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# Maximal Operator, Cotlar's Inequality and Pointwise Convergence for Singular Integral Operators in Dunkl Setting

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**Abstract**
We establish the maximal operator, Cotlar's inequality and pointwise convergence in the Dunkl setting for the (nonconvolution type) Dunkl–Calderón–Zygmund operators introduced recently in Tan et al. (https://arxiv.org/abs/2204.01886). The fundamental geometry of the Dunkl setting involves two nonequivalent metrics: the Euclidean metric and the Dunkl metric deduced by finite reflection groups, and hence the classical methods do not apply directly. The key idea is to introduce truncated singular integrals and the maximal singular integrals by the Dunkl metric and the Euclidean metric. We show that these two kind of truncated singular integrals are dominated by the Hardy–Littlewood maximal function, which yields the Cotlar's inequalities and hence the boundedness of maximal Dunkl–Calderón–Zygmund operators. Further, as applications, two equivalent pointwise convergences for Dunkl–Calderón–Zygmund operators are obtained.

**Keywords** Singular integrals · Cotlar's inequality · Maximal operator

**Mathematics Subject Classification** 42B20

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## 1 Introduction

On the Euclidean space $\mathbb{R}^N$ there is exactly one weight function $\omega(x)$ associated with a normalized root system $R$ and a multiplicity function $\kappa \geq 0$ such that the Dunkl measure is defined by

$$d\omega(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{\kappa(\alpha)} dx,$$

where $dx$ stands for the Lebesgue measure in $\mathbb{R}^N$. We denote by $\mathbf{N} = N + \sum_{\alpha \in R} \kappa(\alpha)$ the homogeneous dimension of the system and by $G$ the reflections $\sigma_\alpha \in G, \alpha \in R$. Let $E(x, y)$ be the associated Dunkl kernel, in [7] Dunkl introduced the Dunkl transform, which enjoys properties similar to the classical Fourier transform, and is defined by

$$\hat{f}(x) = c_\kappa^{-1} \int_{\mathbb{R}^N} E(x, -iy) f(y) d\omega(y),$$

$c_\kappa = \int_{\mathbb{R}^N} e^{-\|x\|^2/2} d\omega(x)$.

Particularly, the Dunkl transform satisfies the Plancherel identity, namely, $\|\hat{f}\|_2 = \|f\|_2$ and if the function $\kappa = 0$, then the Dunkl transform becomes the classical Fourier transform. In [16] the translation operator related to Dunkl transform is defined by

$$\widehat{\tau_y f}(x) = E(y, -ix) \hat{f}(x)$$

for all $x, y \in \mathbb{R}^N$. When the function $f$ is in the Schwartz class $\mathcal{S}(\mathbb{R}^N)$, the above equality holds pointwise. It is possible to define $\tau_x f$ for $L^p(\mathbb{R}^N, d\omega)$-functions, but as a distribution, see [3]. As an operator on $L^2(\mathbb{R}^N, d\omega)$, $\tau_x$ is bounded. However, it is not at all clear whether they are bounded on $L^p(\mathbb{R}^N, d\omega)$ for $p \neq 2$. For $f, g \in L^2(\mathbb{R}^N, d\omega)$, their convolution can be defined in terms of the translation operator by

$$f *_\kappa g(x) = \int_{\mathbb{R}^N} f(y) \tau_x g^\vee(y) d\omega(y),$$

where $g^\vee(y) = g(-y)$.

In the Dunkl setting, the Euclidean metric is defined by $\|x - y\| = \Big\{ \sum\limits_{j=1}^{N} |x_j - y_j|^2 \Big\}^{\frac{1}{2}}$ and the distance between two G-orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ is given by $d(x, y) = \min\limits_{\sigma \in G} \|x - \sigma(y)\|$. Obviously, $d(x, y) = d(y, x)$ and $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{R}^N$. However, $d(x, y) = 0$ when $\sigma(y)$ for $\sigma \in G$ and thus, $d(x, y)$ is not a metric. We still call $d(x, y)$ by the Dunkl metric and note that $d(x, y) \leqslant \|x - y\|$ and hence, $d(x, y)$ and $\|x - y\|$ are Not equivalent.

Consider the Dunkl setting as the Euclidean space $\mathbb{R}^N$, together with the Euclidean metric $\|x - y\|$ and the Dunkl measure $d\omega$. Then $(\mathbb{R}^N, \|\cdot\|, d\omega)$ becomes a space of homogeneous type in the sense of Coifman and Weiss (see [5, 6]), since $d\omega$ satisfies the doubling and reverse doubling properties, that is, there is a constant $C > 0$ such that for all $x \in \mathbb{R}^N$, $r > 0$, $\lambda \geqslant 1$,

$$C^{-1}\lambda^N \omega(B(x,r)) \leqslant \omega(B(x,\lambda r)) \leqslant C\lambda^{\mathbf{N}}\omega(B(x,r)). \tag{1.1}$$

Moreover, $\omega(B(x,r)) \sim \omega(B(y,r))$ when $\|x-y\| \sim r$ and $\omega(B(x,r)) \leqslant \omega(B_d(x,r)) \leqslant |G|\omega(B(x,r))$, where $B(x,r) := \{y \in \mathbb{R}^N : \|x-y\| < r\}$, $B_d(x,r) := \{y \in \mathbb{R}^N : d(x,y) < r\}$, and the notion $a \sim b, 0 < a, b < \infty$, means that there exits two constant $c_1$ and $c_2$ such that $c_1 \leqslant \frac{a}{b} \leqslant c_2$.

The Dunkl operators $T_j$ are defined by

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha \in R^+} \frac{\kappa(\alpha)}{2}\langle \alpha, e_j\rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle},$$

where $e_1, \dots, e_N$ are the standard basis of $\mathbb{R}^N$.

The Dunkl Laplacian related to $R$ and $\kappa$ is defined as $\triangle = \sum_{j=1}^{N} T_j^2$, which is equivalent to

$$\triangle f(x) = \triangle_{\mathbb{R}^N} f(x) + \sum_{\alpha \in R} \kappa(\alpha)\delta_\alpha f(x),$$

where $\delta_\alpha f(x) = \frac{\partial_\alpha f(x)}{\langle \alpha, x\rangle} - \frac{f(x)-f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}$. It is self-adjoint on $L^2(\mathbb{R}^N, d\omega)$ and generates the Dunkl heat semigroup and further the Poisson semigroup follows from the subordination formula. All these Dunkl transform, Laplacian and Poisson integral together with the Dunkl translation and convolution operators opened the door for developing the harmonic analysis related to the Dunkl setting, which includes the Littlewood–Paley theory, Hardy spaces and singular integral operators. See for example [1–3, 8–10, 16] and the references therein.

To be more precise, in [3], the Littlewood–Paley theory was established and the Hardy space $H^1(\mathbb{R}^N)$ was characterized by the area integrals, maximal function and the Riesz transforms, see also [1]. The atomic decomposition of $H^1(\mathbb{R}^N)$ was provided in [8]. The boundedness and the pointwise convergence of the Hörmander multipliers and singular integral convolution operators were given by [9] and [10], respectively.

