [7, Lemma 4.11]:

Lemma 4.3 *The morphism T is pointwise injective.*

We can thus (locally) choose a holomorphic matrix A such that

$$0 \to \mathscr{O}_Z^\nu \xrightarrow{T} \mathscr{O}_Z^r \xrightarrow{A} \mathscr{O}_Z^{r'} \tag{4.4}$$

is pointwise exact, and holomorphic matrices S and B such that

$$I = TS + BA. \tag{4.5}$$

In the same way,we have a natural mapping

$$\mathscr{E}_X^{0,*} \to \mathcal{H}om\big(\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z), \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{C}_\mathcal{U}^Z)\big), \quad \phi \mapsto (\mu \mapsto \phi \wedge \mu). \tag{4.6}$$

If ϕ is in $\mathscr{E}_X^{0,*}$, then the coefficients in the expansion (4.1) of $\phi \wedge \mu$ are in $\mathscr{E}_Z^{0,*}$ so the image of ϕ in (4.6) is represented by an element in $(\mathscr{E}_Z^{0,*})^r$. If X is Cohen–Macaulay,we have the unique representation (3.11) with $\hat{\phi}_\ell$ in $\mathscr{E}_Z^{0,*}$ and hence (4.6) defines an $\mathscr{E}_Z^{0,*}$-linear morphism $(\mathscr{E}_Z^{0,*})^\nu \to (\mathscr{E}_Z^{0,*})^r$ that coincides with T for holomorphic ϕ. Since (4.4) is pointwise exact, we have the exact complex

$$0 \to (\mathscr{E}_Z^{0,*})^\nu \xrightarrow{T} (\mathscr{E}_Z^{0,*})^r \xrightarrow{A} (\mathscr{E}_Z^{0,*})^{r'}. \tag{4.7}$$

We now consider what happens with these representations when we change coordinates.

Lemma 4.4 *Let (ζ, τ) and (ζ', τ') be two coordinate systems in $\mathcal{U}$ as before. There is a matrix L of holomorphic differential operators such that if $\mu \in \mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{C}^Z_{\mathcal{U}})$, and (a_α) and (a'_α) are the coefficients in the associated expansions (4.1), then $(a'_\alpha) = L(a_\alpha)$.*

Proof Let π and π' be the projections $(\zeta, \tau) \mapsto \zeta$ and $(\zeta', \tau') \mapsto \zeta'$, respectively. Fix a multiindex α. Recall that $a'_\alpha \wedge d\zeta' = \pm(2\pi i)^{-1}\pi'_*((\tau')^\alpha\mu)$, cf. (4.2). We can write

$$(\tau')^\alpha = \sum_\rho b_\rho(\zeta)\tau^\rho,$$

where b_ρ are holomorphic. After a preliminary change of coordinates in the ζ-variables, which only affects the coefficients by the factor $d\zeta/d\zeta'$, we may assume that $\zeta' = \zeta$ when $\tau' = 0$ so that

$$\zeta'_j = \zeta_j + \sum_k \tau_k b_{jk}(\zeta, \tau).$$

If $\varphi = \sum'_{|I|=*} \varphi_I(\zeta) d\bar\zeta_I$ is a test form in Z of bidegree $(0, *)$, then

$$\begin{aligned}(\pi')^*\varphi &= \sum_{|I|=*}{}' \varphi_I(\zeta')d\bar\zeta'_I + \mathscr{O}(\bar\tau, d\bar\tau)\\ &= \sum_{|I|=*}{}' \sum_{\gamma,\delta} c_{\gamma,\delta}(\zeta)\frac{\partial^{|\gamma|}\varphi_I}{\partial\zeta^\gamma}(\zeta)d\bar\zeta_I\tau^\delta + \mathscr{O}(\bar\tau, d\bar\tau)\\ &= \sum_{\gamma,\delta} c_{\gamma,\delta}(\zeta)\frac{\partial^{|\gamma|}\pi^*\varphi}{\partial\zeta^\gamma}\tau^\delta + \mathscr{O}(\bar\tau, d\bar\tau).\end{aligned}$$

Since $\bar\tau\mu = 0$ and $d\bar\tau \wedge \mu = 0$, and b_ρ and $c_{\gamma,\delta}$ only depend on ζ,

$$\pi'_*((\tau')^\alpha\mu).\varphi = \mu.(\tau')^\alpha(\pi')^*\varphi = \sum_{\rho,\gamma,\delta} \pm\frac{\partial^{|\gamma|}}{\partial\zeta^\gamma}\left(c_{\gamma,\delta} \wedge b_\rho \wedge \pi_*(\tau^{\rho+\delta}\mu)\right).\varphi$$

which means that

$$a'_\alpha = \sum_{\rho,\gamma,\delta} \pm\frac{\partial^{|\gamma|}}{\partial\zeta^\gamma}\left(c_{\gamma,\delta} \wedge b_\rho \wedge a_{\rho+\delta}\right). \tag{4.8}$$

Note that the expansion of $(\pi')^*\varphi$ is infinite, but it only runs over γ such that $|\gamma| \le |\delta|$. Since $\tau^\delta\mu = 0$ if $|\delta|$ is large enough, the series (4.8) defining a'_α is thus in fact a finite sum. Thus a'_α is obtained from a matrix of holomorphic differential operators applied to (a_β).

Corollary 4.5 *Assume that X is Cohen–Macaulay. Let (ζ, τ) and (ζ', τ') be coordinate systems as above, and let τ^{α_ℓ} and $(\tau')^{\alpha'_\ell}$ be bases as in Sect. 3.1. There is a matrix $\mathcal{L}$ of holomorphic differential operators such that if $(\hat\phi_j) \in (\mathscr{E}^{0,*})^\nu$ and $(\hat\phi'_j) \in (\mathscr{E}^{0,*})^\nu$ are the coefficients with respect to these two bases of the same element $\phi \in \mathscr{E}_X^{0,*}$, then*

$$(\hat\phi'_j) = \mathcal{L}(\hat\phi_j). \tag{4.9}$$

Proof Consider $\phi \in \mathscr{E}_X^{0,*}$ and let Φ be its image in $\mathcal{H}om\big(\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z), \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{C}_\mathcal{U}^Z)\big)$. Given generators $\mu_1, \ldots, \mu_\rho$ for $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}), \mathcal{CH}_\mathcal{U}^Z)$, let T' and S' be the mappings T and S but with respect to the new variables (ζ', τ') and basis $(\tau')^{\alpha'_j}$. Then $T(\hat\phi_j)$ and $T'(\hat\phi'_j)$ are the coefficients with respect to (ζ, τ) and (ζ', τ'), respectively, of $\Phi(\mu_j)$, $j = 1, \ldots, \rho$.

According to Lemma 4.4, $T'(\hat\phi'_j) = \tilde L(T(\hat\phi_j))$, where $\tilde L$ is a matrix of holomorphic differential operators. Thus, cf. (4.5), $(\hat\phi'_j) = S' \circ T'(\hat\phi'_j) = S' \circ \tilde L(T(\hat\phi_j))$, which defines the desired matrix $\mathcal{L}$.

4.4 The Sheaf $\mathcal{C}_X^{0,*}$ of $(0, *)$-Currents

Let us assume now that X is Cohen–Macaulay. We want $\mathcal{C}_X^{0,*}$ to be an $\mathscr{O}_X$-sheaf extension of $\mathscr{E}_X^{0,*}$ so that $\mathscr{E}_X^{0,*}$ is dense in a suitable topology. The idea is to define a $(0, *)$-current ϕ as something that for each choice of coordinates (ζ, τ) and basis τ^{α_ℓ} as in Sect. 3.1 has a representation (3.11) where $(\hat\phi_j)$ are in $(\mathcal{C}_Z^{0,*})^\nu$, and transform by (4.9). However, to get a more invariant definition, we will represent $\mathcal{C}_X^{0,*}$ as a subsheaf of the $\mathscr{O}_X$-sheaf

$$\mathcal{F} := \mathcal{H}om\big(\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z), \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{C}_\mathcal{U}^Z)\big).$$

Let us fix $(\zeta, \tau, \tau^{\alpha_\ell})$. Given an expression (3.11), where $\hat\phi_0, \ldots \hat\phi_{\nu-1}$ are in $\mathcal{C}_Z^{0,*}$, we get a mapping

$$\mathcal{CH}_\mathcal{U}^Z \to \mathcal{C}_\mathcal{U}^Z, \quad \mu \mapsto \hat\phi \wedge \mu, \tag{4.10}$$

by expressing μ as in (4.1) and performing the multiplication formally term by term.

Lemma 4.6 *The mapping (4.10) defines an element in $\mathcal{F}$ that is zero if and only if all $\hat\phi_\ell$ vanish.*

All such images in $\mathcal{F}$ form a coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ that is independent of the local choice $(\zeta, \tau, \tau^{\alpha_\ell})$.

Proof We first claim that $\mathcal{J}(\hat\phi \wedge \mu) = 0$ if $\mathcal{J}\mu = 0$. Let $(\hat\phi_{\ell,\epsilon})$ be tuples in $(\mathscr{E}_Z^{0,*})^\nu$ obtained by regularizing each entry $\hat\phi_\ell$, and let $\hat\phi_\epsilon$ denote the corresponding smooth forms in $\mathscr{E}_X^{0,*}$. Then $\hat\phi_\epsilon \wedge \mu \to \hat\phi \wedge \mu$ as currents, and since $\mathcal{J}(\hat\phi_\epsilon \wedge \mu) = 0$, the claim follows. Thus (4.10) defines an element in $\mathcal{F}$.

Let $\mu_1, \ldots, \mu_\rho$ be generators for $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$. Then the coefficients of $\hat\phi \wedge \mu_j$, $j = 1, \ldots, \rho$, are given by $T(\hat\phi_\ell) \in (\mathcal{C}_Z^{0,*})^r$, where T is the matrix in (4.7).

Indeed, this holds for the smooth $\hat\phi_\epsilon$ and hence for $\hat\phi$. Since T is pointwise injective, the induced mapping is injective as well. If the image of (4.10) vanishes therefore the tuple $\hat\phi_\ell$ vanishes.

For each multiindex γ,

$$\tau^\gamma = \sum_\ell c_\ell(\zeta)\tau^{\alpha_\ell}$$

for unique c_ℓ in $\mathscr{O}_Z$. For ξ in $\mathscr{O}_X$,therefore there is a (unique) matrix A_ξ of $\mathscr{O}_Z$-functions such that

$$(\hat\psi_\ell) = A_\xi(\hat\phi_\ell) \tag{4.11}$$

for any smooth $\hat\phi$ if $\psi = \xi\phi$. Moreover,

$$\xi(\hat\phi \wedge \mu) = \hat\psi \wedge \mu \tag{4.12}$$

since both sides are the equal to the current $\xi\phi \wedge \mu$. If now $(\hat\phi_\ell)$ is in $(\mathcal{C}_Z^{0,*})^\nu$ and $(\hat\psi_\ell)$ is defined by (4.11), then by a regularization as above we see that still (4.12) holds. Thus the image of $(\mathcal{C}_Z^{0,*})^\nu$ is a locally finitely generated $\mathscr{O}_X$-module and hence a coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$.

It remains to check the independence of the choice of $(\zeta, \tau, \tau^{\alpha_\ell})$. Thus assume $(\zeta', \tau', (\tau')^{\alpha'_\ell})$ is another choice, cf. Corollary 4.5. If $(\hat\phi'_\ell) = \mathcal{L}(\hat\phi_\ell)$ and $(\hat\phi'_{j,\epsilon}) = \mathcal{L}a(\hat\phi_{\ell,\epsilon})$, then $\hat\phi'_\epsilon \to \hat\phi$. Since $\hat\phi'_\epsilon \wedge \mu = \hat\phi_\epsilon \wedge \mu$ we conclude that $\hat\phi' \wedge \mu = \hat\phi \wedge \mu$.

Definition 4.7 The sheaf of $(0,*)$-currents $\mathcal{C}_X^{0,*}$ is defined as the sheaf $\mathcal{F}'$.

