The Journal of Geometric Analysis (2023) 33:229
https://doi.org/10.1007/s12220-023-01251-8

Orbifolds, the Regular Solids, and Hurwitz's $84(g - 1)$ Theorem

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Received: 1 February 2023 / Accepted: 15 March 2023 / Published online: 6 May 2023

Abstract
This paper presents an exposition of the analysis via orbifolds of the isometries with isolated fixed points of compact surfaces, giving a view of the classification of the five regular solids, following the treatment by William Thurston. Then it is shown that a closely related method can be used to recover the classical theorem of Hurwitz that the group of holomorphic automorphisms of a compact Riemann surface of genus $g > 1$ has order at most $84(g - 1)$.

Keywords Isometries of compact surfaces · Orbifolds · Gauss Bonnet Theorem for orbifolds · Automorphisms of Riemann surfaces · Genus of Riemann surface

Mathematics Subject Classification 51M20 · 30F10

New developments in mathematics often illuminate things that were known already. The purpose of this article is to show how some special cases of the relatively new idea of an "orbifold" can be used to explain in a different way the familiar fact that there are exactly five regular solids [4]. This explanation of the familiar has the intriguing property that almost the same reasoning yields, as we shall see, the seemingly unrelated result known as Hurwitz's Theorem: A compact Riemann surface of genus $g > 1$ has at most $84(g-1)$ biholomorphic maps to itself. This result turns out to be truly geometric and to be almost the same kind of geometry as the classification of the regular solids.

The orbifold idea was brought to the fore by W. Thurston [4] in the course of his work on the geometrization of 3-manifolds, a project brought to a close by G. Perelman [5] in an historic development. Thus news from the frontier filters back to illuminate the classical–Hurwitz's Theorem is from 1893– and even the truly ancient.

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1 Finite Group Actions on the 2-Sphere

Each of the five regular solids has a natural center, namely the center of gravity of the vertices; and the vertices all lie on a sphere around this center. By projecting the (bounding polyhedron of a) solid radially to this sphere, we obtain a dissection, or tessellation of the sphere into regular “geodesic polygons”, the edges of which are great circles. If you have a soccer ball handy, you can see the icosahedron represented this way by extending the edges of the white hexagons to meet at the centers of the black pentagons.

This idea of projecting on a sphere works the other way around, too: If we have a regular tessellation of the 2-dimensional sphere S^2, that is a dissection into congruent regular geodesic polygons with the same number of polygons meeting at each vertex, then we can reconstruct a regular polyhedron by an obvious process of making the spherical polygons into planar ones. And it is not hard to see *a priori* that every regular polyhedron will arise this way: the properties claimed for the five usual regular polyhedra in the previous paragraph would also hold for any other regular polyhedron, if there were any others. This is important since we are going to prove that there are only the standard five by thinking about the spherical tessellations. You can check this a priori truth just by thinking about moving the solid around to take one vertex to another. The center of gravity of all vertices stays fixed, and you are back in the spherical situation (details left to you).

With a regular spherical tesselation in hand, we look now at the group of rotations around the center that take the tesselation to itself. (There may be reflections in planes that do this, too, but we ignore these). Since the tesselation is regular, a given vertex p can be taken to any other vertex q. And any particular vertex x connected to p by an edge can be taken to any one of the vertices connected to q by an edge.

These two pieces of information, where a given vertex goes and how the sphere is rotated with that image vertex fixed once the vertex gets there, clearly determine the rotation of the sphere completely. This observations make it easy to compute the order of the group of rotations of the sphere that preserve the tesselation. Consider the dodecahedron. There are twenty vertices, and at each vertex there are three rotational positions, since there are three edges at each vertex. So the group of symmetries has order 60. Similarly, the tetrahedral group has order 12, the group for the cube has order 24, for the octahedron also 24 and for the icosahedron order 60.

We are going to study the regular solids from the viewpoint of the associated symmetry groups. But it turns out to be easier to consider the situation a little more generally. Instead of just thinking of rotations of the sphere S^2, we shall consider symmetries of surfaces in general. This added generality is no harder, and it opens up the possibility of using the same method on Riemann surfaces.