Particularly, we would like to recall the Calderón– Zygmund singular integral convolution operators given in [10]. For a positive integer $s$, consider a kernel $K \in C^s\left(\mathbb{R}^N \backslash \{0\}\right)$ such that

$$\sup_{0<a<b<\infty}\left|\int_{a<\|x\|<b} K(x)dw(x)\right| < \infty,$$

and

$$\left|\frac{\partial^\beta}{\partial x^\beta}K(x)\right| \leq C\|x\|^{-\mathrm{N}-|\beta|} \quad \text{for all } |\beta| \leqslant s.$$

Set

$$K^{\{t\}}(x) = K(x)\left(1 - \phi\left(t^{-1}x\right)\right),$$

where $\phi$ is a fixed radial $C^{\infty}$-function supported by the unit ball $B(0, 1)$ such that $\phi(x) = 1$ for $\|x\| < 1/2$. The authors in [10] proved the following:

**Theorem A** ([10, Theorems 4.1, 4.2 ]) *Suppose that* $K(f)(x) = f * K^{\{t\}}(x)$ *with the kernel* $K(x)$ *satisfies the above conditions and the symbol* $*$ *denotes the Dunkl convolution. Then for an* $s$*, the smallest even positive integer bigger than* $\frac{N}{2}$*, then there are constants* $C_p > 0$ *independent of* $t > 0$ *such that*

$$\left\| f * K^{\{t\}} \right\|_{L^p(d\omega)} \leq C_p \|f\|_{L^p(d\omega)} \quad \textit{for } 1 < p < \infty$$

*and*

$$w\left(\left\{x \in \mathbb{R}^N : \left| f * K^{\{t\}}(x) \right| > \lambda \right\}\right) \leqslant C_1 \lambda^{-1} \|f\|_{L^1(d\omega)}.$$

*Moreover, under the additional assumption*

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < 1} K(x) d\omega(x) = L,$$

*where* $L \in \mathbb{C}$*, the limit* $\lim_{t \to 0^+} f * K^{\{t\}}(x)$ *exists and defines a bounded operator on* $L^p(\mathbb{R}^N, d\omega)$ *for* $1 < p < \infty$*, which is of weak type* $(1, 1)$*.*

*The authors introduced the maximal operator*

$$K^* f(x) = \sup_{t > 0} \left| f * K^{\{t\}}(x) \right|$$

*and provided the following estimate for the maximal operator.*

**Theorem B** ([10, Lemma 5.2 ]) *Let* $p \in [1, \infty)$*. There is a constant* $C > 0$ *such that for all* $f \in L^p(\mathbb{R}^N, d\omega) \cap L^{\infty}$ *and* $x \in \mathbb{R}^N$ *we have*

$$K^* f(x) \leqslant C\left(\sum_{\sigma \in G} M(Kf)(\sigma(x)) + \|f\|_{L^{\infty}}\right),$$

*with*

$$Mf(x) = \sup_{x \in B} \frac{1}{\omega(B)} \int_B |f(y)| d\omega(y),$$

*where the supremum is taken over all Euclidean balls* $B$ *which contain* $x$ *and* $M$ *is the noncentred Hardy–Littlewood maximal function defined on the space of homogeneous type* $\left(\mathbb{R}^N, \|\cdot\|, d\omega\right)$*.*

*As a consequence of the above theorem, the boundeness of the operator* $K^*f$ *for* $L^p(\mathbb{R}^N, d\omega)$, $1 < p < \infty$ *and the weak type* $(1, 1)$ *are obtained. See* [10] *for more details.*

Recently, a new class of the Dunkl–Calderón–Zygmund singular integral operators was introduced in [15]. We first introduce the following:

**Definition 1.1** Let $\dot{C}^\eta(\mathbb{R}^N)$ be the Hölder space of continuous functions $f$ with

$$\|f\|_{\dot{C}^\eta} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|^\eta} < \infty.$$

We denote $\dot{C}_0^\eta(\mathbb{R}^N)$ by the set of functions in the Hölder space $\dot{C}^\eta(\mathbb{R}^N)$ with compact supports.

The Dunkl–Calderón–Zygmund singular integral operators is defined by

**Definition 1.2** ([15]) An operator $T : \dot{C}_0^\eta(\mathbb{R}^N) \to (\dot{C}_0^\eta(\mathbb{R}^N))'$ with $\eta > 0$, is said to be a Dunkl–Calderón–Zygmund singular integral operator if $K(x, y)$, the kernel of $T$, satisfies the following estimates: for some $0 < \delta \leqslant 1$,

$$|K(x, y)| \leqslant C \frac{1}{\omega(B(x, d(x, y)))} \Big(\frac{d(x, y)}{\|x - y\|}\Big)^\delta \quad \text{for all} \quad x \neq y; \tag{1.2}$$

$$|K(x, y) - K(x, y')| \leqslant C\Big(\frac{\|y - y'\|}{\|x - y\|}\Big)^\delta \frac{1}{\omega(B(x, d(x, y)))} \quad \text{for} \quad \|y - y'\| \leqslant d(x, y)/2; \tag{1.3}$$

$$|K(x', y) - K(x, y)| \leqslant C\Big(\frac{\|x - x'\|}{\|x - y\|}\Big)^\delta \frac{1}{\omega(B(x, d(x, y)))} \quad \text{for} \quad \|x - x'\| \leqslant d(x, y)/2. \tag{1.4}$$

Moreover, $\langle T(f), g\rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) f(x) g(y) d\omega(x) d\omega(y)$ for $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. $T$ is said to be a Dunkl–Calderón–Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^N, d\omega)$. Here $\dot{C}_0^\eta(\mathbb{R}^N)$ is the classical Hölder space (see Definition 1.1).

We point out that in [15] it was proved that this new class Dunkl–Calderón–Zygmund singular integral operator covers the well-known Dunkl–Riesz transforms and generalizes the classical Calderón–Zygmund singular integrals on spaces of homogeneous type in the sense of Coifman and Weiss.
Thus, it is natural to ask the following:

**Question** *Does the Dunkl–Calderón–Zygmund operator* $Tf$ *exist pointwise for* $f \in L^2(\mathbb{R}^N, d\omega)$ *and for almost every* $x \in \mathbb{R}^N$?

The purpose of this paper is to give a positive answer. Let us first recall the pointwise convergence for the classical Calderón–Zygmund operator, that is, if $K(x, y)$ is the kernel of $T$, whether the following

$$T(f)(x) = \lim_{\varepsilon \to 0^+} \int_{\{y : \|x - y\| > \varepsilon\}} K(x, y) f(y) dy \tag{1.5}$$

holds for $f \in L^2(\mathbb{R}^N, d\omega)$ (or more generally, $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$) and for almost every $x \in \mathbb{R}^N$.

It is well known that in the classical case, (1.5) is proved via the remarkable Cotlar's inequality. See [4] for the classical singular integral convolution operators and [12] for the generalized singular integral operators. See also [13] for more general theory for maximal operators.