Given $(\zeta, \tau, \tau^{\alpha_\ell})$, thus each element ϕ in $\mathcal{C}_X^{0,*}$ has a unique representation (3.11). However, in view of Lemma 4.6, the current $\phi \wedge \mu$ has an invariant meaning. We have natural mappings $\bar\partial : \mathcal{C}_X^{0,q} \to \mathcal{C}_X^{0,q+1}$, defined by $(\hat\phi_\ell) \mapsto (\bar\partial\hat\phi_\ell)$. They are well defined since $\bar\partial$ commutes with the transition matrices $\mathcal{L}$ in the preceding proof. We thus get the complex

$$0 \to \mathscr{O}_X \to \mathcal{C}_X^{0,0} \xrightarrow{\bar\partial} \mathcal{C}_X^{0,1} \xrightarrow{\bar\partial} \cdots . \tag{4.13}$$

Proposition 4.8 *The sheaf complex* (4.13) *is exact.*

Proof First assume that ϕ is in $\mathcal{C}_X^{0,0}$. Given $(\zeta, \tau, \tau^{\alpha_\ell})$, we then have a unique representation (3.11) with $\hat\phi_j \in \mathcal{C}_Z^{0,*}$. If $\bar\partial\phi = 0$, then all $\bar\partial\hat\phi_\ell = 0$, so $\hat\phi_\ell \in \mathscr{O}_Z$, and thus $\phi \in \mathscr{O}_X$. If ϕ is in $\mathcal{C}_X^{0,q+1}$, then $\bar\partial\phi = 0$ means that each $\bar\partial\hat\phi_\ell = 0$, and thus,we have local solutions to $\bar\partial\hat u_\ell = \hat\phi_\ell$ in $\mathcal{C}_Z^{0,q}$. It follows that u defined by $\hat u_\ell$ is a solution to $\bar\partial u = \phi$.

Definition 4.9 A sequence ϕ_k in $\mathcal{C}_X^{0,*}$ converges to ϕ if $\phi_k \wedge \mu \to \phi \wedge \mu$ for all μ in $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$.

Notice that $\bar\partial(\phi \wedge \mu) = \bar\partial\phi \wedge \mu$. Thus $\phi_k \to \phi$ implies that $\bar\partial\phi_k \to \bar\partial\phi$.

Lemma 4.10 *Let $(\zeta, \tau, \tau^{\alpha_\ell})$ be a local basis in $\mathcal{U}$ and assume that*

$$\phi_k = \sum \hat{\phi}_{\ell k} \tau^{\alpha_\ell}, \quad \phi = \sum \hat{\phi}_\ell \tau^{\alpha_\ell}. \tag{4.14}$$

The sequence ϕ_k in $\mathcal{C}_X^{0,*}$ converges to ϕ in $\mathcal{U}$ if and only if $\hat{\phi}_{\ell k} \to \hat{\phi}_\ell$ for each ℓ.

It follows that $\mathscr{E}_X^{0,*}$ is dense in $\mathcal{C}_X^{0,*}$, since $\mathscr{E}_Z^{0,*}$ is dense in $\mathcal{C}_Z^{0,*}$.

Proof If $\hat{\phi}_{\ell k} \to \hat{\phi}_\ell$ for each ℓ, then $\phi_k \wedge \mu \to \phi \wedge \mu$. For the converse, let us choose a generating set $\mu_1, \dots, \mu_\rho$ as above. If $\phi_k \to \phi$, then in particular $\phi_k \wedge \mu_j \to \phi \wedge \mu_j$ for each $j = 1, \dots, \rho$. This means that $T(\hat{\phi}_{\ell k}) \to T(\hat{\phi}_\ell)$ for each ℓ. Since the matrix T is pointwise injective, therefore $\hat{\phi}_{\ell k} \to \hat{\phi}_\ell$ for each ℓ.

Remark 4.11 From the very definition, cf. Sect. 3.2, a sequence $\phi_k \in \mathscr{E}_X^{0,*}$ tends to 0 at a given point x if and only if given a small local embedding $i: X \to \mathcal{U}$ at x there are representatives $\Phi_k \in \mathscr{E}_{\mathcal{U}}^{0,*}$ such that $\Phi_k \to 0$ in $\mathcal{U}$. If τ^{α_ℓ} is a local basis and $\hat{\phi}_{\ell k} \to 0$ for each ℓ, then

$$\Phi_k(\zeta, \tau) := \sum_\ell \hat{\phi}_{\ell k}(\zeta) \tau^{\alpha_\ell} \to 0$$

in $\mathcal{U}$ and hence $\phi_k \to 0$ in $\mathscr{E}^{0,*}(X \cap \mathcal{U})$. Also the converse is true. In fact, if Φ_k are representatives in $\mathcal{U}$ and $\Phi_k \to 0$ in $\mathcal{U}$, then each of the coefficients of $\Phi_k \wedge \mu_j$ in the representation (4.1) tends to 0 in $\mathscr{E}^{0,*}(Z \cap \mathcal{U})$ for each j. This precisely means that $T(\hat{\phi}_{\ell k})$ tend to 0 in $\mathscr{E}^{0,*}(Z \cap \mathcal{U})$. Since T is pointwise injective,this implies that $\hat{\phi}_{\ell k} \to 0$ in $\mathscr{E}^{0,*}(Z \cap \mathcal{U})$ for each ℓ.

Remark 4.12 We only define $\mathcal{C}_X^{0,*}$ on the part where Z is smooth, as we there need to embed $L^p_{0,*}(X)$ into a larger space that allows for more flexibility. We do not know what an appropriate definition of $\mathcal{C}_X^{0,*}$ would be over the singular part of Z. In [7], we introduce a sheaf $\mathcal{W}_X^{0,*}$ of pseudomeromorphic $(0,*)$-currents on X with the so-called standard extension property, also when Z is singular. On the part where Z is smooth, $\mathcal{W}_X^{0,*}$ is a subsheaf of $\mathcal{C}_X^{0,*}$, and consists of currents which admit a representation (3.7), where the $\hat{\psi}_m$ are in $\mathcal{W}_Z^{0,*} \subseteq \mathcal{C}_Z^{0,*}$.

Remark 4.13 We do not know if the embedding $\mathcal{C}_X^{0,*} \to \mathcal{F}$ is an isomorphism, i.e., if $\mathcal{F}' = \mathcal{F}$. For any h in $\mathcal{F}$ that can be approximated by smooth forms h_ϵ in $\mathcal{F}$, it follows as above that h is in $\mathcal{F}'$, but it is not clear that this is possible for an arbitrary h in $\mathcal{F}$. An analogous statement for the subsheaf $\mathcal{W}_X^{0,*}$ is indeed true, see [7, Lemma 7.5], but the proof relies on the fact that elements in $\mathcal{W}_Z^{0,*}$ are in a suitable sense generically smooth and does not generalize to $\mathcal{C}_X^{0,*}$.

5 L^p-Spaces

Assume that X is Cohen–Macaulay and that the underlying manifold $Z = X_{red}$ is smooth. Recall that we have chosen a Hermitian metric on Z and let dV be the associated volume form. Assume $1 \leq p < \infty$. If $K \subset X$ is a compact subset and ϕ is in $\mathscr{E}^{0,*}(X)$ then

$$\left(\int_{K_{red}} |\phi|_X^p dV \right)^{1/p}$$

is finite and defines a semi-norm on $\mathscr{E}^{0,*}(X)$. We define the sheaf $L^p_{loc;0,*}$ as the completion of $\mathscr{E}^{0,*}_X$ with respect to these semi-norms. In particular,we get the spaces $L^p_{0,*}(K)$ for any compact subset $K \subset X$. For a relatively compact open subset $\mathcal{V} \subset\subset X$,we let $L^p_{0,*}(\mathcal{V}) = L^p_{0,*}(\overline{\mathcal{V}})$. Clearly these spaces are independent of the choice of $|\cdot|_X$ and Hermitian structure on Z. In the same way,we define the sheaf $\mathcal{C}^{0,*}_X$ as the completion of $\mathscr{E}^{0,*}_X$ with respect to the semi-norms $\sup_K |\phi|_X$.

Proposition 5.1 *Let $i \colon X \to \mathcal{U} \subset \mathbb{C}^N$ be a local embedding and let $\mathcal{V} = X \cap \mathcal{U}$. If $\phi \in L^p_{0,*}(\mathcal{V})$, then $\phi \in \mathcal{C}^{0,*}(\mathcal{V})$. Given coordinates and basis $(\zeta, \tau, \tau^{\alpha_\ell})$, each $\phi \in L^p_{0,*}(\mathcal{V})$ has a unique representation (3.11) where $\hat{\phi}_\ell \in L^p(\mathcal{V}_{red})$. If $\phi_k \to \phi$ in $L^p_{0,*}(\mathcal{V})$, then $\hat{\phi}_{k\ell} \to \hat{\phi}_\ell$ in $L^p_{0,*}(\mathcal{V}_{red})$ for each ℓ.*

If ϕ_k are smooth and tend to ϕ in $L^p_{0,*}(\mathcal{V})$, it follows that $\phi_k \to \phi$ in $\mathcal{C}^{0,*}(\mathcal{V})$ and hence

$$L^p_{loc;0,*;X} \subset \mathcal{C}^{0,*}_X. \tag{5.1}$$

Proof If $\phi \in L^p_{0,*}(\mathcal{V})$, then by definition, there are smooth ϕ_k such that $\|\phi - \phi_k\|_{L^p(\mathcal{V})} \to 0$. Since we have unique representations

$$\phi_k = \sum_\ell \hat{\phi}_{k\ell}(\zeta)\tau^{\alpha_\ell},$$

it follows from (3.12) and (3.13) that $k \mapsto \hat{\phi}_{\ell k}$ is a Cauchy sequence in $L^p(\mathcal{V}_{red})$ for each ℓ and hence converges to a function $\hat{\phi}_\ell \in L^p(\mathcal{V}_{red})$. Thus we have a representation (3.11) for ϕ, where $\hat{\phi}_\ell \in L^p(\mathcal{V}_{red})$. The last statement now follows from (3.12) and (3.13).

Example 5.2 Let $\hat{X}$ be the space in Example 3.1 and let $\hat{\mathcal{V}} = \mathcal{U}' \cap \hat{X}$, where $\mathcal{U}'$ is a relatively compact subset of $\mathcal{U}$. Let $L^{j,p}(\hat{\mathcal{V}}_{red})$ be the Sobolev space of all $(0,*)$-currents whose holomorphic derivates up to order j are in $L^p(Z)$. It follows from (11.5) that $L^p(\hat{\mathcal{V}})$ can be realized as all expressions of the form (3.7), where $\psi_m \in L^{|M-m|,p}_{0,*}(\hat{\mathcal{V}}_{red})$.

For a general Cohen–Macaulay space X,there is no such simple way to describe $L^p(X)$ locally in terms of a single choice of $(\zeta, \tau, \tau^{\alpha_\ell})$.

Remark 5.3 Assume that $\phi \in \mathcal{C}^{0,*}(\mathcal{V})$ and that its coefficients $\hat{\phi}_{\iota,\ell}$ with respect to each of the bases $(\zeta^\iota, \tau^\iota, (\tau^\iota)^{\alpha^\iota_\ell})$, cf. (3.13), are in $L^p(\mathcal{V}_{red})$. We do not know if this implies that ϕ is in $L^p_{0,*}(\mathcal{V})$. Consider the coefficients $\hat{\phi}_{\iota,\ell}$ with respect to a fixed basis

$(\zeta^\iota, \tau^\iota, (\tau^\iota)^{\alpha^\iota \ell})$. If one approximates these coefficients in L^p by smooth forms, then we get convergence in the norm $|\cdot|_{X,\pi^\iota}$, cf. (3.12). However, there is no reason to believe that they converge in the other norms in the right-hand side of (3.13). The problem is that the transition matrix (4.9) involves derivatives.

