Applications of Nijenhuis Geometry III: Frobenius Pencils and Compatible Non-homogeneous Poisson Structures

Alexey V. Bolsinov[1] · Andrey Yu. Konyaev[2] · Vladimir S. Matveev[3]

Abstract

We consider multicomponent local Poisson structures of the form $\mathcal{P}_3 + \mathcal{P}_1$, under the assumption that the third-order term $\mathcal{P}_3$ is Darboux–Poisson and nondegenerate, and study the Poisson compatibility of two such structures. We give an algebraic interpretation of this problem in terms of Frobenius algebras and reduce it to classification of Frobenius pencils, i.e. linear families of Frobenius algebras. Then, we completely describe and classify Frobenius pencils under minor genericity conditions. In particular, we show that each Frobenuis pencil is a subpencil of a certain *maximal* pencil. These maximal pencils are uniquely determined by some combinatorial object, a directed rooted in-forest with edges and vertices labelled by numerical marks. They are also naturally related to certain pencils of Nijenhuis operators. We show that common Frobenius coordinate systems admit an elegant invariant description in terms of the corresponding Nijenhuis pencils.

Keywords Nijenhuis operators · Compatible Poisson brackets · Integrable PDE systems · Dispersive perturbation · Differential geometric Poisson brackets · Frobenius algebras

Mathematics Subject Classification 37K05 · 37K06 · 37K10 · 37K25 · 37K50 · 53B10 · 53A20 · 53B20 · 53B30 · 53B50 · 53B99 · 53D17

✉ Alexey V. Bolsinov
  A.Bolsinov@lboro.ac.uk

  Andrey Yu. Konyaev
  maodzund@yandex.ru

  Vladimir S. Matveev
  vladimir.matveev@uni-jena.de

[1] Department of Mathematical Sciences, Loughborough University, LE11 3TU Loughborough, UK

[2] Faculty of Mechanics and Mathematics, Moscow State University and Moscow Center for Fundamental and Applied Mathematics, Moscow, Russia 119992

[3] Institut für Mathematik, Friedrich Schiller Universität Jena, 07737 Jena, Germany

1 Introduction

1.1 Foreword

Nijenhuis operator is a (1,1)-tensor field $L = L_j^i$ on an n-dimensional manifold M such that its Nijenhuis torsion vanishes. Nijenhuis geometry, as initiated in [6] (where all necessary definitions can also be found) and further developed in [7, 8, 28], studies Nijenhuis operators and their applications. There are many topics in mathematics and mathematical physics in which Nijenhuis operators appear naturally; this paper is devoted to the study of ∞-dimensional compatible Poisson brackets of type $\mathcal{P}_3 + \mathcal{P}_1$, where the lower index i indicates the order of the homogeneous bracket $\mathcal{P}_i$ (the necessary definitions will be given in Sect. 1.2). Nijenhuis geometry allows us to reformulate the initial problem, originated from mathematical physics, first into the language of algebra and then into the language of differential geometry and finally solve it using the machinery of differential geometry in combination with that of algebra. Translating back the results gives a full description of (nondegenerate) compatible Poisson brackets of type $\mathcal{P}_3 + \mathcal{P}_1$ such that $\mathcal{P}_3$ is Darboux–Poisson.

1.2 Mathematical Setup

The construction below is a special case of the general approach suggested in [25]. For $n = 1$, the construction can be found in [24], see also [12, 35, 42].

We work in an open disc $U \subset \mathbb{R}^n$ with coordinates $u^1, \ldots, u^n$. Our constructions are invariant with respect to coordinate changes so one may equally think of $(u^1, \ldots, u^n)$ as a coordinate chart on a smooth manifold M.

Consider the jet bundles (of curves) over U. Recall that for a point $\mathsf{p} \in U$, the kth jet space $J_\mathsf{p}^k U$ at this point is an equivalence class of smooth curves $c : (-\varepsilon, \varepsilon) \to U$ such that $c(0) = \mathsf{p}$. The parameter of the curves c will always be denoted by x. The equivalence relation is as follows: two curves are equivalent if they coincide at $c(0)$ up to terms of order $k + 1$.

For example, for $k = 0$, the space $J_\mathsf{p}^0 U$ contains only one element and the definition of $J_\mathsf{p}^1 U$ coincides with one of the standard definitions of the tangent space $T_\mathsf{p} U$.

It is known that $J_\mathsf{p}^k U$ is naturally equipped with the structure of a vector space of dimension $n \times k$ with coordinates denoted by

$$(u_x^1, \ldots, u_x^n, u_{x^2}^1, \ldots, u_{x^2}^n, \ldots, u_{x^k}^1, \ldots, u_{x^k}^n). \tag{1}$$

Namely, a curve $c(x) = (u^1(x), \ldots, u^n(x))$ with $c(0) = \mathsf{p}$ viewed as an element of $J_\mathsf{p}^k U$ has coordinates

$$
\begin{aligned}
&\left(u_x^1, \ldots, u_x^n, u_{x^2}^1, \ldots, u_{x^2}^n, \ldots, u_{x^k}^1, \ldots, u_{x^k}^n \right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x}(u^1), \ldots, \frac{\mathrm{d}}{\mathrm{d}x}(u^n), \frac{\mathrm{d}^2}{\mathrm{d}x^2}(u^1), \ldots, \frac{\mathrm{d}^2}{\mathrm{d}x^2}(u^n), \ldots, \frac{\mathrm{d}^k}{\mathrm{d}x^k}(u^1), \ldots, \frac{\mathrm{d}^k}{\mathrm{d}x^k}(u^n) \right)_{|x=0}.
\end{aligned}
\tag{2}
$$

We denote by $J^k U$ the union $\bigcup_{\mathsf{p} \in U} J^k_\mathsf{p} U$. It has a natural structure of a $k \times n$-dimensional vector bundle over U. The coordinates $(u^1, \ldots, u^n)$ on U and (1) on $J^k_\mathsf{p} U$ generate a coordinate system

$$(u^1, \ldots, u^n, u^1_x, \ldots, u^n_x, u^1_{x^2}, \ldots, u^n_{x^2}, \ldots, u^1_{x^k}, \ldots, u^n_{x^k})$$

on $J^k U$ adapted to the bundle structure. Any C^∞ curve $c : [a,b] \to U$, $x \mapsto (u^1(x), \ldots, u^n(x))$ naturally lifts to a curve $\widehat{c}$ on $J^k U$ by

$$\widehat{c} : [a,b] \to J^k U, \quad x_0 \mapsto \left(u^1, \ldots, u^n, \tfrac{\mathrm{d}}{\mathrm{d}x}(u^1), \ldots, \tfrac{\mathrm{d}}{\mathrm{d}x}(u^n), \ldots, \tfrac{\mathrm{d}^k}{\mathrm{d}x^k}(u^1), \ldots, \tfrac{\mathrm{d}^k}{\mathrm{d}x^k}(u^n) \right)_{|x=x_0}. \tag{3}$$

Next, for every $\mathsf{p} \in U$ denote by $\Pi[J^k_\mathsf{p} U]$ the algebra of polynomials in variables (1) on $J^k_\mathsf{p} U$. It has a natural structure of an infinite-dimensional vector bundle over U. Let $\mathfrak{A}_k$ denote the algebra of C^∞-smooth sections of the bundle $\Pi[J^k_\mathsf{p} U]$. Notice that we have natural inclusion $\mathfrak{A}_k \subset \mathfrak{A}_{k+1}$ and set $\mathfrak{A} = \bigcup_{k=0}^\infty \mathfrak{A}_k$. In simple terms, the elements of $\mathfrak{A}$ are finite sums of finite products of coordinates

$$(u^1_x, \ldots, u^n_x, u^1_{x^2}, \ldots, u^n_{x^2}, \ldots, u^1_{x^k}, \ldots, u^n_{x^k}, \ldots) \tag{4}$$

with coefficients being C^∞-functions on U. The summands in this sum, i.e. terms of the form $a^{j_1 \ldots j_n}_{i_1 \ldots i_n}(u)(u^1_{x^{i_1}})^{j_1} \ldots (u^n_{x^{i_n}})^{j_n}$ with $a^{j_1 \ldots j_n}_{i_1 \ldots i_n}(u) \not\equiv 0$ will be called *differential monomials*. The *differential degree* of such a differential monomial is the number $i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$. For example, $f(u) u^1_{x^2}(u^2_x)^2$ has differential degree $1 \times 2 + 2 \times 1 = 4$. Differential degree of an element of $\mathfrak{A}$ is the maximum of the differential degrees of its differential monomials, and it is a nonnegative integer number. Elements of $\mathfrak{A}$ will be called *differential polynomials*.

Generators of this algebra are coordinates $u^i_{x^j}$ and functions on U. Every element of $\mathfrak{A}$ can be obtained from finitely many generators using finitely many summation and multiplication operations.

The following two linear mappings will be important for us. The first one, called the *total x-derivative* and denoted by D (another standard notation used in literature is $\frac{d}{dx}$), is defined as follows. One requires that D satisfies the Leibnitz rule and then defines it on the generators of $\mathfrak{A}$, i.e. on functions $f(u)$ and coordinates (4), by setting

$$D(f) = \sum_{i=1}^n \frac{\partial f}{\partial u^i} u^i_x, \quad D(u^i_{x^j}) = u^i_{x^{j+1}}.$$

Clearly, the operation D increases the differential degree by one at most.

Next, denote by $\tilde{\mathfrak{A}}$ the quotient algebra $\mathfrak{A}/D(\mathfrak{A})$. The tautological projection $\mathfrak{A} \to \tilde{\mathfrak{A}}$ is traditionally denoted by $\mathcal{H} \mapsto \int \mathcal{H} dx \in \tilde{\mathfrak{A}}$. In simple terms, it means that we think that two differential polynomials $\mathcal{H}, \bar{\mathcal{H}}$ are equal, if their difference is a total derivative of a differential polynomial.

Note that by construction, the operation D has the following remarkable property, which explains its name and also the notation $\frac{d}{dx}$ used for D sometimes in literature. For any curve $c : [a, b] \to U$ whose lift (3) will be denoted by $\widehat{c}$ and for any element $\mathcal{H} \in \mathfrak{A}$ we have:

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{H}(\widehat{c})\right) = (D\mathcal{H})(\widehat{c}). \tag{5}$$

The second mapping is the mapping from $\mathfrak{A}$ to an n-tuple of elements of $\mathfrak{A}$. The mapping will be denoted by δ and will be called *the variational derivative*. Its ith component will be denoted by $\frac{\delta}{\delta u^i}$ and for an element $\mathcal{H} \in \mathcal{A}$ it is given by the Euler–Lagrange formula:

$$\frac{\delta \mathcal{H}}{\delta u^i} = \sum_{k=0}^{\infty} (-1)^k D^k \left(\frac{\partial \mathcal{H}}{\partial u^i_{x^k}}\right)$$

(only finitely many elements in the sum are different from zero so the result is again a differential polynomial). It is known, see, e.g. [24], that for an element $\mathcal{H} \in \mathfrak{A}$ we have $\delta \mathcal{H} = 0$ if and only if $\mathcal{H}$ is a total x-derivative. Then, we again see that the variational derivative does not depend on the choice of differential polynomial in the equivalence class $\mathcal{H} \subset \mathfrak{A}$. Then, the mapping δ induces a well-defined mapping on $\widetilde{\mathfrak{A}}$, which will be denoted by the same letter δ. One can think of $\delta \mathcal{H}$ as a covector with entries from $\widetilde{\mathfrak{A}}$, because the transformation rule of its entries under the change of u-coordinates is a natural generalisation of the transformation rule for (0,1)-tensors.

Following [16, 17], let us define a (homogeneous, nondegenerate) Poisson bracket of order 1. We choose a contravariant flat metric $g = g^{ij}$ of any signature whose Levi-Civita connection will be denoted by $\nabla = (\Gamma^i_{jk})$. Next, consider the following operation $\mathcal{A}_g : \widetilde{\mathfrak{A}} \times \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{A}}$: for two elements $\mathcal{H}, \bar{\mathcal{H}} \in \widetilde{\mathfrak{A}}$, we set

$$\mathcal{A}_g(\mathcal{H}, \bar{\mathcal{H}}) = \int \frac{\delta \bar{\mathcal{H}}}{\delta u^\alpha}\left(g^{\alpha\beta} D\left(\frac{\delta \mathcal{H}}{\delta u^\beta}\right) - \Gamma^{\alpha\beta}_\gamma \frac{\delta \mathcal{H}}{\delta u^\beta} u^\gamma_x\right) \mathrm{d}x. \tag{6}$$

In the formula above and later in the text, we sum over repeating indices and assume $\Gamma^{is}_j = \Gamma^s_{pj} g^{pi}$. The components Γ^{is}_j will be called *contravariant Christoffel symbols*, when we speak about different metrics we always raise the index *by the own metric*. A common way to write the operation $\mathcal{A}_g$ which we also will use in our paper assumes applying it to $\frac{\delta \mathcal{H}}{\delta u^\beta}$ and multiplication with $\frac{\delta \bar{\mathcal{H}}}{\delta u^\alpha}$ (and of course summation and projection to $\widetilde{\mathfrak{A}}$):

$$\mathcal{A}_g = g^{\alpha\beta} D - \Gamma^{\alpha\beta}_\gamma u^\gamma_x. \tag{7}$$

It is known, see,e.g. [16–18], that the operation $\mathcal{A}_g$ given by (6) defines a Poisson bracket on $\widetilde{\mathfrak{A}}$, that is, it is skew-symmetric and satisfies the Jacobi identity. Moreover, one can show that the operation constructed by g and Γ via (6) defines a Poisson bracket if and only if g is flat, that is, its curvature is zero, and Γ^i_{jk} is the Levi-Civita

connection of g. It is also known that the construction (6) does not depend on the coordinate system on U.

Next, let us define a (nondegenerate, homogeneous) Darboux–Poisson structure of order 3. We choose a nondegenerate contravariant flat metric $h = h^{ij}$ of arbitrary signature and define the operation $\mathcal{B}_h : \tilde{\mathfrak{A}} \times \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ by the formula:

$$
\begin{aligned}
\mathcal{B}_h &= h^{\alpha q} \left(\delta_q^p D - \Gamma_{qm}^p u_x^m \right) \left(\delta_p^r D - \Gamma_{pk}^r u_x^k \right) \left(\delta_r^\beta D - \Gamma_{rs}^\beta u_x^s \right) \\
&= h^{\alpha\beta} D^3 - 3 h^{\alpha q} \Gamma_{qs}^\beta u_x^s D^2 \\
&\quad + 3 \left(h^{\alpha q} \left(\Gamma_{qs}^p \Gamma_{pr}^\beta - \frac{\partial \Gamma_{qs}^\beta}{\partial u^r} \right) u_x^s u_x^r - h^{\alpha q} \Gamma_{qs}^\beta u_{x^2}^s \right) D \\
&\quad + \left(h^{\alpha q} \left(2 \Gamma_{qs}^a \frac{\partial \Gamma_{ar}^\beta}{\partial u^p} + \frac{\partial \Gamma_{qs}^a}{\partial u^r} \Gamma_{ap}^\beta - \Gamma_{qs}^a \Gamma_{ar}^b \Gamma_{bp}^\beta - \frac{\partial^2 \Gamma_{qs}^\beta}{\partial u^r \partial u^p} \right) u_x^s u_x^r u_x^p \right. \\
&\quad \left. + h^{\alpha q} \left(2 \Gamma_{qs}^a \Gamma_{ar}^\beta + \Gamma_{qr}^a \Gamma_{as}^\beta - 2 \frac{\partial \Gamma_{qr}^\beta}{\partial u^s} - \frac{\partial \Gamma_{qs}^\beta}{\partial u^r} \right) u_x^s u_{x^2}^r - h^{\alpha q} \Gamma_{qs}^\beta u_{x^3}^s \right)
\end{aligned}
\tag{8}
$$

In the formula, we have used the same conventions as above, i.e. assume summation over repeating indices. Moreover, similar to formula (7), we did not write $\mathcal{H}, \bar{\mathcal{H}}$ in the formula. They are assumed there as follows: the differential operator (8) is applied to $\frac{\delta \mathcal{H}}{\delta u^\alpha}$, the result is multiplied by $\frac{\delta \bar{\mathcal{H}}}{\delta u^\beta}$, and then we perform summation with respect to the repeating indices α, β.

As in the case of order 1, the operation $\mathcal{B}_h$ given by (8) defines a Poisson bracket on $\tilde{\mathfrak{A}}$. The construction of this Poisson bracket is independent on the choice of coordinate system on U. However, in contrast to the case of order 1, the form (8) is not the most general form for a local Poisson bracket on $\tilde{\mathfrak{A}}$ of order 3. In fact, the word *Darboux* indicates that in a certain coordinate system (flat coordinate system for h in our case), the coefficients of the Poisson structure are constants.[1] In this *Darboux* coordinate system, the Christoffel symbols Γ_{jk}^i are all zero and formula (8) reduces to[2]

$$
\mathcal{B}_h(\mathcal{H}, \bar{\mathcal{H}}) = \frac{\delta \bar{\mathcal{H}}}{\delta u^\beta} h^{\alpha\beta} D^3 \left(\frac{\delta \mathcal{H}}{\delta u^\alpha} \right),
\tag{9}
$$

with the components $h^{\alpha\beta}$ of the metric h being constants.

Poisson structures $\mathcal{P}_1$ of order 1 are always Darboux–Poisson, but there are examples, see,e.g. [21, 22, 37], of Poisson structures $\mathcal{P}_3$ of order 3 which are not Darboux–Poisson.

Similar to the finite-dimensional case, a Poisson structure $\mathcal{P}$ and choice of a 'Hamiltonian' $\mathcal{H} \in \tilde{\mathfrak{A}}$ allow one to define the Hamiltonian flow, which in our setup is a system of n PDEs on n functions $u^i(t, x)$ of two variables, t and x. It is given by:

$$
\frac{\partial u^\beta}{\partial t} = \mathcal{P}^{\alpha\beta} \left(\frac{\delta \mathcal{H}}{\delta u^\alpha} \right).
\tag{10}
$$

---

[1] The terminology 'Darboux–Poisson' is motivated by [14].

[2] In fact, (8) is just the formula (9) rewritten in an arbitrary (not necessarily 'flat') coordinate system.

For example, in the case of the Poisson structure (6) for a Hamiltonian of degree 0 (i.e. for a function H on U), the Hamiltonian flow is given by

$$\frac{\partial u^\beta}{\partial t} = g^{\beta\alpha} \frac{\partial^2 H}{\partial u^\alpha \partial u^\gamma} u_x^\gamma - \Gamma_\gamma^{\beta\alpha} \frac{\partial H}{\partial u^\alpha} u_x^\gamma = \frac{\partial u^\gamma}{\partial x} \nabla^\beta \nabla_\gamma H. \tag{11}$$

Such systems of PDEs are called *Hamiltonian systems of hydrodynamic type*.

In our paper, we study compatibility of non-homogeneous Poisson structures of type $\mathcal{P}_3 + \mathcal{P}_1$ such that the part of order 3 is Darboux–Poisson. That is, we have 4 nondegenerate Poisson structures: $\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$ constructed by flat metrics g and $\bar{g}$ by (6), and $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ constructed by flat metrics h and $\bar{h}$ by (8). We assume that $\mathcal{A}_g + \mathcal{B}_h$ and $\mathcal{A}_{\bar{g}} + \mathcal{B}_{\bar{h}}$ are (non-homogeneous) Poisson structures and ask the question when these structures are compatible in the sense that any of their linear combinations is a Poisson structure [32]. Since it is automatically skew-symmetric, the compatibility is equivalent to the Jacobi identity for each linear combination of $\mathcal{A}_g + \mathcal{B}_h$ and $\mathcal{A}_{\bar{g}} + \mathcal{B}_{\bar{h}}$.

The meaning of the word 'nondegenerate' relative to the Poisson structures under discussion is as follows: the metrics $g, \bar{g}, h, \bar{h}$ which we used to construct them are nondegenerate, i.e. they are given by matrices with nonzero determinant. Additional nondegeneracy condition, natural from the viewpoint of mathematical physics, is as follows: the operators $R_h = \bar{h}h^{-1}$ and $R_g = \bar{g}g^{-1}$ have n different eigenvalues. Under these conditions, we solve the problem completely: we find explicitly all pairs of such Poisson structures.

Let us comment on the assumption that $\mathcal{P}_3$ is *Darboux–Poisson*. The compatibility of two geometric Poisson brackets $\mathcal{P}_3 + \mathcal{P}_1$ and $\bar{\mathcal{P}}_3 + \bar{\mathcal{P}}_1$ amounts to a highly overdetermined PDE system which is expected to imply additional conditions on the third-order parts, from which *Darboux–Poisson* is a natural candidate. Indeed, in the literature, we have not found any example of compatible Poisson brackets $\mathcal{P}_3 + \mathcal{P}_1$ and $\bar{\mathcal{P}}_3 + \bar{\mathcal{P}}_1$ such that $\mathcal{P}_3$ and $\bar{\mathcal{P}}_3$ are not Darboux–Poisson and nonproportional. Even in the case when $\bar{\mathcal{P}}_3 = 0$, only few examples are known, namely the compatible brackets for WDVV system from [23] (this example is three-dimensional and moreover, $\mathcal{P}_1 = 0$) and a family of compatible brackets $\mathcal{P}_3 + \mathcal{P}_1$ and $\bar{\mathcal{P}}_1$ constructed in [31] in dimensions 1 and 2.

Our main motivation came from the theory of integrable systems, in which many famous integrable systems have been constructed and analysed using compatible Poisson brackets. Those of the form $\mathcal{P}_3 + \mathcal{P}_1$, described and classified in the present paper, generalise compatible Poisson brackets related to KdV, Harry Dym, Camassa-Holm and Dullin-Gottwald-Holm equations. The applications of such new brackets will be developed in a series of separate papers, of which the first one has already appeared, [10]. In this paper, we have generalised the above equations for an arbitrary number of components and have constructed new integrable PDE systems that have no low-component analogues. Note that in [10], we used the simplest pencil of compatible Poisson structures of type $\mathcal{P}_3 + \mathcal{P}_1$ (so-called AFF-pencil from Sect. 2.3); more complicated pencils will lead to more new families of integrable multicomponent PDEs.

Let us also comment on a more physics-oriented approach to the construction above, see, e.g. [14]. Physicists often view x as a space coordinate, and $(u^1, \ldots, u^n)$ as field coordinates. In the simplest situation, the values $u^1, \ldots, u^n$ at x may describe some physical values (e.g. pressure, temperature, charge, density, momenta). The total

energy of the system is the integral over the x variable of some differential polynomial in $u^1, \ldots, u^n$, and the Hamiltonian functions $\mathcal{H} \in \mathfrak{A}$ have then the physical meaning of the density of the energy, i.e. of the integrand in the formula $\mathbf{Energy}(c) = \int \mathcal{H}(\widehat{c}) dx$. Further, it is assumed that the physical system is either periodic in x, or one is interested in fast decaying solutions as $x \to \pm\infty$. The integration by parts implies then that the differential polynomial is defined up to an addition of the total derivative in x which allows one to pass to $\tilde{\mathfrak{A}} = \mathfrak{A}/D(\mathfrak{A})$. The natural analogue of the differential of a function in this setup is the variational derivative $\frac{\delta}{\delta u^\alpha}$, and actually, the equation (10) is the natural analogue of the finite-dimensional equation $\dot{u} = X_H$ (where X_H is the Hamiltonian vector field of a function H; it is given by $X_H^j = P^{ij} \frac{\partial H}{\partial u^i}$ where $P(u)^{ij}$ is the matrix of the Poisson structure; please note similarity with (10)). Generally, it is useful to keep in mind the physical interpretation and analogy with the finite-dimensional case.

1.3 Brief Description of Main Results, Structure of the Paper and Conventions

In this paper, we address the following problems:

(A) *Description of compatible pairs, $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$, of non-homogeneous Poisson brackets in arbitrary dimension n.* In Theorems 1 and 2,we give an algebraic interpretation of this problem in terms of Frobenius algebras and reduce it to classification of Frobenius pencils, i.e. linear families of Frobenius algebras. We do it under the following nondegeneracy assumption: the (1,1)-tensor $R_h = \bar{h}h^{-1}$ (connecting h and $\bar{h}$) has n different eigenvalues.

(B) *Description and classification of Frobenius pencils.* We reduce this purely algebraic problem to a differential geometric one (explicitly formulated in Sect. 6.1) and completely solve it using geometric methods. The nondegeneracy assumption is that the (1,1)-tensor $R_g = \bar{g}g^{-1}$ (connecting g and $\bar{g}$) has n different eigenvalues. Namely, we show that each Frobenius pencil in question is a subpencil of a certain *maximal* pencil. We explicitly describe all maximal pencils, see Theorems 3, 4 and 5.