We now return to our question in the Dunkl setting. Suppose that, as in the Definition 1.2, $T$ is a Dunkl–Calderón–Zygmund operator with the kernel $K(x, y)$ involving the different metrics, the Euclidean metric $\|x - y\|$ and the Dunkl metric $d(x, y)$. As in the classical case, the truncated kernels can be defined for each $\varepsilon > 0$,

$$K_\varepsilon(x, y) = \begin{cases} K(x, y), & \text{when } \|x - y\| > \varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{1.6}$$

The truncated operators $T_\varepsilon$ are defined by

$$T_\varepsilon f(x) = \int_{\mathbb{R}^N} K_\varepsilon(x, y) f(y) d\omega(y) \tag{1.7}$$

and the maximal operators are defined by

$$T_* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)|. \tag{1.8}$$

However, Cotlar's inequality for $T_* f(x)$ does not follow from the classical method since the kernel of $T$ involves the Dunkl metric $d(x, y)$, which causes a difficulty for estimating $T_* f(x)$.

To overcome this problem, we introduce the truncated kernels

$$\widetilde{K}_\varepsilon(x, y) = \begin{cases} K(x, y), & \text{when } d(x, y) > \varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{1.9}$$

and the truncated operators $\widetilde{T}_\varepsilon$ are defined by

$$\widetilde{T}_\varepsilon f(x) = \int_{\mathbb{R}^N} \widetilde{K}_\varepsilon(x, y) f(y) d\omega(y)$$

for $x \in \mathbb{R}^N$. The corresponding maximal operators are defined by

$$\widetilde{T}_* f(x) = \sup_{\varepsilon > 0} |\widetilde{T}_\varepsilon(f)(x)|.$$

The relationship between $T_* f(x)$ and $\widetilde{T}_* f(x)$ gives Cotlar's inequalities for both $T_* f(x)$ and $\widetilde{T}_* f(x)$, which are given by the following:

**Theorem 1.3** *Suppose that $T$ is a Dunkl–Calderón–Zygmund operator as in Definition 1.2. Then for any $r > 0$,*

$$T_* f(x), \widetilde{T}_*(f)(x) \leqslant C_r \big\{ M\big(|T(f)|^r\big)(x)^{1/r} + \widetilde{M} f(x) \big\},$$

*where $C_r$ is a constant depending on $r$ but not on $x$, $Mf(x)$ is the Hardy–Littlewood maximal function, i.e. $Mf(x) = \sup\limits_{t>0} \frac{1}{\omega(B(x,t))} \int_{B(x,t)} |f(y)| d\omega(y)$, and $\widetilde{M} f(x) = \sum\limits_{\sigma \in G} Mf(\sigma(x))$.*

**Theorem 1.4** *Suppose that $T$ is a Dunkl–Calderón–Zygmund operator with the kernel $K(x, y)$ as in Definition 1.2. Then $\lim\limits_{\varepsilon \to 0^+} [T_\varepsilon(f)(x) - \widetilde{T}_\varepsilon(f)(x)] = 0$, for all $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$ and almost all $x \in \mathbb{R}^N$.*

*Furthermore, under an additional condition $\int_{\{y:\, \varepsilon < d(x,y) < M\}} K(x, y) d\omega(y) = 0$ for all $0 < \varepsilon < M < \infty$, we have*

$$\begin{aligned} T(f)(x) &= \lim_{\varepsilon \to 0^+} \int_{\{y:\, \|x-y\| > \varepsilon\}} K(x, y) f(y) d\omega(y) \\ &= \lim_{\varepsilon \to 0^+} \int_{\{y:\, d(x,y) > \varepsilon\}} K(x, y) f(y) d\omega(y) \end{aligned}$$

*for $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$ and almost all $x \in \mathbb{R}^N$.*

The paper is organized as follows. In the next section, we recall the preliminaries for the Dunkl–Calderón–Zygmund singular integral operators. Cotlar's inequality and the pointwise convergence will be given in Sects. 3 and 4, respectively.

## 2 Preliminaries: Dunkl–Calderón–Zygmund Operators

We first remark that the size and regularity conditions of the Dunkl–Calderón–Zygmund singular integral operator as in Definition 1.2 are much weaker than the classical Calderón–Zygmund singular integral operators given in space of homogeneous type in the sense of Coifman and Weiss. Let recall these conditions by the following: for some $0 < \delta \leqslant 1$,

(i) $|K(x, y)| \leqslant \dfrac{C}{\omega(B(x, \|x - y\|))}$ for $x \neq y$;

(ii) $|K(x, y) - K(x, y')| \leqslant \Big(\dfrac{\|y - y'\|}{\|x - y\|}\Big)^\delta \dfrac{C}{\omega(B(x, \|x - y\|))}$ for $\|y - y'\| \leqslant \frac{1}{2}\|x - y\|$;

(iii) $|K(x, y) - K(x', y)| \leqslant \Big(\dfrac{\|x - x'\|}{\|x - y\|}\Big)^\delta \dfrac{C}{\omega(B(x, \|x - y\|))}$ for $\|x - x'\| \leqslant \frac{1}{2}\|x - y\|$.

By the reverse doubling condition in (1.1) on the measure $d\omega$,

$$\omega(B(x,\|x-y\|))=\omega\Big(B\Big(x,\frac{\|x-y\|}{d(x,y)}\cdot d(x,y)\Big)\Big)\geqslant C\Big(\frac{\|x-y\|}{d(x,y)}\Big)^{N}\omega(B(x,d(x,y))).$$

Thus,

$$\frac{1}{\omega(B(x,\|x-y\|))}\leqslant C\Big(\frac{d(x,y)}{\|x-y\|}\Big)^{N}\frac{1}{\omega(B(x,d(x,y)))}\leqslant C\Big(\frac{d(x,y)}{\|x-y\|}\Big)^{\delta}\frac{1}{\omega(B(x,d(x,y)))}$$

and note that $d(x,y)\leqslant\|x-y\|$, thus, $\omega(B(x,\|x-y\|))\geqslant\omega(B(x,d(x,y)))$. If $\|x-x'\|\leqslant\frac{1}{2}d(x,y)\leqslant\frac{1}{2}\|x-y\|$, then,

$$\Big(\frac{\|x-x'\|}{\|x-y\|}\Big)^{\varepsilon}\frac{1}{\omega(B(x,\|x-y\|)}\leqslant C\Big(\frac{\|x-x'\|}{\|x-y\|}\Big)^{\varepsilon}\frac{1}{\omega(B(x,d(x,y)))}.$$

Further, $K(x,y)$ is locally integrable for $x\neq y$. Indeed, for any fixed $x\in\mathbb{R}^N$ and $0<\varepsilon<R<\infty$, by the doubling properties in (1.1) of the measure $d\omega$,

$$\int_{\varepsilon<\|x-y\|<R}|K(x,y)|d\omega(y)\leqslant C\frac{1}{\varepsilon^{\delta}}\int_{d(x,y)<R}\frac{d(x,y)^{\delta}}{\omega(B(x,d(x,y)))}d\omega(y)\leqslant C\Big(\frac{R}{\varepsilon}\Big)^{\delta}<\infty.$$