Notice that if we have an embedding $\iota\colon X \to \hat{X}$ as in Sect. 3, then, with the notation used there,

$$\|\phi\|_{L^p(X\cap\mathcal{U})} \sim \sum_j \|\gamma_j\phi\|_{L^p(\hat{X}\cap\mathcal{U})}.$$

Remark 5.4 Here is a heuristic proof of Theorem 1.1. For simplicity, let us assume that only two submersions π^1 and π^2 are needed in (3.13). Assume that ϕ is in $L^p_{0,1}(\mathcal{V})$. Then we can find a solution u^ι in $\mathcal{C}^{0,0}(\mathcal{V})$ to $\bar\partial u^\iota = \phi$ so that the coefficients with respect to $(\zeta^\iota, \tau^\iota, (\tau^\iota)^{\alpha^\iota \ell})$ of u^ι are in $L^p(\mathcal{V}_{red})$. This means that $|u^\iota|_{X,\pi^\iota}$ is in $L^p(\mathcal{V}_{red})$ for each ι. Now $h = u^2 - u^1$ is $\bar\partial$-closed, thus holomorphic, and hence bounded. It follows that also $|u^2|_{X,\pi^1} = |u^1 + h|_{X,\pi^1}$ is in $L^p(\mathcal{V}_{red})$. In view of (3.13),one might conclude that u^2 actually is in $L^p_{0,0}(\mathcal{V})$ if we disregard the problem pointed out in Remark 5.3. Clearly, this argument breaks down if ϕ has bidegree $(0, q+1)$, $q \geq 1$.

It is not clear to us if it is possible to make this outline into a strict argument. In any case, we will prove Theorem 1.1 by means of an integral formula from [7]. Besides being a closed formula for a solution, it also makes sense at non-Cohen–Macaulay points and offers a possibility to obtain a priori estimates, cf. Sect. 11. Hopefully,it could lead to results for general $(0, q)$-forms.

6 Koppelman Formulas on *X*

6.1 Koppelman Formulas in $\mathbb{C}^N$

Let $\mathcal{U} \subset \mathbb{C}^N$ be a domain, and let $\mathcal{U}' \subset\subset \mathcal{U}$. Moreover, let δ_η be contraction by the vector field

$$2\pi i \sum_{j=1}^N (\zeta_j - z_j)\frac{\partial}{\partial \zeta_j}$$

in $\mathcal{U}_\zeta \times \mathcal{U}'_z$ and let $\nabla_\eta = \delta_\eta - \bar\partial$. Assume that $g = g_{0,0} + \cdots + g_{n,n}$ is a smooth form such that $g_{k,k}$ has bidegree (k, k) and only contains holomorphic differentials with respect to ζ. We say that g is a weight in $\mathcal{U}$ with respect to $\mathcal{U}'$ if $\nabla_\eta g = 0$ and $g_{0,0}$ is 1 when $\zeta = z$. Notice that if g and g' are weights, then $g' \wedge g$ is again a weight. The basic observation is that if g is a weight, then

$$\phi(z) = \int g\phi, \quad z \in \mathcal{U}' \tag{6.1}$$

if ϕ is holomorphic in $\mathcal{U}$, see, [1, Proposition 3.1].

If $\mathcal{U}$ is pseudoconvex, following [2, Example 1], we can find a weight g, with respect to $\mathcal{U}'$, with compact support in $\mathcal{U}$, such that g depends holomorphically on z and has no anti-holomorphic differentials with respect to z. For our purpose, we can assume that these domains are balls with center at $0 \in \mathcal{U}$. Then we can take

$$g = \chi - \bar\partial\chi \wedge \frac{\sigma}{\nabla_\eta \sigma} = \chi - \bar\partial\chi \wedge \sum_{\ell=1}^{N} \frac{1}{(2\pi i)^\ell} \frac{\zeta \cdot d\bar\zeta \wedge (d\zeta \cdot d\bar\zeta)^{\ell-1}}{(|\zeta|^2 - \bar\zeta \cdot z)^\ell}, \tag{6.2}$$

where

$$\sigma = \frac{1}{2\pi i} \frac{\zeta \cdot d\bar\zeta}{|\zeta|^2 - \bar\zeta \cdot z}.$$

Here χ is a cutoff function in $\mathcal{U}$ that is 1 in a neighborhood of $\overline{\mathcal{U}'}$. It is convenient to choose it of the form $\chi = \tilde\chi(|\zeta|^2)$ where $\tilde\chi(t)$ is identically 1 close to 0 and 0 when t is large.

Elaborating this construction, one can obtain Koppelman formulas for $\bar\partial$. Let

$$b = \frac{1}{2\pi i} \frac{\sum_{j=1}^N (\overline{\zeta_j - z_j}) d\zeta_j}{|\zeta - z|^2}$$

so that $\delta_\eta b = 1$ where $\zeta \neq z$, and

$$B = \frac{\nabla_\eta b}{\bar\partial b} = b + b \wedge \bar\partial b + \cdots + b \wedge (\bar\partial b)^{N-1} \tag{6.3}$$

is the full Bochner–Martinelli form, cf. [1, Section 2]. Then $\nabla_\eta B = 1 - [\Delta]'$, where $[\Delta]'$ is the component with full degree in $d\zeta$ of the current of integration over the diagonal $\Delta \subset \mathcal{U} \times \mathcal{U}'$. Now

$$\mathcal{K}\phi = \int_\zeta g \wedge B \wedge \phi \tag{6.4}$$

defines integral operators $\mathscr{E}^{0,*+1}(\mathcal{U}) \to \mathscr{E}^{0,*}(\mathcal{U}')$ such that $\phi = \bar\partial\mathcal{K}\phi + \mathcal{K}(\bar\partial\phi)$ in $\mathcal{U}'$. The integral in (6.4) is, by definition, the pushforward $\pi_*(g \wedge B \wedge \phi)$, where π is the natural projection $\mathcal{U} \times \mathcal{U}' \to \mathcal{U}'$.

6.2 Hefer Morphisms

Let (E, f) be a locally free resolution as in (2.3). As in [2] and elsewhere, we equip $E := \oplus E_k$ with a superstructure, by splitting into the part $\oplus E_{2k}$ of even degree and the part $\oplus E_{2k+1}$ of odd degree. An endomorphism $\alpha \in \mathrm{End}(E)$ is even if it preserves the degree, and odd if it switches the degree. The total degree $\deg\alpha$ of a form-valued morphism α is the sum of the endomorphism degree and the form degree of α. For instance, f is an odd endomorphism. The contraction by δ_η is a derivation (and has odd degree) that takes the total degree into account, so if α and β are two morphisms, then $\delta_\eta(\alpha\beta) = \delta_\eta\alpha + (-1)^{\deg\alpha}\alpha\delta_\eta\beta$.

In order to construct division–interpolation formulas with respect to (E, f), in [2] was introduced the notion of an associated family $H = (H_k^\ell)$ of Hefer morphisms. Here H_k^ℓ are holomorphic $(k-\ell)$-forms with values in $\mathrm{Hom}\,(E_{\zeta,k}, E_{z,\ell})$ so they are even. They are connected in the following way: To begin with, $H_k^\ell = 0$ if $k - \ell < 0$, and H_ℓ^ℓ is equal to I_{E_ℓ} when $\zeta = z$. In general,

$$\delta_\eta H_{k+1}^\ell = H_k^\ell f_{k+1}(\zeta) - f_{\ell+1}(z) H_{k+1}^{\ell+1}. \tag{6.5}$$

Let R and U be the associated currents, see Sect. 2. The basic observation is that $g' = f_1(z)H^1U + H^0R$ is a kind of non-smooth weight so that if Φ is holomorphic, then

$$\Phi(z) = \int_\zeta g' \wedge g\Phi = f_1(z)\int_\zeta H^1 U \wedge g\Phi + \int_\zeta H^0 R \wedge g\Phi, \quad z \in \mathcal{U}'. \tag{6.6}$$

When defining these integral operators, we tacitly understand that only components of the integrands that contribute to the integral should be taken into account.

6.3 Local Koppelman Formulas on *X*

Now assume that our non-reduced space X is locally embedded in a pseudoconvex domain $\mathcal{U}$. Let $\mathcal{V} = X \cap \mathcal{U}$ and $\mathcal{V}' = X \cap \mathcal{U}' \subset\subset \mathcal{V}$. Let (E, f) be a locally free resolution of $\mathscr{O}_X$ as in (2.3). Then $R\Phi = 0$ if $\Phi = 0$, cf. (2.6), and hence (6.6) is an intrinsic representation formula

$$\phi(z) = \int_\zeta p(\zeta, z)\phi(\zeta), \quad z \in \mathcal{V}',$$

for $\phi \in \mathscr{O}(\mathcal{V}')$. Following [9] and [7], one can define operators

$$\mathcal{K}\phi(z) = \int_\zeta g \wedge B \wedge H^0 R \wedge \phi, \quad z \in \mathcal{V}' \tag{6.7}$$

mapping $(0, *+1)$-forms in $\mathcal{V}$ to $(0, *)$-forms in $\mathcal{V}'$. However, not even in 'good' cases,the formula (6.7), as it stands, produces a form that is smooth in $\mathcal{U}'$, cf. [7, Remark 10.4], so the precise definition of $\mathcal{K}\phi$ is somewhat more involved, cf. [7, Section 9]: If $\mu \in \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$ in $\mathcal{U}'$, then $\mu(z) \wedge R(\zeta)$ is a well-defined pseudomeromorphic current in $\mathcal{U} \times \mathcal{U}'$. Moreover, B is almost semi-meromorphic in $\mathcal{U} \times \mathcal{U}'$ and smooth outside the diagonal. Hence $\mu(z) \wedge g \wedge B \wedge H^0 R \wedge \phi$ is well defined in $\mathcal{U} \times \mathcal{U}'$, as the limit of $\mu(z) \wedge g \wedge B^\epsilon \wedge H^0 R \wedge \phi$, where $B^\epsilon = \chi(|\zeta - z|^2/\epsilon)B$, cf. (2.2). The equality (6.7) is to be interpreted as the fact that there is a unique pseudomeromorphic current $u = \mathcal{K}\phi$ in $\mathcal{V}'$ such that

$$\mu \wedge u = \int_\zeta \mu(z) \wedge g \wedge B \wedge H^0 R \wedge \phi,$$

for all $\mu \in \mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z)$ in $\mathcal{U}'$. By [7, Theorem 9.1] the operators so defined satisfy the Koppelman formula

$$\phi = \bar\partial \mathcal{K}\phi + \mathcal{K}(\bar\partial\phi) \tag{6.8}$$

in $\mathcal{V}'$. It turns out, [7, Theorem 10.1], that $\mathcal{K}$ maps $\mathscr{E}^{0,*+1}(\mathcal{V}) \to \mathscr{E}^{0,*}(\mathcal{V}')$ if Z is smooth and X is Cohen–Macaulay.

Remark 6.1 In general, $\mathcal{K}\phi$ is not necessarily smooth in $\mathcal{V}'$, so one has to replace $\mathscr{E}_X^{0,*}$ by the sheaves $\mathscr{A}_X^{0,*}$, cf. Introduction, [7] and Sect. 11.

Let us now assume that $Z = X_{red}$ is smooth. By shrinking $\mathcal{U}$, we can assume that we have coordinates (ζ, τ) in $\mathcal{U}$ as usual, and we let $(z.w)$ be the corresponding 'output' coordinates in $\mathcal{U}'$. If in addition X is Cohen–Macaulay, we can choose (E, f) so that the associated free resolution (2.3) of $\mathscr{O}_\mathcal{U}/\mathcal{J}$ has length $\kappa = N - n$. Then R has just one component R_κ. For a smooth $(0, * + 1)$-form ϕ in $\mathcal{V}$, then

$$\mathcal{K}\phi(z, w) = \int_{\zeta,\tau} (g \wedge B)_n \wedge H_\kappa^0 R_\kappa \wedge \phi, \quad (z, w) \in \mathcal{V}', \tag{6.9}$$

where B is the Bochner-Martinelli form with respect to $(\zeta, \tau; z, w)$, and $(\ )_n$ denotes the component of bidegree $(n, n - * - 1)$ in (ζ, τ).

7 Extension of Koppelman Formulas to Currents

We keep the notation from the preceding section.

Proposition 7.1 *The operator $\mathcal{K}\colon \mathscr{E}^{0,*+1}(\mathcal{V}) \to \mathscr{E}^{0,*}(\mathcal{V}')$ in (6.9) extends to an operator $\mathcal{C}^{0,*+1}(\mathcal{V}) \to \mathcal{C}^{0,*}(\mathcal{V}')$ and the Koppelman formula (6.8) still holds in $\mathcal{V}'$.*

The proposition gives a new proof of the exactness of (4.13).