  (B1) A generic in a certain sense maximal pencil corresponds to the well-known multi-Poisson structure discovered by M. Antonowitz and A. Fordy in [1] and studied by E. Ferapontov and M. Pavlov [20], see also [2, 3, 9]. We refer to it as to Antonowitz-Fordy-Frobenius pencil, *AFF-pencil*. In Theorem 3, we show that, under an additional genericity assumption, every two-dimensional Frobenius pencil is contained in the *AFF-pencil* .

  (B2) Our main result, Theorems 4 and 5 , gives a complete description in the most general case. Theorem 4 constructs all maximal Frobenius pencils using AFF-pencils as building blocks. Theorem 5 states that each Frobenius pencil is a subpencil of a certain *maximal* pencil from Theorem 4. These maximal pencils are uniquely determined by some combinatorial data, directed rooted in-forest F with edges labelled by numbers λ_α's and vertices labelled by natural numbers whose sum is the dimension of the manifold. The AFF-pencil corresponds to the simplest case, when F consists of a single vertex. To

the best of our knowledge, the other Frobenius pencils and the corresponding bi-Poisson structures are new.

In addition, we show that common Frobenius coordinate systems admit an elegant invariant description in terms of the Nijenhuis pencil $\mathcal{L}$, see Theorem 4.

(C) *Dispersive perturbations of compatible Poisson brackets of hydrodynamic type.* The general question is as follows: given two compatible Poisson structures $\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$ of the first order, can one find flat metrics h and $\bar{h}$ such that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible Poisson structures? This passage from a Poisson bracket of hydrodynamic type to a non-homogeneous Poisson bracket of higher order is called *dispersive perturbation* in literature. We study dispersive perturbations of bi-Hamiltonian structures assuming that the third-order terms $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ are Darboux–Poisson.

We describe all such perturbations under the assumption that both $R_h = \bar{h}h^{-1}$ and $R_g = \bar{g}g^{-1}$ have n different eigenvalues, and in particular, answer a question from [20] on dispersive perturbations of the AFF-pencil (Remark 3.2).

The results of this paper also have the following unexpected application. It turns out that the *diagonal* coordinates for the operator $R_g = \bar{g}g^{-1}$ are orthogonal separating coordinates for the metrics g. In [11], we show that every orthogonal separating coordinates for a flat metric g of arbitrary signature can be constructed in this way. Namely, we reduce PDEs that define orthogonal separating coordinates to those studied in the present paper. This leads us to an explicit description of all orthogonal separating coordinates for metrics of constant curvature and thus solves a long-standing and actively studied problem in mathematical physics, see [11] for details.

The structure of the paper is as follows. In Sect. 2, we start with basic facts and constructions related to compatibility of homogeneous Poisson structures of order 1 and 3, then give description of compatible non-homogeneous structures $\mathcal{B}_g + \mathcal{A}_h$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ in terms of Frobenius algebras (Theorems 1 and 2), leading us to the classification problem for the so-called Frobenius pencils. We conclude this section with an example of AFF-pencil. The AFF-pencil plays later a role of a building block in our general construction. Moreover, it provides an answer under a minor nondegeneracy assumption, see Theorem 3 in Sect. 3, where we also discuss a question of Ferapontov and Pavlov. Theorem 3 will be proved in Sect. 6.

In Sect. 4, we formulate the answer to the classification problem in its full generality. Theorem 5 (proved in Sect. 7) gives a description of Frobenius pencils in the 'diagonal' coordinates for $g, \bar{g}$, and Theorem 4 (proved in Sect. 8) describes the corresponding Frobenius coordinates. In Sect. 4.2, we discuss the case of two blocks and give explicit formulas, see Theorem 6.

All objects in our paper are assumed to be of class C^∞; actually, our results show that most of them are necessarily real analytic.

Throughout the paper, we use $\mathcal{A}_g$ and $\mathcal{B}_h$ to denote the Poisson structures of order 1 and 3 given by (7) and (8), respectively. Unless otherwise stated, the metrics we deal with (such as $g, h, \bar{g}, \bar{h}, \dots$) are contravariant.

2 Non-homogeneous Compatible Brackets and Frobenius Algebras

2.1 Basic Facts and Preliminary Discussion

Recall that we study the compatibility of two Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$, constructed by flat metrics $h, \bar{h}, g, \bar{g}$; our goal is to construct all of them. Recall that by definition, it means that for any constants $\lambda, \bar{\lambda}$, the linear combination $\lambda(\mathcal{B}_h + \mathcal{A}_g) + \bar{\lambda}(\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}})$ is a Poisson structure. Using that $\mathcal{B}$ and $\mathcal{A}$ that have different orders, one obtains (see, e.g. [14])

Fact 1 *Let h, $\bar{h}$, g and $\bar{g}$ be flat metrics. If $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible Poisson structures, then the following holds:*

  (i) *$\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$ are compatible,*
  (ii) *$\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ are compatible,*
 (iii) *$\mathcal{A}_g$ and $\mathcal{B}_h$ are compatible (as well as $\mathcal{A}_{\bar{g}}$ and $\mathcal{B}_{\bar{h}}$).*

This Fact naturally leads us to considering pencils (= linear combinations of metrics) $\lambda h + \bar{\lambda}\bar{h}$ and $\lambda g + \bar{\lambda}\bar{g}$. We need the following definition:

Definition 1 (Dubrovin, [18, Definition 0.5]) Two contravariant flat metrics g and $\bar{g}$ are said to be *Poisson compatible*, if for each (nondegenerate) linear combination $\widehat{g} = \lambda g + \bar{\lambda}\bar{g}, \lambda, \bar{\lambda} \in \mathbb{R}$, the following two conditions hold:

  1. $\widehat{g}$ is flat;
  2. the contravariant Christoffel symbols for g, $\bar{g}$ and $\widehat{g}$ are related as

$$\widehat{\Gamma}_s^{\alpha\beta} = \lambda \Gamma_s^{\alpha\beta} + \bar{\lambda}\bar{\Gamma}_s^{\alpha\beta}. \tag{12}$$

In this case, the family of metrics $\{\lambda g + \bar{\lambda}\bar{g}\}_{\lambda,\bar{\lambda}\in\mathbb{R}}$ is said to be a *flat pencil* of metrics.

The next fact explains the relationship between Poisson compatibility of flat metrics and compatibility of the corresponding Poisson structures.

Fact 2 *Let h, $\bar{h}$, g and $\bar{g}$ be flat metrics. Then, the following statements are true:*

  (i) *$\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$ are compatible if and only if g and $\bar{g}$ are Poisson compatible.*
  (ii) *If $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ are compatible, then h and $\bar{h}$ are Poisson compatible.*
 (iii) *If $\mathcal{B}_h$ and $\mathcal{A}_g$ are compatible, then h and g are Poisson compatible.*

The (i)-part of Fact 2 is in [16], see also [19, 33, 34]. In view of formula (7), the two conditions from Definition 1 are nothing else but a geometric reformulation of the compatibility condition for Poisson structures of order one, which explains the name *Poisson compatible*. The (ii)-part is an easy corollary of [14, Theorem 3.2], see also proof of Theorem 3 below. The (iii)-part follows from [29, Theorem 2.2].

Notice that unlike the case of Poisson structures of order 1, not every pair of Poisson-compatible metrics h and $\bar{h}$ (resp. h and g) leads to compatible Poisson structures of higher order $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ as in (ii) (resp. $\mathcal{B}_h$ and $\mathcal{A}_g$ as in (iii)). Some extra conditions are required. These conditions will be explained below in Fact 4 (for h and g leading to compatible $\mathcal{B}_h$ and $\mathcal{A}_g$) and Theorem 3 (for h and $\bar{h}$ leading to compatible $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$).

Let us also recall the relation of compatible metrics to Nijenhuis geometry:

Fact 3 (see [19, 33, 34]) *If g and $\bar{g}$ are Poisson compatible, then the (1,1)-tensor $R = \bar{g}g^{-1}$ is a Nijenhuis operator. Moreover, if g is flat, R is a nondegenerate Nijenhuis operator with n different eigenvalues, and $\bar{g} := Rg$ is flat, then $\bar{g}$ is Poisson compatible with g.*

As already explained, the condition that $\mathcal{B}_h + \mathcal{A}_g$ is a Poisson structure is a nontrivial geometric condition on the flat metrics h and g, stronger than Poisson compatibility in the sense of Definition 1. This condition was studied in literature (see, e.g. [4]) and it was observed that the compatibility of homogeneous Poisson structures of order 3 and 1 is sometimes related to certain algebraic structure. In our case, under the assumption that $\mathcal{B}_h$ is Darboux–Poisson, the algebraic structure which pops up naturally is *Frobenius algebra*.

Definition 2 Let $(\mathfrak{a}, \star)$ be an n-dimensional commutative associative algebra over $\mathbb{R}$ endowed with a nondegenerate symmetric bilinear form $b(\ ,\ )$. The pair $\big((\mathfrak{a}, \star), b\big)$ is called a *Frobenius algebra*, if

$$b(\xi \star \eta, \zeta) = b(\xi, \eta \star \zeta), \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{a}. \tag{13}$$

The form b is then called a *Frobenius form*.

Notice that we do not assume that $\mathfrak{a}$ is unital which makes our version slightly more general than the one used in the theory of Frobenius manifolds (see,e.g. [18]), or in certain branches of Algebra. The bilinear form b may have any signature.

Condition (13) is linear in b, so all Frobenius forms (if we allow some of them to be degenerate) on a given commutative associative algebra form a vector space.

Fix a basis $e^1, \ldots, e^n$ in $\mathfrak{a}$. Below we will interpret $\mathfrak{a}$ as the dual $(\mathbb{R}^n)^*$ and for this reason, we interchange lower and upper indices. Consider the structure constants a_k^{ij} defined by $e^i \star e^j = a_k^{ij} e^k$ and coefficients $b^{ij} := b(e^i, e^j)$ of the Frobenius form b. The algebra $\mathfrak{a}$ is Frobenius if and only if a_k^{ij} and b^{ij} satisfy the following conditions:

$$
\begin{aligned}
a_k^{ij} &= a_k^{ji} && \text{(commutativity)}, \\
a_\alpha^{ij} a_k^{\alpha r} &= a_k^{i\alpha} a_\alpha^{jr} && \text{(associativity)}, \\
b^{\alpha r} a_\alpha^{ij} &= b^{i\alpha} a_\alpha^{jr} && \text{(Frobenius condition)}.
\end{aligned}
\tag{14}
$$

The dual $\mathfrak{a}^*$ has a natural structure of an affine space $\mathbb{R}^n$ with $u^i \simeq e^i$ being coordinates on $\mathfrak{a}^* \simeq \mathbb{R}^n$. Thus, on $\mathfrak{a}^*$, we can introduce the contravariant metric $g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta} u^s$ which is known to be flat (e.g. [29, Lemma 4.1]; the result also follows from [4]). What is special here is not the metric g itself, but the coordinate system $u^1, \ldots, u^n$ which establishes a relationship between g and the Frobenius algebra $\mathfrak{a}$. This leads us to

Definition 3 Let g be a flat metric. We say that $u^1, \ldots, u^n$ is a *Frobenuis coordinate system* for g if

$$g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta} u^s, \tag{15}$$

where $a_s^{\alpha\beta}$ are structure constants of a certain Frobenius algebra $\mathfrak{a}$ and $b = (b^{\alpha\beta})$ is a (perhaps degenerate) Frobenius form for $\mathfrak{a}$.

Frobenius coordinates possess the following important property that can be easily checked.

Fact 4 (see [4] and [29]) *Let g be a contravariant metric and $u^1, \ldots, u^n$ a coordinate system. The following two conditions are equivalent:*

1. *In coordinates $u^1, \ldots, u^n$, the contravariant Christoffel symbols $\Gamma_s^{\alpha\beta}$ of g are constant and symmetric in upper indices.*
2. *$u^1, \ldots, u^n$ are Frobenius coordinates, i.e. g is given by* (15).

If either of these conditions holds, then g is flat and $\Gamma_s^{\alpha\beta} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial s}$.

The relation of Frobenius coordinate systems to our problem is established by the following remarkable and fundamental statement:

Fact 5 [29, Theorem 2.2] *Let g and h be two flat metrics. Then $\mathcal{B}_h + \mathcal{A}_g$ is Poisson if and only if there exists a coordinate system $u^1, \ldots, u^n$ such that the following holds:*

1. *$g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta}u^s$, where $a_s^{\alpha\beta}$ are structure constants of a certain Frobenius algebra $\mathfrak{a}$, and b is a Frobenius form for $\mathfrak{a}$;*
2. *the entries $h^{\alpha\beta}$ of h in this coordinate system are constant;*
3. *$h = (h^{\alpha\beta})$ is a Frobenius form for $\mathfrak{a}$, that is, $h^{\alpha q}a_q^{\beta\gamma} = h^{\gamma q}a_q^{\beta\alpha}$.*

This fact was independently obtained by P. Lorenzoni and R. Vitolo in their unpublished paper. The 'if' part of the statement follows from [41] by I. Strachan and B. Szablikowski, see also [15, Theorem 5.12].

The coordinates $(u^1, \ldots, u^n)$ from Fact 5 will be called *Frobenius coordinates* for the non-homogeneous Poisson structure $\mathcal{B}_h + \mathcal{A}_g$. Of course, Frobenius coordinates are not unique; indeed, they remain to be Frobenius after any affine coordinate change. This is the only freedom since the components of h are constant in Frobenius coordinates.

2.2 Reduction of our Problem to an Algebraic One and Frobenius Pencils

Definition 4 Let $(\mathfrak{a}, \star)$ and $(\bar{\mathfrak{a}}, \bar{\star})$ be Frobenius algebras defined on the same vector space V and $h, \bar{h} : V \times V \to \mathbb{R}$ the corresponding Frobenius forms. We will say that $(\mathfrak{a}, h)$ and $(\bar{\mathfrak{a}}, \bar{h})$ are *compatible* if the operation

$$\xi, \eta \mapsto \xi \star \eta + \xi \bar{\star} \eta, \qquad \xi, \eta \in V, \tag{16}$$

defines the structure of a Frobenius algebra with the Frobenius form $h + \bar{h}$.

Similarly, if $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ are Frobenius algebras each of which is endowed with two Frobenius forms b, h and $\bar{b}, \bar{h}$, respectively, then we say that the triples $(\mathfrak{a}, b, h)$ and $(\bar{\mathfrak{a}}, \bar{b}, \bar{h})$ are *compatible* if (16) defines a Frobenius algebra for which $b + \bar{b}$ and $h + \bar{h}$ are both Frobenius forms.

Formally, the definition requires that $b + \bar{b}$ and $h + \bar{h}$ are nondegenerate. It is not essential. Indeed, if the operations $\star$ and $\bar{\star}$ are associative, and also the operation

$\widehat{\star} := \star + \bar{\star}$ given by (16) is associative, then any linear combination $\lambda\star + \bar\lambda\bar\star$ is associative. Moreover, if $\widehat{b} := b + \bar{b}$, possibly degenerate, satisfies the condition (13) for $\widehat{\star}$, then the linear combination $\lambda b + \bar\lambda\bar b$ also satisfies the condition (13) with respect to $\lambda\star + \bar\lambda\bar\star$. Thus, passing to a suitable linear combination, we can make $\widehat{b}$ and also $\widehat{h}$ nondegenerate.

In view of Facts 4 and 5, compatible Frobenius triples $(\mathfrak{a}, b, h)$ and $(\bar{\mathfrak{a}}, \bar{b}, \bar{h})$ naturally define compatible Poisson structures $\mathcal{B}_g + \mathcal{A}_h$ and $\mathcal{B}_{\bar g} + \mathcal{A}_{\bar h}$. The next theorem shows that the converse is also true under the assumption that $R_h = \bar{h}h^{-1}$ has n different eigenvalues.

Theorem 1 *Consider two non-homogeneous Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g}$ and suppose that $R_h = \bar h h^{-1}$ has n different eigenvalues.*

Then, they are compatible if and only if (g, h) and $(\bar g, \bar h)$ admit a common Frobenius coordinate system $u^1, \ldots, u^n$ in which

1. *$h^{\alpha\beta}$ and $\bar h^{\alpha\beta}$ are constant,*
2. *$g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta} u^s$ and $\bar g^{\alpha\beta}(u) = \bar b^{\alpha\beta} + \bar a_s^{\alpha\beta} u^s$,*
3. *$(\mathfrak{a}, b, h)$ and $(\bar{\mathfrak{a}}, \bar b, \bar h)$ are compatible Frobenius triples (here $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ denote the algebras with structure constants $a_s^{\alpha\beta}$ and $\bar a_s^{\alpha\beta}$, respectively).*

Corollary 2.1 *In more explicit terms, compatibility of $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g}$ such that $R_h = \bar h h^{-1}$ has n different eigenvalues, is equivalent to reducibility of these operators, in an appropriate coordinate system $u^1, \ldots, u^n$, to the following simultaneous canonical form*

$$\mathcal{B}_h + \mathcal{A}_g = h^{\alpha\beta} D^3 + b^{\alpha\beta} D + a_s^{\alpha\beta} u^s D + \tfrac{1}{2} a_s^{\alpha\beta} u_x^s,$$
$$\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g} = \bar h^{\alpha\beta} D^3 + \bar b^{\alpha\beta} D + \bar a_s^{\alpha\beta} u^s D + \tfrac{1}{2} \bar a_s^{\alpha\beta} u_x^s,$$

where $h^{\alpha\beta}, \bar h^{\alpha\beta}, b^{\alpha\beta}, \bar b^{\alpha\beta}, a_s^{\alpha\beta}, \bar a_s^{\alpha\beta}$ are constants symmetric in upper indices and satisfying the conditions:

$$a_q^{\alpha\beta} a_s^{q\gamma} = a_q^{\gamma\beta} a_s^{q\alpha}, \quad \bar a_q^{\alpha\beta} \bar a_s^{q\gamma} = \bar a_q^{\gamma\beta} \bar a_s^{q\alpha}, \quad \bar a_q^{\alpha\beta} a_s^{q\gamma} + a_q^{\alpha\beta} \bar a_s^{q\gamma} = \bar a_q^{\gamma\beta} a_s^{q\alpha} + a_q^{\gamma\beta} \bar a_s^{q\alpha},$$
$$h^{\alpha q} a_q^{\beta\gamma} = h^{\gamma q} a_q^{\beta\alpha}, \quad b^{\alpha q} a_q^{\beta\gamma} = b^{\gamma q} a_q^{\beta\alpha}, \quad \bar h^{\alpha q} \bar a_q^{\beta\gamma} = \bar h^{\gamma q} \bar a_q^{\beta\alpha}, \quad \bar b^{\alpha q} \bar a_q^{\beta\gamma} = \bar b^{\gamma q} \bar a_q^{\beta\alpha},$$
$$\bar h^{\alpha q} a_q^{\beta\gamma} + h^{\alpha q} \bar a_q^{\beta\gamma} = \bar h^{\gamma q} a_q^{\beta\alpha} + h^{\gamma q} \bar a_q^{\beta\alpha}, \quad \bar b^{\alpha q} a_q^{\beta\gamma} + b^{\alpha q} \bar a_q^{\beta\gamma} = \bar b^{\gamma q} a_q^{\beta\alpha} + b^{\gamma q} \bar a_q^{\beta\alpha}.$$
$$\tag{17}$$

Notice that the coordinates $u^1, \ldots, u^n$ from Theorem 1 are just flat coordinates for h (or equivalently, for $\bar h$ as these metrics have common flat coordinates by Theorem 1).

We see that Theorem 1 reduces the problem of description and classification of pairs of compatible Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g}$ such that $R_g = \bar h h^{-1}$ has n different eigenvalues to a purely algebraic problem. As we announced above, we will reformulate it in differential geometric terms in Sect. 6.1, and solve it under the assumption that $R_g = \bar g g^{-1}$ has n different eigenvalues.

We have not succeeded in solving the problem by purely algebraic means. Like many other problems in Algebra, it reduces to a system of quadratic and linear equations (see

relations ([17])). For example, classification of Frobenius algebras is a problem of the same type. This problem is solved under the additional assumption that the Frobenius form is positive definite in [5] and, in our opinion, is out of reach otherwise. Of course, for a fixed dimension, one can find complete or partial answers. In particular, in [36], it is shown that up to dimension 6, there is a finite number of isomorphism classes of commutative associative algebras and for $n > 6$, the number of classes is infinite. In [27], the classification of nilpotent commutative associative algebras up to dimension 6 is given. See also [31, 41].

In the situation discussed in Theorem 1, consider the pencil of first-order Poisson structures $\mathcal{A}_{\lambda g + \mu \bar{g}}$, which is sometimes referred to as quasiclassical limit [20] of the non-homogeneous pencil $\mathcal{B}_{\lambda h + \mu \bar{h}} + \mathcal{A}_{\lambda g + \mu \bar{g}}$. We can ask the inverse question: *Given a flat pencil $\{\lambda g + \mu \bar{g}\}$, does the corresponding Poisson pencil $\{\mathcal{A}_{\lambda g + \mu \bar{g}}\}$ admit a perturbation with nondegenerate Darboux–Poisson structures of order three of general position?*

Theorem 1 basically shows that the main condition for the related quadruple of metrics $(h, \bar{h}, g, \bar{g})$ is the existence of a common Frobenius coordinate system for g and $\bar{g}$. Indeed, if this condition holds true and this Frobenuis coordinate system is given, then the other two metrics h and $\bar{h}$ can be 'reconstructed' by solving a system of linear equations. More precisely, we have the following

Theorem 2 *Let g and $\bar{g}$ be Poisson compatible flat metrics that admit a common Frobenius coordinate system $u^1, \ldots, u^n$, that is*

$$g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta} u^s \quad and \quad \bar{g}^{\alpha\beta}(u) = \bar{b}^{\alpha\beta} + \bar{a}_s^{\alpha\beta} u^s,$$

where $(\mathfrak{a}, b)$ and $(\bar{\mathfrak{a}}, \bar{b})$ are Frobenius pairs (here $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ denote the algebras with structure constants $a_s^{\alpha\beta}$ and $\bar{a}_s^{\alpha\beta}$, respectively). Then

  (i) *the corresponding Frobenius algebras are compatible,*
 (ii) *there exist nondegenerate metrics h and $\bar{h}$ (with $h^{\alpha\beta}$ and $\bar{h}^{\alpha\beta}$ being constant in coordinates $u^1, \ldots, u^n$), such that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible Poisson structures,*
(iii) *in Frobenius coordinates $u^1, \ldots, u^n$, the (constant) metrics h and $\bar{h}$ can always be chosen in the form*

$$h^{\alpha\beta} = m^0 b^{\alpha\beta} + a_s^{\alpha\beta} m^s \quad and \quad \bar{h}^{\alpha\beta}(u) = m^0 \bar{b}^{\alpha\beta} + \bar{a}_s^{\alpha\beta} m^s,$$
$$(m^1, \ldots, m^n) \in \mathbb{R}^n, \ m^0 \in \mathbb{R}. \tag{18}$$

2.3 AFF-Pencil

Consider a real affine space $V \simeq \mathbb{R}^n$ with coordinates $u^1, \ldots, u^n$ and define the (Nijenhuis) operator L and contravariant metric g_0 on it by:

$$L = \begin{pmatrix} u^1 & 1\,0\ldots 0 \\ u^2 & 0\,1\ldots 0 \\ & \ddots \\ u^{n-1} & 0\,0\ldots 1 \\ u^n & 0\,0\ldots 0 \end{pmatrix}, \quad g_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & -u^1 \\ 0 & 0 & \ldots & 1 & -u^1 & -u^2 \\ & & \ddots & & & \\ 0 & 1 & \ldots & -u^{n-4} & -u^{n-3} & -u^{n-2} \\ 1 & -u^1 & \ldots & -u^{n-3} & -u^{n-2} & -u^{n-1} \end{pmatrix}. \quad (19)$$

Next, introduce $n+1$ contravariant metrics $g_i = L^i g$ for $i = 0, \ldots, n$. In matrix form, we have

$$g_i = \begin{pmatrix} a_{n-i} & 0 \\ 0 & b_i \end{pmatrix}, \quad (20)$$

where a_{n-i} is a $(n-i) \times (n-i)$ matrix

$$a_{n-i} = \begin{pmatrix} 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & 1 & -u^1 \\ 0 & \ldots & 1 & -u^1 & -u^2 \\ & \ldots & & & \\ 1 & \ldots & -u^{n-i-3} & -u^{n-i-2} & -u^{n-i-1} \end{pmatrix}$$

and b_i is $i \times i$ matrix of the form

$$b_i = \begin{pmatrix} u^{n-i+1} & u^{n-i+2} & \ldots & u^{n-1} & u^n \\ u^{n-i+2} & u^{n-i+3} & \ldots & u^n & 0 \\ & & \ldots & & \\ u^{n-1} & u^n & \ldots & 0 & 0 \\ u^n & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

In particular, $g_0 = a_n$ and $g_n = b_n$.