To recall results in [15], we need to extend the definition of the Dunkl–Calderón–Zygmund operators to functions in $\dot{C}_b^{\eta}(\mathbb{R}^N)$, the bounded Hölder functions. The idea for doing this is to define $T(f)$ for $f\in\dot{C}_b^{\eta}(\mathbb{R}^N)$ as a distribution on $\dot{C}_{0,0}^{\eta}(\mathbb{R}^N)=\{g\in\dot{C}_0^{\eta}:\int_{\mathbb{R}^N}g(x)d\omega(x)=0\}$. To this end, given $g\in\dot{C}_{0,0}^{\eta}(\mathbb{R}^N)$ with the support contained in the ball $B(x_0,R)$ for some $x_0\in\mathbb{R}^N$ and $R>0$. Let $\xi(x)=1$ for $x\in B_d(x_0,2R)$ and $\xi(x)=0$ for $x\in\big(B_d(x_0,4R)\big)^c$. Write $f(x)=\xi(x)f(x)+(1-\xi(x))f(x)$ and formally, $\langle Tf,g\rangle=\langle T(f\xi),g\rangle+\langle T[(1-\xi)f],g\rangle$. The first term $\langle T(f\xi),g\rangle$ is well defined. By the cancellation condition of $g$, we can write

$$\langle T[(1-\xi)f],g\rangle=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[K(x,y)-K(x_0,y)](1-\xi(y))f(y)g(x)d\omega(y)d\omega(x).$$

Observe that if $x\in B(x_0,R)$ and $y\notin B_d(x_0,2R)$,then $\|x-x_0\|\leqslant R\leqslant\frac{1}{2}d(x_0,y)$. Thus,

$$\int_{\mathbb{R}^N}\int_{\{y:d(x_0,y)\geqslant 2R\}}|K(x,y)-K(x_0,y)||g(x)|d\omega(y)d\omega(x)\leqslant C\|g\|_1$$

and hence, $\langle Tf, g\rangle$ is well defined. Therefore, $T(f)$ is a distribution on $\left(\dot{C}^{\eta}_{0,0}(\mathbb{R}^N)\right)'$.

The *weak boundedness property* (WBP) in the Dunkl setting is defined by the following:

**Definition 2.1** The Dunkl–Calderón–Zygmund singular integral operator $T$ with the distribution kernel $K(x, y)$ is said to have the weak boundedness property (WBP) if there exist $\eta > 0$ and $C < \infty$ such that

$$|\langle K, f\rangle| \leqslant C \max\{\omega(B(x_0, r)), \omega(B(y_0, r))\}$$

for all $f \in \dot{C}^{\eta}_0(\mathbb{R}^N \times \mathbb{R}^N)$ with $\mathrm{supp}(f) \subseteq B(x_0, r) \times B(y_0, r), x_0, y_0 \in \mathbb{R}^N$, $\|f\|_{L^\infty(\mathbb{R}^N)} \leqslant 1$, $\|f(\cdot, y)\|_{\dot{C}^\eta(\mathbb{R}^N)} \leqslant r^{-\eta}$ for all $y \in \mathbb{R}^N$ and $\|f(x, \cdot)\|_{\dot{C}^\eta(\mathbb{R}^N)} \leqslant r^{-\eta}$ for all $x \in \mathbb{R}^N$.

The following $T(1)$ theorem for the Dunkl–Calderón–Zygmund singular integral operators was provided in [15].

**Theorem 2.2** *Suppose that $T$ is a Dunkl–Calderón–Zygmund singular integral operator. Then $T$ extends to a bounded operator on $L^2(\mathbb{R}^N, d\omega)$ if and only if* (a) $T(1) \in BMO(\mathbb{R}^N, d\omega)$; (b) $T^*(1) \in BMO(\mathbb{R}^N, d\omega)$; (c) *$T$ has WBP.*

In [15], they also show the following:

**Theorem 2.3** *Suppose $T$ is a Dunkl–Calderón–Zygmund operator. Then $T$ extends to a bounded operator from $L^p(\mathbb{R}^N, d\omega)$, $1 < p < \infty$, to itself. Moreover, there exists a constant $C$ such that*

$$\|Tf\|_{L^p(\mathbb{R}^N, d\omega)} \leqslant C\|f\|_{L^p(\mathbb{R}^N, d\omega)}.$$

We remark that applying the $L^2$-boundeness of $T$ and the Calderón–Zygmund decomposition on space of homogeneous type $(\mathbb{R}^N, \|x - y\|, d\omega)$ as in [9, 10], the weak type (1,1) estimate of Theorem 2.3 also holds. See [9, 10] for details.

## 3 Proof of Cotlar's Inequality

***Proof*** We need to show that if $f \in L^2(\mathbb{R}^N, d\omega)$ and for any fixed $\varepsilon > 0$, then $\int_{\{y:\|x-y\|\geqslant\varepsilon\}} K(x, y)f(y)d\omega(y)$ converges absolutely for almost all $x \in \mathbb{R}^N$, where $K(x, y)$ is the kernel of the Dunkl–Calderón–Zygmund operator $T$. Instead of showing this, we would like to first prove that $\int_{\{y:d(x,y)\geqslant\varepsilon\}} K(x, y)f(y)d\omega(y)$ converges absolutely for almost all $x \in \mathbb{R}^N$ and for any fixed $\varepsilon$. Indeed, for almost all $x \in \mathbb{R}^N$ and any fixed $\varepsilon > 0$,

$$\begin{aligned}\int_{\{y:d(x,y)\geqslant\varepsilon\}} |K(x, y)|^2 d\omega(y) &\lesssim \sum_{j=0}^{\infty} \int_{\{y:2^j\varepsilon\leqslant d(x,y)\leqslant 2^{j+1}\varepsilon\}} \frac{1}{\omega(B(x, d(x, y)))^2} d\omega(y)\\ &\lesssim \sum_{j=0}^{\infty} \int_{\{y:2^j\varepsilon\leqslant d(x,y)\leqslant 2^{j+1}\varepsilon\}} \frac{1}{\omega(B(x, 2^j\varepsilon))^2} d\omega(y)\end{aligned}$$

$$\lesssim \sum_{j=0}^{\infty} \frac{\omega(B(x, 2^{j+1}\varepsilon)}{\omega(B(x, 2^j\varepsilon))^2} d\omega(y)$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-jN} \frac{1}{\omega(B(x, \varepsilon))} < \infty,$$

where the last two inequalities follow from the doubling property in (1.1) and fact that $\inf_{x\in\mathbb{R}^N} \omega(B(x, \varepsilon)) > 0$, respectively. The notation $a \lesssim b$ means that there exists a constant C such that $a \leqslant Cb$.

Observe that the above estimate does not work for $\int_{\{y:\|x-y\|\geqslant\varepsilon\}} |K(x, y)|^2 d\omega(y)$.

We now show the following relationship between truncated operators $T_\varepsilon(f)(x)$ and $\widetilde{T}_\varepsilon(f)(x)$, which is one of the main reasons why we introduce the truncated operator $\widetilde{T}_\varepsilon(f)(x)$.