Proof Let us choose a basis τ^{α_ℓ} for $\mathscr{O}_X$ in $\mathcal{U}$, as in Sect. 3.1. If we represent $\phi \in \mathcal{C}^{0,*+1}(\mathcal{V})$ by $\Phi = \sum \hat\phi_\ell(\zeta)\tau^{\alpha_\ell}$, where $\hat\phi_\ell(\zeta) \in \mathcal{C}^{0,*+1}(Z \cap \mathcal{U})$, and regularize each $\hat\phi_\ell$ by $\hat\phi_\ell^\epsilon$, we obtain smooth Φ^ϵ, representing smooth ϕ^ϵ that tend to ϕ. Note that the weight g defining $\mathcal{K}$ has support in the ζ-variable in a fixed compact set $K \subset \mathcal{U}$, and thus $\mathcal{K}\phi^\epsilon$ is defined when ϵ is small enough. We want to show that $\mathcal{K}\phi := \lim_{\epsilon\to 0} \mathcal{K}\phi^\epsilon$ is a well-defined object in $\mathcal{C}^{0,*}(\mathcal{V}')$.

By assumption, B is of the form (6.3), where

$$b = \frac{1}{2\pi i} \frac{\sum_{j=1}^n \overline{(\zeta_j - z_j)} d\zeta_j + \sum_{i=1}^\kappa \overline{(\tau_i - w_i)} d\tau_i}{|\zeta - z|^2 + |\tau - w|^2}. \tag{7.1}$$

Take $\mu = \mu(z, w) \in \mathcal{H}om(\mathscr{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}^Z_{\mathcal{U}})$. Since R is annihilated by $\bar{\tau}$ and $d\bar{\tau}$, and μ is annihilated by $\bar{w}$ and $d\bar{w}$, see Sect. 2, we have that

$$\mu(z, w) \wedge \mathcal{K}\phi^\epsilon = \mu(z, w) \wedge \left(\int_{\zeta,\tau} g(\zeta, z) \wedge B(\zeta, z) \wedge H^0_\kappa R_\kappa \wedge \phi^\epsilon \right), \tag{7.2}$$

where $B(\zeta, z)$ is the Bochner-Martinelli kernel with respect to the variables ζ, z, and $g(\zeta, z)$ only depends on ζ and z (provided that it is chosen as in (6.2), but for (ζ, τ) and (z, w), however, this special choice of g is not important). More precisely, in view of the representation (4.1) of R_κ, its action involves holomorphic derivatives with respect to τ followed by evaluation at $\tau = 0$, cf. (3.4). Therefore,all terms involving $\bar{\tau}$ can be canceled without affecting the integral. For the same reason, all terms involving $\bar{w}$ disappear.

Therefore, H is the only factor in the integral that depends on w. Using the expansions of the form (3.7) of ϕ together with the fact that R_κ is annihilated by $\mathcal{J}$, and the expansion (4.1) of R_κ, and evaluating the τ-integral in the right-hand side of (7.2) we get

$$\mu(z, w) \wedge \left(\int_\zeta g(\zeta, z) \wedge B(\zeta, z) \wedge \sum_{\ell'=0}^{\nu-1} h_{\ell'}(\zeta, z, w)\hat{\phi}^\epsilon_{\ell'} \right),$$

for appropriate holomorphic functions $h_{\ell'}$. If we express each occurrence of w in the basis w^{α_ℓ} as in (3.11) modulo $\mathcal{J}$ (with w instead of τ) and using that μ is annihilated by $\mathcal{J}$, we get

$$\mu \wedge \mathcal{K}\phi^\epsilon = \mu(z, w) \wedge \sum_{\ell=0}^{\nu-1} w^{\alpha_\ell} \int_\zeta g(\zeta, z) \wedge B(\zeta, z) \wedge \sum_{\ell'=0}^{\nu-1} h_{\ell',\ell}(\zeta, z)\hat{\phi}^\epsilon_{\ell'},$$

where $h_{\ell',\ell}$ are polynomials in ζ, z. Thus

$$\mu(z, w) \wedge \mathcal{K}\phi^\epsilon = \mu(z, w) \wedge \sum_\ell \mathcal{K}_\ell(\hat{\phi}^\epsilon) w^{\alpha_\ell},$$

where the $\mathcal{K}_\ell(\hat{\phi}^\epsilon)$ is the result of multiplying the tuple $(\hat{\phi}^\epsilon_{\ell'})$ by a matrix of smooth forms in ζ, z followed by convolution by the Bochner-Martinelli form $B(\zeta)$. Therefore, each limit $\lim_{\epsilon\to 0} \mathcal{K}_\ell(\hat{\phi}^\epsilon) =: \mathcal{K}_\ell(\phi)$ exists in the sense of currents on Z and is independent of the regularization $\hat{\phi}^\epsilon$, and we see that $\mathcal{K}(\phi) = \sum_\ell \mathcal{K}_\ell(\phi) w^{\alpha_\ell} = \lim \mathcal{K}(\phi^\epsilon)$ is well defined. Since the Koppelman formula holds for ϕ^ϵ, it follows that it also holds for ϕ by letting $\epsilon \to 0$.

8 Comparison of Hefer Mappings

We will use an instance of the following general result.

Lemma 8.1 *Let $a : (\hat{E}, \hat{f}) \to (E, f)$ be a morphism of complexes, and let $\hat{H}$ and H denote holomorphic Hefer mappings associated to $(\hat{E}, \hat{f})$ and (E, f), respectively.*

Then (locally) there exist holomorphic $(k-\ell+1)$-forms C_k^ℓ with values in $Hom(\hat{E}_{\zeta,k}, E_{z,\ell})$ such that

$$C_k^\ell = 0, \quad k < \ell, \tag{8.1}$$

$$\delta_\eta C_\ell^\ell = H_\ell^\ell a_\ell(\zeta) - a_\ell(z)\hat{H}_\ell^\ell, \tag{8.2}$$

and

$$\delta_\eta C_k^\ell = H_k^\ell a_k(\zeta) - a_\ell(z)\hat{H}_k^\ell - C_{k-1}^\ell \hat{f}_k(\zeta) - f_{\ell+1}(z)C_k^{\ell+1}. \tag{8.3}$$

Here, just as in [22], we consider a as a morphism in $\mathrm{End}(\oplus(\hat{E}_k \oplus E_k))$, and thus a is a morphism of even degree, cf. Sect. 6.2.

Proof Since H_ℓ^ℓ and $\hat{H}_\ell^\ell$ are the identity mappings on $E_{\ell,z}$ and $\hat{E}_{\ell,z}$, respectively, when $\zeta = z$, one can solve the equation (8.2) by [2, Lemma 5.2]. We now proceed by induction over $k-\ell$. We know the lemma holds if $k-\ell \le 0$ so let us assume that it is proved for $k-\ell \le m$ and assume $k-\ell = m+1$. By [2, Lemma 5.2], it is then enough to see that the right-hand side of (8.3) is δ_η-closed. To simplify notation,we suppress indices and variables. By (6.5), $\delta H = Hf - fH$ and $\delta\hat{H} = \hat{H}\hat{f} - \hat{f}\hat{H}$. In addition, $fa = a\hat{f}$ and since f is of odd degree, while a is of even degree, $\delta f = -f\delta$ and $\delta a = a\delta$. We then have, using that $ff = 0$ and $\hat{f}\hat{f} = 0$,

$$\begin{aligned}&\delta\big(Ha - a\hat{H} - (C\hat{f} - f(z)C)\big)\\ &\quad = (Hf - fH)a - a(\hat{H}\hat{f} - \hat{f}\hat{H}) - (Ha - a\hat{H} - fC)\hat{f} + f(Ha - a\hat{H} - C\hat{f})\end{aligned}$$

and using the relations above,it is readily verified that the right-hand side vanishes.

9 L^p-Estimates in Special Cases

In this section,we consider the space $\hat{X}$, $\mathscr{O}_{\hat{X}} = \mathscr{O}_{\mathcal{U}}/\mathcal{I}$, in Example 3.1 where, in a local embedding and suitable coordinates (ζ, τ) in $\mathcal{U}$, $\mathcal{I} = \langle \tau^{\mathbf{M}+\mathbf{1}} \rangle$.

Since $\mathcal{I}$ is a complete intersection, the Koszul complex provides a resolution of $\mathscr{O}_{\mathcal{U}}/\mathcal{I}$. That is, if $e_1, \dots, e_\kappa$ is a nonsense basis for the trivial vector bundle $\hat{E}_1 \simeq \mathbb{C}^\kappa \times \mathcal{U}$, then the resolution is generated by $(\hat{E}, \hat{f})$, where $\hat{E}_k = \Lambda^k \hat{E}_1$, each $\hat{f}_k$ is contraction by

$$\tau_1^{M_1+1} e_1^* + \cdots + \tau_\kappa^{M_\kappa+1} e_\kappa^*,$$

and e_j^* is the dual basis. The associated residue current is

$$\hat{R}_\kappa = \bar{\partial}\frac{1}{\tau_1^{M_1+1}} \wedge \ldots \wedge \bar{\partial}\frac{1}{\tau_\kappa^{M_\kappa+1}} \wedge e_1 \wedge \cdots \wedge e_\kappa,$$

see for example [3, Corollary 3.5]. In $\mathcal{U} \times \mathcal{U}'$ we use the coordinates $(\zeta, \tau; z, w)$. If

$$h = \frac{1}{2\pi i} \sum_j \sum_{0 \le \alpha_j \le M_j} \tau_j^{\alpha_j} w_j^{M_j - \alpha_j} d\tau_j \wedge e_j^*,$$

then it is readily checked that a choice of Hefer forms $\hat{H}_k^\ell$ is given by contraction by $\wedge^{k-\ell} h$. In particular,

$$\hat{H}_\kappa^0 = \pm \frac{1}{(2\pi i)^\kappa} \sum_{0 \le \alpha \le M} w^\alpha \tau^{M-\alpha} d\tau_1 \wedge \dots d\tau_\kappa \wedge (e_1 \wedge \dots \wedge e_\kappa)^*,$$

where we use the multiindex notation $w^\alpha = w_1^{\alpha_1} \cdots w_\kappa^{\alpha_\kappa}$.