The metrics $g_0, g_1, \ldots, g_n$ are flat and pairwise compatible, so that they generate an $n+1$-dimensional flat pencil with remarkable properties, see,e.g. [9, 20]. We can write this pencil as

$$\{P(L)g_0\}, \quad \text{where } P(\cdot) \text{is an arbitrary polynomial of degree } \le n \quad (21)$$

and L and g_0 are given by (19). We will refer to it as an *AFF-pencil*. This pencil was discovered, in the form (19) and (20), by M. Antonowicz and A. Fordy [1]. As we see, the components of each metric g_i are affine functions, moreover, the coordinates $(u^1, \ldots, u^n)$ are common Frobenius coordinates for all of them.

The corresponding Frobenius algebras are easy to describe. Consider two well-known examples:

- the algebra $\mathfrak{a}_n$ of dimension n with basis $e_1, e_2, \ldots, e_n$ and relations

$$e_i \star e_j = \begin{cases} e_{i+j}, & \text{if } i + j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\mathfrak{a}_n$ can be modelled as the matrix algebra $\mathrm{Span}(J, J^2, \ldots, J^n)$, where J is the nilpotent Jordan block of size $(n+1) \times (n+1)$. It contains *no multiplicative unity* element.

- the algebra $\mathfrak{b}_n$ of dimension n with basis $e_1, e_2, \ldots, e_n$ and relations

$$e_i \star e_j = \begin{cases} e_{i+j-1}, & \text{if } i + j - 1 \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

This algebra can be understood as the *unital* matrix algebra $\mathrm{Span}(\mathrm{Id}, J, J^2, \ldots, J^{n-1})$ where J is the nilpotent Jordan block of size $n \times n$. The difference from the previous example is that $\mathfrak{b}_n$, by definition, contains the identity matrix. Equivalently, we can define $\mathfrak{b}_n$ as the algebra of truncated polynomials $\mathbb{R}[x]/\langle x^n \rangle$ (similarly $\mathfrak{a}_n \simeq \langle x \rangle / \langle x^{n+1} \rangle$).

It is straightforward to see that the metric $g_n = b_n$ is related to the Frobenius algebra $\mathfrak{b}_n$. Similarly $g_0 = a_n$ is related to the Frobenius algebra $\mathfrak{a}_n$ (this becomes obvious if we reverse the order of basis vectors and multiply each of them by -1). Hence, formula (20) shows that the Frobenius algebra associated with g_i is isomorphic to the direct sum $\mathfrak{a}_{n-i} \oplus \mathfrak{b}_i$.

It is interesting that a generic metric $g = P(L)g_0$ from the AFF-pencil (21), i.e. such that $P(L)$ has n distinct roots, corresponds to the direct sum $\mathbb{R} \oplus \cdots \oplus \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, where each copy of $\mathbb{R}$ relates to a real root and each copy of $\mathbb{C}$ relates to a pair of complex conjugate roots of $P(\cdot)$.

It is a remarkable fact that for each g_i, we can find a partner h_i such that $\mathcal{B}_{h_i} + \mathcal{A}_{g_i}$ is a Poisson structure and all these structures are pairwise compatible. The (constant) metrics h_i take the form

$$h_i = (g_i)_{\bar{m}, m^0}, \quad \bar{m} = (m^1, \ldots, m^n) \in \mathbb{R}^n, m^0 \in \mathbb{R}, \tag{22}$$

where $(g_i)_{\bar{m}, m^0}$ is obtained from the matrix $g_i(u)$ by replacing u^s with m^s and all 1's with m^0. In this way, we obtain an $(n+1)$-dimensional pencil of non-homogeneous Poisson structures generated by $\mathcal{B}_{h_i} + \mathcal{A}_{g_i}$:

$$\left\{ \sum_{i=0}^{n} c_i \left(\mathcal{B}_{h_i} + \mathcal{A}_{g_i} \right) \right\}_{c_i \in \mathbb{R}} \tag{23}$$

Alternatively, the pencil (23) can be described as follows. Fix $\bar{m} = (m^1, \ldots, m^n) \in \mathbb{R}^n$, $m^0 \in \mathbb{R}$ and let $L(\bar{m})$ denote the operator with constant entries obtained from $L = L(u)$ by replacing u^i with constants $m^i \in \mathbb{R}$. Similarly, $g_0(\bar{m})$ denotes the metric with constant coefficients obtained from g_0 by replacing u^i with the same constants $m^i \in \mathbb{R}$.

Then for $g = P(L)g_0$, we can define its partner h (metric with constant entries) as

$$h = m^0 P \left(L \left(\tfrac{1}{m^0} \bar{m} \right) \right) g_0 \left(\tfrac{1}{m^0} \bar{m} \right)$$

It can be easily checked that the correspondence $(m^0, m^1, \ldots, m^n) \mapsto h$ defined by this formula is linear so that it makes sense for $m_0 = 0$ (the denominators cancel out). Then the pencil (23) can be, equivalently, defined as

$$\left\{ \mathcal{B}_{m^0 P\left(L\left(\frac{1}{m^0}\bar{m}\right)\right)g_0\left(\frac{1}{m^0}\bar{m}\right)} + \mathcal{A}_{P(L)g_0} \right\}_{\deg P(\cdot) \leq n}. \tag{24}$$

Notice that such a pencil is not unique, as the above construction depends on $n + 1$ arbitrary parameters $m^0, m^1, \ldots, m^n$. In other words, in (24), the polynomial $P(\cdot)$ serves as a parameter of the bracket within the AFF-pencil, whereas $(m_0, \bar{m})$ parametrise dispersive perturbations of this pencil.

Remark 2.1 For our purposes below, it will be convenient to rewrite this pencil in another coordinate system by taking the eigenvalues of L as local coordinates $x^1, \ldots, x^n$. In these coordinates, g_0 and L from (19) take the following diagonal form (see, e.g. [20, p. 214] or [6, §6.2])[3]:

$$g_{\mathsf{LC}} = \sum_{i=1}^{n} \left(\prod_{s \neq i} (x^i - x^s) \right)^{-1} \left(\frac{\partial}{\partial x^i} \right)^2, \quad L = \mathrm{diag}(x^1, \ldots, x^n), \tag{25}$$

so that the AFF-pencil (21) becomes diagonal too:

$$\{P(L)g_{\mathsf{LC}}\}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree } \leq n. \tag{26}$$

We also notice that the transition from the diagonal coordinates x to Frobenius coordinates u is quite natural: the coordinates u^i are the coefficients σ_i of the characteristic polynomial $\chi_L(t) = \det(t \cdot \mathrm{Id} - L) = t^n - \sigma_1 t^{n-1} - \sigma_2 t^{n-2} - \cdots - \sigma_n$, so that, up to sign, u_i are elementary symmetric polynomials in $x^1, \ldots, x^n$.

The AFF-pencil provides a lot of examples of compatible flat metrics g and $\bar{g}$ that admit a common Frobenius coordinate system: one can take any two metrics from the pencil (21) or, equivalently, (26).

3 Compatible Flat Metrics with a Common Frobenius Coordinate System: Generic Case

Theorems 1 and 2 reduce the compatibility problem for two Poisson structures of the form $\mathcal{B}_h + \mathcal{A}_g$ to a classification of all pairs of metrics g and $\bar{g}$ admitting a common Frobenius coordinate system. The next theorem solves this problem under the standard assumption that $R_g = \bar{g}g^{-1}$ has n different eigenvalues and one minor additional condition.

---

[3] The letters LC in g_{LC} refer to Levi-Civita. The metric g_{LC} played the key role in his classification of geodesically equivalent metrics [30]. See also [9] for discussion on the relationship between geodesically equivalent and Poisson-compatible metrics.

Theorem 3 *Let g and $\bar{g}$ be compatible flat metrics that admit a common Frobenius coordinate system. Assume that the eigenvalues of the operator $R_g = \bar{g}g^{-1}$ are all different and in the diagonal coordinates (such that R_g is diagonal), every diagonal component of g depends on all variables. Then the flat pencil $\lambda g + \mu\bar{g}$ is contained in the AFF-pencil, in other words, there exists a coordinate system $(x^1, \ldots, x^n)$ such that*

$$g = P(L)g_{\text{LC}} \quad and \quad \bar{g} = Q(L)g_{\text{LC}}.$$

for some polynomials $P(\cdot)$ and $Q(\cdot)$ of degree $\leq n$ and g_{LC} and L defined by (25).

Moreover, if $n \geq 2$ and $P(\cdot)$ and $Q(\cdot)$ are not proportional, then the common Frobenius coordinate system for $g = P(L)g_{\text{LC}}$ and $\bar{g} = Q(L)g_{\text{LC}}$ is unique up to an affine coordinate change.

Theorem 3 will be proved in Sect. 6. The uniqueness part will be explained in Sect. 7.3, see Remark 7.1.

Remark 3.1 In Theorem 3, we allow some of the eigenvalues R_g to be complex. In this case, we think that a part of the *diagonal* coordinates $(x^1, \ldots, x^n)$ is also complex-valued. For example, the coordinates $x^1, \ldots, x^k$ may be real-valued, and the remaining coordinates $x^{k+1} = z^1, x^{k+2} = \bar{z}^1, \ldots, x^{n-1} = z^{\frac{n-k}{2}}, x^n = \bar{z}^{\frac{n-k}{2}}$, where ' ¯ ' means complex conjugation, are complex-valued. In this case, (26) gives us a well-defined (real) metric g_{LC} and a (real) Nijenhuis operator L.

The genericity condition in Theorem 3 is that every diagonal component of g depends on all variables. In Theorems 4, 5 below, we will solve the problem in full generality, without assuming this or any other genericity condition.

Remark 3.2 In [20, §5] E. Ferapontov and M. Pavlov asked whether dispersive perturbations of the pencil (21) with g_0 and L given by (25) other than those described in Sect. 2.3 are possible. Theorem 3 leads to a negative answer under the additional assumption that the dispersive perturbation is in the class of nondegenerate Darboux–Poisson structures of order 3. Indeed, according to Theorem 1, every dispersive perturbation $\lambda(\mathcal{B}_{\bar{h}} + \mathcal{A}_g) + \mu(\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}})$ of the pencil $\lambda\mathcal{A}_g + \mu\mathcal{A}_{\bar{g}}$ can be reduced to a simple normal form in a common Frobenius coordinate system for g and $\bar{g}$ (assuming that $R_h = \bar{h}h^{-1}$ has different eigenvalues). Moreover, in this coordinate system, h and $\bar{h}$ are constant and represent Frobenius forms for the corresponding Frobenius algebras $\mathfrak{a}$ and $\bar{\mathfrak{a}}$. Since by Theorem 3, such a coordinate system is unique, it remains to solve a Linear Algebra problem of choosing suitable forms h and $\bar{h}$, satisfying three conditions (cf. (17)):

$$h(\xi \star \eta, \zeta) = h(\xi, \eta \star \zeta),$$
$$\bar{h}(\xi \,\bar{\star}\, \eta, \zeta) = \bar{h}(\xi, \eta \,\bar{\star}\, \zeta), \tag{27}$$
$$\bar{h}(\xi \star \eta, \zeta) + h(\xi \,\bar{\star}\, \eta, \zeta) = \bar{h}(\xi, \eta \star \zeta) + h(\xi, \eta \,\bar{\star}\, \zeta),$$

It is straightforward to show for a generic pair $g, \bar{g}$ of metrics from the AFF-pencil, the forms h and $\bar{h}$ are defined by $n + 1$ parameters $m^0, m^1, \ldots, m^n$ as in (24). No other

solutions exist. In particular, formula (24) describes all possible dispersive perturbations of the AFF-pencil by means of nondegenerate Darboux–Poisson structures of order 3. Moreover, this conclusion holds for any generic two-dimensional subpencil.

4 Compatible Flat Metrics with a Common Frobenius Coordinate System: General Case

4.1 General Multi-block Frobenius Pencils

Let us now discuss the general case without assuming that in diagonal coordinates, every diagonal component of g depends on all variables.

Similar to Theorem 3, the metrics g and $\bar{g}$ will belong to a large Frobenius pencil built up from several blocks each of which has a structure of an (extended) AFF pencil. We start with constructing a series of such pencils.

We first divide our diagonal coordinates into B blocks of positive dimensions $n_1, \ldots, n_B$ with $n_1 + \cdots + n_B = n$:

$$(\underbrace{x_1^1, \ldots, x_1^{n_1}}_{X_1}, \ldots, \underbrace{x_B^1, \ldots, x_B^{n_B}}_{X_B}). \tag{28}$$

Next, we consider a collection of n_α-dimensional Levi-Civita metrics g_α^{LC} and n_α-dimensional operators L_α (as in Theorem 3 but now for each block separately):

$$g_\alpha^{\mathsf{LC}} = \sum_{s=1}^{n_\alpha} \left(\prod_{j \neq s} (x_\alpha^s - x_\alpha^j) \right)^{-1} \left(\frac{\partial}{\partial x_\alpha^s} \right)^2 \ , \quad L_\alpha = \mathrm{diag}(x_\alpha^1, \ldots, x_\alpha^{n_\alpha}). \tag{29}$$

Then we introduce a new block-diagonal metric $\widehat{g}$

$$\widehat{g} = \mathrm{diag}(\widehat{g}_1, \ldots, \widehat{g}_B) \quad \text{with } \widehat{g}_\alpha = \prod_{s < \alpha} \left(\frac{1}{\det(\lambda_{s\alpha} \cdot \mathrm{Id} - L_s)} \right)^{c_{s\alpha}} g_\alpha^{\mathsf{LC}}, \tag{30}$$

where $c_{s\alpha} = 0$ or 1. The values of the discrete parameters $c_{s\alpha}$ and numbers $\lambda_{s\alpha}$ are determined by some combinatorial data as explained below.
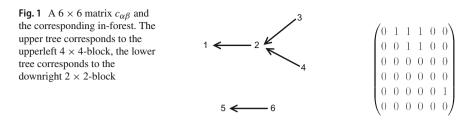
Finally, we consider the pencil of (contravariant) metrics of the form

$$\{\widehat{L}\,\widehat{g} \mid \widehat{L} \in \mathcal{L}\} \tag{31}$$

where $\mathcal{L}$ is a family (pencil) of block-diagonal operators of the form

$$\widehat{L} = \mathrm{diag}\big(P_1(L_1), P_2(L_2), \ldots, P_B(L_B)\big).$$

where $P_\alpha(\cdot)$ are polynomials with $\deg P_\alpha \leq n_\alpha + 1$ treated as parameters of this family. The coefficients of the polynomials P_α are not arbitrary but satisfy a collection

Fig. 1 A 6×6 matrix $c_{\alpha\beta}$ and the corresponding in-forest. The upper tree corresponds to the upperleft 4×4-block, the lower tree corresponds to the downright 2×2-block

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

of linear relations involving coefficients from different polynomials so that this pencil, in general, is not a direct sum of blocks (although, direct sum is a particular example). Notice that $\mathcal{L}$ is a Nijenhuis pencil whose algebraic structure is quite different from that of the pencil $\{P(L)\}$ from Theorem 3.

The numbers $c_{s\alpha}$, $\lambda_{s\alpha}$ and relations on the coefficients of P_α's are determined by a combinatorial object, an oriented graph F with special properties, namely, a directed rooted in-forest (see [26] for a definition) whose edges are labelled by numerical marks λ_α. This graph may consists of several connected components, each of which is a rooted tree whose edges are oriented from its *leaves* to the *root*. An example is shown in Fig. 1.

Each vertex of F is associated with a certain block of the above decomposition (28) and labelled by an integer number $\alpha \in \{1, \ldots, B\}$. The structure of a directed graph defines a natural strict partial order (denoted by $\prec$) on the set $\{1, \ldots, B\}$: for two numbers $\alpha \neq \beta \in \{1, \ldots, B\}$, we set $\alpha \prec \beta$, if there exists an oriented way from β to α. For instance, for the graph shown on Fig. 1, we have $1 \prec 3$, $2 \prec 4$, $5 \prec 6$. Without loss of generality, we can and will always assume that the vertices of F are labelled in such a way that $\alpha \prec \beta$ implies $\alpha < \beta$.

Notice that the vertices of degree one are of two types, roots and leaves: α is a *root* if there is no β such that $\beta \prec \alpha$ and, conversely, β is a *leaf* if there is no β such that $\alpha \prec \beta$. Notice that roots of degree ≥ 2 are also allowed, whereas all leaves have degree 1. We say $\alpha = \text{next}(\beta)$, if $\alpha \prec \beta$ and there is no γ with $\alpha \prec \gamma \prec \beta$. In the upper tree of Fig. 1,the root is 1, the leaves are 3 and 4 and we have: $1 = \text{next}(2)$ and $2 = \text{next}(3)$, $2 = \text{next}(4)$.

The numbers $c_{s\alpha}$ in (30) are now defined from F as follows:

$$c_{s\alpha} = \begin{cases} 1, & \text{if } s \prec \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Recall that in our assumptions, $s \prec \alpha$ implies $s < \alpha$ so that the $B \times B$-matrix $c_{s\alpha}$ is upper triangular with zeros on the diagonal, see Fig. 1.

The parameters $\lambda_{s\alpha}$ are defined as follows. For each vertex α that is not a root, there is exactly one out-going edge which we will denote by e_α. Notice that the correspondence $\alpha \mapsto e_\alpha$ is a bijection between the set of edges of F and the (sub)set of vertices which are not roots. To each edge e_α,we now assign a number λ_α (these numbers will serve as parameters of our construction) and set

$$\lambda_{s\alpha} = \lambda_\beta, \quad \text{where } s \prec \beta \prec \alpha \text{ and } s = \text{next}(\beta) \text{ (or } \beta = \alpha, \text{ if } s = \text{next}(\alpha)\text{). (33)}$$

Such β exists and is unique, if $s \prec \alpha$, i.e. $c_{s\alpha} = 1$. Otherwise, $c_{s\alpha} = 0$ and the value of $\lambda_{s\alpha}$ plays no role in (30).

Remark 4.1 The above definitions of parameters $c_{s\alpha}$ and $\lambda_{s\alpha}$ are convenient to make our formulas shorter, but do not quite clarify the meaning of (30) in terms of the graph F. Equivalently, formula (30) can be rewritten as follows:

$$\widehat{g} = \mathrm{diag}(\widehat{g}_1, \ldots, \widehat{g}_B) \quad \text{with } \widehat{g}_\alpha = f_\alpha \cdot g_\alpha^{\mathsf{LC}},$$

where the function f_α is defined by the oriented path from the vertex α to a certain root β:

$$\beta = \alpha_0 \xleftarrow{\lambda_{\alpha_1}} \alpha_1 \xleftarrow{\lambda_{\alpha_2}} \ldots \xleftarrow{\lambda_{\alpha_{k-1}}} \alpha_{k-1} \xleftarrow{\lambda_{\alpha_k}} \alpha_k = \alpha,$$

each edge of which is endowed with a number λ_{α_i}, $i = 1, \ldots, k$. Namely, we set

$$f_\alpha = \prod_{i=1}^{k} \frac{1}{\det(\lambda_{\alpha_i} \cdot \mathrm{Id} - L_{\alpha_{i-1}})},$$

which coincides with the factor in front of g_α^{LC} in formula (30) written in terms of $c_{s\alpha}$ and $\lambda_{s\alpha}$.

Finally, for a vertex α, we denote the coefficients of the corresponding polynomial P_α by $P_\alpha(t) = \overset{\alpha}{a}_0 + \overset{\alpha}{a}_1 t + \cdots + \overset{\alpha}{a}_{n_\alpha+1} t^{n_\alpha+1}$. Then the conditions on the coefficients $P_\alpha(t)$ are

(i) If α is a root, then $\overset{\alpha}{a}_{n_\alpha+1} = 0$, i.e. $\deg P_\alpha \le n_\alpha$.

(ii) If $\alpha = \mathrm{next}(\beta)$, then λ_β is a root of P_α and $\overset{\beta}{a}_{n_\beta+1} = P'(\lambda_\beta)$, where $P'(t)$ denotes the derivative of $P(t)$.

(iii) If $\alpha = \mathrm{next}(\beta)$ and $\alpha = \mathrm{next}(\gamma)$ with $\lambda_\beta = \lambda_\gamma = \lambda$, $\beta \ne \gamma$, then λ is a double root of P_α (in view of (ii) this automatically implies $\overset{\beta}{a}_{n_\alpha+1} = \overset{\gamma}{a}_{n_\gamma+1} = 0$).

Remark 4.2 Each of the above conditions is linear in the coefficients of P_α's. However, (i)–(iii) may imply that $P_\alpha = 0$ for some α. This happens, for instance, if the vertex α has too many neighbours γ_i such that $\alpha = \mathrm{next}(\gamma_i)$, $i = 1, \ldots, k$. Then all λ_{γ_i} must be roots of P_α due to (ii). However, in view of (i), $\deg P_\alpha \le n_\alpha + 1$. If $n_\alpha + 1 < k$ and λ_{γ_i} are all different, then P_α cannot have k different roots unless $P_\alpha = 0$. Strictly speaking, such a situation should be excluded as the corresponding metrics turn out to be degenerate at every point. However, from the algebraic point of view, we still obtain an example of a good Frobenius pencil.

We also notice that the shift $L_\alpha \mapsto L_\alpha + c_\alpha \,\mathrm{Id}$ in any individual block leads to an isomorphic pencil. In particular, if at a certain vertex α of F we add the same number c_α simultaneously to all numerical parameters λ_β on the incoming edges, we get an isomorphic pencil.

This completes the description of the pencil (31) of (contravariant) metrics and we can state our next result.

Theorem 4 *The pencil* (31) *(with $c_{s\alpha}$ defined by* (32), *$\lambda_{s\alpha}$ defined by* (33) *and coefficients of P_α satisfying* (i)–(iii)) *is Frobenius. In other words, all the metrics*

$$g = \widehat{L}\,\widehat{g} = \mathrm{diag}\big(P_1(L_1)\widehat{g}_1, \ldots, P_B(L_B)\widehat{g}_B\big) \quad with \ \widehat{g}_\alpha = \prod_{s<\alpha}\left(\frac{1}{\det(\lambda_{s\alpha}\cdot\mathrm{Id}-L_s)}\right)^{c_{s\alpha}} g_\alpha^{\mathsf{LC}},$$

(34)

are flat, Poisson compatible and admit a common Frobenius coordinate system

$$(u^1, \ldots, u^n) = (\underbrace{u_1^1, \ldots, u_1^{n_1}}_{U_1}, \ldots, \underbrace{u_B^1, \ldots, u_B^{n_B}}_{U_B})$$

(35)

which is defined as follows. Let $\sigma_\alpha^1, \ldots, \sigma_\alpha^{n_i}$ denote the coefficients of the characteristic polynomial of L_α

$$\chi_{L_\alpha}(t) := \det\big(t\,\mathrm{Id}_{n_\alpha\times n_\alpha}-L_\alpha\big) = t^{n_\alpha} - \sigma_\alpha^1 t^{n_\alpha-1} - \sigma_\alpha^2 t^{n_\alpha-2} - \cdots - \sigma_\alpha^{n_\alpha}, \quad \alpha = 1, \ldots, B.$$

Then

$$\begin{aligned}
u_1^k &= \sigma_1^k, & k &= 1, \ldots, n_1,\\
u_2^k &= \big(\det(\lambda_{12}\,\mathrm{Id}-L_1)\big)^{c_{12}}\,\sigma_2^k, & k &= 1, \ldots, n_2,\\
u_3^k &= \big(\det(\lambda_{13}\,\mathrm{Id}-L_1)\big)^{c_{13}}\big(\det(\lambda_{23}\,\mathrm{Id}-L_2)\big)^{c_{23}}\,\sigma_3^k, & k &= 1, \ldots, n_3,\\
&\cdots\\
u_B^k &= \prod_{s<B}\big(\det(\lambda_{sB}\,\mathrm{Id}-L_s)\big)^{c_{sB}}\,\sigma_B^k, & k &= 1, \ldots, n_B.
\end{aligned}$$

(36)

Theorem 4 will be proved in Sect. 8.