**Lemma 3.1** *Suppose that the kernel $K(x, y)$ satisfies the following size condition*

$$|K(x, y)| \leqslant C \frac{1}{\omega(B(x, d(x, y)))} \Big(\frac{d(x, y)}{\|x-y\|}\Big)^{\delta}, \quad 0 < \delta \leqslant 1.$$

*Then*

$$|T_\varepsilon(f)(x) - \widetilde{T}_\varepsilon(f)(x)| \leqslant C\widetilde{M}(f)(x).$$

Indeed, $\{y : \|x-y\| > \varepsilon\} = \{y : d(x, y) > \varepsilon\} \cup \{y : \|x-y\| > \varepsilon \geqslant d(x, y)\}$ and hence,

$$\int_{\{y:\ \|x-y\|>\varepsilon\}} K(x, y) f(y) d\omega(y)$$
$$= \int_{\{y:\ d(x,y)>\varepsilon\}} K(x, y) f(y) d\omega(y) + \int_{\{y:\ \|x-y\|>\varepsilon\geqslant d(x,y)\}} K(x, y) f(y) d\omega(y).$$

We estimate $\int_{\{y:\ \|x-y\|>\varepsilon\geqslant d(x,y)\}} K(x, y) f(y) d\omega(y)$ as follows:

$$\Big|\int_{\{y:\ \|x-y\|>\varepsilon\geqslant d(x,y)\}} K(x, y) f(y) d\omega(y)\Big|$$
$$\lesssim \frac{1}{\varepsilon^\delta} \int_{\{y:\ \varepsilon\geqslant d(x,y)\}} \frac{d(x, y)^\delta}{\omega(B(x, d(x, y)))} |f(y)| d\omega(y)$$
$$= \frac{1}{\varepsilon^\delta} \sum_{k=1}^{\infty} \int_{\{y:\ 2^{-k}\varepsilon\leqslant d(x,y)<2^{-k+1}\varepsilon\}} \frac{d(x, y)^\delta}{\omega(B(x, d(x, y)))} |f(y)| d\omega(y)$$
$$\lesssim \frac{1}{\varepsilon^\delta} \sum_{k=1}^{\infty} \int_{\{y: d(x,y)<2^{-k+1}\varepsilon\}} \frac{(2^{-k}\varepsilon)^\delta}{\omega(B(x, 2^{-k}\varepsilon))} |f(y)| d\omega(y)$$
$$\lesssim \widetilde{M} f(x),$$

where the last inequality follows from the doubling condition in (1.1) and the fact that for any $t > 0$,

$$\begin{aligned}
&\frac{1}{\omega(B(x,t))}\int_{\{y:d(x,y)\leqslant t\}}|f(y)|d\omega(y)\\
&\leqslant \sum_{\sigma\in G}\frac{1}{\omega(B(\sigma(x),t))}\int_{\{y:\|\sigma(x)-y\|\leqslant t\}}|f(y)|d\omega(y)\\
&\leqslant \sum_{\sigma\in G} Mf(\sigma(x))\\
&= \widetilde{M}f(x).
\end{aligned}$$

As a direct consequence of the above estimate, we obtain that

$$T_*(f)(x) \leqslant \widetilde{T}_*(f)(x) + C\widetilde{M}f(x)$$

and

$$\widetilde{T}_*(f)(x) \leqslant T_*(f)(x) + C\widetilde{M}f(x),$$

where $C$ is a constant. Therefore, we just need to show Cotlar's inequalities for $\widetilde{T}_* f$ only.

Let us fix an $\bar{x} \in \mathbb{R}^N$ and $\varepsilon > 0$ and write $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)$ for $d(x,\bar{x}) \leqslant \varepsilon$ and $f_2(x) = f(x)$ when $d(x,\bar{x}) > \varepsilon$.

First we show that $|Tf_2(x) - Tf_2(\bar{x})| \leqslant C\widetilde{M}f(\bar{x})$, whenever $\|x-\bar{x}\| < \frac{\varepsilon}{2}$. Observe that if $\|x-\bar{x}\| < \frac{\varepsilon}{2}$ then the smoothness condition (1.4) in Definition 1.2 on the kernel $K(x,y)$ yields

$$\begin{aligned}
|Tf_2(x) - Tf_2(\bar{x})| &\leqslant \int_{\{y:d(\bar{x},y)>\varepsilon\}}|K(x,y)-K(\bar{x},y)||f(y)|d\omega(y)\\
&\lesssim \int_{\{y:d(\bar{x},y)>\varepsilon\}}\left(\frac{\|x-\bar{x}\|}{\|\bar{x}-y\|}\right)^{\delta}\frac{1}{\omega(B(\bar{x},d(\bar{x},y)))}\cdot|f(y)|d\omega(y).
\end{aligned}$$

We split the range of integration into the dyadic shells $\{y : 2^{k+1}\varepsilon \geqslant d(\bar{x},y) > 2^k\varepsilon\}$, $k \in \mathbb{N}$. It carries out the estimate of the last term about by the following:

$$\begin{aligned}
|Tf_2(x) - Tf_2(\bar{x})| &\lesssim \sum_{k=0}^{\infty}\int_{\{y:2^{k+1}\varepsilon\geqslant d(\bar{x},y)>2^k\varepsilon\}}\left(\frac{\|x-\bar{x}\|}{\|\bar{x}-y\|}\right)^{\delta}\frac{1}{\omega(B(\bar{x},d(\bar{x},y)))}|f(y)|d\omega(y)\\
&\lesssim \sum_{k=0}^{\infty}\int_{\{y:d(\bar{x},y)\leqslant 2^{k+1}\varepsilon\}}\left(\frac{1}{2^{k+1}}\right)^{\delta}\frac{1}{\omega(B(\bar{x},2^k\varepsilon))}|f(y)|d\omega(y)\\
&\lesssim \sum_{\sigma\in G}\sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}}\right)^{\delta}\frac{1}{\omega(B(\sigma(\bar{x}),2^{k+1}\varepsilon))}\int_{\{y:\|\sigma(\bar{x})-y\|\leqslant 2^{k+1}\varepsilon\}}|f(y)|d\omega(y)\\
&\leqslant \sum_{\sigma\in G} Mf(\sigma(\bar{x}))
\end{aligned}$$

$$= \widetilde{M} f(\bar{x}).$$

Therefore

$$|\widetilde{T}_\varepsilon f(\bar{x})| = |Tf_2(\bar{x})| \leqslant |Tf_2(x)| + C \cdot \widetilde{M} f(\bar{x}) \leqslant |Tf(x)| + |Tf_1(x)| \\ +C \cdot \widetilde{M} f(\bar{x}), \tag{3.1}$$

whenever $\|x - \bar{x}\| < \frac{\varepsilon}{2}$.