In particular, with the notation (3.5), and the formula (3.6),

$$\hat{H}_\kappa^0 \hat{R}_\kappa = \frac{1}{(2\pi i)^\kappa} \sum_{\mathbf{0} \le \alpha \le M} w^\alpha \bar{\partial} \frac{d\tau}{\tau^{\alpha + \mathbf{1}}}.$$

Using the notation from Sect. 6.3 and Sect. 7, we consider the operators

$$\hat{\mathcal{K}}\psi = \int_{\zeta, \tau} g \wedge B \wedge \hat{H}_\kappa^0 \hat{R}_\kappa \wedge \psi \tag{9.1}$$

for $\psi \in \mathscr{E}^{0,*+1}(\mathcal{U} \cap \hat{X})$. As was noted in Sect. 8, only the parts of B and g depending on z, ζ are relevant. In view of (3.4) we therefore get

$$\hat{\mathcal{K}}\psi(z, w) = \sum_{\alpha \le M} w^\alpha \int_\zeta g(\zeta, z) \wedge B(\zeta, z) \wedge \frac{1}{\alpha!} \frac{\partial \psi}{\partial \tau^\alpha}(\zeta, 0).$$

Since $B(\zeta, z)$ only depends on $\zeta - z$, by a change of variables, we see that

$$\frac{\partial}{\partial w^m \partial z^\gamma} \hat{\mathcal{K}}\psi(z, 0) = \sum_{\beta' + \beta'' + \delta = \gamma} \int_\zeta B(\zeta, z) \wedge c_{\beta', \beta''} \frac{\partial}{\partial z^{\beta'} \partial \zeta^{\beta''}} g(\zeta, z) \wedge \frac{\partial \psi}{\partial \zeta^\delta \partial \tau^m}(\zeta, 0) \tag{9.2}$$

for appropriate constants $c_{\beta', \beta''}$. Since $B(\zeta, z)$ is uniformly integrable in ζ and z, and g is smooth, it follows by, e.g., [30, Appendix B], that

$$\left\| \frac{\partial \hat{\mathcal{K}}\psi}{\partial w^m \partial z^\gamma}(z, 0) \right\|_{L^p(Z \cap \mathcal{U}')} \lesssim \sum_{\delta \le \gamma} \left\| \frac{\partial \psi}{\partial \zeta^\delta \partial \tau^m}(\zeta, 0) \right\|_{L^p(Z \cap \mathcal{U})}. \tag{9.3}$$

From (9.3) and (3.8) it follows that there is a constant C_p such that

$$\|\hat{\mathcal{K}}\psi\|_{L^p(\hat{X} \cap \mathcal{U}')} \le C_p \|\psi\|_{L^p(\hat{X} \cap \mathcal{U})}, \quad 1 \le p \le \infty. \tag{9.4}$$

Example 9.1 Let $X = \mathbb{C}^n \times X_0$ be an analytic space which is the product of $\mathbb{C}^n$ with a space X_0 whose underlying reduced space is a single point $0 \in \mathbb{C}^\kappa$, i.e., such that $\mathscr{O}_X = \mathscr{O}_{\mathbb{C}^n_\zeta \times \mathbb{C}^\kappa_\tau}/\mathcal{J}$, where $\mathcal{J} = \pi^*\mathcal{J}_0$, and $\mathcal{J}_0 \subset \mathscr{O}_{\mathbb{C}^\kappa_\tau}$ is an ideal such that $Z(\mathcal{J}_0) = 0$ and π is the projection $\pi(\zeta, \tau) = \tau$. Note in particular that this includes the basic examples $\hat{X}$ as in Example 3.1.

Since the operator $\hat{\mathcal{K}}$ maps τ^α to w^α, it maps $\mathcal{J}$ to $\mathcal{J}_w$, where $\mathcal{J}_w$ denotes the ideal $\mathcal{J}$ in the (z, w)-coordinates. Furthermore, it maps $\bar\tau_k$ and $d\bar\tau_k$ to 0, so it descends to an operator $\hat{\mathcal{K}} : \mathscr{E}^{0,*+1}(\mathcal{U} \cap X) \to \mathscr{E}^{0,*}(\mathcal{U}' \cap X)$. Note that one may choose $\gamma_1, \dots, \gamma_\rho$ in (3.9) that only depend on τ. Thus, if $\psi \in \mathscr{E}^{0,*+1}(\mathcal{U} \cap X)$, then

$$\hat{\mathcal{K}}(\gamma_k(\tau)\psi) = \gamma_k(w)\hat{\mathcal{K}}\psi. \tag{9.5}$$

By (3.10), (9.4),and (9.5), it follows that

$$\|\hat{\mathcal{K}}\psi\|_{L^p(X\cap\mathcal{U}')} \le C_p\|\psi\|_{L^p(X\cap\mathcal{U})}, \quad 1 \le p \le \infty. \tag{9.6}$$

We can now prove

Proposition 9.2 *Let X be a space of the form $\mathbb{C}^n \times X_0$ as in Example 9.1. The operators $\mathcal{K}\colon \mathscr{E}^{0,q+1}(X \cap \mathcal{U}) \to \mathscr{E}^{0,q}(X \cap \mathcal{U}')$ extend to bounded operators $L^p_{0,q+1}(X \cap \mathcal{U}) \to L^p_{0,q}(X \cap \mathcal{U}')$, $q \ge 0$, $1 \le p < \infty$, so that the Koppelman formula (6.8) holds. The same statements hold for $C^{0,q}$ instead of $L^p_{0,q}$.*

In particular, if $\psi \in L^p_{0,q+1}(X \cap \mathcal{U})$ and $\bar\partial\psi = 0$, then $\bar\partial\mathcal{K}\psi = \psi$ in $X \cap \mathcal{U}'$ by (6.8). Thus Theorem 1.1 holds for all q when X is of the form as in Example 9.1.

Proof If $\psi \in L^p_{0,q+1}(X \cap \mathcal{U})$, then by definition there is a sequence $\psi_k \in \mathscr{E}^{0,q+1}(X \cap \mathcal{U})$ such that $\|\psi - \psi_k\|_{L^p(X\cap\mathcal{U})} \to 0$. It follows from (9.4) that $\mathcal{K}\psi_k$ is a Cauchy sequence in $L^p_{0,q}(X \cap \mathcal{U}')$ and hence has a limit $\mathcal{K}\psi$. Clearly this limit satisfies (9.6). Moreover, it is in $\mathcal{C}^{0,q}(X \cap \mathcal{U}')$. Thus these extended operators satisfy the Koppelman formula, see Proposition 7.1. The statements about $C^{0,q}$ follow in exactly the same way.

Remark 9.3 We use the intrinsic integral formulas on $\hat{X} \cap \mathcal{U}$ here for future reference. To obtain the theorem, one can just as well solve the $\bar\partial$-equation with relevant L^p-Sobolev norms in $X \cap \mathcal{U}$ for each coefficient in the expansion (3.7). However, this is naturally done by an integral formula on $Z \cap \mathcal{U}$, and the required computations are basically the same.

We finish this section with an example showing that the spaces in Example 9.1 may not necessarily be written in the simple form as in Example 3.1 after a change of coordinates, even if $\mathcal{J}$ is a complete intersection.

Example 9.4 Let $\mathcal{J}$ be generated by $(w_1^3, w_1^2 + w_2^3)$. Then we claim that one cannot find local coordinates τ_1, τ_2 near 0 such that $\mathcal{J}$ is generated by (τ_1^ℓ, τ_2^m). Indeed, since the multiplicity of $\mathcal{J}$ is 9, $\ell m = 9$. The assumptions imply that

$$\begin{bmatrix} w_1^3 \\ w_1^2 + w_2^3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \tau_1^\ell \\ \tau_2^m \end{bmatrix} \text{ and } \tau_j = a_{j1}w_1 + a_{j2}w_2, \text{ for } j = 1, 2,$$

where the a_{jk} and b_{jk} are holomorphic. One may exclude the case $\ell = m = 3$ since the above equations would imply that w_2^2 belongs to the ideal generated by $(w_1, w_2)^3$. The case $\ell = 1$, $m = 9$ may be excluded as that would imply that $\tau_1 = c_1 w_1^3 + c_2(w_1^2 + w_2^3)$ for some holomorphic functions c_j, which would contradict the fact that τ_1 is part of a coordinate system near 0.

10 L^p-Estimates at Cohen–Macaulay Points

Assume that we have a local embedding $X \to \mathcal{U}$ where $Z \cap \mathcal{U}$ is smooth and X is Cohen–Macaulay. Moreover, assume that we have coordinates (ζ, τ) in $\mathcal{U}$ such that $Z = \{\tau_1 = \cdots = \tau_\kappa = 0\}$, and a basis τ^{α_ℓ} for $\mathscr{O}_X$ over $\mathscr{O}_Z$. We may also assume that we have a Hermitian resolution (E, f) of $\mathscr{O}_X = \mathscr{O}_{\mathcal{U}}/\mathcal{J}$ of minimal length, so that its associated residue current is $R = R_\kappa$.

In general, if X is Cohen–Macaulay, and the underlying space Z is smooth, it is not possible to choose coordinates so that X becomes a product space as in Example 9.1, even if the space is defined by a complete intersection.

Example 10.1 Let $\mathcal{J} \subset \mathscr{O}_{\mathbb{C}^3_{z,w_1,w_2}}$ be generated by $g = (w_1^2, zw_1 + w_2^2)$, and $\mathscr{O}_X = \mathscr{O}/\mathcal{J}$. Then $Z(\mathcal{J}) = \{w = 0\}$, so $\mathcal{J}$ is a complete intersection ideal, and X is Cohen–Macaulay. We claim that one cannot choose new local coordinates (ζ, τ_1, τ_2) near 0 such that $\mathcal{J} = \pi^*\mathcal{J}_0$, where $\mathcal{J}_0 \subseteq \mathbb{C}^2_\tau$ is an ideal such that $Z(\mathcal{J}_0) = \{\tau = 0\}$ and $\pi(\zeta, \tau) = \tau$.

Indeed, assume that there are such coordinates. First of all, from any set of generators of an ideal, one may select among them a minimal subset of generators, and the number is independent of the choice of generators. Thus, one may assume that $\mathcal{J}$ is generated by $f_1(\tau), f_2(\tau)$. Since f and g generate $\mathcal{J}$, there is an invertible matrix A of holomorphic functions such that $f = Ag$ and $g = A^{-1}f$. Note that if $\mathfrak{m}$ is the maximal ideal of functions vanishing at $\{z = w = 0\}$, then g belongs to $\mathfrak{m}\mathcal{J}_Z$. Since $f = A^{-1}g$, the same must hold for f. Since $\{\tau = 0\} = \{w = 0\}$, one may write $\tau = Bw$ for some holomorphic matrix B. Note also that since f only depends on τ, $f = C\tau \mod \mathcal{J}_Z^2$ for some constant matrix C. Since f belongs to $\mathfrak{m}\mathcal{J}_Z$, we must have that $C = 0$, i.e., $f = 0 \mod \mathcal{J}_Z^2$. Thus, also $g = 0 \mod \mathcal{J}_Z^2$, which yields a contradiction.

Let us assume that we have coordinates (ζ, τ) in $\mathcal{U}$ and choose a simple ideal $\mathcal{I}$ as in Sect. 9, such that $\mathcal{I} \subset \mathcal{J}$, and hence, as in Sect. 3, get the embedding

$$\iota \colon X \to \hat{X}, \tag{10.1}$$

where $\mathscr{O}_{\hat{X}} = \mathscr{O}_{\mathcal{U}}/\mathcal{I}$. Let $\mathcal{V} = X \cap \mathcal{U}$ and $\mathcal{V}' = X \cap \mathcal{U}'$ as before and let $\hat{\mathcal{V}} = \hat{X} \cap \mathcal{U}$ and $\hat{\mathcal{V}}' = \hat{X} \cap \mathcal{U}'$. Here is our principal result.

Proposition 10.2 *Let $\mathcal{V}$ and $\mathcal{V}'$ be as above and $\mathcal{K}$ as in Sect. 6.3.*
(i) *There are constants C_p, $1 \le p \le \infty$, such that if ϕ is a smooth* $(0,1)$*-form and* $\bar\partial\phi = 0$*, then*

$$\|\mathcal{K}\phi\|_{L^p(\mathcal{V}')} \le C_p \|\phi\|_{L^p(\mathcal{V})}. \tag{10.2}$$

(ii) *If ϕ is in $L^p_{0,1}(\mathcal{V})$, $p < \infty$, and $\bar\partial\phi = 0$. Then $\mathcal{K}\phi$ is in $L^p_{0,0}(\mathcal{V}')$, $\bar\partial\mathcal{K}\phi = \phi$, and* (10.2) *holds. If $\phi \in C_{0,1}(\mathcal{V})$ and $\bar\partial\phi = 0$, then $\mathcal{K} \in C_{0,0}(\mathcal{V}')$, $\bar\partial\mathcal{K}\phi = \phi$, and*

$$\|\mathcal{K}\phi\|_{C(\mathcal{V}')} \leq C_\infty \|\phi\|_{C(\mathcal{V})}.$$

Clearly Theorem 1.1 follows from this proposition. The rest of this section is devoted to its proof.

Proof Choose an embedding (10.1) as above. Since the proposition is local we can assume that we have a basis τ^{α_ℓ} in $\mathcal{U}$. Let ϕ be a smooth $(0,*)$-form in $\mathcal{V}$. As in Sect. 9, let $(\hat{E}, \hat{f})$ be the Koszul complex of $\mathcal{I} = \langle \tau^{\mathbf{M+1}} \rangle$ in $\mathcal{U}$. Let us choose a morphism $a\colon (\hat{E}, \hat{f}) \to (E, f)$ of complexes that extends the natural surjection $\mathscr{O}_\mathcal{U}/\mathcal{I} \to \mathscr{O}_\mathcal{U}/\mathcal{J}$ and such that a_0 is the identity morphism $\hat{E}_0 \simeq E_0$, see, e.g., [22, Proposition 3.1]. By (3.10), we are to estimate the $L^p(\hat{\mathcal{V}}')$-norm of

$$\gamma\mathcal{K}\phi = \gamma(z,w)\int_{\zeta,\tau} g \wedge B \wedge H^0_\kappa R_\kappa \wedge \phi,$$

where γ is any of the functions in (3.9). (By the way, one can choose γ_j as the components of a_κ, cf. [7, Example 6.9]).