2 Finite Groups of Isometries of Surfaces

Suppose M is a compact, oriented surface with a Riemannian metric given. The essential points will be clear if one thinks about compact surfaces in euclidean space $\mathbb{R}^3$. But later we shall want to use “abstract" surfaces which have metries that do not arise as surfaces in $\mathbb{R}^3$. Let G be a finite group of orientation-preserving isometries of M, that is invertible (smooth) mappings from M to M that preserve the metric and the orientation. Our prototypical examples are the rotational symmetry groups of the regular solids acting on S^2.

If γ is any orientation-preserving isometry of such an M and if γ is not the identity mapping, then the fixed points of γ are isolated; and since M is compact, there only finitely many of them. To see this, suppose $\gamma(p) = p$ for some $p \in M$. Then the action of γ in a neighborhood of p is by a rotation (in M's Riemannian metric) around p. This is a familiar fact from differential geometry. The isometry γ take geodesics to geodesics, so what happens near the fixed point is determined by how the geodesics are permuted among themselves. But the only possibilities are that this permutation happens by a rotation or by a rotation then a reflection in a line. Since γ is orientation-preserving, no reflection is involved. So every geodesic is rotated and hence every point near p except p itself is moved by γ.

Since each $\gamma \in G$ has only finitely many fixed points, the set B (B for branching) of points of M that are fixed by some non-identity element of G is finite. The set B is obviously G invariant: $\gamma(q) \in B$ if $q \in B$ and $\gamma \in G$. Moreover on $M \backslash B$, G acts "freely": If $\gamma \neq \text{id}$ and $q \in M \backslash B$, then $\gamma(q) \neq q$.

Our next goal is to look hard at the "quotient space" of M by G. For this, we define an equivalence relation on M, that $p_1 - p_2$ if and only if there is a $\gamma \in G$ with $\gamma(p_1) = p_2$. We denote the set of equivalence classes by $\mathcal{O}$ ($\mathcal{O}$ for orbits, since these equivalence classes are usually called the orbits of G). There is a natural topology on $\mathcal{O}$: define a set $\mathcal{U} \subset \mathcal{O}$ to be open if and only if $\{x \in M : [x] \in \mathcal{U}\}$ is open in M, where $[x]$ denotes the equivalence class of $x \in M$.

If one makes these definitions for the general situation of a possibly infinite group of isometries of a surface, the space $\mathcal{O}$ can be complicated. But in our situation, $\mathcal{O}$ is a simple object topologically. It is actually another compact surface!

To see this, remember that B is G-invariant, so if $x \in B$ and $y \in [x]$ then $y \in B$. Thus we can classify "points" of $\mathcal{O}$ according to whether they arise from B or from $M \backslash B$. If $[x]$ is from $M \backslash B$, then there is a neighborhood V of x such that the "projection" map $\pi : V \to \mathcal{O}$ defined by $\pi(y) = [y]$ is one-to-one onto a neighborhood of $[x]$ in $\{\mathcal{O}\}$. Thus $\mathcal{O}$ near $[x]$ is indeed locally euclidean.

If $[x] \in \mathcal{O}$ comes from B, then the elements $x_1, x_2, \ldots, x_m$ of $[x]$ are all fixed points of some non-identity elements of G. Let $I_{x_1} = \{\gamma \in G : \gamma(x_1) = x_1\}$. The subgroup I_{x_1} of G is called the isotropy group of x_1. And x_1 being in B is exactly equivalent to I_{x_1} being not just the identity by itself. Also if $\gamma(x_1) = x_2$, then $I_{x_1} = \gamma \cdot I_{x_1} \cdot \gamma^{-1}$: the isotropy groups of points of an orbit are conjugate.

If one takes a small enough open disc D around x_1, say, then π maps D onto an open neighborhood of $[x]$ in an explicitly describable way: x_1 is the only element of $\pi^{-1}([x])$ in D, but every other point of $\pi(D)$ has exactly as many preimages in $\pi|D$ as the order of I_{x_1}. For notational convenience let i_{x_1} = the order of I_{x_1}. Then $\pi|D$ can be thought of in this way: Divide D into i_{x_1} sectors, each with angle $2\pi/i_{x_1}$. Then π is one-to-one on each sector except that the edges of the sector are identified to each other. Thus $\pi(D)$ is homeomorphic to a cone. Since a cone is itself locally euclidean, so is $\mathcal{O}$ in a neighborhood of $[x]$. There is a "singularity" at $[x]$ but it is not topological, it is metric in a sense we shall now make a little more precise.