Now for any $\alpha > 0$ and $r > 0$, we have

$$\omega\big(\big\{x \in B(\bar{x}, \frac{\varepsilon}{2}) : |Tf(x)| > \alpha\big\}\big) \leqslant \alpha^{-r} \int_{B(\bar{x}, \frac{\varepsilon}{2})} |Tf(x)|^r d\omega(x) \\ \leqslant \alpha^{-r} \omega(B(\bar{x}, \frac{\varepsilon}{2})) \cdot M\big(|Tf|^r\big)(\bar{x}).$$

And by the week type (1,1) estimate of $T$ we have

$$\omega\big(\big\{x \in B(\bar{x}, \frac{\varepsilon}{2}) : |Tf_1(x)| > \alpha\big\}\big) \lesssim \alpha^{-1} \int_{\mathbb{R}^N} |f_1(x)| d\omega(x) \\ = \alpha^{-1} \int_{\{x:\, d(\bar{x},x) \leqslant \varepsilon\}} |f(x)| d\omega(x) \\ \lesssim \alpha^{-1} \omega(B(\bar{x}, \frac{\varepsilon}{2})) \cdot \widetilde{M} f(\bar{x}).$$

Let $\alpha = C_0\big\{M\big(|Tf|^r\big)(\bar{x})^{1/r} + \widetilde{M} f(\bar{x})\big\}$, where $C_0$ is a large constant such that $\omega\big(\big\{x \in B(\bar{x}, \frac{\varepsilon}{2}) : |Tf(x)| > \alpha\big\}\big) \leqslant \frac{1}{4}\omega(B(\bar{x}, \frac{\varepsilon}{2}))$ and $\omega\big(\big\{x \in B(\bar{x}, \frac{\varepsilon}{2}) : |Tf_1(x)| > \alpha\big\}\big) \leqslant \frac{1}{4}\omega(B(\bar{x}, \frac{\varepsilon}{2}))$.

As a consequence there exists an $x \in B\Big(\bar{x}, \dfrac{\varepsilon}{2}\Big)$ so that $|Tf(x)| \leqslant \alpha$ and $|Tf_1(x)| \leqslant \alpha$. Hence by (3.1), we have

$$|\widetilde{T}_\varepsilon f(\bar{x})| \leqslant 2\alpha + C \cdot \widetilde{M} f(\bar{x}) \leqslant (2C_0 + C) \cdot \big\{M\big(|Tf|^r\big)(\bar{x})^{1/r} + \widetilde{M} f(\bar{x})\big\}.$$

The proof of the Theorem 1.3 is complete. □

As a direct consequence of Theorem 1.3 and Theorem 2.3, we obtain the following:

**Corollary 3.2** *Suppose that $T$ is a Dunkl–Calderón–Zygmund operator. Then the maximal operator $T_*$ ( and $\widetilde{T}_*$) is bounded on $L^p(\mathbb{R}^N, d\omega)$ and is of the weak type $(1, 1)$. Moreover, there exists a constant $C$ such that*

$$\|T_* f\|_{L^p(\mathbb{R}^N, d\omega)} \leqslant C\|f\|_{L^p(\mathbb{R}^N, d\omega)}$$

*for $1 < p < \infty$ and*

$$\omega\big\{x \in \mathbb{R}^N : |T_* f(x)| > \alpha\big\} \lesssim \alpha^{-1} \int_{\mathbb{R}^N} |f(x)| d\omega(x)$$

*for all $\alpha > 0$.*

## 4 Pointwise Convergence of Truncated Operators

We first show the following boundedness of the truncated operator without using the smoothness conditions on the kernel. See a similar result in [14].

**Theorem 4.1** *Suppose that the operator $T$ with the kernel $K(x, y)$ is bounded on $L^p(\mathbb{R}^N, d\omega)$ for some $1 < p < \infty$, i.e. $\|T(f)\|_{L^p(\mathbb{R}^N,d\omega)} \leqslant C\|f\|_{L^p(\mathbb{R}^N,d\omega)}$. Moreover, $K(x, y)$ satisfies the following size condition only:*

$$|K(x, y)| \leqslant C\frac{1}{\omega(B(x, d(x, y)))}\left(\frac{d(x, y)}{\|x-y\|}\right)^{\delta} \quad \textit{for all} \quad x \neq y, \ \textit{and some } 0 < \delta \leqslant 1,$$

*and*

$$Tf(x) = \int_{\mathbb{R}^N} K(x, y)f(y)d\omega(y),$$

*for a.e. $x$ outside the support of $f$.*

*Then there exists a constant $C'$ such that*

$$\|T_\varepsilon(f)\|_{L^p(\mathbb{R}^N,d\omega)} \leqslant C'\|f\|_{L^p(\mathbb{R}^N,d\omega)}$$

*and*

$$\|\widetilde{T}_\varepsilon(f)\|_{L^p(\mathbb{R}^N,d\omega)} \leqslant C'\|f\|_{L^p(\mathbb{R}^N,d\omega)},$$

*where $C'$ is independent of $\varepsilon$.*

***Proof*** According to the proof of Theorem 1.3, $|T_\varepsilon(f)(x)-\widetilde{T}_\varepsilon(f)(x)| \leqslant C\widetilde{M}f(x)$, we just need to show $\|\widetilde{T}_\varepsilon(f)\|_{L^p(\mathbb{R}^N,d\omega)} \leqslant C'\|f\|_{L^p(\mathbb{R}^N,d\omega)}$ only. Let the collections of the balls $\{B(\bar{x}_k, \frac{1}{4}\varepsilon)\}$ satisfy $\bigcup_{k=1}^{\infty} B(\bar{x}_k, \frac{1}{4}\varepsilon) = \mathbb{R}^N$, and $\{B(\bar{x}_k, 2\varepsilon)\}$ have the bounded overlapping property: There exists an integer $M$, such that no point in $\mathbb{R}^N$ belongs to more than $M$ of $B(\bar{x}_k, 2\varepsilon)$. Let $\chi_{k,\delta}$ be the characteristic function of $B_d(\bar{x}_k, \delta)$. It is easy to see that $\sum_{k=1}^{\infty} \chi_{k,2\varepsilon}(x) \leqslant |G| \cdot M$, for all $x \in \mathbb{R}^N$. By writing $\widetilde{\widetilde{T}}_\varepsilon = T - \widetilde{T}_\varepsilon$, we just need to show that $\|\widetilde{\widetilde{T}}_\varepsilon(f)\|_{L^p(\mathbb{R}^N,d\omega)} \leqslant C'\|f\|_{L^p(\mathbb{R}^N,d\omega)}$. Observe that

$$\begin{aligned}
\|\widetilde{\widetilde{T}}_\varepsilon(f)\|^p_{L^p(\mathbb{R}^N,d\omega)} &\lesssim \sum_{k=1}^{\infty} \int_{B_d(\bar{x}_k, \frac{1}{4}\varepsilon)} |\widetilde{\widetilde{T}}_\varepsilon(f)(x)|^p d\omega(x) \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}^N} |\chi_{k,\frac{1}{4}\varepsilon}(x) \cdot \widetilde{\widetilde{T}}_\varepsilon(f)(x)|^p d\omega(x).
\end{aligned}$$

And $\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon(f)(x) = \chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon(\chi_{k,2\varepsilon}\cdot f)(x)$, since $\widetilde{\widetilde{T}}_\varepsilon[(1-\chi_{k,2\varepsilon})f](x) = 0$, for all $x \in B_d(\bar{x}_k, \frac{1}{4}\varepsilon)$.