Since $\gamma\mathcal{K}\phi$ is to be considered as an element in $\mathscr{E}^{0,*}(\hat{\mathcal{V}}')$, it is determined by $\hat\mu \wedge \gamma\mathcal{K}\phi$, where

$$\hat\mu(z,w) = \bar\partial\frac{dw}{w^{\mathbf{M+1}}} \wedge dz.$$

since $\hat\mu$ is a generator for $\mathcal{H}om(\mathscr{O}_\mathcal{U}/\mathcal{I}, \mathcal{CH}^Z_\mathcal{U})$ in $\mathcal{U}$, cf. Sect. 6.3.

To ϕ, we associate the representative $\Phi = \sum \hat\phi_\ell(\zeta)\tau^{\alpha_\ell}$ in $\mathscr{E}^{0,*}(\mathcal{U})$, where $\hat\phi_\ell$ are in $\mathscr{E}^{0,*}(Z \cap \mathcal{U})$, as in (3.11).

Lemma 10.3 *We have that*

$$\hat\mu \wedge \gamma\mathcal{K}\phi = \hat\mu \wedge \gamma\int_{\zeta,\tau} g \wedge B \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \Phi. \tag{10.3}$$

Proof Recall from Sect. 6.3 that $\hat\mu \wedge \gamma\mathcal{K}\phi$ is defined as the limit of

$$\hat\mu \wedge \gamma\int_{\zeta,\tau} \chi_\epsilon g \wedge B \wedge H^0_\kappa R_\kappa \wedge \Phi, \tag{10.4}$$

where χ is a cutoff function and $\chi_\epsilon = \chi(|(\zeta,\tau)-(z,w)|^2/\epsilon)$. By [22, Theorem 4.1], $R_\kappa a_0 = a_\kappa \hat{R}_\kappa$. Using Lemma 8.1, the fact that a_0 is the identity, and that $\hat{f}_\kappa \hat{R}_\kappa = 0$ by (2.5), we get

$$H^0_\kappa R_\kappa = H^0_\kappa a_\kappa \hat{R}_\kappa = (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa + f_1(z,w)C^1_\kappa \hat{R}_\kappa. \tag{10.5}$$

Since $\gamma\mathcal{J} \subseteq \mathcal{I}$ and $\hat{\mu}$ is annihilated by $\mathcal{I}$, we have that $\gamma(z)f_1(z,w)\hat{\mu} = 0$ so by (10.5), (10.4) is equal to

$$\hat{\mu} \wedge \gamma \int_{\zeta,\tau} \chi_\epsilon g \wedge B \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \Phi.$$

Taking the limit as $\epsilon \to 0$, we obtain (10.3).

Let us choose a holomorphic 1-form Γ in $\mathcal{U}$ such that

$$\delta_\eta \Gamma = \gamma(\zeta,\tau) - \gamma(z,w). \tag{10.6}$$

From (10.3) and (10.6) we get

$$\hat{\mu} \wedge \gamma \mathcal{K}\phi = \hat{\mu} \wedge \int_{\zeta,\tau} g \wedge B \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \gamma\phi$$
$$+ \hat{\mu} \wedge \int_{\zeta,\tau} g \wedge B \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \delta_\eta \Gamma \Phi =: \hat{\mu} \wedge T_1\phi + \hat{\mu} \wedge T_2\phi.$$

Notice that we can write ϕ rather than Φ in $\hat{\mu} \wedge T_1\phi$, since $\hat{R}_\kappa\gamma$ annihilates $\mathcal{J}$. Now $T_1\phi = T_{11}\phi + T_{12}\phi$, where

$$T_{11}\phi = \int_{\zeta,\tau} g \wedge B \wedge \hat{H}^0_\kappa \hat{R}_\kappa \wedge \gamma\phi$$

and

$$T_{12}\phi = \int_{\zeta,\tau} g \wedge B \wedge (\delta_\eta C^0_\kappa)\hat{R}_\kappa) \wedge \gamma\phi.$$

Lemma 10.4 *Let A be a holomorphic $(\kappa+1,0)$-form in $d\zeta, d\tau$, $\psi = \psi(\zeta,\tau)$ a smooth $(0,*)$-form on $\mathcal{U}$. Then*

$$\hat{\mu} \wedge \int_{\zeta,\tau} g \wedge B \wedge (\delta_\eta A)\hat{R}_\kappa \wedge \psi = \hat{\mu} \wedge \int_{\zeta,\tau} g \wedge A\hat{R}_\kappa \wedge \psi$$
$$- \hat{\mu} \wedge \int_{\zeta,\tau} g \wedge B \wedge A\hat{R}_\kappa \wedge \bar{\partial}\psi - \hat{\mu} \wedge \bar{\partial}_{z,w} \int_{\zeta,\tau} g \wedge B \wedge A\hat{R}_\kappa \wedge \psi.$$

Proof As in the proof of Lemma 10.3,

$$\hat{\mu} \wedge \int_{\zeta,\tau} g \wedge B \wedge (\delta_\eta A)\hat{R}_\kappa \wedge \psi = \lim_{\epsilon\to 0} \hat{\mu} \wedge \int_{\zeta,\tau} \chi_\epsilon g \wedge B \wedge (\delta_\eta A)\hat{R}_\kappa \wedge \psi.$$

Let $(\ )_k$ denote the component of degree k in $d\zeta, d\tau$. Then

$$\begin{aligned}(\nabla_\eta(\chi_\epsilon g \wedge B \wedge A\hat{R}_\kappa \wedge \psi))_N \\ = -\bar\partial_{\zeta,\tau}\chi_\epsilon \wedge (g\wedge B)_{n-1} \wedge A\hat{R}_\kappa \wedge \psi + \chi_\epsilon g_{n-1} \wedge A\hat{R}_\kappa \wedge \psi \\ - \chi_\epsilon (g\wedge B)_n \wedge \delta_\eta A\hat{R}_\kappa \wedge \psi - \chi_\epsilon (g\wedge B)_{n-1} \wedge A\hat{R}_\kappa \wedge \bar\partial\psi \\ - \bar\partial_{z,w}\big(\chi_\epsilon (g\wedge B)_{n-1} A\hat{R}_\kappa \wedge \psi\big),\end{aligned} \tag{10.7}$$

where we have used that $\kappa + n = N$, $\nabla_\eta g = 0$ since g is a weight, $\chi_\epsilon \nabla_\eta B = \chi_\epsilon$, $\nabla_\eta A = \delta_\eta A$ since A is holomorphic, $\hat{R}_\kappa$ is $\bar\partial$-closed $(0,\kappa)$-current so that $\nabla_\eta \hat{R}_\kappa = 0$, and finally that g, B and A are the only terms containing differentials in $d\zeta, d\tau$, and A has degree $\kappa + 1$ in $d\zeta, d\tau$.

We claim that

$$\lim_{\epsilon\to 0} \hat\mu \wedge \bar\partial\chi_\epsilon \wedge (g\wedge B)_{n-1} \wedge A\hat{R}_\kappa \wedge \psi = 0. \tag{10.8}$$

In fact, let us write $B = \sum B_k$. Since B has only holomorphic differentials in $d\zeta, d\tau$, B_k has bidegree $(k, k-1)$, and so

$$(g\wedge B)_{n-1} \wedge A = \sum_{k\le n-1} g_{n-k-1} \wedge B_k \wedge A.$$

In particular, it suffices to show that

$$\lim_{\epsilon\to 0} \hat\mu \wedge \bar\partial\chi_\epsilon \wedge B_k \wedge \hat{R}_\kappa = 0$$

for $k \le n-1$. The limit of such a term on the left-hand side is a pseudomeromorphic current of bidegree $(*, k+2\kappa)$, see the comment after (2.2). Since the support of $\bar\partial\chi_\epsilon$ tends to Δ, the limits have support on $\Delta \cap (Z\times Z) \cong Z\cap\{pt\}$, which has codimension $\kappa + (n+\kappa) = n + 2\kappa$. By the dimension principle, Proposition 2.1, therefore the limit of each such term is 0 since $k + 2\kappa < n + 2\kappa$. Thus the claim holds.

The lemma follows from the claim by applying $\hat\mu \wedge \int_{\zeta,\tau}$ to (10.7) and letting $\epsilon \to 0$ since

$$\begin{aligned}-(\nabla_\eta(\chi_\epsilon g \wedge B \wedge A\hat{R}_\kappa \wedge \psi \wedge \hat\mu))_N = \bar\partial(\chi_\epsilon g \wedge B \wedge A\hat{R}_\kappa \wedge \psi \wedge \hat\mu)_N \\ = d(\chi_\epsilon g \wedge B \wedge A\hat{R}_\kappa \wedge \psi \wedge \hat\mu)_N.\end{aligned}$$

so that, by Stokes' theorem,

$$\hat\mu \wedge \int_{\zeta,\tau} (\nabla_\eta(\chi_\epsilon g \wedge B \wedge A\hat{R}_\kappa \wedge \psi \wedge \hat\mu))_N = 0.$$

Using Lemma 10.4 with $A = C^0_\kappa$, we get that $T_{12}\phi = T_{121}\phi + T_{122}\phi + T_{123}\phi$, where

$$T_{121}\phi = \int_{\zeta,\tau} g_{n-1} \wedge C^0_\kappa \hat{R}_\kappa \wedge \gamma\phi,$$
$$T_{122}\phi = -\int_{\zeta,\tau} (g \wedge B)_{n-1} \wedge C^0_\kappa \hat{R}_\kappa \wedge \gamma\bar{\partial}\phi$$

and

$$T_{123}\phi = \pm\bar{\partial}_{z,w} \int_{\zeta,\tau} (g \wedge B)_{n-1} \wedge C^0_\kappa \hat{R}_\kappa \wedge \gamma\phi.$$

Note that since $\hat{\mu} f_1(z) = 0$ and $\hat{f}_\kappa \hat{R} = 0$, we get that

$$T_2\phi = \hat{\mu} \wedge \int_{\zeta,\tau} g \wedge B \wedge \delta_\eta((\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\Gamma)\hat{R}_\kappa \wedge \Phi.$$

Thus, by applying Lemma 10.4 with $A = (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa) \wedge \Gamma$, we get that $T_2\phi = T_{21}\Phi + T_{22}\Phi + T_{23}\Phi$, where

$$T_{21}\Phi = \int_{\zeta,\tau} g_{n-1,*} \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \Gamma \wedge \Phi,$$
$$T_{22}\Phi = \int_{\zeta,\tau} g \wedge B \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \Gamma \wedge \bar{\partial}\Phi,$$

and

$$T_{23}\Phi = \pm\bar{\partial}_{z,w} \int_{\zeta,\tau} (g \wedge B)_{n-1} \wedge (\hat{H}^0_\kappa + \delta_\eta C^0_\kappa)\hat{R}_\kappa \wedge \Gamma \wedge \Phi.$$

We can now prove (i). If $\bar{\partial}\phi = 0$, then clearly $T_{22}\phi$ and $T_{122}\phi$ vanish. If ϕ has bidegree (0, 1), then $T_{123}\phi$ and $T_{23}\phi$ vanish for degree reasons since $(g \wedge B)_{n-1}$ and ϕ are the only terms containing $d\bar{\zeta}, d\bar{\tau}$. Therefore,

$$\gamma\mathcal{K}\phi = T_{11}\phi + T_{121}\phi + T_{21}\Phi. \tag{10.9}$$

The main term $T_{11}\phi$ is precisely $\hat{\mathcal{K}}(\gamma\phi)$, so from (9.4) and (3.10),

$$\|T_{11}\phi\|_{L^p(\hat{\mathcal{V}}')} \le C_p\|\gamma\phi\|_{L^p(\hat{\mathcal{V}})} \le C'_p\|\phi\|_{L^p(\mathcal{V})}$$

as desired.