Since G is a group of isometries of M, the metric on M gives rise to a smooth Riemannian metric on $\pi(M\backslash B)$; a tangent vector at a point of $\pi(M-B)$ is the image under the differential of π of a tangent vector at any one of the order (G) preimages of the point of $\pi(M\backslash B)$. All these order (G) tangent vectors have the same length since they are all transformations of each other by elements of G.

The metric situation at points of $\pi(B)$, however, looks not like a smooth point of a surface, but like the vertex of a cone. A metric circle of radius r around a point $[x]$ of $\pi(B)$ has length not $2\pi r$ + higher order terms, as at a smooth point, but instead $\frac{1}{i_x}(2\pi r)$ + higher order terms, as $r \to 0^+$ (where i_x= the order of isotropy at any $x \in [x]$, as before).

The metric structure on $\mathcal{O}$ is useful and important, in part because there is a version of the Gauss–Bonnet Theorem that applies. Actually, this generalized Gauss-Bonnet Theorem works for any surface with "cone-point" metric singularities even if the surface as a whole did not come from a group action quotient. Such spaces look locally like the quotient of a manifold by a group action, but they may not be of the form M/G globally. They are "orbifolds". Cone Point (Orbifold) Gauss-Bonnet Theorem: Let N be a compact surface with a Riemannian metric having a finite number of cone-point singularities $p_1 \cdots , p_k$ of (integer) orders $i_1, \cdots , i_k$. Then

$$\frac{1}{2\pi}\int_{N-\{p_1,\cdots,p_k\}} (\text{Gauss curvature }) \tag{2.1}$$

$$= \chi(N) + \sum_{\ell=1}^{k}\left(\frac{1}{i_k} - 1\right) . \tag{2.2}$$

Here $\chi(N) =$ the topological Euler characteristic of N, and a point p is a cone point singularity of order i if a neighborhood of p is isometric to the quotient of a disc with a smooth Riemannian metric by a group of rotations of order i around the center that preserve the metric.

To see why this Gauss–Bonnet formula holds, excise (small) discs of radius r around each of $p_1, \cdots , p_k$. This reduces the Euler characteristic by k. So the Gauss–Bonnet Theorem for surfaces with boundary then gives

$$2\pi(\chi(N) - k) = \int_{N-\cup \text{ discs}} (\text{Gauss curvature}) - \sum_{\ell=1}^{k}\int_{C_\ell} \text{geodesic curvature}$$

where $C_\ell =$ the circle of radius r around p_ℓ (here geodesic curvature is taken positive). Since p_ℓ is a cone point singularity, the length of C_ℓ is $(\frac{1}{i_\ell}2\pi r)$ + higher order terms, as $r \to 0^+$. Also, the geodesic curvature is $\frac{1}{r}$ + higher order terms, also as $r \to 0^+$. Thus

$$\lim_{r\to 0^+}\int_{C_\ell} (\text{geodesic curvature}) = 2\pi/i_\ell.$$

Letting $r \to 0^+$ yields

$$2\pi(\chi(N) - k) = \int_{N-\cup\{p_\ell\}} (\text{geodesic curvature}) - 2\pi \sum_{\ell=1}^{k} \frac{1}{i_\ell},$$

which is equivalent to the formula as stated in the theorem.

In the case that N is a quotient M/G of a surface by a group action, as we were previously discussing, the cone-point Gauss–Bonnet Formula is closely related to the topological counting result that is known as the Riemann-Hurwitz formula of classical Riemann surface theory. First note that in this case,

$$\int_{N-\{p_1,\cdots,p_k\}} \text{Gauss curvature} = \frac{1}{\text{order}(G)} \int_M \text{Gauss curvature} \tag{2.3}$$

$$= \frac{\lambda\pi}{\text{order}(G)}\, \chi(M)\,. \tag{2.4}$$

Thus

$$\frac{1}{\text{order}(G)}\, \chi(M) = \chi(M/G) + \sum_{p\in B} \left(\frac{1}{i_p} - 1\right).$$