Now we write

$$\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon(\chi_{k,2\varepsilon}\cdot f)(x)=\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon(\chi_{k,\frac{1}{2}\varepsilon}\cdot f)(x)$$
$$+\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon[(\chi_{k,2\varepsilon}-\chi_{k,\frac{1}{2}\varepsilon})\cdot f](x).$$

Note that $\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon(\chi_{k,\frac{1}{2}\varepsilon}\cdot f)(x)=\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot T(\chi_{k,\frac{1}{2}\varepsilon}\cdot f)(x)$, since $\widetilde{T}_\varepsilon(\chi_{k,\frac{1}{2}\varepsilon}\cdot f)(x)=0$ for all $x\in B_d(\bar{x}_k,\frac{1}{4}\varepsilon)$. Hence

$$\begin{aligned}\int_{\mathbb{R}^N}|\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon(\chi_{k,\frac{1}{2}\varepsilon}\cdot f)(x)|^p d\omega(x)&=\int_{\mathbb{R}^N}|\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot T(\chi_{k,\frac{1}{2}\varepsilon}\cdot f)(x)|^p d\omega(x)\\&\leqslant C\|\chi_{k,\frac{1}{2}\varepsilon}\cdot f\|^p_{L^p(\mathbb{R}^N,d\omega)}\\&\leqslant C\|\chi_{k,2\varepsilon}\cdot f\|^p_{L^p(\mathbb{R}^N,d\omega)}.\end{aligned}$$

And

$$\begin{aligned}&\int_{\mathbb{R}^N}|\chi_{k,\frac{1}{4}\varepsilon}(x)\cdot\widetilde{\widetilde{T}}_\varepsilon[(\chi_{k,2\varepsilon}-\chi_{k,\frac{1}{2}\varepsilon})\cdot f](x)|^p d\omega(x)\\&=\int_{B_d(\bar{x}_k,\frac{1}{4}\varepsilon)}\left|\int_{\{y:\ \frac{1}{4}\varepsilon\leqslant d(x,y)<\varepsilon\}}K(x,y)(\chi_{k,2\varepsilon}(y)-\chi_{k,\frac{1}{2}\varepsilon}(y))\cdot f(y)d\omega(y)\right|^p d\omega(x)\\&\lesssim\int_{B_d(\bar{x}_k,\frac{1}{4}\varepsilon)}\left|\int_{\mathbb{R}^N}\frac{1}{\omega(B(x,\frac{1}{4}\varepsilon))}\chi_{k,2\varepsilon}(y)\cdot|f(y)|d\omega(y)\right|^p d\omega(x)\\&\lesssim\int_{B_d(\bar{x}_k,\frac{1}{4}\varepsilon)}\left|\int_{\mathbb{R}^N}\frac{1}{\omega(B(\bar{x}_k,\frac{1}{4}\varepsilon))}\chi_{k,2\varepsilon}(y)\cdot|f(y)|d\omega(y)\right|^p d\omega(x)\\&=\frac{\omega(B_d(\bar{x}_k,\frac{1}{4}\varepsilon))}{\omega(B(\bar{x}_k,\frac{1}{4}\varepsilon))^p}\|\chi_{k,2\varepsilon}\cdot f\|^p_{L^1(\mathbb{R}^N,d\omega)}\\&\lesssim\frac{1}{\omega(B(\bar{x}_k,\frac{1}{4}\varepsilon))^{p-1}}\|\chi_{k,2\varepsilon}\cdot f\|^p_{L^p(\mathbb{R}^N,d\omega)}\cdot\|\chi_{k,2\varepsilon}\|^p_{L^q(\mathbb{R}^N,d\omega)}\quad\left(\text{here }\frac{1}{p}+\frac{1}{q}=1\right)\\&\lesssim\|\chi_{k,2\varepsilon}\cdot f\|^p_{L^p(\mathbb{R}^N,d\omega)}.\end{aligned}$$

Based on the above results, we have

$$\begin{aligned}\|\widetilde{\widetilde{T}}_\varepsilon(f)\|^p_{L^p(\mathbb{R}^N,d\omega)}&\lesssim\sum_{k=1}^{\infty}\|\chi_{k,2\varepsilon}\cdot f\|^p_{L^p(\mathbb{R}^N,d\omega)}\\&=\sum_{k=1}^{\infty}\int_{\mathbb{R}^N}\chi_{k,2\varepsilon}(x)|f(x)|^p d\omega(x)\\&\lesssim\int_{\mathbb{R}^N}|f(x)|^p d\omega(x).\end{aligned}$$

The proof of Theorem 4.1 is complete. □

Now we show the Theorem 1.4.

***Proof of Theorem 1.4*** We first show that

$$T_\varepsilon f(x) - \widetilde{T}_\varepsilon f(x) = \int_{\{y:\ \|x-y\|>\varepsilon\geqslant d(x,y)\}} K(x,y)f(y)d\omega(y) \to 0,$$

as $\varepsilon \to 0^+$, for $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$, and almost every $x \in \mathbb{R}^N$.

Indeed, if we let the set $E = \{x | \langle x, \alpha\rangle = 0, \text{ for some } \alpha \in R\}$, then it is easy to see that $\omega(E) = 0$. Now for any $x \in \mathbb{R}^N \backslash E$, we let

$$d_x = \inf_{\substack{\sigma\in G\\ \sigma\neq id}} \|x - \sigma(x)\|,$$

then $d_x > 0$. Note that if $\varepsilon < d_x/2$ and $\|x-y\| > \varepsilon > d(x,y)$, then $\|x-y\| \geqslant d_x/2$. Applying the size condition of $K(x,y)$ implies that

$$\begin{aligned}
&\int_{\{y:\ \|x-y\|>\varepsilon\geqslant d(x,y)\}} |K(x,y)f(y)|d\omega(y)\\
&\lesssim \int_{\{y:\ \varepsilon\geqslant d(x,y)\}} \left(\frac{d(x,y)}{d_x}\right)^\delta \frac{1}{\omega(B(x,d(x,y)))}|f(y)|d\omega(y)\\
&= \sum_{k=1}^\infty \int_{\{y:\ 2^{-k}\varepsilon<d(x,y)\leqslant 2^{-k+1}\varepsilon\}} \left(\frac{d(x,y)}{d_x}\right)^\delta \frac{1}{\omega(B(x,d(x,y)))}|f(y)|d\omega(y)\\
&\lesssim \sum_{k=1}^\infty \int_{\{y:\ d(x,y)\leqslant 2^{-k+1}\varepsilon\}} \left(\frac{2^{-k}\varepsilon}{d_x}\right)^\delta \frac{1}{\omega(B(x,2^{-k}\varepsilon))}|f(y)|d\omega(y)\\
&\lesssim \sum_{k=1}^\infty \left(\frac{2^{-k}\varepsilon}{d_x}\right)^\delta \widetilde{M}f(x)\\
&\lesssim \left(\frac{\varepsilon}{d_x}\right)^\delta \widetilde{M}f(x) \to 0, \text{ as } \varepsilon \to 0^+, \text{ for almost every } x \in \mathbb{R}^N.
\end{aligned}$$

Therefore $\lim_{\varepsilon\to 0^+} \int_{\{y:\ \|x-y\|>\varepsilon\geqslant d(x,y)\}} K(x,y)f(y)d\omega(y) = 0$, for almost every $x \in \mathbb{R}^N$.