The advantage of the formulas for Frobenius coordinates in Theorem 4 is that they are invariant in the sense they do not depend on the choice of coordinates in blocks, but use coefficients of the characteristic polynomials of blocks L_i.

Let us explain how one can use this property to check algorithmically (say, using computer algebra software) that the coordinates in Theorem 4 are indeed Frobenius for the metric g.

In each block (with number α), we change from diagonal coordinates $X_\alpha = (x_\alpha^1, \ldots, x_\alpha^{n_\alpha})$ to the coordinates $Y_\alpha = (y_\alpha^1, \ldots, y_\alpha^{n_\alpha})$ given as follows:

$$\chi_{L_\alpha}(t) = t^{n_\alpha} - y_\alpha^1\,t^{n_\alpha-1} - y_\alpha^2\,t^{n_\alpha-2} - \cdots - y_\alpha^{n_\alpha}.$$

(37)

Note that in the coordinates Y_α, the metric g_α^{LC} and the operator L_α have the form (19) with $u^1, \ldots, u^n$ replaced by $y_\alpha^1, \ldots, y_\alpha^{n_\alpha}$. The iterated warped product metric $g = (g^{ij})$ is given by the following easy algebraic formula

$$g = g_1 + \left(\frac{1}{\chi_{L_1}(\lambda_{12})}\right)^{c_{12}} g_2 + \left(\frac{1}{\chi_{L_1}(\lambda_{13})}\right)^{c_{13}} \left(\frac{1}{\chi_{L_1}(\lambda_{23})}\right)^{c_{23}} g_3 + \cdots ,$$

with $g_\alpha = P_\alpha(L_\alpha)g_\alpha^{\mathsf{LC}}$ and g_α^{LC} and L_α explicitly given by (19).

In order to check whether the coordinates u given by (36) are Frobenius, one needs to perform the multiplication

$$J g J^\top ,$$

where $J = \left(\frac{\partial u^i}{\partial y^j}\right)$ is the Jacobi matrix of the coordinate transformation[4] $(y^1, \ldots, y^n)$ $\to (u^1, \ldots, u^n)$ and check whether the entries of the resulting matrix $J g J^\top$ are affine functions in u^i and conditions (17) are fulfilled. All these operations can be realised by standard computer algebra packages.

The next result gives a description of two-dimensional Frobenius pencils in the general case.

Theorem 5 *Let g and $\bar{g}$ be compatible flat metrics that admit a common Frobenius coordinate system. If the eigenvalues of the operator $R_g = \bar{g}g^{-1}$ are all different at a point p, then in a neighbourhood of this point, the pencil $\lambda g + \mu \bar{g}$ is isomorphic to a two-dimensional subpencil of the Frobenius pencil (31) with suitable parameters, i.e. in a certain coordinate system these metrics take the form*

$$g = \mathrm{diag}\,(P_1(L_1)\widehat{g}_1, \ldots, P_B(L_B)\widehat{g}_B) \quad and \quad \bar{g} = \mathrm{diag}\,(Q_1(L_1)\widehat{g}_1, \ldots, Q_B(L_B)\widehat{g}_B)$$
(38)

(with parameters $c_{\alpha\beta}$ defined by (32), $\lambda_{s\alpha}$ defined by (33) and coefficients of P_α and Q_α satisfying (i)–(iii)).

Theorem 5 will be proved in Sect. 7.

Remark 4.3 In Theorem 5,we allow complex eigenvalues of R_g. The corresponding part of diagonal coordinates is then complex. Moreover, the polynomials P_α and Q_α may have complex coefficients, and also the numbers $\lambda_{s\alpha}$ may be complex. The only condition is that the metrics given by (30) should be well-defined real metrics. It is easy to see that this condition implies in particular that every block $(g_\alpha^{\mathsf{LC}}, L_\alpha)$ is either real or pure complex (= all coordinates are complex; the coefficients of the polynomials P_α and Q_α may be complex as well), and that a pure complex block comes together with a complex conjugate one. See also [6, §3] for discussion on Nijenhuis operators some of whose eigenvalues are complex.

In certain special cases, a common Frobenius coordinate system for g and $\bar{g}$ is not unique (up to affine transformations). This is the case when $n_\alpha = 1$, $c_{\alpha\beta} = 0$ for all β (i.e. this block represents a leaf of the corresponding in-forest) and the diagonal component of $R_g = \bar{g}g^{-1}$ corresponding to this block is constant; in other words, the

---

[4] This transformation is given by (36) as $y_\alpha^i = \sigma_\alpha^i$ and $\det(\lambda_{\alpha\beta}\,\mathrm{Id} - L_\alpha) = \chi_{L_\alpha}(\lambda_{\alpha\beta})$.

(quadratic) polynomials P_α and Q_α are proportional. The restrictions g_α and $\bar{g}_\alpha$ onto these blocks are then as follows

$$
\begin{aligned}
g_\alpha &= f \cdot \left(a_2(x^\alpha)^2 + a_1 x^\alpha + a_0 \right) \left(\tfrac{\partial}{\partial x^\alpha} \right)^2 \quad \text{and} \quad \bar{g}_\alpha = c\, g_\alpha \\
&= c\, f \cdot \left(a_2(x^\alpha)^2 + a_1 x^\alpha + a_0 \right) \left(\tfrac{\partial}{\partial x^\alpha} \right)^2 ,
\end{aligned}
$$

where f is some function of the remaining coordinates, $c \in \mathbb{R}$ and $L_\alpha = (x^\alpha)$ (diagonal 1×1 matrix). However, we can do coordinate transformation $x^\alpha \mapsto \tilde{x}^\alpha = \tilde{x}^\alpha(x^\alpha)$ that changes the coefficients a_1 and a_0 (the highest coefficient a_2 is fixed by condition (ii)).

$$
\begin{aligned}
g_\alpha &= f \cdot \left(a_2(\tilde{x}^\alpha)^2 + \tilde{a}_1 \tilde{x}^\alpha + \tilde{a}_0 \right) \left(\tfrac{\partial}{\partial \tilde{x}^\alpha} \right)^2 \quad \text{and} \quad \bar{g}_\alpha \\
&= c\, g_\alpha = c\, f \cdot \left(a_2(\tilde{x}^\alpha)^2 + \tilde{a}_1 \tilde{x}^\alpha + \tilde{a}_0 \right) \left(\tfrac{\partial}{\partial \tilde{x}^\alpha} \right)^2 ,
\end{aligned}
$$

Hence, with a new operator $L_\alpha^{\text{new}} = (\tilde{x}^\alpha)$ and new polynomials $P_\alpha^{\text{new}}(t) = a_2 t^2 + \tilde{a}_1 t + \tilde{a}_0$, $Q_\alpha^{\text{new}}(t) = c(a_2 t^2 + \tilde{a}_1 t + \tilde{a}_0)$, we still remain in the framework of our construction and (38) still holds. This transformation will lead to another Frobenius coordinate system. In Sect. 7.3, we explain that only this situation allows ambiguity in the choice of Frobenius coordinates up to affine transformations.

Remark 4.4 In [20, Theorem 2], it was claimed that *under some general assumptions for $n > 2$, there is only one equivalence class of $(n + 1)$-Hamiltonian hydrodynamic systems* (in the sense of [20]) and $n + 1$ is the best possible. The corresponding multi-Hamiltonian structure comes from the $(n + 1)$-dimensional AFF-pencil. In this view, it is interesting to notice that multi-block pencils from Theorem 5 also provide such a structure, which may have even higher dimension.

4.2 Case of Two Blocks

In the case of two blocks, i.e. $B = 2$, the construction explained in the previous section gives a natural and rather simple answer. We have two cases: $c_{12} = 0$ and $c_{12} = 1$. The first case is trivial being a direct product of two blocks (possibly complex conjugate) each of which is as in Theorem 3; in (34) we set $\widehat{g}_i = g_i^{\mathsf{LC}}$ and take P_i to be arbitrary polynomials of degrees $\leq n_i$ ($i = 1, 2$).

Theorem below is a special case of Theorem 4 in the nontrivial case $c_{12} = 1$.

Theorem 6 *Suppose $B = 2$, $c_{12} = 1$ and consider the metric g given by the construction from Sect. 4.1:*

$$
g = g_1 + \frac{1}{\det(-L_1)}\, g_2, \quad \text{with } g_i = P_i(L_i) g_i^{\mathsf{LC}}. \tag{39}
$$

Following this construction, assume that the polynomials P_1 and P_2 have degrees no greater than n_1 and $n_2 + 1$, respectively: $P_1 = \sum_{s=0}^{n_1} a_s t^s$ and $P_2 = \sum_{s=0}^{n_2+1} b_s t^s$.

Then the coordinates from Theorem 4 are Frobenius for g if and only if $a_0 = 0$ and $a_1 = b_{n_2+1}$.

Example 4.1 In Theorem 6, take $n_1 = n_2 = 2$. In diagonal coordinates x^1, x^2, x^3, x^4, the metric $g = (g^{ij})$ is as follows:

$$g = \text{diag} \left(\frac{P_1(x^1)}{x^1 - x^2}, \frac{P_1(x^2)}{x^2 - x^1}, \frac{P_2(x^3)}{x^1 x^2 (x^3 - x^4)}, \frac{P_2(x^4)}{x^1 x^2 (x^4 - x^3)} \right),$$

where $P_1(t) = a_1 t + a_2 t^2$ and $P_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$ with $b_3 = a_1$. Recall that $L = L_1 \oplus L_2$ with $L_1 = \text{diag}(x^1, x^2)$, $L_2 = \text{diag}(x^3, x^4)$, and the relation between the diagonal coordinates x^i and the Frobenius coordinates u^i given by Theorem 4 are as follows:

$$
\begin{aligned}
u^1 &= \text{tr}\, L_1 = x^1 + x^2, \\
u^2 &= -\det L_1 = -x^1 x^2, \\
u^3 &= \det L_1 \cdot \text{tr}\, L_2 = x^1 x^2 (x^3 + x^4), \\
u^4 &= -\det L_1 \cdot \det L_2 = -x^1 x^2 x^3 x^4 = -\det L.
\end{aligned}
$$

In these Frobenius coordinates, the metric $g = (g^{ij})$ has the following form:

$$
g = \begin{pmatrix}
a_2 u^1 + a_1 & a_2 u^2 & a_2 u^3 & a_2 u^4 \\
a_2 u^2 & a_1 u^2 & a_1 u^3 & a_1 u^4 \\
a_2 u^3 & a_1 u^3 & -a_1 u^4 - b_1 u^2 - b_2 u^3 & -b_0 u^2 - b_2 u^4 \\
a_2 u^4 & a_1 u^4 & -b_0 u^2 - b_2 u^4 & b_0 u^3 - b_1 u^4
\end{pmatrix}.
$$

This formula defines a 5-dimensional pencils of metrics (with parameters a_1, a_2, b_0, b_1, b_2). For any choice of the parameters such that g is nondegenerate, the coordinates u^i are Frobenius for it in the sense of Definition 3.

From the algebraic viewpoint, we may equivalently think of this formula as 5-parametric family (pencil) of Frobenius algebras $(\mathfrak{a}, b)$. The entries of g define the structure constants of $\mathfrak{a}$. For instance, $g^{11} = a_2 u^1 + a_1$ and $g^{34} = -b_0 u^2 - b_2 u^4$ imply

$$e^1 \star e^2 = a_2 e^1 \quad \text{and} \quad e^3 \star e^4 = -b_0 e^2 - b_2 e^4$$

for a basis e^1, e^2, e^3, e^4 of $\mathfrak{a}$. The matrix (b^{ij}) of the corresponding Frobenius form b is obtained from that of g by assigning to u^i any constant values $u^i = m^i \in \mathbb{R}$ (such that b is nondegenerate for generic choice of a_1, a_2, b_0, b_1, b_2). To get a Frobenius pencil, the constants m^i should be the same for all parameters a_1, a_2, b_0, b_1, b_2.

In the coordinates $(u^1, \ldots, u^4)$ the operators L_1 and L_2 are given by the matrices

$$L_1 = \begin{pmatrix} u^1 & 1 & 0 & 0 \\ u^2 & 0 & 0 & 0 \\ u^3 & 0 & 0 & 0 \\ u^4 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-u^4 u^2 - (u^3)^2}{(u^2)^2} & \frac{u^3}{u^2} & 1 \\ 0 & -\frac{u^4 u^3}{(u^2)^2} & \frac{u^4}{u^2} & 0 \end{pmatrix}.$$

The matrices of g_1^{LC} and g_2^{LC} are

$$g_1^{\mathsf{LC}} = \begin{pmatrix} 0 & 1 & \frac{u^3}{u^2} & \frac{u^4}{u^2} \\ 1 & -u^1 & -\frac{u^3 u^1}{u^2} & -\frac{u^1 u^4}{u^2} \\ \frac{u^3}{u^2} & -\frac{u^3 u^1}{u^2} & -\frac{(u^3)^2 u^1}{(u^2)^2} & -\frac{u^1 u^4 u^3}{(u^2)^2} \\ \frac{u^4}{u^2} & -\frac{u^1 u^4}{u^2} & -\frac{u^1 u^4 u^3}{(u^2)^2} & -\frac{u^1 (u^4)^2}{(u^2)^2} \end{pmatrix}, \quad g_2^{\mathsf{LC}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u^2 \\ 0 & 0 & -u^2 & u^3 \end{pmatrix}.$$

5 Proof of Theorems 1 and 2

Proof of Theorem 1 We assume that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible with the additional condition that the eigenvalues of $R_h = \bar{h}h^{-1}$ are pairwise different. We also assume that $h + \bar{h}$ is nondegenerate.

Recall that Theorem 7.1 from [14] implies that $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ are compatible Poisson structures (item (i) in Fact 1). Let $\Gamma_s^{\alpha\beta}$ and $\bar{\Gamma}_s^{\alpha\beta}$ denote the contravariant Levi-Civita connections of h and $\bar{h}$. From Theorem 3.2 in [14] applied to $\mathcal{B}_h + \mathcal{B}_{\bar{h}}$, it follows that the connection $\widehat{\Gamma}_{qs}^{\beta}$ defined from

$$\Gamma_s^{\alpha\beta} + \bar{\Gamma}_s^{\alpha\beta} = (h + \bar{h})^{\alpha q} \widehat{\Gamma}_{qs}^{\beta}$$

is symmetric and flat.

By direct computation $\widehat{\nabla}(h + \bar{h}) = \nabla h + \bar{\nabla}\bar{h} = 0$, so that $\widehat{\Gamma}$ is the Levi-Civita connection for $h + \bar{h}$ and moreover, $h + \bar{h}$ is flat. According to Theorem 6.2 in [14], this implies that $\mathcal{B}_h + \mathcal{B}_{\bar{h}}$ is Darboux–Poisson (i.e. is given by (8)). Hence, in our notations, we obtain the formula

$$\mathcal{B}_h + \mathcal{B}_{\bar{h}} = \mathcal{B}_{h+\bar{h}}. \tag{40}$$

Setting $\widehat{\Gamma}_s^{\alpha\beta} = (h + \bar{h})^{\alpha q} \widehat{\Gamma}_{qs}^{\beta}$ to be the contravariant Levi-Civita connection of $h + \bar{h}$, we get

$$\Gamma_s^{\alpha\beta} + \bar{\Gamma}_s^{\alpha\beta} = \widehat{\Gamma}_s^{\alpha\beta}, \tag{41}$$

and conclude that h and $\bar{h}$ are Poisson compatible in the sense of Definition 1 (in particular, this proves the (ii)-part of Fact 2). Hence, $R_h = \bar{h}h^{-1}$ is a Nijenhuis operator (Fact 3).

For a pair of flat metrics h and $\bar{h}$, introduce the so-called obstruction tensor

$$S^{\beta}_{rq} = \Gamma^{\beta}_{rq} - \bar{\Gamma}^{\beta}_{rq}.$$

It vanishes if and only if $h, \bar{h}$ can be brought to constant form simultaneously (thus, the name). It is obviously symmetric in lower indices. Condition (41) can be written in equivalent form ([33], Lemma 3.1 and Theorem 3.2) in terms of only $\Gamma^{\alpha\beta}_s, \bar{\Gamma}^{\alpha\beta}_s, h, \bar{h}$

$$\bar{\Gamma}^{\alpha\beta}_q h^{q\gamma} - \bar{\Gamma}^{\gamma\beta}_q h^{q\alpha} + \Gamma^{\alpha\beta}_q \bar{h}^{q\gamma} - \Gamma^{\gamma\beta}_q \bar{h}^{q\alpha} = 0. \tag{42}$$

After lowering both indices with h and rearranging the terms, we get

$$S^{\beta}_{pq} R^q_s - R^q_p S^{\beta}_{qs} = 0. \tag{43}$$

For a given metric h and its Levi-Civita connection, define

$$c^{\alpha\beta}_{rs} = h^{\alpha q}\left(\Gamma^a_{qr}\Gamma^{\beta}_{as} - \frac{\partial \Gamma^{\beta}_{qs}}{\partial u^r}\right).$$

The $\bar{c}^{\alpha\beta}_{rs}, \widehat{c}^{\alpha\beta}_{rs}$ for $\bar{h}$ and $h + \bar{h}$ are defined in a similar way. This formula is one 'half' of the formula for Riemann curvature tensor and the flatness of the metrics implies that $c^{\alpha\beta}_{rs} = c^{\alpha\beta}_{sr}$ (and similarly for metrics $\bar{h}, h + \bar{h}$). Using this symmetry in lower indices, we apply the general formula (8) to the Poisson structures in (40) and collect coefficients in front of D^2 to get

$$3c^{\alpha\beta}_{rs} u^r_x u^s_x - 3\Gamma^{\alpha\beta}_s u^s_{xx} + 3\bar{c}^{\alpha\beta}_{rs} u^r_x u^s_x - 3\bar{\Gamma}^{\alpha\beta}_s u^s_{xx} = 3\widehat{c}^{\alpha\beta}_{rs} u^r_x u^s_x - 3\widehat{\Gamma}^{\alpha\beta}_s u^s_{xx}.$$

Collecting all the terms with $u^r_x u^s_x$ in this differential polynomial, in turn, implies

$$\widehat{c}^{\alpha\beta}_{rs} - c^{\alpha\beta}_{rs} - \bar{c}^{\alpha\beta}_{rs} = 0.$$

Using the characteristic property of the Levi-Civita connection

$$\frac{\partial h^{\alpha\beta}}{\partial u^s} + h^{\alpha q}\Gamma^{\beta}_{qs} + \Gamma^{\alpha}_{qs} h^{q\beta} = 0$$

we rewrite $c^{\alpha\beta}_{rs}$ as

$$c^{\alpha\beta}_{rs} = h^{\alpha q}\left(\Gamma^a_{qr}\Gamma^{\beta}_{as} - \frac{\partial \Gamma^{\beta}_{qs}}{\partial u^r}\right) = -\frac{\partial}{\partial u^r}\left[h^{\alpha q}\Gamma^{\beta}_{qs}\right] - h^{pq}\Gamma^{\alpha}_{qr}\Gamma^{\beta}_{ps} \tag{44}$$

Applying (41), (42) and (44) yields

$$
\begin{aligned}
0 &= \big(\widehat{c}^{\alpha\beta}_{rs} - c^{\alpha\beta}_{rs} - \bar{c}^{\alpha\beta}_{rs}\big)(h+\bar{h})^{s\gamma} \\
&= \Big(h^{pq}\Gamma^{\alpha}_{qr}\Gamma^{\beta}_{ps} + \bar{h}^{pq}\bar{\Gamma}^{\alpha}_{qr}\bar{\Gamma}^{\beta}_{ps} - (h+\bar{h})^{pq}\widehat{\Gamma}^{\alpha}_{qr}\widehat{\Gamma}^{\beta}_{ps}\Big)(h+\bar{h})^{s\gamma} \\
&= h^{pq}\Gamma^{\alpha}_{qr}\Gamma^{\beta}_{ps}h^{s\gamma} + h^{pq}\Gamma^{\alpha}_{qr}\Gamma^{\beta}_{ps}\bar{h}^{s\gamma} + \bar{h}^{pq}\bar{\Gamma}^{\alpha}_{qr}\bar{\Gamma}^{\beta}_{ps}h^{s\gamma} + \bar{h}^{pq}\bar{\Gamma}^{\alpha}_{qr}\bar{\Gamma}^{\beta}_{ps}\bar{h}^{s\gamma} \\
&\quad - \big(h^{pq}\Gamma^{\alpha}_{qr} + \bar{h}^{pq}\bar{\Gamma}^{\alpha}_{qr}\big)\big(\Gamma^{\beta}_{ps}h^{s\gamma} + \bar{\Gamma}^{\beta}_{ps}\bar{h}^{s\gamma}\big) \\
&= \Gamma^{\alpha}_{qr}\big(h^{pq}\Gamma^{\beta}_{ps}\bar{h}^{s\gamma} - h^{pq}\bar{\Gamma}^{\beta}_{ps}\bar{h}^{s\gamma}\big) - \bar{\Gamma}^{\alpha}_{qr}\big(\bar{h}^{pq}\Gamma^{\beta}_{ps}h^{s\gamma} - \bar{h}^{pq}\bar{\Gamma}^{\beta}_{ps}h^{s\gamma}\big) \\
&= \big(S^{\alpha}_{qr} + \bar{\Gamma}^{\alpha}_{qr}\big)\big(h^{pq}\Gamma^{\beta}_{ps}\bar{h}^{s\gamma} - h^{pq}\bar{\Gamma}^{\beta}_{ps}\bar{h}^{s\gamma}\big) - \bar{\Gamma}^{\alpha}_{qr}\big(\bar{h}^{pq}\Gamma^{\beta}_{ps}h^{s\gamma} - \bar{h}^{pq}\bar{\Gamma}^{\beta}_{ps}h^{s\gamma}\big) \\
&= S^{\alpha}_{qr}h^{pq}S^{\beta}_{ps}\bar{h}^{s\gamma} + \bar{\Gamma}^{\alpha}_{qr}\big(h^{pq}\Gamma^{\beta}_{ps}\bar{h}^{s\gamma} - h^{pq}\bar{\Gamma}^{\beta}_{ps}\bar{h}^{s\gamma} - \bar{h}^{pq}\Gamma^{\beta}_{ps}g^{s\gamma} + \bar{h}^{pq}\bar{\Gamma}^{\beta}_{ps}h^{s\gamma}\big) \\
&= S^{\alpha}_{qr}h^{pq}S^{\beta}_{ps}\bar{h}^{s\gamma}.
\end{aligned}
$$

Now consider the coordinate system in which the Nijenhuis operator R_h is diagonal. As R_h by definition is self-adjoint with respect to both h and $\bar{h}$, we get that both h and $\bar{h}$ are also diagonal. Condition (43) implies that for given β the only nonzero elements of S^{β}_{pq} are the ones that stand on the diagonal. The previous calculation yields

$$
S^{\alpha}_{qr}h^{pq}S^{\beta}_{ps}\bar{h}^{s\gamma} = 0
$$

which, for fixed α and β, is just the product of four diagonal matrices, two of which are nondegenerate. Taking $\alpha = \beta$, we see that the matrix S^{α}_{qr} must be zero. As α is arbitrary, this implies that the obstruction tensor vanishes and $h, \bar{h}$ have common Darboux coordinates.

Fix the coordinates in which both h and $\bar{h}$ are flat. Applying Fact 5, we see that these coordinates are Frobenius for both g and $\bar{g}$. Using (40) and applying Fact 5 to the sum of our Poisson structures, we get that $a^{\alpha\beta}_s + \bar{a}^{\alpha\beta}_s$ define a commutative associative algebra, while $b^{\alpha\beta}_s + \bar{b}^{\alpha\beta}_s$ and $h^{\alpha\beta}_s + \bar{h}^{\alpha\beta}_s$ are Frobenius forms for this algebra, as required.

The inverse statement immediately follows from Facts 4 and 5 . $\square$

Proof of Theorem 2 Consider a pair of compatible flat metrics $g, \bar{g}$ in common Frobenius coordinates $u^1, \ldots, u^n$

$$
g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s \quad \text{and} \quad \bar{g}^{\alpha\beta}(u) = \bar{b}^{\alpha\beta} + \bar{a}^{\alpha\beta}_s u^s,
$$

Fact 4 implies that $-\frac{1}{2}a^{\alpha\beta}_s$ and $-\frac{1}{2}\bar{a}^{\alpha\beta}_s$ are the contravariant Christoffel symbols for g and $\bar{g}$, respectively. Compatibility of g and $\bar{g}$ means that the contravariant Levi-Civita symbols for the flat metric $g + \bar{g}$ are the sum of the corresponding symbols for g and $\bar{g}$, that is, $-\frac{1}{2}a^{\alpha\beta}_s - \frac{1}{2}\bar{a}^{\alpha\beta}_s$. At the same time, these symbols are constant and symmetric in upper indices. Hence the coordinates $u^1, \ldots, u^n$ are Frobenius for $g + \bar{g}$ (Fact 4).