The usual Riemann–Hurwitz formula (cf. [2], p. 216ff) is that if $\pi M \twoheadrightarrow N$ is a "branched covering" of one compact Riemann surface by another, then

$$\chi(M) = \text{order}(\pi)\chi(N) - \Sigma(\text{ branching order} - 1)$$

where the sum is over all points of M at which the map π is "ramified" or "branches". If the "branched covering" is of the sort $\pi : M \twoheadrightarrow M/G = N$ of the sort we have been considering, then the order of the covering $=$ order(G). And a point $x \in M$ is a branch point in the Riemann-Hurwitz sense precisely when the isotropy at x $i_x = \{g \in G : g(\lambda) = x\}$ is not the set consisting of the identity alone. Its branching order is order i_x. Note that such points x occur as elements of $\pi^{-1}(p)$, $p \in B$, in our previous notation; and the number of points in $\pi^{-1}(p) = \text{order}(G)/i_p$. Thus the sum over branch points in M

$$\sum_{x\in M, x \text{ a branch point}} (\text{branching order } - 1) = \sum_{p\in B} (i_p - 1) \left(\frac{\text{order}(G)}{i_p}\right) \tag{2.5}$$

$$= (\text{order}(G) \sum_{p\in B} \left(1 - \frac{1}{i_p}\right). \tag{2.6}$$

Thus the Riemann-Hurwitz formula becomes

$$\frac{\chi(M)}{\text{order}(\pi)} = \chi(M/G) + \sum_{p\in B} \left(\frac{1}{i_p} - 1\right)$$

as in our cone-point derived result. It is worth noting, however, that the cone-point Gauss-Bonnet Theorem also covers “orbifolds” which do not arise in the form M/G, but have only the local structure of a group quotient.

3 The Application to the Regular Solids

Let us return to the situation where M is the 2-sphere S^2 and $N = M/G$ with G the rotation-symmetry group of a regular solid. Gauss curvature in this case is everywhere positive, so the Gauss-Bonnet result gives

$$\chi(N) + \sum_{\ell=1}^{k} \left(\frac{1}{i_{p_\ell}} - 1 \right) > 0.$$

Since N is itself a compact oriented surface. $\chi(N)$ is 2 or 0 or negative. Since the terms $\frac{1}{i_p} - 1$ are negative with absolute value at least $\frac{1}{2}$, the positivity of the left-hand side implies that $\chi(N) = 2$ and that there are at most three $\frac{1}{i_p} - 1$ terms, i.e., $k \leq 3$. Also, $k \geq 3$ because the vertices the centers of gravity of the faces, and the midpoints of the edges are all in B but belong to three different equivalence classes. These have orders i_1= number of faces meeting at a vertex > 2, i_2 = number of edges in each face > 2, and $i_3 = 2$ (since two faces meet at and edge by definition of a polyhedron). Thus

$$-1 + \frac{1}{i_1} + \frac{1}{i_2} + \frac{1}{2} > 0 \quad \text{or} \quad \frac{1}{i_1} + \frac{1}{i_2} > \frac{1}{2}.$$

The only possibilities then are

(a) $i_1 = 3, \quad i_2 = 3$
(b) $i_1 = 3, \quad i_2 = 4$
(c) $i_1 = 4, \quad i_2 = 3$
(d) $i_1 = 3, \quad i_2 = 5$
(e) $i_1 = 5, \quad i_2 = 3$.

For each of these possibilities, we can go on to compute the order of the associated group. For example, for possibility (c),

$$\frac{1}{2\pi} \frac{1}{\text{order}(G)} \int_{S^2} 1 = 2 + \left(+ \frac{1}{5} - 1 \right) + \left(\frac{1}{3} - 1 \right) + \left(\frac{1}{2} - 1 \right)$$

or

$$\frac{2}{\text{order}(G)} = \frac{1}{30}$$

so the order $(G) = 60$. Therefore there are $60/i_1 = 12$ vertices, $60/i_2 = 20$ faces, and $60/i_3 = 30$ edges. The regular solid is totally determined by this information:

The twenty (geodesic) equilateral triangles on S^2 each have area $\frac{1}{20}$ (area of S^2), so their side length is determined, and the pattern in which they are joined together is determined by knowing that five meet at a vertex.

Thus the entire tessellation of S^2 and hence the solid itself are determined: it is the icosahedron. Similar argument shows that each of the other four possibilities gives rise to a unique regular solid.