Now under the additional assumption $\int_{\{y:\ \varepsilon<d(x,y)<M\}} K(x,y)d\omega(y) = 0$, for all $0 < \varepsilon < M$ and $x \in \mathbb{R}^N$, we will show that $\lim_{\varepsilon\to 0^+} \int_{\{y:d(x,y)>\varepsilon\}} K(x,y)f(y)d\omega(y)$ exists for $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$ and almost all $x \in \mathbb{R}^N$.

First we claim that for each $C_0^1(\mathbb{R}^N)$ function $f$ with compact support,

$$\lim_{\varepsilon\to 0^+} \int_{\{y:d(x,y)>\varepsilon\}} K(x,y)f(y)d\omega(y)$$

exists for all $x \in \mathbb{R}^N$. Indeed, the integral $\int_{\{y:\ d(x,y)>\varepsilon\}} K(x,y)f(y)d\omega(y)$ can be written as the sum of $\int_{\{y:\ \varepsilon<d(x,y)<M\}} K(x,y)[f(y) - f(x)]d\omega(y)$ and

$\int_{\{y:\, d(x,y)\geqslant M\}} K(x,y)f(y)d\omega(y)$. Obviously, the second integral converges absolutely for any fixed $M > \varepsilon$. By the size condition (1.2) in Definition 1.2 on the kernel $K(x,y)$ and the smoothness condition on the function $f$, the first integral is also converges absolutely. This is because

$$\begin{aligned}
&\int_{\{y:\, \varepsilon<d(x,y)<M\}} |K(x,y)[f(y)-f(x)]|d\omega(y)\\
&\lesssim \int_{\substack{\varepsilon<d(x,y)<M\\ \|x-y\|\geqslant 1}} \frac{1}{\omega(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^{\delta} d\omega(y)\\
&\quad+\int_{\substack{\varepsilon<d(x,y)<M\\ \|x-y\|<1}} \frac{1}{\omega(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^{\delta} \|x-y\|d\omega(y)\\
&\lesssim \int_{d(x,y)<M} \frac{d(x,y)^{\delta}}{\omega(B(x,d(x,y)))}d\omega(y) < \infty.
\end{aligned}$$

Now we show $\lim_{\varepsilon\to 0^+}\int_{\{y:\, d(x,y)>\varepsilon\}} K(x,y)f(y)d\omega(y)$ exists for $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$ and almost all $x \in \mathbb{R}^N$. To this end, recall $\widetilde{T}_\varepsilon(f)(x) = \int_{\{y:\, d(x,y)>\varepsilon\}} K(x,y)f(y)d\omega(y)$ and

$$\Omega(f;x) = \lim_{\varepsilon\to 0^+}\left(\sup_{0<t<s<\varepsilon} |\widetilde{T}_t(f)(x) - \widetilde{T}_s(f)(x)|\right),$$

for any $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$.

Observe that $\Omega(f;x)$ satisfies the following obvious properties:

$$\begin{aligned}
\Omega(f_1+f_2;x) &\leqslant \Omega(f_1;x) + \Omega(f_2;x);\\
\Omega(f;x) &\leqslant 2\widetilde{T}_* f(x);
\end{aligned}$$

and

$$\Omega(f;x) = 0 \tag{4.1}$$

for all $x \in \mathbb{R}^N$ and $f \in C_0^1(\mathbb{R}^N)$. Indeed, if $f \in C_0^1(\mathbb{R}^N)$ and $0 < t < s < \varepsilon < 1$, then

$$\begin{aligned}
|\widetilde{T}_t(f)(x) - \widetilde{T}_s(f)(x)| &= \left|\int_{t<d(x,y)\leqslant s} K(x,y)[f(y)-f(x)]d\omega(y)\right|\\
&\lesssim \int_{d(x,y)<\varepsilon} \frac{d(x,y)^{\delta}}{\omega(B(x,d(x,y)))}d\omega(y)\\
&\lesssim \sum_{k=1}^{\infty}\int_{2^{-k}\varepsilon\leqslant d(x,y)<2^{-k+1}\varepsilon} \frac{d(x,y)^{\delta}}{\omega(B(x,d(x,y)))}d\omega(y)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{k=1}^{\infty}\sum_{\sigma\in G}\int_{2^{-k}\varepsilon\leqslant\|\sigma(x)-y\|<2^{-k+1}\varepsilon}\frac{2^{-k\delta}\varepsilon^{\delta}}{\omega(B(x,2^{-k}\varepsilon))}d\omega(y)\\
&\lesssim \sum_{k=1}^{\infty}\sum_{\sigma\in G}2^{-k\delta}\varepsilon^{\delta}\frac{\omega(B(\sigma(x),2^{-k+1}\varepsilon)}{\omega(B(x,2^{-k}\varepsilon))}\\
&\lesssim \sum_{k=1}^{\infty}\#G\cdot 2^{-k\delta}\varepsilon^{\delta}\\
&\lesssim \varepsilon^{\delta},
\end{aligned}
$$

which implies (4.1) holds.

Let us suppose that $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$. We fix $\alpha > 0$ and verify that $\omega(\{x \in R^N : \Omega(f; x) > \alpha\}) = 0$. Indeed, let $\beta > 0$ be a real number and let $g \in C_0^1(\mathbb{R}^N)$ be a function such that $\|f - g\|_{L^p(\mathbb{R}^N,d\omega)} \leqslant \beta$. Then

$$
\Omega(f; x) \leqslant \Omega(f - g; x) + \Omega(g; x) = \Omega(f - g; x)
$$

for all $x \in \mathbb{R}^N$ and hence, by the Corollary 3.2 we get

$$
\begin{aligned}
\omega(\{x \in R^N : \Omega(f; x) > \alpha\}) &\leqslant \omega(\{x \in \mathbb{R}^N : 2\widetilde{T}_*(f - g)(x) > \alpha\})\\
&\leqslant C_p\alpha^{-p}\|f - g\|^p_{L^p(\mathbb{R}^N,d\omega)}\\
&\leqslant C^p\alpha^{-p}\beta^p.
\end{aligned}
$$

Letting $\beta$ tends 0 yields $\omega\{x \in \mathbb{R}^N : \Omega(f; x) > \alpha\} = 0$, and hence

$$
\lim_{\varepsilon\to 0^+}\int_{\{y:d(x,y)>\varepsilon\}} K(x, y) f(y)d\omega(y)
$$

exists for $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$ and almost all $x \in \mathbb{R}^N$. By Lemma 3.1, we also have

$$
\lim_{\varepsilon\to 0^+}\int_{\{y:\|x-y\|>\varepsilon\}} K(x, y) f(y)d\omega(y)
$$

exists for $f \in L^p(\mathbb{R}^N, d\omega)$, $1 \leqslant p < \infty$ and almost all $x \in \mathbb{R}^N$.

The proof of the Theorem 1.4 is complete. □

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