This, in turn, implies that $a_s^{\alpha\beta} + \bar{a}_s^{\alpha\beta}$ are the structure constants of a commutative associative algebra and $b_s^{\alpha\beta} + \bar{b}_s^{\alpha\beta}$ is one of its Frobenius form. Thus, the corresponding Frobenius algebras are compatible.

As g and $\bar{g}$ are both nondegenerate metrics, this implies that for a generic collection of constants $m^0, m^1, \ldots, m^n$, the bilinear forms

$$h^{\alpha\beta} = m^0 b^{\alpha\beta} + a_s^{\alpha\beta} m^s \text{ and } \bar{h}^{\alpha\beta} = m^0 \bar{b}^{\alpha\beta} + \bar{a}_s^{\alpha\beta} m^s$$

are both nondegenerate too. At the same time, each of them is the sum of a Frobenius form ($m^0 b^{\alpha\beta}$ and resp. $m^0 \bar{b}^{\alpha\beta}$) and trivial form ($a_s^{\alpha\beta} m^s$ and resp. $\bar{a}_s^{\alpha\beta} m^s$), which corresponds to $m \in \mathfrak{a}^*$ with coordinates $m^1, \ldots, m^n$ and, thus, is also Frobenius.[5] As a result, h and $\bar{h}$ lead us to Frobenius triples $(h, b, \mathfrak{a})$ and $(\bar{h}, \bar{b}, \bar{\mathfrak{a}}.)$.

By construction, $h + \bar{h}$ defines (if nondegenerate) a Frobenius form for the 'sum' of the algebras. Thus, we get compatible Frobenius triples, which yield compatible non-homogeneous Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$.   □

6 Proof of Theorem 3

6.1 Rewriting the Existence of Frobenius Coordinates in a Differential Geometric Form

We start with the following observation related to Frobenius coordinate systems (Fact 4): $(u^1, \ldots, u^n)$ is a Frobenius coordinate system for a metric g if and only if the contravariant Christoffel symbols $\Gamma_k^{ij} = \sum_s g^{si} \Gamma_{sk}^j$ in this coordinate system are constant and symmetric in upper indices.

We denote by $\Gamma, \bar{\Gamma}$ the Levi-Civita connections of g and $\bar{g}$. Assuming that a common Frobenius coordinate system $u^1, \ldots, u^n$ exists, we let $\widehat{\Gamma}$ be the flat connection whose Christoffel symbols identically vanish in this coordinate system. Let $R^i{}_{jk\ell}$, $\bar{R}^i{}_{jk\ell}$, and $\widehat{R}^i{}_{jk\ell}$ denote the corresponding curvature tensors. We assume $n \geq 2$, the case $n = 1$ is trivial.

Consider the tensors

$$S^{ij}{}_k := \sum_s g^{si} \left(\Gamma_{sk}^j - \widehat{\Gamma}_{sk}^j \right)$$

$$\bar{S}^{ij}{}_k := \sum_s \bar{g}^{si} \left(\bar{\Gamma}_{sk}^j - \widehat{\Gamma}_{sk}^j \right).$$

In terms of these tensors, the necessary and sufficient conditions that the connection $\widehat{\Gamma}$ determines Frobenius coordinates are:

$$0 = \widehat{R}^i{}_{jk\ell} = R^i{}_{jk\ell} = \bar{R}^i{}_{jk\ell} \tag{45}$$

---

[5] Here we use a well-known fact for any $m \in \mathfrak{a}^*$, the form $\xi, \eta \mapsto \langle \xi \star \eta, m \rangle$ is Frobenius, perhaps degenerate. If $\mathfrak{a}$ has a unity element, then every Frobenius form is of this kind. Otherwise, there might exist other (nontrivial) Frobenius forms.

$$S^{ij}_{k} = S^{ji}_{k} \tag{46}$$

$$\bar{S}^{ij}_{k} = \bar{S}^{ji}_{k} \tag{47}$$

$$0 = \widehat{\nabla}_m S^{ij}_{k} = \widehat{\nabla}_m \bar{S}^{ij}_{k}. \tag{48}$$

Indeed, if $(u^1, \ldots, u^n)$ is a common Frobenius coordinate system for g and $\bar{g}$, then in these coordinates $\widehat{\Gamma}^j_{sk} = 0$, and $\Gamma^{ij}_k = g^{is}\Gamma^j_{sk}$ and $\bar{\Gamma}^{ij}_k = \bar{g}^{is}\bar{\Gamma}^j_{sk}$ are both constant and symmetric in upper indices by Fact 4. Hence, (45)–(48) obviously follow.

Conversely, if (45)–(48) hold, then in the flat coordinates for $\widehat{\Gamma}^j_{sk}$ we see that $\Gamma^{ij}_k = S^{ij}_{k}$ and $\bar{\Gamma}^{ij}_k = \bar{S}^{ij}_{k}$ are both symmetric in upper indices due to (46) and (47) and are also constant due to (48). Therefore, by Fact 4, $(u^1, \ldots, u^n)$ are Frobenius coordinates for both g and $\bar{g}$.

6.2 General Form of the Metric in Diagonal Coordinates

We work in the coordinates $(x^1, \ldots, x^n)$ such that

$$R_g = \bar{g}g^{-1} = \mathrm{diag}(\ell_1(x^1), \ldots, \ell_n(x^n)), \quad g_{ij} = \mathrm{diag}(\varepsilon_1 e^{g_1}, \ldots, \varepsilon_n e^{g_n}), \quad (49)$$

where g_i are local functions on our manifold and $\varepsilon_i \in \{-1, 1\}$. Local existence of such coordinates follows from Facts 2 and 3 which imply that R_g is a Nijenhuis operator and therefore, according to Haantjes theorem, is diagonalisable and ℓ_i depends on x^i only (see also various versions of diagonalisability theorems in [6] which, in particular, allows us to include the case of complex eigenvalues too). We assume that all $\ell_i(x^i)$ are different and never vanish.

Remark 6.1 We allow some of the *diagonal* variables x^i to be complex. Note that if a variable x^i is complex then by [6, §3], we may assume that the corresponding eigenvalue ℓ_i is a holomorphic function of x^i. In the first read, we recommend to think of all the eigenvalues as real and then to carefully check that our proofs are based on algebraic manipulations and differentiations, which are perfectly defined over complex coordinates, so that generalisation of the proofs to complex eigenvalues requires no change in formulas. See also a discussion at the end of [9, §7].

Note that the results we use (e.g. [38, 40]) are also based on algebraic manipulations (essentially, on a careful calculation of the curvature tensor and connection coefficients) and are applicable if a part of eigenvalues is complex.

Note also that when we work over complex numbers, we may think that the numbers ε_i are all equal to 1. If all the eigenvalues of R_g are real, objects we will introduce in the proof will automatically be real as well.

Let us first consider the conditions (46, 47). We view them as linear (algebraic) system of equations with unknown $\widehat{\Gamma}^i_{jk}$'s (satisfying also $\widehat{\Gamma}^i_{jk} = \widehat{\Gamma}^i_{kj}$) whose coefficient matrix is constructed from the entries of g and L and the free terms are constructed from $g, \bar{g}, \Gamma, \bar{\Gamma}$. Being rewritten in such a way that unknowns are on the left-hand side

and free terms are on the right-hand side, it has the following form:

$$
\begin{aligned}
\varepsilon_i e^{-g_i}\widehat{\Gamma}^j_{ik} - \varepsilon_j e^{-g_j}\widehat{\Gamma}^i_{jk} &= \varepsilon_i e^{-g_i}\Gamma^j_{ik} - \varepsilon_j e^{-g_j}\Gamma^i_{jk}, \\
\ell_i \varepsilon_i e^{-g_i}\widehat{\Gamma}^j_{ik} - \ell_j \varepsilon_j e^{-g_j}\widehat{\Gamma}^i_{jk} &= \ell_i \varepsilon_i e^{-g_i}\bar{\Gamma}^j_{ik} - \ell_j \varepsilon_j e^{-g_j}\bar{\Gamma}^i_{jk}.
\end{aligned}
\tag{50}
$$

We see that for fixed $i = j = k$, the system bears no information. For fixed $i \neq j$, the coefficient matrix $\begin{pmatrix} \varepsilon_i e^{-g_i} & -\varepsilon_j e^{-g_j} \\ \ell_i \varepsilon_i e^{-g_i} & -\ell_j \varepsilon_j e^{-g_j} \end{pmatrix}$ of the linear system (50) is nondegenerate, since the eigenvalues ℓ_i are all different, and therefore the system has a unique solution.

The entries of the connections Γ and $\bar{\Gamma}$ of the diagonal metrics g_{ij} and $\bar{g}_{ij} := gL^{-1}$ were calculated many times in the literature, see, e.g.[9, Lemma 7.1], and are given by the following formulas:

- $\Gamma^k_{ij} = \bar{\Gamma}^k_{ij} = 0$ for pairwise different i, j and k,
- $\Gamma^k_{kj} = \frac{1}{2}\frac{\partial g_k}{\partial x^j}$ for arbitrary k, j,
- $\Gamma^k_{jj} = -\frac{\varepsilon_j}{\varepsilon_i}\frac{e^{g_j-g_k}}{2}\frac{\partial g_j}{\partial x^k}$ for arbitrary $k \neq j$,
- $\bar{\Gamma}^k_{kj} = \Gamma^k_{kj}$ for arbitrary $k \neq j$,
- $\bar{\Gamma}^i_{ii} = \Gamma^i_{ii} - \frac{\ell'_i}{2\ell_i}$
- $\bar{\Gamma}^k_{jj} = \frac{\ell_k}{\ell_j}\Gamma^k_{jj}$ for arbitrary $k \neq j$.

By direct calculations using these formulas, we obtain that the solution of the system (50) is as follows:

(A) $\widehat{\Gamma}^i_{ii} = u_i$ for all i, where u_i's are (yet unknown) functions on the manifold.
(B) $\widehat{\Gamma}^i_{ij} = \widehat{\Gamma}^i_{ji} = \frac{\partial g_i}{\partial x^j}$ for $i \neq j$
(C) $\widehat{\Gamma}^i_{jk} = 0$ for all $i \neq j$ and $k \neq i$ (we allow the case $k = j$).

Combining these with the formulas for S^{ij}_{k}, we obtain:

- $S^{ii}_{i} = \varepsilon_i e^{-g_i}\left(u_i - \frac{1}{2}\frac{\partial g_i}{\partial x^i}\right)$ for all i,
- $S^{ii}_{j} = \frac{\varepsilon_i}{2}e^{-g_i}\frac{\partial g_i}{\partial x^j}$, for all $i \neq j$,
- $S^{ij}_{i} = S^{ji}_{i} = \frac{\varepsilon_j}{2}e^{-g_j}\frac{\partial g_i}{\partial x^j}$ for all $i \neq j$,
- $S^{ij}_{k} = 0$ for all $i \neq j \neq k \neq i$.

By direct calculations we see that for any $i \neq j \neq k \neq i$ we have $\widehat{\nabla}_k S^{ij}_{j} = \frac{\varepsilon_i}{2}\frac{\partial^2 g_i}{\partial x^j \partial x^k}$ implying

$$
0 = \frac{\partial^2 g_i}{\partial x^j \partial x^k}.
\tag{51}
$$

Next, consider the terms of the form $\widehat{\nabla}_j S^{ii}_{i}$ and $\widehat{\nabla}_i S^{ii}_{j}$ with $i \neq j$. They are given by

$$
\widehat{\nabla}_j S^{ii}_{i} = \varepsilon_i \frac{e^{-g_i}}{2}\left(\frac{\partial g_i}{\partial x^j}\frac{\partial g_j}{\partial x^i} + \frac{\partial^2 g_i}{\partial x^i \partial x^j}\right),
$$

$$\widehat{\nabla}_i S^{ii}{}_j = -\varepsilon_i \frac{e^{-g_i}}{2} \left(\frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^i} + \frac{\partial^2 g_i}{\partial x^i \partial x^j} - 2 \frac{\partial u_i}{\partial x^j} \right).$$

Since $\widehat{\nabla}_j S^{ii}{}_i = \widehat{\nabla}_i S^{ii}{}_j = 0$, the formulas above imply

$$\frac{\partial u_i}{\partial x^j} = 0 \tag{52}$$

so each u_i is a function of x^i only.

Next, we prove the following Lemma. Denote by $U_i = U_i(x^i)$ and $U_j = U_j(x^j)$ the primitive functions for $e^{\tilde{u}_i}$ and $e^{\tilde{u}_j}$, where $\tilde{u}_i$ and $\tilde{u}_j$ are primitive functions for u_i and u_j. By their definition, $U_i' \neq 0$ and $U_j' \neq 0$.

Lemma 6.1 *There exist constants C_{ij} and $E_{ij} = E_{ji}$ such that for the constants $\alpha_{ij} \in \{0, 1\}$ given by the formula*

$$\alpha_{ij} = \alpha_{ji} = \begin{cases} 0 & \text{if } C_{ij} = C_{ji} = 0 \\ 1 & \text{otherwise} \end{cases}$$

and for any $i \neq j$, the function

$$g_i - \ln \left(|C_{ij} U_i + C_{ji} U_j + E_{ij}|^{\alpha_{ij}} \right)$$

does not depend on x^j (we use the convention that $0^0 = 1$).

Proof We consider the curvature tensor $\widehat{R}^i{}_{jk\ell}$ of the connection $\widehat{\Gamma}$. To compute it, we need to substitute $\widehat{\Gamma}$ given by (A,B,C) above into the standard formula for the curvature

$$\widehat{R}^\ell{}_{ijk} = \frac{\partial}{\partial x^j} \widehat{\Gamma}^\ell_{ik} - \frac{\partial}{\partial x^k} \widehat{\Gamma}^\ell_{ij} + \sum_s \left(\widehat{\Gamma}^\ell_{js} \widehat{\Gamma}^s_{ik} - \widehat{\Gamma}^\ell_{ks} \widehat{\Gamma}^s_{ij} \right). \tag{53}$$

We obtain for $i \neq j$:

$$0 = \widehat{R}^i{}_{iji} = -\frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^i} - \frac{\partial^2 g_i}{\partial x^j \partial x^i} \tag{54}$$

$$0 = \widehat{R}^i{}_{jji} = -\frac{\partial g_i}{\partial x^j} u_j + \left(\frac{\partial g_i}{\partial x^j} \right)^2 + \frac{\partial^2 g_i}{\partial x^{j2}} \tag{55}$$

$$0 = \widehat{R}^j{}_{jji} = \frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^i} + \frac{\partial^2 g_j}{\partial x^i \partial x^j} \tag{56}$$

$$0 = \widehat{R}^j{}_{iji} = -\left(\frac{\partial g_j}{\partial x^i} \right)^2 + u_i \frac{\partial g_j}{\partial x^i} - \frac{\partial^2 g_j}{\partial x^{i2}}. \tag{57}$$

We view 4 equations above a system of PDEs on the unknown functions

$$a = \frac{\partial g_i}{\partial x^j} \text{ and } b = \frac{\partial g_j}{\partial x^i}. \tag{58}$$

The condition (52) implies that the coefficients of the system depend on x^i and x^j and we may temporarily 'forget' all other variables. The system then has the following form:

$$\frac{\partial a}{\partial x^i} = -ab \,, \quad \frac{\partial a}{\partial x^j} = -a^2 + au_j \,, \quad \frac{\partial b}{\partial x^i} = -b^2 + bu_i \,, \quad \frac{\partial b}{\partial x^j} = -ab. \quad (59)$$

This system is of Cauchy-Frobenius type (in the sense that all first derivatives of unknown functions are explicit expressions of the unknown functions and variables). By direct computation, we check that its integrability conditions hold identically. Then, its solution depends on an arbitrary choice of the values of a and b at one arbitrarily chosen point p_0. Note that if $a(p_0) = 0$ then a is identically 0, the same is true for b.

By direct substitution we see that for any constants C_i, C_j, E, the pair of functions

$$a = \frac{C_j U_j'}{C_i U_i + C_i U_j + E}, \quad b = \frac{C_i U_i'}{C_i U_i + C_j U_j + E} \quad (60)$$

satisfies the equation. In the case $C_i = C_j = 0$, we think that a and b given by (60) vanish identically. The functions $U_i = U_i(x^i)$ and $U_j = U_j(x^j)$ used in (60) are as defined before Lemma 6.1. By varying the constants C_i, C_j, E, one can get any nonzero initial values $a(p_0)$, $b(p_0)$ so this is indeed a general solution.

In the case $C_i \neq 0$ or $C_j \neq 0$, using (58), we obtain

$$g_i = \ln(|C_i U_i + C_j U_j + E|) + D_i \quad \text{and} \quad g_j = \ln(|C_i U_i + C_j U_j + E|) + D_j \quad (61)$$

with D_i independent of x^j and D_j independent of x^i. In the case $C_i = 0 = C_j$ we obtain that g_i is independent of x^j and g_j is independent of x^i automatically.

Note that if we 'remember' all the coordinates, then C_i, D_i, E may also depend on all other variables x^k, $k \notin \{i, j\}$.

Let us study the dependence of C_i, C_j and E on the variable x^k with $k \notin \{i, j\}$. We first consider the case when $C^i \neq 0$ and $C^j \neq 0$. We observe that by (51), for $i \neq j \neq k \neq i$, we have

$$0 = \frac{\partial^2 g_i}{\partial x^j \partial x^k} \overset{(58, 60)}{=} \frac{\partial}{\partial x^k} \frac{C_j U_j'}{C_i U_i + C_j U_j + E} = -\frac{U_j' \left(U_i \frac{\partial}{\partial x^k} \frac{C_i}{C_j} + \frac{\partial}{\partial x^k} \frac{E}{C_j} \right)}{(\frac{C_i}{C_j} U_i + U_j + \frac{E}{C_j})^2}$$

implying that the ratios C_i/C_j and E/C_j are constant. Note that $U_j' \neq 0$ since it is a primitive function for a nonvanishing function. Then, we may assume that C_i, C_j and E are constants, since in the formula (61) their dependence on other variables can be hidden in D_i and D_j.

In the cases $C_i = 0$ or $C_j = 0$ (but not $C_i = 0 = C_j$ both), the dependence of C_i, C_j and E on other variables can similarly be hidden in D_i and D_j. In the remaining case, when $C_i = 0 = C_j$, we already know that g_i is independent of x^j and g_j is independent of x^i.

Thus, in all cases, $g_i - \ln(|C_i U_i + C_j U_j + E|^{\alpha_{ij}})$ does not depend on x^j and $g_j - \ln(|C_i U_i + C_j U_j + E|^{\alpha_{ij}})$ does not depend on x^i. Since constants C_i, C_j and E are constructed for fixed i and j, in what follows we denote them C_{ij}, C_{ji} and $E_{ij} = E_{ji}$, respectively. In this notation, the function $g_i - \ln(|C_{ij} U_i + C_{ji} U_j + E_{ij}|^{\alpha_{ij}})$ does not depend on x^j, as stated. $\square$

Consequently applying the Lemma, we see that

$$g_i - \sum_{s \neq i} \ln \left(|C_{ij} U_i + C_{is} U_s + E_{ij}|^{\alpha_{is}} \right)$$

depends on x^i only.

Therefore, the ith diagonal component g_{ii} of the metric g is as follows:

$$g_{ii} = \varepsilon_i e^{g_i} = h_i(x^i) \left(\prod_{s \neq i} \left(C_{is} U_i(x^i) + C_{si} U_s(x^s) + E_{is} \right)^{\alpha_{is}} \right) \qquad (62)$$

for some functions h_i of one variable.

6.3 Last Step of the Proof of Theorem 3: Making All C_{ij} Equal ± 1

In the previous section, we have proved that the metric g is given by (62). Observe that by the assumptions of Theorem 3, the diagonal coordinates depend on all variables, so all $\alpha_{ij} = 1$ and all $C_{ij} \neq 0$ for $i \neq j$. First note that in the case when all $C_{ij} = 1$ for $i < j$, $C_{ij} = -1$ for $i > j$ and $E_{ij} = 0$ for $i \neq j$, the diagonal metric g is in the so-called *Levi-Civita* form:

$$g_{ii} = \left(\prod_{j \neq i} (U_i(x^i) - U_j(x^j)) \right) h_i(x^i). \qquad (63)$$

In this section, we show that one can bring the metric to the form (63) by certain 'admissible' operations which include only coordinate transformations and renaming of functions. Combining this with a result of A. Solodovnikov (Fact 6) will prove Theorem 3.

We will use the condition $\widehat{\nabla}_i S^{ij}_{k} = 0$. Assuming $i \neq j \neq k \neq i$, this condition reads

$$-\frac{\varepsilon_j}{2} e^{-g_j} \left(\frac{\partial g_i}{\partial x^j} \frac{\partial g_i}{\partial x^k} - \frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^k} - \frac{\partial g_i}{\partial x^k} \frac{\partial g_k}{\partial x^j} \right) = 0. \qquad (64)$$

Substituting (62) in it, we see that the following relation holds:

$$0 = C_{ik} C_{ji} C_{kj} - C_{ij} C_{jk} C_{ki} = \det \begin{pmatrix} 0 & C_{ij} & C_{ik} \\ -C_{ji} & 0 & C_{jk} \\ -C_{ki} & -C_{kj} & 0 \end{pmatrix}. \qquad (65)$$

Let us now use the relation (65) and 'make' $C_{ij} = 1$ for $i < j$ and $C_{ij} = -1$ for $i > j$. We will use the following operations for it:

(a) We can multiply the factor $(C_{ij}U_i + C_{ji}U_j + E_{ij})$ in the ith and jth diagonal components of g (given by (62)) by a nonzero constant and correspondingly change h_i and h_j by dividing them by the same constant.

(b) For every fixed pair $i \neq j$ if $C_{ij} \neq 0$ we can rename $C_{ij}U_i$ by U_i (which we will do if $i < j$) or by $-U_i$ (if $i > j$).

By applying the operation (a), we make $C_{1i} = 1$ for all $i \neq 1$. By applying operation (b), we make and $C_{i1} = -1$ for all $i \neq 1$. In addition, by applying operation (a) we make $C_{2i} = 1$ with $i > 2$. Then, the first two rows and the first column of the matrix C_{ij} are as we want. Then, the condition (65) with $i = 1$, $j = 2$ arbitrary $k > 2$ reads $C_{k2} = -1$ so the second column is automatically as we want. Then, applying operation (a), we make all C_{3k} with $k > 3$ equal to 1. The condition (65) with $i = 1$, $j = 3$ arbitrary $k > 3$ reads $C_{k3} = -1$ and implies that the third column is as we want. Repeating the procedure, we bring the metrics in the Levi-Civita form (63).

Let us now take $U_i(x^i)$ as a local coordinate system: $x^i_{\text{new}} := U_i(x^i_{\text{old}})$. We can do it because the derivative of U_i is not zero. In the new coordinates, R_g is still diagonal and the ith diagonal component depends on the variable i only. In these coordinates, the metric (63) is diagonal with

$$g_{ii} = \left(\prod_{j \neq i}^{n} (x^i - x^j + E_{ij}) \right) \frac{1}{H_i(x^i)},$$

where E_{ij} is now a skew-symmetric constant matrix. In these coordinates the condition (64) is equivalent to $E_{jk} + E_{ki} + E_{ij} = 0$. An easy exercise in Linear Algebra shows the existence of constants $E_1, \ldots, E_n$ such that $E_{ij} = E_i - E_j$. Moreover, the constants E_i are uniquely defined up to adding a common constant E to all of them. Hence, after the coordinate change $x^i_{\text{new}} = x^i_{\text{old}} + E_i$, the metric is diagonal with

$$g_{ii} = \left(\prod_{j \neq i}^{n} (x^i - x^j) \right) \frac{1}{H_i(x^i)}. \tag{66}$$

Therefore, the metric $\bar{g} = L\bar{g}$ is also diagonal with

$$\bar{g}_{ii} = \left(\prod_{j \neq i}^{n} (x^i - x^j) \right) \frac{1}{\bar{H}_i(x^i)} \tag{67}$$

Fact 6 *The diagonal metric of form (66) (resp. (67)) in dimension at least two has constant curvature if and only if there exists a polynomial P (resp. Q) of degree $\leq n + 1$ such that $H_i(x^i) = P(x^i)$ (resp. $\bar{H}_i(x^i) = Q(x^i)$). Moreover, the curvature of the metric vanishes if and only if the polynomial P (resp. Q) has degree $\leq n$.*

Fact 6 was proved in [40, §5] and easily follows from calculations in [9, §7].