4 The Geometric Hurwitz Theorem

While Hurwitz's Theorem as stated is about complex analysis, it is really a special case of a purely geometric result: Recall that a diffeomorphism of a surface is a homeomorphism which is (C^∞) differentiable and the inverse of which is (C^∞) differentiable.

THE GEOMETRIC HURWITZ THEOREM: If M is a compact oriented surface of genus $g > 1$ and if G is a finite group of orientation-preserving diffeomorphisms of M, then G contains at most $84(g - 1)$ elements. To put this in our isometry group picture, first note that there is always a Riemannian metric on M for which G acts as a group of isometries. This may have to be an "abstract" Riemannian metric, one that does not necessarily arise from embedding the surface M in euclidean 3-space. But we can always find such a metric by an averaging process: Choose an arbitrary Riemannian inner product $\langle , \rangle$ on M and define a new inner product by

$$\langle\langle v, w\rangle\rangle = \sum_{\gamma\in G} \langle d\gamma(v), d\gamma(w)\rangle_{\gamma(p)}$$

where v, w are tangent vectors at any point $p \in M$ and $d\gamma$ is the differential of γ at p. The fact that we have summed over G makes this new inner product invariant under G automatically.

Now we go back to the "cone point" Gauss–Bonnet Theorem: With $p_1, \ldots, p_k$ being the cone point singularities of $N = M/G$ with orders $i_1, \cdots, i_k$ just as before, we have again

$$\frac{1}{2\pi}\frac{1}{\text{order}(G)} \int_M \text{ Gauss curvature} = \chi(N) + \sum_{\ell=1}^{k} \left(\frac{1}{i_k} - 1\right).$$

Since $\int_M$ Gauss curvature $= 2\pi(2 - 2\,g)$ with $g =$ genus of $M > 1$, this yields, taking negative reciprocals

$$\frac{\text{order}(G)}{2g - 2} = \frac{1}{2g_n - 2 + \sum_{\ell=1}^{k}\left(1 - \frac{1}{i_\ell}\right)}$$

where $g_N =$ the genus of N. Since order (G) and $2g - 2$ are both positive, so is the denominator on the right. Hence

$$\text{order}(G) \leq (g-1)\left(\frac{1}{\text{minimum possible value of denominator}}\right).$$

To find the minimum possible positive value of $2g_N - 2 + \sum(1 - \frac{1}{i_\ell})$, note first that if $g_N > 1$, then the value is at least 2, since all $1 - \frac{1}{i_\ell}$ terms are positive. If $g_N = 1$, then $2g_N - 2 + \sum(1 - \frac{1}{i_\ell}) \geq \frac{1}{2}$ since for $2g_N - 2 + \Sigma\left(1 - \frac{1}{i_\ell}\right)$ to be positive, at least one $1 - \frac{1}{i_\ell}$ term must occur (with $i_\ell \geq 2$). Finally, if $g_N = 0$, then we are looking for the minimum possible positive value of $-2 + \sum_{\ell=1}^{k}\left(1 - \frac{1}{i_\ell}\right)$. Note that there must be at least three terms of the form $1 - \frac{1}{i_{i_\ell}}$ for positivity. If there are more than four terms then, since each is $\geq \frac{1}{2}$,

$$-2 + \sum\left(1 - \frac{1}{i_\ell}\right) \geq -2 + 5\left(\frac{1}{2}\right) = \frac{1}{2}.$$

If there are exactly four, not all i_ℓ can be 2, again by positivity, so one $i_\ell \geq 3$ and

$$-2 + \sum\left(1 - \frac{1}{i_\ell}\right) \geq -2 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{1}{6}.$$

If there are exactly three i_ℓ terms then $-2 + \sum\left(1 - \frac{1}{i_\ell}\right)$ can have the value

$$-2 + (1 - \frac{1}{2}) + (1 - \frac{1}{3}) + (1 - \frac{1}{7}) = \frac{1}{42}.$$

This is actually the minimum possible in this (and hence all) cases:

With exactly three i_ℓ, positivity precludes two of the i_ℓ being $= 2$. If all i_ℓ are ≥ 3, then

$$-2 + \sum(1 - \frac{1}{i_\ell}) \geq -2 + 3(\frac{2}{3}) = 1.$$

So we need only consider the case $i_1 = 2$, $i_2, i_3 \geq 3$. Then if i_2 and i_3 are ≥ 4, we obtain, because $-2 + \frac{1}{2} + \frac{3}{4} + \frac{3}{4} = 0$, that $i_3 \geq 5$ and hence

$$-2 + \sum(1 - \frac{1}{i_\ell}) \geq -2 + \frac{1}{2} + \frac{3}{4} + \frac{4}{5} = \frac{1}{20}.$$

Finally, we have the case $i_1 = 2, i_2 = 3$, and $i_3 \geq 3$. Since $i_3 \leq 6$ violates positivity, the $i_1 = 2$, $i_2 = 3$, $i_3 = 7$ case gives the minimum $\frac{1}{42}$.

From this elementary, (and classical) argument, we have obtained that the minimum possible positive value of $-2 + \sum(1 - \frac{1}{i_\ell})$ over all cases is $\frac{1}{42}$. Thus

$$\frac{\text{order}(G)}{2g-2} \leq 1/\left(\frac{1}{42}\right)$$

or

$$\text{order}(G) \leq 84(g-1).$$

The number 84 here arose from what looked like fooling around with arithmetic. So it is rather surprising that the estimate is sharp: for example, the "Klein quartic" Riemann surface which will be defined later has genus 3 and 168 biholomorphic diffeomorphisms. Hence, as we shall see in the next section, it has a Riemann metric which has exactly 168 orientation-preserving isometries. The number 84 is thus something very like a fundamental constant of nature, just as the number 5 of regular solids is.

5 How the Complex Analytic Hurwitz Theorem Follows from the Geometric Hurwitz Theorem

The occurrence of $84(g-1)$ in the geometric Hurwitz Theorem we have proved strongly suggests that the ordinary, complex analytic theorem is somehow the geometric one in disguise. This is in fact true: the two versions are essentially equivalent.

To go from the geometric theorem to the complex analytic one, we need to know first of all that, given a compact Riemann surface M of genus $g > 1$, there is a Riemannian metric on M which is invariant under biholomorphic self-mappings of M. We need this just to be in a geometric setting.

Fortunately, complex analysis provides such a metric: the Riemann surface M is a covering-space quotient of the unit disc Δ via a holomorphic covering map $\pi : \Delta \to M$. The deck transformations of this covering are biholomorphic maps of Δ to itself, and they are hence isometries of the Poincare metric $(dx^2 + dy^2)/(1 - x^2 - y^2)^2$ of Δ. So the Poincare metric can be "pushed down" to M. It is then straightforward to check that the biholomorphic maps of M to M are isometries of this pushed-down metric. (Such a biholomorphic map lifts to a biholomorphic map from Δ to Δ and is thus an isometry of the Poincare metric and hence of the pushed-down metric.)

The next and final step from geometric to complex analytic is actually itself geometric. Namely, we need the following:

Finiteness Lemma: If M is a compact orient surface of genus $g > 1$ with a Riemannian metric given, then the group of orientation-preserving isometries of M is finite and hence of order $\leq 84(g-1)$.

The essential point here is the finiteness, since, with that known, our Geometric Hurwitz Theorem gives the $84(g-1)$ estimate.

To establish the finiteness, note first that the isometry group is compact. Indeed, the isometry group of any compact Riemannian manifold is compact: this is a consequence of the Arzela–Ascoli Theorem, in effect. So to show the group is finite, it is enough to show that it is discrete: there is a neighborhood of the identity that contains only the identity.

To check this, recall from differential geometry that there is an $\varepsilon > 0$ such that any two points of M with distance less than ε are joined by a unique minimal geodesic (of length $< \varepsilon$). Suppose an isometry γ moves all points of M by distances less than ε. If $\gamma \neq$ identity, then we can define a vector field on M everywhere except at the fixed points of γ, by assigning to each $x \in M$ with $x \neq \gamma(x)$ the unit vector tangent to the unique minimal geodesic from x to $\gamma(x)$. From our earlier description of how γ behaves at fixed points, one sees that the index of this vector field at a fixed point is $+1$. By the famous theorem of Poincare, the sum of these indices is the Euler characteristic of $M = 2 - 2g < 0$. This contradiction shows that γ must be the identity.

Thus the complex analytic, usual form of Hurwitz's Theorem is established.