The remaining two terms $T_{121}\phi$ and $T_{21}\Phi$ in (10.9) are simpler since their integrands do not contain the factor B. We now use that Φ has the form (3.11) and $\hat{R}_\kappa$ only depends on τ. Integrating with respect to τ therefore does not give rise to any derivates with respect to ζ. Thus, the $L^p(\mathcal{V}')$-norms of these two terms are bounded by integrals of the form

$$\sum_{\ell=0}^{\nu-1}\left(\int_z\left|\int_\zeta |\xi_\ell(\zeta,z)\hat\phi_\ell(\zeta)|\right|^p\right)^{1/p},$$

where $\xi_j(\zeta,z)$ are smooth forms with compact support in $Z\cap\mathcal{U}$. It follows from (3.12) and (3.13) that these terms are $\lesssim \|\phi\|_{L^p(\mathcal{V})}$. Thus part (i) is proved.

We now consider part (ii), so assume that $\phi\in L^p_{0,1}(\mathcal{V})$, $p<\infty$ and $\bar\partial\phi=0$. We cannot deduce (ii) directly from (i). The problem is that we do not know whether it is possible to regularize ϕ so that the smooth approximands are $\bar\partial$-closed, cf. Remarks 5.3 and 5.4.

By Proposition 7.1,we know that $\bar\partial\mathcal{K}\phi=\phi$ in the current sense. We must show that actually $\mathcal{K}\phi$ is in $L^p(\mathcal{V}')$ and that (10.2) holds. Let ϕ_k be a sequence of smooth $(0,1)$-forms in $\mathcal{V}$ that converge to ϕ in $L^p(\mathcal{V})$ and let Φ_k denote the representatives in $\mathcal{U}$ given by (3.11). Since $T_{123}\phi_k$ and $T_{23}\phi_k$ vanish for degree reasons, we have

$$\gamma\mathcal{K}\phi_k = G\Phi_k + G'(\bar\partial\Phi_k), \tag{10.10}$$

where

$$G\Phi_k = T_{11}\phi_k + T_{121}\phi_k + T_{21}\Phi_k,\quad G'_\gamma(\bar\partial\Phi_k) = T_{122}\Phi_k + T_{22}\Phi_k.$$

The proof of part (i) gives the a priori estimate

$$\|G\tilde\Phi\|_{L^p(\hat{\mathcal{V}}')} \le C_p\|\tilde\phi\|_{L^p(\mathcal{V})}$$

for $\tilde\phi$ in $\mathscr{E}^{0,1}(\mathcal{V})$. We conclude that $G\Phi_k$ has a limit $G\Phi$ in $L^p(\hat{\mathcal{V}}')$ and that

$$\|G\Phi\|_{L^p(\hat{\mathcal{V}}')} \le C_p\|\phi\|_{L^p(\mathcal{V})} \tag{10.11}$$

Next we claim that $\hat\mu\wedge G'(\bar\partial\Phi_k)\to 0$. In fact,

$$\bar\partial\Phi_k = \sum_\ell (\bar\partial\hat\phi_{k,\ell})\tau^{\alpha_\ell},$$

so arguing as in the proof of Proposition 7.1 the claim follows, since $\bar\partial\hat\phi_{k,\ell}\to 0$ for each ℓ.

Since $\gamma\mathcal{K}\phi_k\to\gamma\mathcal{K}\phi$ in $\mathcal{C}^{0,1}(\mathcal{V}')$, it follows from (10.10) that $\gamma\mathcal{K}\phi = G\Phi$. Thus $\gamma\mathcal{K}\phi$ is indeed in $L^p(\hat{\mathcal{V}}')$ and, cf. (10.11),

$$\|\gamma\mathcal{K}\phi\|_{L^p(\hat{\mathcal{V}}')} \le C_p\|\phi\|_{L^p(\mathcal{V})}.$$

Since this estimate holds for any $\gamma = \gamma_j$,we get cf. (3.10),

$$\|\mathcal{K}\phi\|_{L^p(\mathcal{V}')} \sim \sum_{j=1}^{\rho} \|\gamma_j \mathcal{K}\phi\|_{L^p(\hat{\mathcal{V}}')} \leq C_p \|\phi\|_{L^p(\mathcal{V})}.$$

Thus part (ii) holds for $p < \infty$. The case $p = \infty$ follows in precisely the same way. Thus the proposition is proved. □

Note that if we drop the assumption that ϕ be a (0, 1)-form, then the terms $T_{123}\phi$ and $T_{23}\phi$ no longer vanish, and it is not clear to us how to estimate them. It is also not clear to us whether the estimate (10.2) holds if ϕ is not $\bar{\partial}$-closed.

In the case of product spaces as in Example 9.1, then one may choose C^0_κ, $\hat{H}^0_\kappa$ and Γ such that they only contain holomorphic differentials $d\tau$. In that case, all terms but $T_{11}\phi$ vanish for any $(0, q)$-form ϕ, since all the other terms involve integrals of forms of degree $\kappa + 1$ in $d\tau$, which thus vanish for degree reasons. Thus, one in fact has that $\gamma\mathcal{K}\phi = T_{11}\phi = \hat{\mathcal{K}}(\gamma\phi)$, cf. the proof of Proposition 9.2.

11 An Example Where *X* is not Cohen–Macaulay

In this section, we consider an example where $Z = X_{red}$ is smooth but X is not Cohen–Macaulay. Since X_{red} is smooth, it is still possible to define $L^p_{loc}(X)$ as in Sect. 5. However, our solutions $\mathcal{K}\phi$ are not smooth at the non-Cohen–Macaulay point. In view of works on L^p-estimates on non-smooth reduced spaces, it therefore might be natural to define $L^p(X)$ as the completion of the space of smooth forms with support on the Cohen–Macaulay part of X. In any case, we do not pursue this question here, but just discuss an a priori estimate of the solutions.

Let $\Omega = \mathbb{C}^4_{z,w}$ and $\mathcal{J} = \mathcal{J}(w_1^2, w_1w_2, w_2^2, z_2w_1 - z_1w_2)$, and let X have the structure sheaf $\mathscr{O}_\Omega/\mathcal{J}$. Then $Z = \mathbb{C}^2_z$, and X has the single non-Cohen–Macaulay point (0, 0). Outside that point X is locally of the form discussed in Sect. 9 so that we have local L^p-estimates for $\bar{\partial}$ for all $(0, *)$-forms there. Thus the crucial question is what happens at (0, 0). The structure sheaf $\mathscr{O}_X$ has the free resolution (E, f)

$$0 \to \mathscr{O}_\Omega \xrightarrow{f_3} \mathscr{O}_\Omega^4 \xrightarrow{f_2} \mathscr{O}_\Omega^4 \xrightarrow{f_1} \mathscr{O}_\Omega \to \mathscr{O}_\Omega/\mathcal{J} \to 0,$$

where

$$f_3 = \begin{bmatrix} w_2 \\ -w_1 \\ z_2 \\ -z_1 \end{bmatrix}, \ f_2 = \begin{bmatrix} z_2 & 0 & -w_2 & 0 \\ -z_1 & z_2 & w_1 & -w_2 \\ 0 & -z_1 & 0 & w_1 \\ -w_1 & -w_2 & 0 & 0 \end{bmatrix}$$
$$\text{and } f_1 = \begin{bmatrix} w_1^2 & w_1w_2 & w_2^2 & z_2w_1 - z_1w_2 \end{bmatrix}.$$

We equip the vector spaces E_k with the trivial metrics. Consider also the Koszul complex $(F, \delta_{\mathbf{w}^2})$ generated by $\mathbf{w}^2 := (w_1^2, w_2^2)$, which is a free resolution of $\mathscr{O}/\mathcal{I}$, where $\mathcal{I} = \langle w_1^2, w_2^2\rangle$. If $\hat{X}$ has structure sheaf $\mathscr{O}_{\hat{X}} = \mathscr{O}/\mathcal{I}$, we thus have an embedding $\iota\colon X \to \hat{X}$.

We take the morphism of complexes $a : F_\bullet \to E_\bullet$ given by

$$a_2 = \begin{bmatrix} 0 \\ 0 \\ w_2 \\ w_1 \end{bmatrix}, a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } a_0 = \begin{bmatrix} 1 \end{bmatrix}.$$

Let R and $\hat{R}$ be the residue associated with (E, f) and $(F, \delta_{\mathbf{w}^2})$, respectively. It is well known, see, e.g., [7], that $\hat{R} = \hat{R}_2$ is equal to the Coleff–Herrera product

$$\mu_0 = \bar\partial(1/w_1^2) \wedge \bar\partial(1/w_2^2).$$

11.1 The Current *R*

In [7, Example 6.9], we found that

$$\mu_1 = \bar\partial\frac{1}{w_1} \wedge \bar\partial\frac{1}{w_2} \text{ and } \mu_2 = (z_1 w_2 + z_2 w_1)\bar\partial\frac{1}{w_1^2} \wedge \bar\partial\frac{1}{w_2^2}$$

(times $dz \wedge dw$) generate $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$. Here we intend to calculate $R = R_2 + R_3$. Using a comparison with the current $\hat{R}$,it follows from [22, Theorem 3.2, Lemma 3.4 and (3.10)] that

$$R_2 = (I - f_3\sigma_3)a_2\mu_0, \tag{11.1}$$

where

$$\sigma_3 = \frac{1}{|z|^2 + |w|^2}\begin{bmatrix} \bar w_2 & -\bar w_1 & \bar z_2 & -\bar z_1 \end{bmatrix}$$

is the minimal left-inverse to f_3. Since μ_0 is pseudomeromorphic with support on $\{w = 0\}$, $\bar w_i \mu_0 = 0$, and therefore

$$R_2 = \frac{1}{|z|^2}\begin{bmatrix} * * & -w_2\bar z_2 & w_2\bar z_1 \\ * * & w_1\bar z_2 & -w_1\bar z_1 \\ * * & |z|^2 - z_2\bar z_2 & z_2\bar z_1 \\ * * & z_1\bar z_2 & |z|^2 - z_1\bar z_1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ w_2 \\ w_1 \end{bmatrix}\mu_0 = \frac{1}{|z|^2}\begin{bmatrix} \bar z_1\mu_1 \\ \bar z_2\mu_1 \\ \bar z_1\mu_2 \\ \bar z_2\mu_2 \end{bmatrix}. \tag{11.2}$$

Since X has pure dimension $R_3 = \bar\partial\sigma_3 \wedge R_2$, where the left-hand side is the product of the almost semi-meromorphic current $\bar\partial\sigma_3$ and the pseudomeromorphic current R_2, cf. (2.2) and [7, Section 2]. Since f_3 is injective, $\sigma_3 = (f_3^* f_3)^{-1} f_3^* = f_3^*/(|z|^2+|w|^2)$. Thus, $f_3^*(I - f_3\sigma_3) = 0$, so in view of (11.1), $R_3 = (|z|^2+|w|^2)^{-1} f_3^* R_2$. Furthermore,

$\bar{w}_j R_2 = d\bar{w}_j \wedge R_2 = 0$, so we get

$$R_3 = \frac{1}{|z|^2} \left[0 \; 0 \; d\bar{z}_2 \; -d\bar{z}_1 \right] R_2 = \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{|z|^4} \mu_2. \tag{11.3}$$