Taking $L = \text{diag}(x^1, \ldots, x^n)$ and the (contravariant) Levi-Civita metric

$$g_{\text{LC}} = \sum_i \left(\prod_{j \neq i}^n (x^i - x^j) \right)^{-1} \left(\tfrac{\partial}{\partial x^i} \right)^2, \tag{68}$$

we see that $g = P(L)g_{\text{LC}}$ and $\bar{g} = Q(L)g_{\text{LC}}$, which completes the proof of Theorem 3. (The 'uniqueness' part of Theorem 3 will be explained in Remark 7.1.)

7 Proof of Theorem 5

7.1 Upperblockdiagonal Structure of the Matrix C_{ij}

We assume that the metric g is diagonal and its diagonal elements have the form

$$g_{ii} = h_i(x^i) \prod_{s \neq i} \left(C_{is}x^i + C_{si}x^s + E_{is} \right)^{\alpha_{is}}, \tag{69}$$

which is the form (62) in the 'new' coordinate system $x^i_{\text{new}} := U_i(x^i_{\text{old}})$. We view C_{ij} as entries of an $n \times n$-matrix and E_{ij} as entries of an $n \times n$ symmetric matrix. Since the diagonal elements C_{ii}, E_{ii} do not come into the formula for g, we assume they are zero.

We will first re-arrange the coordinates $x^1, \ldots, x^n$ and make the matrix C upperblockdiagonal such that in every diagonal block, all nondiagonal entries are different from zero. We will need the following Lemma:

Lemma 7.1 *If $C_{ji} = 0$ for certain different $i, j \in \{1, \ldots, n\}$, then for any $k \in \{1, \ldots, n\}$ we have $C_{jk}C_{ki} = 0$. Moreover, if in addition $C_{ij} = 0$, then for any $k \in \{1, \ldots, n\}$, we have $C_{ik}C_{jk} = 0$.*

Proof For $k = i$ or $k = j$, the statement follows from our convention $C_{ii} = C_{jj} = 0$, further we assume $i \neq k \neq j(\neq i)$.

We consider the equation (64): under the assumption $C_{ji} = 0$ the terms $\frac{\partial g_i}{\partial x^j} \frac{\partial g_i}{\partial x^k}$ and $\frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^k}$ vanish. Then, the equation reads $\frac{\partial g_i}{\partial x^k} \frac{\partial g_k}{\partial x^j} = 0$ and implies that $\frac{\partial g_i}{\partial x^k} = 0$ (which in turn implies $C_{ki} = 0$) or $\frac{\partial g_k}{\partial x^j} = 0$ (which in turn implies $C_{jk} = 0$). This proves the first statement of the lemma. Next, observe that under the assumption $C_{jk} = C_{kj} = 0$, the equation (64) reads $\frac{\partial g_i}{\partial x^j} \frac{\partial g_i}{\partial x^k} = 0$ implying $C_{ki}C_{ji} = 0$. Renaming $i \leftrightarrow k$ finishes the proof.   □

Next, consider $i \in \{1, \ldots, n\}$ such that the ith column of the matrix C_{ij} contains the maximal number of zero entries. We assume without loss of generality that $i = 1$, that the elements $C_{21}, \ldots, C_{d1}$ are not zero and the other elements of the first column are zero. Applying Lemma 7.1 to the element $C_{d'1}$ with $d' > d$, we obtain that $C_{d'k}C_{k1} = 0$. Since $C_{k1} \neq 0$ for $k \leq d$, we obtain $C_{d'k} = 0$ for such k.

Thus, all elements of the matrix C_{ij} staying under the upper left $d \times d$ block are zero. If $C_{ij} = 0$ with $i \neq j \in \{1, \ldots, d\}$, we obtain a contradiction with the assumption that the ith column of the matrix C_{ij} contains the maximal number of zeros. Thus, all C_{ij} with $i \neq j \in \{1, \ldots, d\}$ are not zero. Thus, the first d columns of C_{ij} are as in the upperblockdiagonal matrix with the first block of dimension $d \times d$. We further have that all nondiagonal components of the first block are different from zero.

Next, consider the index $i \in \{d+1, \ldots, n\}$ such that the number of zero entries in the columns of lower right $(n-d) \times (n-d)$ block is maximal. We may assume without loss of generality that $i = d + 1$, that the components $C_{d+1\ d+2}, \ldots, C_{d+1\ d+d'}$ are not zero and the components $C_{d+1\ d+d'+1}, \ldots, C_{d+1\ n}$ are zero. Arguing as above, using Lemma 7.1, we obtain that for any $k \in \{d + 1, \ldots, d + d'\}$, the components $C_{d+k\ d+d'+1}, \ldots, C_{d+k\ n}$ are zero. Thus, the first $d + d'$ columns of C_{ij} are as in the upperblockdiagonal matrix with the first block of dimension $d \times d$ and the second block of dimension $d' \times d'$. Moreover, by the 'maximality' condition in our choice of the first column of the second block, all nondiagonal elements of the second block are nonzero.

We can repeat the procedure further and further and obtain that the matrix C_{ij} is as we claimed: it is upperblockdiagonal and in every block, all nondiagonal entries are different from zero. Let us explain now that by the operations (a,b) from Sect. 6.3, we can make C_{ij} in every block equal 1 for $i < j$ and -1 for $j > i$. Indeed, by applying the operations (a) and (b), we can make the first two rows and the first column of every block to be as we claimed. The condition (65) automatically implies that the second column of the block is as we want. Next, applying operation (a), we make the third row as we want. Then, (65) implies that the third column is as we want and so on. Note that these operations with one block do not affect other blocks.

Next, let us construct a $B \times B$ matrix $(c_{\alpha\beta})$. We will denote by $B_{\alpha\beta}$ the blocks of the matrix C (corresponding to the decomposition $n = n_1 + \cdots + n_B$), the block $B_{\alpha\beta}$ has dimension $n_\alpha \times n_\beta$. Above we achieved that if $\alpha \neq \beta$, then either all entries of $B_{\alpha\beta}$ are zero or are equal to 1. In the first case, we put $c_{\alpha\beta} = 0$, in the second case $c_{\alpha\beta} = 1$. If $\alpha > \beta$ then all entries of the block $B_{\alpha\beta}$ are zero, so such $c_{\alpha\beta} = 0$. We put $c_{\alpha\alpha} = 0$.

Next we show that if such a block $B_{\alpha\beta}$ with $\alpha < \beta$ is zero and the block $B_{\alpha\alpha'}$ is not, then all the blocks $B_{\alpha'\beta}$ with $\alpha' \neq \beta$ are also zero. In order to do it, we take an element C_{ij} of this block. By Lemma 7.1, if $c_{\alpha\beta} = 0$ with $\alpha \neq \beta$, then for any $s \notin \{\alpha, \beta\}$, we have $c_{\alpha s} c_{s\beta} = 0$ implying the claim. Analogously, one shows, using the second statement of Lemma 7.1, that $c_{\alpha\beta} = 0$ with $\alpha < \beta$ implies $c_{\alpha s} c_{\beta s} = 0$.

Let us summarise the properties of the $B \times B$ matrix $(c_{\alpha\beta})$:

(a) $c_{\alpha\beta} = 0$ for $\alpha \geq \beta$, i.e. the matrix is upper triangular with zeros on the diagonal.
(b) If $c_{\alpha\beta} = 0$, then for every $s \in \{1, \ldots, B\}$ we have $c_{s\beta} c_{\alpha s} = 0$.
(c) If $c_{\alpha\beta} = 0$ for certain $\alpha < \beta$, then for every $s \in \{1, \ldots, B\}$ we have $c_{\beta s} c_{\alpha s} = 0$.

Let us show that any such matrix can be constructed from a directed rooted in-forest by a procedure described in Sect. 4.1. To see this, we introduce the relation $\prec$ on the set $\{1, \ldots, B\}$: we define $\alpha \prec \beta$ if and only if $c_{\alpha\beta} = 1$. Clearly, $\alpha \prec \beta$ implies that the number α is smaller than the number β. The relation $\prec$ is a strict partial order. Indeed, $\alpha \nprec \alpha$ because of (a), so the relation is irreflexive. If $\alpha \prec \beta$, then $\alpha < \beta$ by

(a) implying $\beta \not\prec \alpha$, so the relation is asymmetric. If $\alpha \prec \beta$ and $\beta \prec \gamma$ then by (b), we have $\alpha \prec \gamma$, so the relation is transitive.

Moreover, for every $s \in \{1, \ldots, B\}$, the set $S_s := \{\alpha \mid \alpha \prec s\}$ is a *chain*, i.e. is totally ordered. Indeed, for $\alpha < \beta \in S_s$, we have $c_{\alpha s} = c_{\beta s} = 1$ implying $c_{\alpha \beta} = 1$ in view of (c).

Next, it is easy to see that every strict partially ordered finite set such that every S_s is a chain can be described by a directed rooted in-forest. The vertices of the forest are the numbers $1, \ldots, B$, and two vertices α, γ are connected by the oriented edge if $\alpha \prec \gamma$ and if there is no β such that $\alpha \prec \beta \prec \gamma$, each connected component of this oriented graph is a directed rooted in-tree. The forest clearly reconstructs the order '$\prec$' and therefore the matrix $(c_{\alpha \beta})$: for two numbers $\alpha \neq \beta \in \{1, \ldots, B\}$, we have $\alpha \prec \beta$ if there exists an oriented way from β to α, see example on Fig. 1.

The converse is also true: every directed in-forest F (with appropriately labelled vertices) defines a matrix $c_{\alpha \beta}$ with properties (a), (b) and (c) by (32).

Let us now deal with E_{ij}. Consider $\alpha \neq \beta \in 1, \ldots, B$. We say that the index j belongs to the α-block (of coordinates), if $n_1 + \cdots + n_{\alpha-1} < j \leq n_1 + \cdots + n_\alpha$. Consider a pair (j, i) such that j belongs to the α-block and i to the β-block. If $c_{\alpha \beta} = 0$, then the component E_{ji} is irrelevant for our formulas since when we build g by (69) the corresponding term $(C_{ji} x^j + C_{ij} x^i + E_{ji})$ equals $(C_{ij} x^i + E_{ji})$ and can be hidden in the factor h_i of g_{ii}. Assume now $c_{\alpha \beta} = 1$ and let us show that then the corresponding numbers E_{ji} do not depend on j from α-block and i from β-block, i.e. all entries of the whole (α, β)-block of the matrix (E_{ij}) are equal to each other.

We will use formula (64). If the β-block has more than one entry, take $k \neq i$ from the β-block. Then, (64)

$$\frac{1}{(x^j + E_{ji})(x^k + E_{ki})} - \frac{1}{(x^j + E_{ji})(x^j - x^k)} + \frac{1}{(x^k + E_{ki})(x^j - x^k)} = 0,$$

implying $E_{ji} = E_{ki}$. Thus, elements of every column of the (α, β) block of the matrix E_{ij} are equal to each other.

Similarly, by taking $k \neq j$ from the α-block, one proves that the elements of every raw of the (α, β) block of the matrix E_{ij} are equal to each other. Thus, provided $c_{\alpha \beta} = 1$, all elements of the (α, β) block of E_{ij} are equal to each other, we call them $-e_{\alpha \beta}$. The matrix $e = (e_{\alpha \beta})$ is a $B \times B$ constant matrix; its element $e_{\alpha \beta}$ is relevant for us (in the sense that it is included in the formulas for the metrics) if the corresponding $c_{\alpha \beta} = 1$.

Finally, let us show that $e_{\alpha \beta} = e_{\alpha \gamma}$ provided $c_{\alpha \beta} = c_{\alpha \gamma} = c_{\beta \gamma} = 1$. We again use (64) assuming that j relates to the α-block, k to the β-block and i to the γ-block, to get

$$\frac{1}{(x^j - e_{\alpha \gamma})(x^k - e_{\beta \gamma})} - 0 - \frac{1}{(x^k - e_{\beta \gamma})(x^j - e_{\alpha \beta})} = 0$$

implying $e_{\alpha \gamma} = e_{\beta \gamma}$.

The matrices $c_{\alpha \beta}$ and $e_{\alpha \beta}$ clearly determine the components of the matrices C_{ij} and components of the matrix E_{ij} which are relevant for (69). Plugging these C_{ij} and E_{ij}

into (69) and taking in account that $\prod_{s=1}^{n^\alpha}(e_{\alpha\beta} - x_\alpha^s) = \det(e_{\alpha\beta} \operatorname{Id}_{n_\alpha} - L_\alpha)$, we obtain the form similar to that in Theorem 5. The difference is as follows: we did not prove yet that the diagonal factors $H_j(x^j) := \frac{1}{h_j(x^j)}$ are polynomials $P_\alpha(x^j)$ of degrees $\leq n_\alpha + 1$ and we did not obtained the additional conditions on these polynomials

Let us do these: first we notice that the metric g has the iterated warped product structure:

$$g = g_1 + \sigma_1(X_1)g_2 + \sigma_2(X_1, X_2)g_3 + \cdots + \sigma_{B-1}(X_1, \ldots, X_{B-1})g_B,$$

where the metric g_α is as follows: take the metric g_α^{LC} given by (29) and multiply its jth diagonal element by $h_j(x^j)$ for every j related to the α-block. The functions σ_β are as follows:

$$\sigma_\beta = \left(\prod_{\alpha < \beta} \det \left(e_{\alpha\beta} \operatorname{Id}_{n_\beta} - L_\beta \right)^{c_{\alpha\beta}} \right)^{-1}.$$

Since g is flat, g_1 must be flat and $g_2, \ldots, g_B$ of constant curvature. Applying the result of [40, §5] (see Fact 6 above) shows that blocks g_α of dimension greater than one are given by $P_\alpha(L_\alpha)g_\alpha^{\mathsf{LC}}$, where P_α is a polynomial of degree $\leq n_\alpha + 1$ (in Sect. 7.2 we will show that coefficients of these polynomials satisfy the conditions (i–iii) from Sect. 4.1 and also consider 1-dimensional blocks).

7.2 Conditions on the Coefficients of P_α

To complete the proof of Theorem 5, it remains to explain that in the case of one-dimensional blocks, the corresponding function $H_j = 1/h_j$ is a polynomial of degree at most 2 and that conditions (i)–(iii) on the coefficients of P_α stated before Theorem 4 are fulfilled. We will need some facts and preliminary work.

Recall that a *Casimir* of a Poisson structure is defined by the property that Poisson structure applied to it gives zero. For a Poisson structure $\mathcal{A}_g$ given by (7) and corresponding to a (flat) metric g of dimension ≥ 2, a Casimir (of the lowest order) can be understood as a function f satisfying $\nabla_i \nabla_j f = 0$. Of course, any constant is a Casimir and n functionally independent Casimirs give us flat coordinates for g in which the components g^{ij} are all constants. For a metric g of constant curvature K, we define *Casimir* as a function $\widehat{f}$ satisfying the equation[6] $\nabla_i \nabla_j \widehat{f} + K \widehat{f} g_{ij} = 0$. The space of Casimirs of a constant curvature metric (on a simply connected manifold of dimension ≥ 2) is a vector space of dimension $n + 1$.

Dimension 1 is special. In this case, we will define a Casimir as a function satisfying $\nabla_i \nabla_j \widehat{f} + K \widehat{f} g_{ij} = 0$ for some constant K, so that for each K, the Casimirs form a two-dimensional space.

Fact 7 *Consider the $n \geq 2$-dimensional metric $g = P(L)g_{\mathsf{LC}}$, where L and g_{LC} are as in (29), and P is a polynomial $P(t) = a_0 + a_1 t + \cdots + a_{n+1}t^{n+1}$. Then, the metric*

---

[6] The functions satisfying this equation are indeed Casimirs of the (nonlocal) Poisson structure corresponding to the constant curvature metric g, see e.g. [9, §2] and references therein.

has constant curvature $-\frac{1}{4}a_{n+1}$. *Moreover, in any dimension $n \geq 1$, the function* $\sqrt{\det(\lambda \operatorname{Id} -L)}$ *is a Casimir of g if and only if $P(\lambda) = 0$ and in this case we have:*

$$4g\left(\mathrm{d}\sqrt{(\lambda \operatorname{Id} -L)}, \mathrm{d}\sqrt{\det(\lambda \operatorname{Id} -L)}\right) = -\frac{\mathrm{d}\,P(t)}{\mathrm{d}\,t}\Big|_{t=\lambda} + a_{n+1}\det(\lambda \operatorname{Id} -L). \quad (70)$$

Proof The metric g_{LC} and operator L are explicitly given (and w.l.o.g. one may assume $\lambda = 0$) so the proof is an exercise in the Vandermonde identities and is left to the reader. $\qquad \square$

Below, we work with warped product metrics for which the 'covariant language' is more convenient. For this reason, starting from Fact 8 and till the end of the current Sect. 7.2, g and g_i will denote covariant metrics. For the corresponding contravariant metrics, we use g^* and g_i^*.

Fact 8 *Suppose a warped product metric $g = g_1 + f(X_1)^2 g_2$ has constant curvature. Then g_1 and g_2 have constant curvatures. Moreover, the following statements hold:*

1. *If g is flat, then g_1 is flat.*
2. *If g is flat and g_2 is $n_2 \geq 2$-dimensional, then $K_2 = g_1^*(\mathrm{d}\,f, \mathrm{d}\,f)$, where K_2 is the curvature of g_2.*
3. *$f(X_1)$ is g_1-Casimir.*

Proof The first statement is well known and immediately follows from geometric arguments. The second statement follows from the second formula in the first line of [38, (4.2)]. The third statement follows, under the additional assumption that the curvature is zero, from the second line of [38, (4.2)]. If the curvature is not zero, we may assume that it is equal to 1. Then, we employ the conification construction: we consider the $(n + 1)$-dimensional metric $\widehat{g} = (dx^0)^2 + (x^0)^2\big(g_1 + f(X_1)^2 g_2\big)$. The metric $\widehat{g}$ is flat and can be viewed as a warped product metric with base $(dx^0)^2 + (x^0)^2 g_1$ and warping function $\widehat{f}^2 := \big(x^0 f(X_1)\big)^2$. Then, the function $\widehat{f} := x^0 f(X_1)$ is a Casimir of $\widehat{g}_1 := (dx^0)^2 + (x^0)^2 g_1$ implying that $f(X_1)$ is a Casimir of g_1. $\qquad \square$

Next, we need the following technical lemma:

Lemma 7.2 *Suppose the warped product metric $g_1 + f(X_1)^2 g_2$ is flat and $f(X_1)$ is g_1-Casimir. Then, the following holds:*

1. *Every g_1-Casimir $F(X_1)$ such that $g_1^*(\mathrm{d}\,f, \mathrm{d}\,F) = 0$ is a Casimir of g.*
2. *If g_2 has constant nonzero curvature K_2 or is one-dimensional, then for any function $\phi(X_2)$ satisfying $\nabla^{g_2}\nabla^{g_2}\phi + K_2\phi g_2 = 0$ the function $f\phi$ is a g-Casimir.*
3. *If g_2 is flat, then for any function $\phi(X_2)$ satisfying $\nabla^{g_2}\nabla^{g_2}\phi(X_2) = \mathrm{const}\, g_2$ and for any g_1-Casimir $\tilde{f}$ such that $g_1^*(\mathrm{d}\,f, \mathrm{d}\,\tilde{f}) = 1$ we have that $f\phi - \mathrm{const}\,\tilde{f}$ is a g-Casimir.*

Notice that the g-Casimirs described in Lemma 7.2 span a $n+1$-dimensional vector space and, therefore, form a basis of the space of g-Casimirs.

Proof By direct calculations (done many times in the literature, see e.g. [38, (4.1)]), one sees that the Christoffel symbols of the warped product metric g are given by the following formulas:

$$\Gamma_{bc}^a = \overset{1}{\Gamma}_{bc}^a \,, \quad \Gamma_{\beta\gamma}^\alpha = \overset{2}{\Gamma}_{\beta\gamma}^\alpha \,, \quad \Gamma_{\beta\gamma}^a = -f \sum_s f_{,s} \overset{1}{g}^{sa} \overset{2}{g}_{\beta\gamma} \,, \quad \Gamma_{\alpha\beta}^\alpha = \frac{1}{f} f_{,a} \delta_\beta^\alpha \,,$$

$$\Gamma_{\alpha b}^a = \Gamma_{b\alpha}^a = \Gamma_{ab}^\alpha = 0.$$

Here $\overset{1}{\Gamma}, \overset{2}{\Gamma}$ relate to the Christoffel symbols of the metrics $g_1 = \overset{1}{g}$ and $g_2 = \overset{2}{g}$, respectively; a, b, c, s run from 1 to n_1 and α, β, γ from $n_1 + 1$ to n. The notation $f_{,s}$ means $\frac{\partial f}{\partial x^s}$.

Therefore, for any function $F(X_1)$, we have:

$$\nabla_a \nabla_b F = \nabla_a^{g_1} \nabla_b^{g_2} F \,, \quad \nabla_\alpha \nabla_a F = 0 \,, \quad \nabla_\alpha \nabla_\beta F = f g_1^*(dF, df) \overset{2}{g}_{\alpha\beta}. \quad (71)$$

In particular, if F is a g_1-Casimir, then $\nabla_i^{g_1} \nabla_j^{g_1} F = f g_1^*(dF, df) \overset{2}{g}_{ij}$ which implies the first statement. Also, if g_2 is flat, then $\mathrm{grad}_{g_1} f$ is light-like (Fact 8, item 2), so f is a g-Casimir.

Next, for any function $\phi(X_2)$, we have

$$\nabla_a \nabla_b \phi = 0 \,, \quad \nabla_a \nabla_\beta \phi = -\frac{1}{f} f_{,a} \phi_{,\beta} \,, \quad \nabla_\alpha \nabla_\beta \phi = \nabla_\alpha^{g_2} \nabla_\beta^{g_2} \phi.$$

In particular, if ϕ satisfies $\nabla_\alpha^{g_2} \nabla_\beta^{g_2} \phi = \mathrm{const}\, \overset{2}{g}_{\alpha\beta}$ and g_2 is flat, then

$$\nabla_a \nabla_b(\phi f) = 0 \,, \quad \nabla_\alpha \nabla_b(\phi f) = 0, \quad \nabla_\alpha \nabla_\beta(\phi f) = \mathrm{const}\, \overset{2}{g}_{\alpha\beta} f.$$

Combining this with (71), we see that $\phi f - \mathrm{const}\, \tilde{f}$ is a Casimir. If ϕ satisfies $\nabla^{g_2} \nabla^{g_2} \phi + K_2 \phi\, g_2 = 0$, then $\nabla_\alpha \nabla_\beta(\phi f) = -K_2 \phi \overset{2}{g}_{\alpha\beta} f - \phi f g_1^*(\mathrm{d}f, \mathrm{d}f) \overset{2}{g}_{\alpha\beta} = 0.$ □

Now, we are able to describe conditions on the polynomials P_α that are necessary for the flatness of the metric g from Theorem 5 given by (38), and also to finish the case of one-dimensional blocks: We need to show that if an α-block is one-dimensional, then the corresponding function $H_j(x^j) = \frac{1}{h_j(x^j)}$ from (66) is a polynomial of degree ≤ 2.

First we consider the most important case, when $c_{\alpha\beta} = 1$ for all $1 \leq \alpha < \beta \leq B$. The corresponding graph in this case is just a 'path' $1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow B$ from leaf B to root 1, so that in view of (69) the metric g_{ij} is given by the warped product of the form

$$g = g_1 + f_1(X_1)^2 g_2 + f_1(X_1)^2 f_2(X_2)^2 g_3 + \cdots + \left(\prod_{s=1}^{B-1} f_s(X_s)^2\right) g_B \quad (72)$$

with $f_\alpha(X_\alpha)^2 = \det(\lambda_\alpha \operatorname{Id}_{n_\alpha} - L_\alpha)$ for $\lambda_\alpha = e_{\alpha\,\alpha+1} = e_{\alpha\,\alpha+2} = \cdots = e_{\alpha\,B}$. The metrics g_α are n_α-dimensional metrics of the form (66).