To go from complex analytic Hurwitz Theorem to our geometric one, it is only necessary to recall that an oriented surface with a Riemannian metric given has a uniquely determined complex structure in which the Riemannian metric is Hermitian and that for this complex structure, isometries of the given metric that preserve orientation are necessarily holomorphic with holomorphic inverse.

6 A Note on the Klein Quartic

The Klein quartic alluded to earlier is the (complex) curve in complex projective space $\mathbb{C}P^2$ defined by

$$xy^3 + yz^3 + zx^3 = 0$$

where $x, y, z \in \mathbb{C}$ are the usual "homogeneous coordinates" on $\mathbb{C}P^2$. This is a curve in the complex sense: complex dimension 1, real dimension 2; so it is a Riemann surface in our geometric language. By a standard formula of algebraic geometry this degree 4 "curve", considered as a Riemann surface, has genus $= (1/2)(4 - 1)(4 - 2) = 3$. (It is not hard to check that this curve is nonsingular and irreducible so the general formula applies that being of degree d gives genus $(1/2)(d - 1)(d - 2)$. e.g. [2], p. 221.)

Some automorphisms of this curve are apparent "by inspection", or at least after not too much experimentation:

The cyclic permutation automorphism of $\mathbb{C}P^2$ determined by

$$(x, y, z) \to (z, x, y)$$

preserves the equation of our curve, and hence acts on the curve itself. That one was easy. It has order 3. Since we are aiming for a group of order $168 = 7 \cdot 3 \cdot 2^3$, we should look for an element of order 7. It is natural to experiment with $\omega = e^{2\pi i/7}$, a primitive 7th root of unity. Trial and error gives that

$$(x, y, z) \to (x, \omega y, \omega^3 z)$$

preserves the equation: each term is multiplied by ω^3 (since $\omega \cdot (\omega^3)^3 = \omega^3$). So this automorphism of $\mathbb{C}P^2$ also acts on our curve. Clearly, it has order 7.

Surprisingly, perhaps, finding an order 2 automorphism of the curve is not so easy. The source of the following automorphism of $\mathbb{C}P^2$ is almost sure to be unclear, but once the automorphism is in hand, it is only a matter of determined calculation to see that it preserves the equation of the Klein quartic up to a factor (of - 7). Thus it acts as an automorphism of the Klein quartic itself, and another calculation shows that this automorphism has order 2. The automorphism of $\mathbb{C}P^2$ we want is

$$(x, y, z) \to (Ax + By + C_z,\ Bx + Cy + Az,\ Cx + Ay + Bz)$$

where $A = \omega^5 - \omega^2$, $B = \omega^3 - \omega^4$, and $C = \omega^6 - \omega$. The details of this topic and much additional information can be found in the volume [3], includes an English translation of Klein's original paper in which the 168 automorphisms are explicitly determined.

If a finite group of diffeomorphisms of a compact oriented surface contains both orientation-preserving and orientation-reversing elements, then the orientation-preserving elements form a subgroup of index 2. Our Geometric Hurwitz Theorem thus implies that the whole group has order no more than $168(g-1)$, when the genus $g > 1$. The Klein quartic has a natural orientation-reversing diffeomorphism, namely the one unduced by complex conjugation on $\mathbb{C}P^2$, $(x, y, z) \to (\bar{x}, \bar{y}, \bar{z})$: the equation of the quartic is real, so this acts on the quartic itself. This, along with group of 168 holomorphic automorphisms, generates a finite group of 336 diffeomorphisms, the maximum size of finite group allowed.

Acknowledgements My interest in biholomorphic self-mappings of complex manifolds, an interest which led to his article, has developed over many years of joint work with S. Krantz and K.T. Kim. P. Petersen and I have had many discussion of orbifold ideas, arising from W. Thurston's famous notes on geometrization [[4]. Those notes in fact contain the orbifold viewpoint on the regular solids expressed here, though not the Hurwitz Theorem observations. J. Ralston was so kind as to provide me with elegant write-up of the proof of Hurwitz's estimate via the Riemann-Hurwitz formula, a classical proof closely related to the view of the matter here. R. Brown read the present paper and offered helpful suggestions. No claim is expressed or implied that the contents of this article are new mathematics. The only novelty is the unification in this particular form and the geometric emphasis in the Hurwitz result.

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