11.2 Hefer Forms for (*E*, *f*)

Recall that a family $H_k^\ell : E_k \to E_\ell$ of Hefer morphisms are to satisfy, cf. (6.5), $H_\ell^\ell = I_{E_\ell}$ and

$$\delta_{(\zeta,\tau)-(z,w)} H_k^\ell = H_{k-1}^\ell f_k(\zeta,\tau) - f_{\ell+1}(z,w) H_k^{\ell+1} \tag{11.4}$$

for $k > \ell$. Due to the superstructure, when considering H and f as matrices, (11.5) means

$$\delta_{(\zeta,\tau)-(z,w)} H_k^\ell = H_{k-1}^\ell f_k(\zeta,\tau) - (-1)^{k-\ell-1} f_{\ell+1}(z,w) H_k^{\ell+1}, \tag{11.5}$$

cf. [24, (2.12)]. By hands-on calculations, or with the help of Macaulay2, one can check that

$$H_1^0 = \frac{1}{2\pi i} \begin{bmatrix} (\tau_1 + w_1) d\tau_1 + w_1 d\tau_1 \\ \tau_1 d\tau_2 + w_2 d\tau_1 \\ (\tau_2 + w_2) d\tau_2 \\ -\zeta_1 d\tau_2 + \zeta_2 d\tau_1 + w_1 d\zeta_2 - w_2 d\zeta_1 \end{bmatrix}^t,$$

$$H_2^1 = \frac{1}{2\pi i} \begin{bmatrix} d\zeta_2 & 0 & -d\tau_2 & 0 \\ -d\zeta_1 & d\zeta_2 & d\tau_1 & -d\tau_2 \\ 0 & -d\zeta_1 & 0 & d\tau_1 \\ -d\tau_1 & -d\tau_2 & 0 & 0 \end{bmatrix}$$

$$H_3^2 = \frac{1}{2\pi i} \begin{bmatrix} d\tau_2 \\ -d\tau_1 \\ d\zeta_2 \\ -d\zeta_1 \end{bmatrix}$$

$$H_2^0 = \frac{1}{(2\pi i)^2} \begin{bmatrix} w_1 d\zeta_2 \wedge d\tau_1 - w_2 d\zeta_1 \wedge d\tau_1 \\ \zeta_2 d\tau_1 \wedge d\tau_2 + w_1 d\zeta_2 \wedge d\tau_2 - w_2 d\zeta_1 \wedge d\tau_2 \\ (\tau_1 + w_1) d\tau_1 \wedge d\tau_2 \\ w_2 d\tau_1 \wedge d\tau_2 \end{bmatrix}^t$$

$$H_3^1 = \frac{1}{(2\pi i)^2} \begin{bmatrix} -d\zeta_2 \wedge d\tau_2 \\ d\zeta_1 \wedge d\tau_2 + d\zeta_2 \wedge d\tau_1 \\ -d\zeta_1 \wedge d\tau_1 \\ d\tau_1 \wedge d\tau_2 \end{bmatrix}$$

$$H_3^0 = \frac{1}{(2\pi i)^3} \left[w_1 d\zeta_2 \wedge d\tau_1 \wedge d\tau_2 - w_2 d\zeta_1 \wedge d\tau_1 \wedge d\tau_2 \right]$$

(where H_1^0 and H_2^0 are written as transposes of matrices just for space reasons) indeed satisfy (11.5) and are thus components of a Hefer morphism.

11.3 Estimates of Integral Operators

Now choose balls $\mathcal{U}' \subset\subset \mathcal{U} \subset\subset \Omega = \mathbb{C}^4_{\zeta,\tau}$ with center at $(0, 0)$ and consider the integral operator

$$\mathcal{K}\phi = \int_{\zeta,\tau} g \wedge B \wedge HR \wedge \phi$$

as in Sect. 6.3 for smooth $(0, 1)$-forms in $\mathcal{V}' = X \cap \mathcal{U}'$. We have that $\mathcal{K}\phi = \mathcal{K}_2\phi + \mathcal{K}_3\phi$, where

$$\mathcal{K}_2\phi = \int_{\zeta,\tau} (g_0 B_2 + g_1 \wedge B_1) \wedge H_2^0 R_2 \wedge \phi \tag{11.6}$$

and

$$\mathcal{K}_3\phi(z) = \int_{\zeta,\tau} \chi B_1 \wedge H_3^0 R_3 \wedge \phi. \tag{11.7}$$

Here $g_0 = \chi$ is a cutoff function in $\mathcal{U}$ with compact support that is equal to 1 on $\mathcal{U}'$, and g_1 contains the factor $\bar{\partial}\chi$, cf. (6.2). Moreover, $B_1 = b$, $B_2 = b \wedge \bar{\partial} b$, where b is given by (7.1), cf. (6.3). Notice, however, that since $\bar{\tau}\mu_i = 0$, $d\bar{\tau} \wedge \mu_i = 0$ and that $\bar{w}_i = 0$ considered as a smooth form on X, precisely as in Sect. 9, we can replace b by

$$\frac{1}{2\pi i} \frac{\sum_{j=1}^2 (\overline{\zeta_j - z_j}) d\zeta_j}{|\zeta - z|^2}$$

in the formula, and we may assume that g_0 and g_1 only depend on ζ and z.

For smooth $(0, *)$-forms we have, see [5, Section 6], that

$$|\phi(z, w)|_X \sim |\phi(z, 0)| + |z| \left| \frac{\partial}{\partial z} \phi(z, 0) \right| + |\mathcal{L}\phi(z, 0)|, \tag{11.8}$$

where

$$\mathcal{L} = z_1 \frac{\partial}{\partial w_1} + z_2 \frac{\partial}{\partial w_2}.$$

Since $B \wedge g$ has no differentials $d\tau_j$, for degree reasons,we only have to take into account terms of H that contain the factor $d\tau_1 \wedge d\tau_2$. By (11.2), but with (ζ, τ) instead of (z, w), and the formula above for H_2^0 the relevant part of $(2\pi i)^2 H_2^0 R_2$ therefore is

$$\frac{1}{|\zeta|^2} \big(|\zeta_2|^2 \mu_1 + \bar{\zeta}_1 (\tau_1 + w_1) \mu_2 + \bar{\zeta}_2 w_2 \mu_2 \big) = \mu_1 + \frac{w_1 \overline{\zeta}_1 + w_2 \overline{\zeta}_2}{|\zeta|^2} \mu_2,$$

where in the second equality,we have used that $\tau_1\mu_2 = \zeta_1\mu_1$. Thus

$$\mathcal{K}_2\phi = \frac{1}{(2\pi i)^2}\int_{\zeta,\tau}(g_0B_2 + g_1\wedge B_1)\wedge\left(\mu_1 + \frac{w_1\bar\zeta_1 + w_2\bar\zeta_2}{|\zeta|^2}\mu_2\right)\wedge\phi\wedge d\tau_1\wedge d\tau_2.$$

Integrating with respect to τ and using that $(2\pi i)^{-2}\mu_2\wedge\phi\wedge d\tau_1\wedge d\tau_2 = \mathcal{L}\phi\wedge[\tau = 0]$,we get

$$\mathcal{K}_2\phi = \int_\zeta (g_0B_2 + g_1\wedge B_1)\wedge\left(\phi + \frac{w_1\bar\zeta_1 + w_2\bar\zeta_2}{|\zeta|^2}\wedge\mathcal{L}\phi\right). \tag{11.9}$$

From (11.3) and the formula for H_3^0,we get

$$\begin{aligned}\mathcal{K}_3\phi &= \pm\frac{1}{(2\pi i)^3}\int\chi\frac{w_1\overline{(\zeta_1 - z_1)} + w_2\overline{(\zeta_2 - z_2)}}{|\zeta - z|^2}\frac{\bar\zeta_1 d\bar\zeta_2 - \bar\zeta_2 d\bar\zeta_1}{|\zeta|^4}\\ &\quad\wedge d\zeta_1\wedge d\zeta_2\wedge\phi\wedge\mu_2\wedge d\tau_1\wedge d\tau_2\\ &= \pm\frac{1}{(2\pi i)^2}\int\chi\\ &\quad\frac{w_1\overline{(\zeta_1 - z_1)} + w_2\overline{(\zeta_2 - z_2)}}{|\zeta - z|^2}\frac{\bar\zeta_1 d\bar\zeta_2 - \bar\zeta_2 d\bar\zeta_1}{|\zeta|^4}\wedge d\zeta_1\wedge d\zeta_2\wedge(\mathcal{L}\phi)(\zeta, 0).\end{aligned} \tag{11.10}$$

We now estimate $\mathcal{K}_2\phi$ by considering the various parts of the norm, cf. (11.8), letting $K = \operatorname{supp}\chi\cap Z$ and keeping in mind that $z\in X\cap\mathcal{U}'$ so that $|g_1|$ is bounded. To begin with

$$|(\mathcal{K}_2\phi)(z, 0)| = \left|\int_\zeta(\chi B_2 + g_1B_1)\wedge\phi(\zeta, 0)\right| \lesssim \int_{\zeta\in K}\frac{1}{|\zeta - z|^3}|\phi(\zeta)|_X. \tag{11.11}$$

Next we have, cf. (9.2),

$$\begin{aligned}|z|\left|\left(\frac{\partial}{\partial z_i}\mathcal{K}_2\phi\right)(z, 0)\right| &= |z|\left|\textstyle\int B_2\wedge\frac{\partial}{\partial\zeta_i}(\chi\phi(\zeta, 0)) + \cdots\right|\\ &\lesssim |z|\textstyle\int_{\zeta\in K}\frac{1}{|\zeta - z|^3}\frac{1}{|\zeta|}|\phi(\zeta)|_X.\end{aligned} \tag{11.12}$$

Finally,

$$\begin{aligned}|\mathcal{L}\mathcal{K}_2\phi)(z, 0)| &= \left|\int_\zeta(\chi B_2 + g_1B_1)\wedge\left(\frac{z_1\bar\zeta_1 + z_2\bar\zeta_2}{|\zeta|^2}\right)(\mathcal{L}\phi)(\zeta, 0)\right|\\ &\lesssim |z|\int_{\zeta\in K}\frac{1}{|\zeta - z|^3}\frac{1}{|\zeta|}|\phi(\zeta)|_X.\end{aligned} \tag{11.13}$$

Since $\mathcal{K}_3\phi$ vanishes when $w = 0$, the two first terms in the norm (11.8) vanish, and thus we get the estimate

$$
\begin{aligned}
|\mathcal{K}_3\phi|_X &\sim |(\mathcal{L}\mathcal{K}_3\phi)(z,0)| \\
&\sim \left| \int_{\zeta,z} \chi \frac{z_1\overline{(\zeta_1 - z_1)} + z_2\overline{(\zeta_2 - z_2)}}{|\zeta - z|^2} \frac{\overline{\zeta}_1 d\overline{\zeta}_2 - \overline{\zeta}_2 d\overline{\zeta}_1}{|\zeta|^4} \wedge d\zeta_1 \wedge d\zeta_2 \wedge (\mathcal{L}\phi)(\zeta,0) \right| \\
&\lesssim |z| \int_{\zeta\in K} \frac{1}{|\zeta - z|} \frac{1}{|\zeta|^3} |\phi(\zeta)|_X .
\end{aligned}
$$

Thus we have proved

$$
|\mathcal{K}_2\phi(z)|_X \le C \int_{\zeta\in K} \left(1 + \frac{|z|}{|\zeta|}\right) \frac{1}{|\zeta - z|^3}, \ |\mathcal{K}_3\phi(z)|_X \le C \int_{\zeta\in K} \frac{1}{|\zeta - z|} \frac{|z|}{|\zeta|^3} |\phi(\zeta)|_X . \tag{11.14}
$$

By [23, Theorem 4.1], $\|\mathcal{K}_2\phi(z)\|_{L^p(\mathcal{V}')} \le C\|\phi\|_{L^p(\mathcal{V})}$ if $p > 4/3$. Following the argument of that proof, but where $\|\zeta - z\|^{2n-1}$ is everywhere replaced by $\|\zeta - z\|$, it follows that $\|\mathcal{K}_3\phi(z)\|_{L^p(\mathcal{V}')} \le C\|\phi\|_{L^p(\mathcal{V})}$ if $p > 4$. We thus obtain the following estimate.

Proposition 11.1 *Let X be the space above and let ϕ be a smooth $(0,*)$-form in $\mathcal{V}$. We have the a priori estimate*

$$
\|\mathcal{K}\phi\|_{L^p(\mathcal{V}')} \le C\|\phi\|_{L^p(\mathcal{V})} \tag{11.15}
$$

for $4 < p \le \infty$.

If ϕ has bidegree $(0, 2)$, then $\mathcal{K}_3\phi$ vanishes for degree reasons, so then (11.15) in fact holds for $p > 4/3$.

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