Suppose that a g_α-component is one-dimensional and denote the corresponding coordinate by x^j. By direct calculations, using the formulas for S_k^{ij} and $\widehat{\Gamma}_{jk}^i$ from Sect. 6.2 and assuming $u_j = 0$, we see that the condition $\widehat{\nabla}_j S_j^{jj} = 0$ becomes

$$\frac{\partial^2 h_j(x^j)}{\partial x^{j\,2}} + 8\,g^*\left(\mathrm{d}\prod_{s=1}^{\alpha-1} f_s(X_s),\, \mathrm{d}\prod_{s=1}^{\alpha-1} f_s(X_s)\right) = 0. \tag{73}$$

Note that the function $\prod_{s=1}^{\alpha-1} f_s(X_s)$ is a Casimir of the metric $g_1 + f_1(X_1)^2 g_2 + \cdots + \left(\prod_{s=1}^{\alpha-1} f_s(X_s)^2\right) g_\alpha$ by Fact 8 so $g^*\left(\mathrm{d}\prod_{s=1}^{\alpha-1} f_s(X_s),\, \mathrm{d}\prod_{s=1}^{\alpha-1} f_s(X_s)\right)$ is a constant and h_j is a polynomial of degree ≤ 2 whose leading coefficient is this constant with a minus sign. Note that the case $\alpha = 1$ is also covered by the argumentation above by assuming that this constant is zero; in this case, h_j is a linear polynomial.

Thus, our metric g is given by (72) with each $g_\alpha = g_\alpha^{\mathsf{LC}}\left(P_\alpha(L_\alpha)\right)^{-1}$ and $f_\alpha^2 = \det(\lambda_\alpha \operatorname{Id}_{n_\alpha} - L_\alpha)$, where P_α has degree at most $n_\alpha + 1$. We denote the coefficients of the polynomials by

$$P_\alpha(t) = \overset{\alpha}{a}_0 + \overset{\alpha}{a}_1 t + \cdots + \overset{\alpha}{a}_{n_1+1} t^{n_1+1}.$$

Lemma 7.3 *In the above notation, we have the following relations:*

$$\begin{aligned}
&\overset{1}{a}_{n_1+1} = 0 \\
&P_\alpha(\lambda_\alpha) = 0 && \textit{for } \alpha = 1, \ldots, B - 1. \\
&\frac{\mathrm{d}\,P_\alpha}{\mathrm{d}\,t}\Big|_{t=\lambda_\alpha} = \overset{\alpha+1}{a}_{n_{\alpha+1}+1} && \textit{for } \alpha = 1, \ldots, B - 1
\end{aligned} \tag{74}$$

Note that in view of Theorem 4, conditions (74) are sufficient for the flatness of g and existence of Frobenius coordinates.

Proof We view g as a warped product metric over the $n_1 + n_2$ dimensional base equipped with the metric $g_1 + f_1(X_1)^2 g_2$. Then, the metric $g_1 + f_1(X_1)^2 g_2$ is flat. If $n_1 > 1$ (the case $n_1 = 1$ was already discussed above), combining Facts 7 and 8 we obtain $a_{n_1+1} = 0$ implying the first line of (74). Next, from Fact 8 we know that $f_1(X_1)$ is a g_1-Casimir implying $P_1(\lambda_1) = 0$ in view of Fact 7.

Let us now show that

$$\frac{\mathrm{d}\,P_1}{\mathrm{d}\,t}\Big|_{t=\lambda_1} = \overset{2}{a}_{n_2+1}. \tag{75}$$

Since $g_1 + f_1(X_1)^2 g_2$ is flat, by Fact 8 we have (we denote by K_2 the curvature of g_2 and assume $n_2 > 1$)

$$\tfrac{1}{4} g^*\left(\mathrm{d}\,f_1,\, \mathrm{d}\,f_1\right) = K_2. \tag{76}$$

Combining (76) with Fact 7, we obtain (75). In the case $n_2 = 1$, (75) follows from (73) and Fact 7.

Next, we view g as a warped product metric over the $n_1 + n_2 + n_3$ dimensional base equipped with the metric $g_1 + f_1(X_1)^2 g_2 + f_1(X_1)^2 f_2(X_2)^2 g_3$. Then, the metric $g_1 + f_1(X_1)^2 g_2 + f_1(X_1)^2 f_2(X_2)^2 g_3$ is flat. But it is itself a warped product metric over the $n_1 + n_2$ dimensional base with the metric $g_1 + f_1(X_1)^2 g_2$. By Fact 8, this implies that $f_1 f_2$ is a Casimir of this metric. By Lemma 7.2, f_2 is a Casimir of g_2 so $P_2(\lambda_2) = 0$. Moreover, by Fact 8 and using $g^*(\mathrm{d}\, f_1, \mathrm{d}\, f_2) = 0$, we have

$$\tfrac{1}{4} g^*\big(\mathrm{d}\,(f_1 f_2), \mathrm{d}\,(f_1 f_2)\big) = K_3 \ \ (= -\tfrac{1}{4}\overset{3}{a}_{n_3+1}). \tag{77}$$

On the other hand

$$
\begin{aligned}
g^*\big(\mathrm{d}\,(f_1 f_2), \mathrm{d}\,(f_1 f_2)\big) &= f_2{}^2 g_1^*(\mathrm{d}\, f_1, \mathrm{d}\, f_1) + g_2^*(\mathrm{d}\, f_2, \mathrm{d}\, f_2)\\
&\overset{(70)}{=} -\tfrac{1}{4} f_2{}^2 \tfrac{\mathrm{d}\, P_1}{\mathrm{d}\, t}\big|_{t=\lambda_1} - \tfrac{1}{4} \tfrac{\mathrm{d}\, P_2}{\mathrm{d}\, t}\big|_{t=\lambda_2} + \overset{2}{a}_{n_2+1} f_2{}^2 \\
&\overset{(75)}{=} -\tfrac{1}{4} \tfrac{\mathrm{d}\, P_2}{\mathrm{d}\, t}\big|_{t=\lambda_2}.
\end{aligned}
\tag{78}
$$

Combining this with (77), we obtain $\overset{3}{a}_{n_3+1} = \tfrac{\mathrm{d}\, P_2}{\mathrm{d}\, t}\big|_{t=\lambda_2}$, as claimed. Iterating this procedure we obtain (74). $\qquad\qquad\square$

Lemma 7.3 completes the proof of Theorem 5 under the additional assumption that for every $\alpha \prec \beta$, we have $c_{\alpha\beta} = 1$. We now reduce the general case to this situation.

We assume without loss of generality that the combinatorial data are given by a directed rooted in-tree with B vertices, i.e. the graph F is connected. Otherwise, we have the direct product situation, i.e. the metric and all other relevant objects are direct products of lower-dimensional metrics and relevant lower dimensional objects.

We denote by $1, 2, \ldots, B$ the vertices of the in-tree F in such a way that $\alpha \prec \beta$ implies $\alpha < \beta$; of course, the vertex 1 is then the root. Other vertices of degree one are called *leaves*. Recall that $\alpha = \mathrm{next}(\beta)$, if $\alpha \prec \beta$ and there is no γ with $\alpha \prec \gamma \prec \beta$.

For every leaf β, we define the *chain* S_β (oriented path to the root) as the sub-tree with vertices β, $\mathrm{next}(\beta)$, $\mathrm{next}(\mathrm{next}(\beta))$,…,1. For example, the upper tree of Fig. 1 has two chains, one with vertices 3, 2, 1 and another with vertices 4, 2, 1.

Next, for the chain S_β and for any fixed point p, we consider the following sub-manifold M_β passing through p: in the coordinates $(X_1, \ldots, X_B) = (x^1, \ldots, x^n)$, it is defined by the system of equations

$$X_\alpha = X_\alpha(p) \ \text{ for every } \alpha \notin S_\beta.$$

This is a totally geodesic submanifold with respect to the connections Γ, $\bar{\Gamma}$ and $\widehat{\Gamma}$. For $\widehat{\Gamma}$, this follows from formulas (A,B,C) of Sect. 6.2. For Γ and $\bar{\Gamma}$, it follows from (62).

Therefore, the restriction of g and $\bar{g}$ onto M_β satisfies the assumptions of Theorem 5. Moreover, the components $c_{\alpha\beta}$ corresponding to this restriction are equal to 1 for $\alpha < \beta$.

For example, for $\beta = 3$, the metric g corresponding to the upper tree of Fig. 1 is given by

$$g = g_1 + \det(e_{12}\,\mathrm{Id}_{n_1} - L_1) \cdot g_2 + \det(e_{12}\,\mathrm{Id}_{n_2} - L_1) \cdot ((e_{23}\,\mathrm{Id}_{n_3} - \det L_2)$$
$$\cdot g_3 + \det(e_{24}\,\mathrm{Id}_{n_4} - L_2) \cdot g_4)$$

and the restriction of the metric g onto M_3 is

$$g_1 + \det(e_{12}\,\mathrm{Id}_{n_1} - L_1) \cdot g_2 + \det(e_{12}\,\mathrm{Id}_{n_2} - L_1)(e_{23}\,\mathrm{Id}_{n_3} - \det L_2) \cdot g_3.$$

The case when $c_{\alpha\beta} = 1$ for all $\alpha < \beta$ is described in Lemma 7.3 and it has been proved that the metric is as in Theorem 5. This implies that the metrics g and $\bar{g}$ are constructed as in Sect. 4.1 and the coefficients of P_α and Q_α satisfy the conditions (i, ii) from Sect. 4.1. It remains to show that they also satisfy condition (iii). In order to do this, suppose that $\alpha = \mathrm{next}(\beta) = \mathrm{next}(\gamma)$ with $\beta \neq \gamma$. We consider the sub-tree with vertices α, β, $\gamma := \mathrm{next}(\alpha) = \mathrm{next}(\beta)$ and the corresponding warped product metric with the base metric g_α and fibre metric $g_\beta + g_\gamma$. For example, for $\alpha = 2$ in the case of the upper tree of Fig. 1, we consider the warped product metric $g_2 + \det(\lambda_\alpha\,\mathrm{Id}_{n_2} - L_2) \cdot (g_3 + g_4)$.

We know that it must be of constant curvature which implies that the direct product metric $g_\beta + g_\gamma$ must be of constant curvature which in turn implies that it is flat. Then, by Fact 7, the coefficients $\overset{\beta}{a}_{n_\beta+1}$ and $\overset{\gamma}{a}_{n_\gamma+1}$ vanish implying $\frac{d P_\alpha}{d t}_{|t=\lambda_\alpha} = 0$. Theorem 5 is proved.

7.3 On the Uniqueness of Frobenius Coordinates for a Pair of Metrics

We consider two flat metrics g, $\bar{g}$ possessing a common Frobenius coordinate system and discuss the uniqueness of this coordinate system. As before, we assume that $R_g = \bar{g}g^{-1}$ has n different eigenvalues. We know that g, $\bar{g}$ are as described in Theorem 5. In particular, in the corresponding coordinates, the connection $\widehat{\Gamma}$ defining the Frobenius coordinate system (i.e. the flat connection that vanishes in Frobenius coordinates) is given by the formulas from Sect. 6.2. That is,

- For every $i \neq j$ and $k \neq i$, $\widehat{\Gamma}^i_{jk} = 0$.
- For the indices $i \neq j$ from one block, $\widehat{\Gamma}^i_{ij} = \frac{1}{x^j - x^i}$.
- For the index i from the block number α and j from the block number $\beta \neq \alpha$, we have $\widehat{\Gamma}^j_{ji} = \widehat{\Gamma}^j_{ij} = \frac{c_{\alpha\beta}}{x^i}$.
- For every i, the component $\widehat{\Gamma}^i_{ii} := u_i$ depends on x^i only.

However, in view of (58) and (59), the function u_i is also uniquely defined in terms of the metric g (in fact, $u_i \equiv 0$ in the diagonal coordinates from Theorem 5) unless the only component of g which can depend on x^i is the component g_{ii}. Such an exceptional situation is possible if and only if g_{ii} represents a one-dimensional block which is a leaf in terms of the in-forest F.

Clearly, the Frobenius coordinate system is determined up to affine change of coordinates by the flat connection $\widehat{\Gamma}$. Therefore, non-uniqueness of this coordinate

system only appears in the above exceptional situation. It is easy to see that in this case the freedom in choosing it is the same as discussed in Sect. 4.1 after Theorem 5.

Remark 7.1 Coming back to Theorem 3 (uniqueness part), we notice that for the metrics $g = P(L)g_{LC}$ and $\bar{g} = Q(L)g_{LC}$, every diagonal component g_{ii} depends on all variables $x^1, \ldots, x^n$. Then, the connection $\widehat{\Gamma}$ is unique implying that Frobenius coordinates are unique up to an affine coordinate change, as required.

8 Pro-Frobenius Coordinates and Multi-block Frobenius Pencils. Proof of Theorem 4

8.1 Extended AFF-Pencils and Pro-Frobenius Coordinates

As seen from Sect. 4.1, the main ingredients in multi-block Frobenius pencils (Theorem 5) are metrics of the form

$$P(L)g_{LC}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree } n + 1. \tag{79}$$

If $\deg P \leq n$, then such metrics are flat and form the AFF-pencil (26). However, if $\deg P = n+1$, i.e. $P(t) = a_{n+1}t^{n+1} + \ldots$, then $g = P(L)g_{LC}$ has constant curvature $K = -\frac{1}{4}a_{n+1}$ (see Fact 8). All together, the metrics (79) form a pencil of compatible constant curvature metrics, which can be thought of as one-dimensional extension of the AFF-pencil (26), see details in [9, 20]. In Frobenuis coordinates $u^1, \ldots, u^n$ from Sect. 2.3, the coefficients $g^{\alpha\beta}$ of the metric $g = P(L)g_{LC}$ with $\deg P = n + 1$ are not affine functions anymore. In particular, in the notation from Sect. 2.3, for $P(t) = t^{n+1}$ we get:

$$g_{n+1} = L^{n+1}g_0 = \begin{pmatrix} u^2 & u^3 & \ldots & u^n & 0 \\ u^3 & \ldots & u^n & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ u^n & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} + \begin{pmatrix} u^1u^1 & u^1u^2 & \ldots & u^1u^n \\ u^2u^1 & u^2u^2 & \ldots & u^2u^n \\ u^3u^1 & u^3u^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ u^nu^1 & u^nu^2 & \ldots & u^nu^n \end{pmatrix}.$$

All the other metrics from the extended AFF-pencil (79), in coordinates $u^1, \ldots, u^n$, take the form

$$g^{\alpha\beta} = b^{\alpha\beta} + a_s^{\alpha\beta}u^s - 4Ku^\alpha u^\beta, \tag{80}$$

which looks as a *quadratic perturbation* of (15). These metrics (more precisely, the corresponding coordinates $u^1, \ldots, u^n$) still possess remarkable properties, similar to those from Fact 4. We use one of them to introduce the following

Definition 5 We say that $u^1, \ldots, u^n$ is a *pro-Frobenius coordinate system* for a metric g given by (80), if the contravariant Christoffel symbols $\Gamma_s^{\alpha\beta}$ of g are symmetric in upper indices.

The Frobenuis coordinates $u^1, \ldots, u^n$ for the metrics from the AFF-pencil are pro-Frobenius for all the metrics (79) from the extended AFF-pencil (recall that these coordinates are the coefficients of the characteristic polynomial of the Nijenhuis operator L).

Notice that the symmetry of $\Gamma_s^{\alpha\beta}$ in upper indices is a strong geometric condition that naturally appears in many problems and admits several equivalent interpretations. The following statement summarises some of them (the proof is straightforward and is left to the reader).

Lemma 8.1 *Let g_{ij} be a metric written in a certain coordinate system $u^1, \ldots, u^n$. The following conditions are equivalent*

- *The contravariant Christoffel symbols $\Gamma_k^{ij} = g^{is}\Gamma_{sk}^j$ are symmetric in upper indices,*
- $\Gamma_k^{ij} = -\frac{1}{2}\frac{\partial g^{ij}}{\partial u^k}$,
- *The Christoffel symbols (of the first kind) $\Gamma_{ijk} = g_{ks}\Gamma_{ij}^s$ are totally symmetric in lower indices,*
- $\Gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial u^k}$,
- g_{ij} *is a Hessian metric, i.e. $g_{ij} = \frac{\partial^2 F}{\partial u^i \partial u^j}$ for a certain function $F(u^1, \ldots, u^n)$.[7]*

This lemma immediately implies the following explicit formula for the contravariant Christoffel symbols of g in pro-Frobenius coordinates:

$$\Gamma_s^{\alpha\beta} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial u^s} = -\frac{1}{2}a_s^{\alpha\beta} + 2Ku^\alpha \delta_s^\beta + 2K\delta_s^\alpha u^\beta. \tag{81}$$

Remark 8.1 Notice that for $K = 0$, a pro-Frobenius coordinate system is Frobenius (see Fact 4). Also it can be checked that a pro-Frobenius metric (80) has constant curvature K. For this reason, pro-Frobenius metrics can naturally be understood as generalisation of Frobenius metrics (15) to the case of constant curvature metrics.

8.2 Warp Product of Pro-Frobenius Metrics

The proof of Theorem 4 is based on the following key statement.

Proposition 8.1 *Let g and $\widehat{g}$ be two constant curvature metrics written in pro-Frobenius coordinates $u = (u^1, \ldots, u^{n_1})$ and $v = (v^1, \ldots, v^{n_2})$, respectively. Let $f(u)$ be a non-homogeneous linear function satisfying the following relations*

- (b1) $\operatorname{grad} f(u) = f(u)(\alpha - 4Ku)$, *where $\alpha = \sum \alpha^i \frac{\partial}{\partial u^i}$ is a constant vector field and* $u = \sum u^i \frac{\partial}{\partial u^i}$,
- (b2) $\frac{1}{f(u)}g(\mathrm{d}f, \mathrm{d}f) = 4\widehat{K} - 4Kf(u)$, *where K and $\widehat{K}$ are the curvatures of g and $\widehat{g}$, respectively.*

---

[7] Following [39], we will refer to this function F as the *potential*.

Then the warp product metric[8] $g_W = g + \frac{1}{f}\widehat{g}$ has constant curvature and the coordinate system $u^1, \ldots, u^{n_1}, y^1 = fv^1, \ldots, y^{n_2} = fv^{n_2}$ is pro-Frobenius for g_W.

Proof The matrix of g_W in coordinates u, y takes the form

$$
g_W = \begin{pmatrix} \mathrm{Id} & 0 \\ v \cdot \mathrm{d}f & f \cdot \mathrm{Id} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & \frac{1}{f}\widehat{g} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & \mathrm{d}f^\top \cdot v^\top \\ 0 & f \cdot \mathrm{Id} \end{pmatrix}
$$
$$
= \begin{pmatrix} g & \mathrm{grad}\, f \cdot v^\top \\ v \cdot (\mathrm{grad}\, f)^\top & g(\mathrm{d}f, \mathrm{d}f)v \cdot v^\top + f\,\widehat{g} \end{pmatrix}
$$

Hence the off-diagonal block has the following form

$$
\mathrm{grad}\, f \cdot v^\top = f(u)(\alpha - 4Ku) \cdot v^\top = (\alpha - 4Ku) \cdot y^\top.
$$

The lower diagonal block is

$$
g(\mathrm{d}f, \mathrm{d}f)v \cdot v^\top + f\,\widehat{g} = (4\widehat{K}f - 4Kf^2)v \cdot v^\top + f\,\widehat{g}
$$

Recall that $\widehat{g}$ consists of three parts, constant, linear and quadratic

$$
\widehat{g} = \widehat{b} + \widehat{a}(v) - 4\widehat{K}v \cdot v^\top \quad \text{or, in more detail,} \quad \widehat{g}^{ij} = \widehat{b}^{ij} + \widehat{a}^{ij}_s v^s - 4\widehat{K}v^i v^j.
$$

Substitution gives:

$$
(4\widehat{K}f - 4Kf^2)v \cdot v^\top + f\,\widehat{g} = (4\widehat{K}f - 4Kf^2)v \cdot v^\top + f(\widehat{b} + \widehat{a}(v) - 4\widehat{K}v \cdot v^\top)
$$
$$
= f\widehat{b} + \widehat{a}(y) - 4Ky \cdot y^\top
$$

Summarising (and denoting $g = b + a(u) - 4Ku \cdot u^\top$), we get

$$
g_W = \begin{pmatrix} b & 0 \\ 0 & f(0)\widehat{b} \end{pmatrix} + \begin{pmatrix} a(u) & \alpha \cdot y^\top \\ y \cdot \alpha^\top & \widehat{a}(y) + m(u)\widehat{b} \end{pmatrix} - 4K \begin{pmatrix} u \\ y \end{pmatrix} \begin{pmatrix} u^\top & y^\top \end{pmatrix},
$$

where $m(u) = f(u) - f(0)$ is the (homogeneous) linear part of the function f. This shows that g_W takes form (80) in coordinates (u, y).

To complete the proof, we only need to show that the (covariant) metric $(g_W)_{ij}$ is Hessian (see Lemma 8.1). This fact follows from

Lemma 8.2 *Suppose $g_{ij}(u) = \frac{\partial^2 F}{\partial u^i \partial u^j}$ is a Hessian metric in coordinates $u^1, \ldots, u^{n_1}$ and $\widehat{g}_{\alpha\beta}(v) = \frac{\partial^2 \widehat{F}}{\partial v^\alpha \partial v^\beta}$ is a Hessian metric in coordinates $v^1, \ldots, v^{n_2}$. Then the (covariant) warp product metric*

$$
(g_W)_{ij} = g_{ij}(u)\mathrm{d}u^i \mathrm{d}u^j + f(u)\widehat{g}_{\alpha\beta}(v)\mathrm{d}v^\alpha \mathrm{d}v^\beta \tag{82}
$$

---

[8] This metric was also discussed in Sect. 7, see Fact 8 et seq. Note the difference in notation and settings used. Unlike Fact 8, now we are working with contravariant metrics, and our *new* function f was previously denoted by f^2.

where $f(u)$ is an arbitrary linear function in u, is Hessian in coordinates $u^1, \dots, u^{n_1}, y^1, \dots, y^{n_2}$, where $y^i = f(u)v^i$. The corresponding potential is $F_w = F + f \cdot \widehat{F} = F(u^1, \dots, u^{n_1}) + f(u)\widehat{F}\left(\frac{y^1}{f(u)}, \dots, \frac{y^{n_2}}{f(u)}\right)$.

Proof Let us rewrite the warp product metric (82) in coordinates u, y:

$$
g_{ij}\mathrm{d}\,u^i\mathrm{d}\,u^j + f\,\widehat{g}_{\alpha\beta}\,\mathrm{d}\,v^\alpha\mathrm{d}\,v^\beta = g_{ij}\mathrm{d}\,u^i\mathrm{d}\,u^j + f\,\widehat{g}_{\alpha\beta}\,\mathrm{d}\left(\frac{y^\alpha}{f}\right)\mathrm{d}\left(\frac{y^\beta}{f}\right) =
$$

$$
g_{ij}\mathrm{d}\,u^i\mathrm{d}\,u^j + f\,\widehat{g}_{\alpha\beta}\left(\frac{\mathrm{d}\,y^\alpha}{f} - \frac{y^\alpha\mathrm{d}\,f}{f^2}\right)\left(\frac{\mathrm{d}\,y^\beta}{f} - \frac{y^\beta\mathrm{d}\,f}{f^2}\right) =
$$

$$
\left(g_{ij} + \frac{1}{f^3}\,\widehat{g}_{\alpha\beta}\,y^\alpha y^\beta \frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}\right)\mathrm{d}\,u^i\mathrm{d}\,u^j - 2\frac{1}{f^2}\widehat{g}_{\alpha\beta}y^\alpha\frac{\partial f}{\partial u^j}\mathrm{d}\,y^\beta\mathrm{d}\,u^j + \frac{1}{f}\widehat{g}_{\alpha\beta}\mathrm{d}\,y^\alpha\mathrm{d}\,y^\beta.
$$

$$(83)$$

Then we compute the second derivatives of the function $F + f \cdot \widehat{F}$ (here we use $v^\alpha = \frac{y^\alpha}{f(u)}$ and $\frac{\partial \widehat{F}}{\partial u^j} = \sum_\alpha \frac{\partial \widehat{F}}{\partial v^\alpha}\frac{\partial v^\alpha}{\partial u^j} = -\sum_\alpha \frac{\partial \widehat{F}}{\partial v^\alpha}\frac{y^\alpha}{f^2}\frac{\partial f}{\partial u^j}$):

$$
\frac{\partial^2}{\partial u^i \partial u^j}(F + f \cdot \widehat{F}) = \frac{\partial^2 F}{\partial u^i \partial u^j} + \frac{\partial}{\partial u^i}\left(\frac{\partial f}{\partial u^j}\cdot\widehat{F} + f\cdot\frac{\partial \widehat{F}}{\partial u^j}\right)
$$

$$
= \frac{\partial^2 F}{\partial u^i \partial u^j} + \frac{\partial}{\partial u^i}\left(\frac{\partial f}{\partial u^j}\cdot\widehat{F} - \frac{1}{f}\sum_\alpha\frac{\partial \widehat{F}}{\partial v^\alpha}y^\alpha\frac{\partial f}{\partial u^j}\right) = \text{(we use } \frac{\partial f}{\partial u^j} = \text{const as } f \text{ is linear)}
$$

$$
\frac{\partial^2 F}{\partial u^i \partial u^j} + \frac{\partial f}{\partial u^j}\left(-\sum_\alpha\frac{\partial \widehat{F}}{\partial v^\alpha}\frac{y^\alpha}{f^2}\frac{\partial f}{\partial u^j} + \frac{1}{f^2}\frac{\partial f}{\partial u^i}\cdot\sum_\alpha\frac{\partial \widehat{F}}{\partial v^\alpha}y^\alpha - \frac{1}{f}\sum_\alpha\frac{\partial}{\partial u^i}\left(\frac{\partial \widehat{F}}{\partial v^\alpha}\right)y^\alpha\right)
$$

$$
= \frac{\partial^2 F}{\partial u^i \partial u^j} - \frac{\partial f}{\partial u^j}\frac{1}{f}\sum_{\alpha,\beta}\frac{\partial^2 \widehat{F}}{\partial v^\beta \partial v^\alpha}\left(-\frac{1}{f^2}\right)\frac{\partial f}{\partial u^i}y^\alpha y^\beta = g_{ij} + \frac{1}{f^3}\sum_{\alpha,\beta}\widehat{g}_{\alpha\beta}y^\alpha y^\beta\frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}.
$$

Next,

$$
\frac{\partial^2}{\partial y^\beta \partial u^j}(F + f \cdot \widehat{F}) = \frac{\partial}{\partial y^\beta}\left(\frac{\partial f}{\partial u^j}\cdot\widehat{F} - \frac{1}{f}\sum_\alpha\frac{\partial \widehat{F}}{\partial v^\alpha}y^\alpha\frac{\partial f}{\partial u^j}\right)
$$

$$
= \frac{\partial f}{\partial u^j}\cdot\frac{\partial}{\partial y^\beta}\left(\widehat{F} - \frac{1}{f}\sum_\alpha\frac{\partial \widehat{F}}{\partial v^\alpha}y^\alpha\right)
$$

$$
= \frac{\partial f}{\partial u^j}\left(\frac{\partial \widehat{F}}{\partial v^\beta}\frac{1}{f} - \frac{1}{f}\frac{\partial \widehat{F}}{\partial v^\beta} - \frac{1}{f}\sum_\alpha\frac{\partial^2 \widehat{F}}{\partial v^\alpha \partial v^\beta}y^\alpha\frac{1}{f}\right) = -\frac{\partial f}{\partial u^j}\frac{1}{f^2}\sum_\alpha\widehat{g}_{\alpha\beta}y^\alpha.
$$

And finally,

$$
\frac{\partial^2}{\partial y^\alpha \partial y^\beta}(F + f \cdot \widehat{F}) = f\frac{\partial^2 \widehat{F}}{\partial y^\alpha \partial y^\beta} = f\frac{\partial^2 \widehat{F}}{\partial v^\alpha \partial v^\beta}\frac{1}{f^2} = \frac{1}{f}\widehat{g}^{\alpha\beta}.
$$

Comparing $\frac{\partial^2}{\partial u^i \partial u^j}(F + f \cdot \widehat{F})$, $\frac{\partial^2}{\partial y^\alpha \partial u^j}(F + f \cdot \widehat{F})$ and $\frac{\partial^2}{\partial y^\alpha \partial y^\beta}(F + f \cdot \widehat{F})$ with (83) shows that g_w is Hessian in coordinates (u, y), as stated.   □

This completes the proof of Proposition 8.1.   □

Our goal is to show that every metric from Theorem 4 is Frobenius in the corresponding coordinate system. It is seen from the construction of such a metric that it can be obtained step-by-step by iterating the warp product construction (following the rules prescribed by the graph F). At each step,we have two constant curvature metrics g and $\widehat{g}$ written in pro-Frobenius coordinate systems and then we 'glue' them by using a certain linear function f as described in Proposition 8.1. This function must satisfy the property (b1). (The second property (b2) should be understood as a condition on the curvature $\widehat{K}$ of the metric $\widehat{g}$.) In order to iterate the warp product procedure, we need to describe those linear functions $f_w(u, y)$ which satisfy (b1) for the metric g_w. The next statement provides such a description.

Proposition 8.2 *In the notation of Proposition 8.1, consider the warp product metric* $g_w = g + \frac{1}{f}\widehat{g}$.

1. *Let $h(u)$ be another function (independent on f) satisfying condition (b1) for g, that is $\mathrm{grad}_g\, h = h(\beta - 4Ku)$ for some constant vector β. Then $h(u)$ satisfies (b1) for g_w, namely*

$$\mathrm{grad}_{g_w}\, h = h\left(\begin{pmatrix} \beta \\ 0 \end{pmatrix} - 4K \begin{pmatrix} u \\ y \end{pmatrix}\right).$$

2. *The function $f(u)$ itself satisfies (b1) for g_w under the additional condition that $\langle \alpha, \mathrm{d}\, f \rangle + 4Kf(0) = 0$, namely*

$$\mathrm{grad}_{g_w}\, f = f\left(\begin{pmatrix} \alpha \\ 0 \end{pmatrix} - 4K \begin{pmatrix} u \\ y \end{pmatrix}\right).$$

3. *Let $\widehat{h}(v)$ be a function satisfying conditions (b1) for the metric $\widehat{g}$, that is, $\mathrm{grad}_{\widehat{g}}\, \widehat{h} = \widehat{h}(\widehat{\beta} - 4\widehat{K}v)$. Then $f(u)\widehat{h}(v) = f(u)\widehat{h}(\frac{y}{f})$ satisfies (b1) for g_w, namely*

$$\mathrm{grad}_{g_w}\left(f(u)\widehat{h}(v)\right) = f(u)\widehat{h}(v)\left(\begin{pmatrix} \alpha \\ \widehat{\beta} \end{pmatrix} - 4K \begin{pmatrix} u \\ y \end{pmatrix}\right).$$

(Notice that $f(u)\widehat{h}(v)$ is linear in coordinates (u, y). Indeed, if $f(u) = m_0 + \sum m_i u^i$ and $\widehat{h}(v) = \widehat{m}_0 + \sum \widehat{m}_j v^j$, then $f(u)\widehat{h}(v) = f(u)(\widehat{m}_0 + \sum \widehat{m}_j v^j) = \widehat{m}_0 f(u) + \sum \widehat{m}_j y^j$.)

Proof 1. We first notice that $g(\mathrm{d}\, f, \mathrm{d}\, h) = \langle \mathrm{d}\, h, \mathrm{grad}_g\, f \rangle = \langle \mathrm{d}\, h, f(\alpha - 4Ku) \rangle$ and, on the other hand, $g(\mathrm{d}\, f, \mathrm{d}\, h) = \langle \mathrm{d}\, f, \mathrm{grad}_g\, h \rangle = \langle \mathrm{d}\, f, h(\beta - 4Ku) \rangle$. Since f and h are linear (perhaps non-homogeneous), we have $\langle \mathrm{d}\, h, u \rangle = h - h_0$ and similarly $\langle \mathrm{d}\, f, u \rangle = f - f_0$. Hence,

$$g(\mathrm{d}\, f, \mathrm{d}\, h) = \langle \mathrm{d}\, f, h(\beta - 4Ku) \rangle = h\langle \mathrm{d}\, f, \beta \rangle - 4Kh(f - f_0)$$

$$= h(\langle \mathrm{d}\, f, \beta \rangle + 4K f_0) - 4K h f,$$

and, similarly,

$$g(\mathrm{d}\, f, \mathrm{d}\, h) = f(\langle \mathrm{d}\, h, \alpha \rangle + 4K h_0) - 4K h f.$$

It follows from this that $h(\langle \mathrm{d}\, f, \beta \rangle + 4K f_0) = f(\langle \mathrm{d}\, h, \alpha \rangle + 4K h_0)$. Notice that the expressions in brackets are some constants. Since h and f are not proportional, we conclude that these expressions are identically zero implying $g(\mathrm{d}\, f, \mathrm{d}\, h) = -4K f h$ and $\langle \mathrm{d}\, h, \alpha \rangle + 4K h_0 = 0$.

We now compute $\mathrm{grad}_{g_w} h$:

$$\mathrm{grad}_{g_w} h = \begin{pmatrix} g & (\alpha - 4Ku)\cdot y^\top \\ y\cdot(\alpha-4Ku)^\top & * \end{pmatrix} \begin{pmatrix} \mathrm{d}\, h \\ 0 \end{pmatrix} = \begin{pmatrix} h(\beta - 4Ku) \\ y\cdot(\alpha - 4Ku)^\top \mathrm{d}\, h \end{pmatrix},$$

where

$$y\cdot(\alpha - 4Ku)^\top \mathrm{d}\, h = y\cdot(\alpha\,\mathrm{d}\, h + 4K h(0)) - y\cdot 4K(h(0) + (u^\top \mathrm{d}\, h))$$
$$= y\cdot 0 - y\cdot 4K h = -y\cdot 4K h,$$

which gives $\mathrm{grad}_{g_w} h = h\left(\begin{pmatrix} \beta \\ 0 \end{pmatrix} - 4K \begin{pmatrix} u \\ y \end{pmatrix} \right)$, as required.

2. The proof is just the same, but we need to use $\alpha\,\mathrm{d}\, f + 4K f(0) = 0$ as an additional condition.

3. We now compute the gradient of $f(u)\widehat{h}(v) = f(u)\widehat{h}\left(\frac{y}{f}\right) = f(u)(\widehat{m}_0 + \sum \widehat{m}_j v^j) = \widehat{m}_0 f(u) + \sum \widehat{m}_j y^j = \widehat{h}(0) f(u) + \widehat{h}(y) - \widehat{h}(0)$.

$$\mathrm{grad}_{g_w}\left(\widehat{h}(0) f(u) + \widehat{h}(y) - \widehat{h}(0)\right) = \begin{pmatrix} g & \mathrm{grad}_g f\cdot v^\top \\ v\cdot(\mathrm{grad}_g f)^\top & g(\mathrm{d}\, f, \mathrm{d}\, f)v\cdot v^\top + f\widehat{g} \end{pmatrix} \begin{pmatrix} \widehat{h}(0)\mathrm{d}\, f \\ \mathrm{d}\,\widehat{h} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\widehat{h}(0) + v^\top \mathrm{d}\,\widehat{h}\right)\mathrm{grad}_g f \\ g(\mathrm{d}\, f, \mathrm{d}\, f)\left(\widehat{h}(0) + v^\top \mathrm{d}\,\widehat{h}\right)v + f\,\mathrm{grad}_{\widehat{g}} h(v) \end{pmatrix}$$

$$= \begin{pmatrix} h(v) f(u)(\alpha - 4Ku) \\ g(\mathrm{d}\, f, \mathrm{d}\, f)h(v)v + f(u)h(v)(\widehat{\beta} - 4\widehat{K}v) \end{pmatrix}$$

$$= \begin{pmatrix} h(v) f(u)(\alpha - 4Ku) \\ (4\widehat{K} f - 4K f^2)h(v)v + f(u)h(v)(\widehat{\beta} - 4\widehat{K}v) \end{pmatrix} = h(v) f(u)\begin{pmatrix} \alpha - 4Ku \\ \widehat{\beta} - 4Ky \end{pmatrix},$$

as required. □

Propositions 8.1 and 8.2 allow us to construct pro-Frobenius metrics using metrics from extended AFF-pencils as building blocks. We only need to describe appropriate function f satisfying properties (b1) and (b2) from Proposition 8.1 for these metrics.

Proposition 8.3 *Consider metrics $g = P(L)g_{\mathsf{LC}}$ and $\widehat{g} = \widehat{P}(\widehat{L})\widehat{g}_{\mathsf{LC}}$ from two different extended AFF-pencils (79) with pro-Frobenius coordinates $u^1, \ldots, u^{n_1}$ and $v^1, \ldots, v^{n_2}$, respectively. Let λ be a root of $P(\cdot)$, then*

1. $f = \det(\lambda \operatorname{Id} - L)$ *satisfies condition* (b1) , *namely* $\operatorname{grad} f = f(\alpha - 4Ku)$, *where* $K = \frac{1}{4}b_{n_1}$, $\alpha = (b_{n_1-1}, \ldots, b_1, b_0)$ *and* b_i's *are defined from* $P(t) = (t - \lambda)(b_0 + b_1 t + \cdots + b_{n_1} t^{n_1})$.

2. *If* $\widehat{P}(t) = \widehat{a}_{n_2+1} t^{n_2+1} + \widehat{a}_{n_2} t^{n_2} + \ldots$, *then condition* (b2) *for* f *takes the form* $\widehat{a}_{n_2+1} = P'(\lambda)$.

3. *Moreover, if* λ *is a double root of* $P(\cdot)$, *then* $\langle \alpha, \operatorname{d} f \rangle + 4Kf(0) = 0$ (*cf. item 2 of Proposition* 8.2).

Proof The first part of this statement can be verified by a straightforward computation. We also notice that $f(u)$ is linear in variables $u^1, \ldots, u^{n_1+1}$.

The second statement easily follows from Fact 7. Indeed, from (70), we get

$$\frac{1}{f} g(\operatorname{d} f, \operatorname{d} f) = -P'(\lambda) - 4Kf. \tag{84}$$

Therefore, (b2) can be written as $4\widehat{K} = -P'(\lambda)$. In view of Fact 7, $\widehat{K} = -\frac{1}{4}\widehat{a}_{n_2+1}$. Hence, (b2) amounts to $\widehat{a}_{n_2+1} = P'(\lambda)$, as stated.

Finally, $\operatorname{grad} f = f(\alpha - 4Ku)$ implies

$$\frac{1}{f} g(\operatorname{d} f, \operatorname{d} f) = \frac{1}{f}\langle \operatorname{grad} f, \operatorname{d} f \rangle = \langle \alpha, \operatorname{d} f \rangle - 4K\langle u, \operatorname{d} f \rangle$$
$$= \langle \alpha, \operatorname{d} f \rangle - 4K(f - f(0)) = \langle \alpha, \operatorname{d} f \rangle + 4Kf(0) - 4Kf.$$

Comparing this relation with (84), we see that $\langle \alpha, \operatorname{d} f \rangle + 4Kf(0) = -P'(\lambda)$. Hence, the l.h.s. of this relation vanishes if and only if λ is a double root of $P(t)$. $\qquad \square$

We now 'return' to the notation from Sect. 4.1. It follows from Propositions 8.1 and 8.3 that the metric $P_\alpha(L_\alpha)g_\alpha^{\mathsf{LC}} + \frac{1}{\det(\lambda_\beta \operatorname{Id} - L)} P_\beta(L_\beta)g_\beta^{\mathsf{LC}}$ is pro-Frobenius in coordinates $\sigma_\alpha^1, \ldots, \sigma_\alpha^{n_\alpha}, \det(\lambda \operatorname{Id} - L) \cdot \sigma_\beta^1, \ldots, \det(\lambda \operatorname{Id} - L) \cdot \sigma_\beta^{n_\beta}$, if λ_β is a root of $P_\alpha(t)$ and $\overset{\beta}{a}_{n_\beta+1} = (-1)^{n_\alpha} P_\alpha'(\lambda_\beta)$. These two conditions exactly coincide with condition (ii) used in the construction of multi-block pencils from Theorem 4 (cf. Sect. 4.2 where we discuss the two-block case in slightly different notation). We can next repeat this construction by 'adding' one more similar block $P_\gamma(L_\gamma)g_\gamma^{\mathsf{LC}}$ to $P_\alpha(L_\alpha) g_\alpha^{\mathsf{LC}}$:

$$P_\alpha(L_\alpha) g_\alpha^{\mathsf{LC}} + \frac{1}{\det(\lambda_\beta \operatorname{Id} - L_\alpha)} P_\beta(L_\beta) g_\beta^{\mathsf{LC}} + \frac{1}{\det(\lambda_\gamma \operatorname{Id} - L_\alpha)} P_\gamma(L_\gamma) g_\gamma^{\mathsf{LC}}.$$

Of course, two conditions should be fulfilled, namely $P_\alpha(\lambda_\gamma) = 0$ and $\overset{\gamma}{a}_{n_\gamma+1} = (-1)^{n_\alpha} P_\alpha'(\lambda_\gamma)$. Moreover, in view of Proposition 8.2 (item 2) and Proposition 8.2 (item 3), if $\lambda_\beta = \lambda_\gamma$, we need to require that λ_β is a double root of P_α (which is exactly condition (iii) used in the construction of multi-block pencils from Theorem 4).

To complete the proof of Theorem 4, we should now iterate the above procedure step-by-step following the combinatorial data provided by the graph F (starting from leaves and moving towards the root). At each step of this construction, we obtain a pencil of pro-Frobenius metrics, leading finally to a Frobenius pencil (the condition (i) will guarantee the flatness of the final metric).

Declarations

References

1. Antonowicz, M., Fordy, A.: Coupled KdV equations with multi-Hamiltonian structures. Physica D **28**(3), 345–357 (1987)
2. Antonowicz, M., Fordy, A.: Coupled Harry Dym equations with multi-Hamiltonian structures. J. Phys. A **21**(5), L269–L275 (1988)
3. Antonowicz, M., Fordy, A.: Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems. Commun. Math. Phys. **124**, 465–486 (1989)
4. Balinskii, A., Novikov, S.: Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. Doklady Ak. Nauk. **283**(5), 1036–1039 (1985)
5. Boralevi, A., Draisma, J., Horobet, E., Robeva, E.: Orthogonal and unitary tensor decomposition from an algebraic perspective. Israel J. Math. **222**(1), 223–260 (2017)
6. Bolsinov, A.V., Konyaev, AYu., Matveev, V.S.: Nijenhuis geometry. Adv. Math. **394**, 108001 (2021). https://doi.org/10.1016/j.aim.2021.108001
7. Bolsinov, A.V., Konyaev, AYu., Matveev, V.S.: Nijenhuis Geometry III: gl-regular Nijenhuis operators. Rev. Mat. Iberoam. (2023). https://doi.org/10.4171/RMI/1416
8. Bolsinov, A.V., Konyaev, AYu., Matveev, V.S.: Applications of Nijenhuis geometry: nondegenerate singular points of Poisson-Nijenhuis structures. Eur. J. Math. **8**, 1355–1376 (2022). https://doi.org/10.1007/s40879-020-00429-6
9. Bolsinov, A.V., Konyaev, AYu., Matveev, V.S.: Applications of Nijenhuis geometry II: maximal pencils of multihamiltonian structures of hydrodynamic type. Nonlinearity **34**(8), 5136–5162 (2021)
10. Bolsinov, A.V., Konyaev, AYu., Matveev, V.S.: Applications of Nijenhuis geometry IV: multicomponent KdV and Camassa-Holm equations. Dyn. Partial Differ. Equ. **20**(1), 73–98 (2023)
11. Bolsinov, A.V., Konyaev, A.Yu., Matveev, V.S.: Orthogonal separation of variables for spaces of constant curvature, arXiv:2212.01605
12. Casati, M., Lorenzoni, P., Vitolo, R.: Three computational approaches to weakly nonlocal Poisson brackets. Stud. Appl. Math. **144**(4), 412–448 (2020)
13. de Vries, H.L.: Über Riemannsche Räume, die infinitesimale konforme Transformation gestatten. Math. Z. **60**, 328–347 (1954)
14. Doyle, P.W.: Differential geometric Poisson bivectors in one space variable. J. Math. Phys. **34**, 1314–1338 (1993)
15. Dorfman, I.: Dirac Structures and Integrability of Nonlinear Evolution Equations. Wiley, New York (1993)
16. Dubrovin, B., Novikov, S.: Hamiltonian formalism of one-dimensional systems of hydrodynamic type, and the Bogolyubov "Whitman averaging method". Dokl. Akad. Nauk SSSR **270**(4), 781–785 (1983)

17. Dubrovin, B., Novikov, S.: Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory. Uspekhi Mat. Nauk, **44**(6), 29–98 (1989)
18. Dubrovin, B.: Flat pencils of metrics and Frobenius manifolds. In: Saito, M.-H., Shimizu, Y., Ueno, K. (eds.) Integrable Systems and Algebraic Geometry: Proceedings of the Taniguchi Symposium 1997, Kobe, June 30–July 4, 1997 and Research Institute for Mathematical Sciences, Kyoto University, July 7–11, (1997)- Sing, ISBN 981-02-3266-7
19. Ferapontov, E.V.: Compatible Poisson brackets of hydrodynamic type. J. Phys. A **34**, 2377–2388 (2001)
20. Ferapontov, E.V., Pavlov, M.V.: Quasiclassical limit of coupled KdV equations. Riemann invariants and multi-Hamiltonian structure, Physica D **52**, 211–219 (1991)
21. Ferapontov, E.V., Pavlov, M.V., Vitolo, R.F.: Towards the classification of homogeneous third-order Hamiltonian operators. IMRN **22**, 6829–6855 (2016). https://doi.org/10.1093/imrn/rnv369
22. Ferapontov, E.V., Pavlov, M.V., Vitolo, R.F.: Systems of conservation laws with third-order Hamiltonian structures. Lett. Math. Phys. **108**(6), 1525–1550 (2018)
23. Ferapontov, E.V., Galvao, C.A.P., Mokhov, O., Nutku, Y.: Bi-Hamiltonian structure of equations of associativity in 2-d topological field theory. Commun. Math. Phys. **186**, 649–669 (1997)
24. Gelfand, I., Dikii, L.: Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations. (Russian) Uspehi Mat. Nauk, **30**, no. 5(185), 67-100 (1975). English tanslation in: Russian Mathematical Surveys, **30**, 77-113 (1975)
25. Gelfand, I., Dorfman, I.: Hamiltonian operators and algebraic structures related to them. Funct. Anal. Appl. **13**(4), 248–262 (1979)
26. https://en.wikipedia.org/wiki/Tree_(graph_theory)
27. Kaygorodov, I., Rakhimov, I., Said Husain, Sh.K.: The algebraic classification of nilpotent associative commutative algebras. J. Algeb. Appl. **19**(11), 2050220 (2020)
28. Konyaev, AYu.: Nijenhuis geometry II: Left-symmetric algebras and linearization problem for Nijenhuis operators. Diff. Geom. App. **74**, 101706 (2021)
29. Konyaev, AYu.: Geometry of inhomogeneous Poisson brackets, multicomponent Harry Dym hierarchies and multicomponent Hunter-Saxton equations. Russian J. Math. Phys. **29**, 518–541 (2022)
30. Levi-Civita, T.: Sulle trasformazioni delle equazioni dinamiche. Ann. di Mat., serie 2^a. **24**, 255–300 (1896)
31. Lorenzoni, P., Savoldi, A., Vitolo, R.: Bi-Hamiltonian structures of KdV type. J. Phys. A **51**, 045202 (2018)
32. Magri, F.: A simple model of the integrable Hamiltonian equation. J. Math. Phys. **19**, 1156 (1978)
33. Mokhov, O.: Compatible flat metrics. J. Appl. Math. **2**(7), 337–370 (2002)
34. Mokhov, O.: Pencils of compatible metrics and integrable systems. Uspekhi Mat. Nauk, **72**(5)(437), 113–164 (2017), Russian Math. Surv. **72**(5), 889-937 (2017)
35. Olver, P.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
36. Poonen, B.: Isomorphism types of commutative algebras of finite rank over an algebraically closed field. In: Computational Arithmetic Geometry, Contemp. Math., vol. 463, pp. 111-120. Amer. Math. Soc., Providence, RI (2008)
37. Potemin, G.: On third-order Poisson brackets of differential geometry. Russ. Math. Surv. **52**(3), 617–618 (1997)
38. Prvanovic, M.: On warped product manifolds. Filomat **9**(2), 169–185 (1995)
39. Shima, H.: The Geometry of Hessian Structures. World Scientific Publishing Co. Pte. Ltd., Hackensack (2007)
40. Solodovnikov, A.S.: Geometric description of possible presentations of Riemannian metrics in the Levi-Civita form (Russian) Tr. Semin. Vektor. Tenzor. Anal. **12**, 131–173 (1963)
41. Strachan, I.A.B., Szablikowski, B.M.: Novikov Algebras and a classification of multicomponent Camassa-Holm equations. Stud. Appl. Math. **133**(1), 84–117 (2014)
42. Vitolo, R.: Computing with Hamiltonian operators. Comput. Phys. Commun. **244**, 228–245 (2019)