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Closed Embedded Self-shrinkers of Mean Curvature Flow

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Abstract
In this article, we show the existence of closed embedded self-shrinkers in $\mathbb{R}^{n+1}$ that are topologically of type $S^1 \times M$, where $M \subset S^n$ is any isoparametric hypersurface in S^n for which the multiplicities of the principle curvatures agree. This yields new examples of closed self-shrinkers, for example self-shrinkers of topological type $S^1 \times S^k \times S^k \subset \mathbb{R}^{2k+2}$ for any k. If the number of distinct principle curvatures of M is one, the resulting self-shrinker is topologically $S^1 \times S^{n-1}$ and the construction recovers Angenent's shrinking doughnut (Angenent in Shrinking doughnuts, Birkhäuser, Boston, pp 21–38).

Keywords Mean curvature flow · Self-shrinker · Isoparametric foliations

1 Introduction

Consider a compact n-dimensional manifold Σ that is smoothly immersed in $\mathbb{R}^{n+1}$ via a map $F_0 : \Sigma \to \mathbb{R}^{n+1}$. A *mean curvature flow of* $F_0(\Sigma)$ is a family of smooth immersions $F_t : \Sigma \to \mathbb{R}^{n+1}$ where $t \in \mathbb{R}$ varies over some interval and for which

$$\partial_t F_t(x) = \boldsymbol{H}_t(x)$$

holds for all t. Here $\boldsymbol{H}_t(x)$ is the mean curvature of $F_t(\Sigma)$ at $F_t(x)$. In other words, $F_t(\Sigma)$ flows along its mean curvature vector in $\mathbb{R}^{n+1}$. Due to compactness of Σ, such a flow necessarily becomes singular in finite time, see, e.g. [13].

By the work of Huisken [13], Ilmanen [14] and White [21], rescaling $F_t(M)$ near the singular time in an appropriate way leads to weak limits that are so-called *self-shrinkers*, that is immersed manifolds whose mean curvature flow is given by dilations. These self-shrinkers then take a special role in the singularity theory of the mean curvature flow.

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In this paper, we use the theory of isoparametric foliations of the sphere S^n to construct new examples of closed embedded self-shrinkers. Concretely we show:

Theorem A *For any isoparametric hypersurface M in S^n, $n \geq 2$, for which the multiplicities m_1 and m_2 of the principal curvatures agree, there is a closed embedded self-shrinker of topological type $S^1 \times M$ in $\mathbb{R}^{n+1}$. This hypersurface is a union of homothetic copies of the leaves of the isoparametric foliation of S^n associated to M.*

The theory of isoparametric hypersurfaces of the sphere S^n is very rich, so the above theorem can be used to produce self-shrinkers of novel topology (for example $S^1 \times S^k \times S^k \subset \mathbb{R}^{2k+2}$ for $k \in \mathbb{N}$ or $S^1 \times SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2) \subset \mathbb{R}^5$). These hypersurfaces have previously been applied to the problem of mean curvature self-shrinkers by Chang and Spruck [6], who constructed for any isoparametric hypersurface M of the sphere S^n a self-shrinking end that is asymptotic to the cone $C(M)$.

The terminology of isoparametric hypersurfaces will be recalled in Sect. 2.1. The proof of Theorem A works via a reduction of the shrinker condition to a geodesic equation in a two-dimensional manifold. Simple periodic solutions of this ordinary differential equation are then established by shooting methods very similar to [2], although the equation itself is quite different.

Denoting with g the number of principal curvatures of a regular leaf of the isoparametric foliation, one has in the case $g = 1$ that the leaves become the latitudes of a sphere. The hypersurfaces found by Theorem A are then rotationally invariant under the $O(n)$ action on $\mathbb{R}^{n+1}$ and topologically of type $S^1 \times S^{n-1}$, the same topological type as the "shrinking donut" found by Angenent [2]. It is currently an open question whether there exist embedded rotationally invariant self-shrinkers of type $S^1 \times S^{n-1}$ in $\mathbb{R}^{n+1}$ other than Angenent's example, so we also remark:

Proposition B *In the case $g = 1$, the construction of Theorem A gives Angenent's shrinking doughnut* [2].

The structure of this article is as follows: In Sect. 2, we recall the necessary facts about isoparametric foliations, explain the reduction of the self-shrinker problem to an ordinary differential equation, and remark on some elementary properties of the resulting equation. In Sect. 3, the shooting argument is presented and Theorem A is shown with the exception of a technical proposition. Proposition 1 is also proved in Sect. 3. The aforementioned technical proposition is shown in Sect. 4.

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2 Reduction and Geodesic Equation

In SubSect. 2.1, we recall some basic definitions and facts about isoparametric foliations. Subsection 2.2 explains the reduction procedure: a result due to Angenent [2] gives that a hypersurface is a self-shrinker if and only if it is minimal in some metric g_{Ang}. The reduction theorem of Palais and Terng [17] is then applied in order to reduce the shrinker property to a geodesic equation on an open subset of $\mathbb{R}^2$ equipped with a special metric. In SubSect. 2.3, we present this geodesic equation and simplify its form.

2.1 Isoparametric Foliations on Spheres

Definition 2.1 Let M be a smooth Riemannian manifold. A smooth function $f : M \to \mathbb{R}$ is called *isoparametric* if there are smooth functions $a, b : f(M) \to \mathbb{R}$ so that

$$\|\nabla f\|^2 = a \circ f, \tag{1}$$
$$\Delta f = b \circ f. \tag{2}$$

The geometric meaning of condition (1) is that the fibres of f form a (singular) transnormal system,[1] in particular they are all equidistant to each other. Condition (2) implies that the regular fibres of the foliation are of constant mean curvature in M (cf. [7]). If M is a space-form, one even has that the individual principal curvatures of such a fibre are constant along the fibre. Foliations that arise from the fibres of an isoparametric function are called *isoparametric foliations*. A hypersurface is called an *isoparametric hypersurface* if it is a regular leaf of an isoparametric foliation.

The classification of isoparametric foliations in spheres was initiated by Cartan in [3–5]. This has proven to be a difficult problem and, despite a long and active history of research, it is in part still open. A significant part of the structure theory of these foliations was developed by Münzner in two seminal papers [15, 16].

We review now some structural facts of isoparametric foliations of S^n, cf. [7, 11, 15, 16, 20] for proofs and further information:

(i) The principal curvatures of any regular fibre are constant along the fibre.
(ii) The number of distinct principal curvatures of a regular fibre is the same for any two regular fibres. Denoting this number by g, one has that $g \in \{1, 2, 3, 4, 6\}$.
(iii) There are precisely two singular fibres V_1 and V_2. One has $\operatorname{dist}(V_1, V_2) = \frac{\pi}{g}$. These singular fibres are closed and minimal submanifolds of S^n.
(iv) Any regular fibre is of the form $M_\varphi := \{x \in S^n \mid \operatorname{dist}(x, V_1) = \varphi\}$, where $\varphi \in (0, \frac{\pi}{g})$. These fibres are all diffeomorphic to one another.
(v) The principal curvatures of a regular fibre M_φ are of the form $\cot(\varphi), \cot(\varphi + \frac{\pi}{g}), \ldots, \cot(\varphi + \frac{(g-1)\pi}{g})$. Denoting with $m_1 := \operatorname{codim}(V_1) - 1, m_2 := \operatorname{codim}(V_2) - 1$ the multiplicities of these principle curvatures are $m_1, m_2, m_1, \ldots$. In particular $n - 1 = \frac{g}{2}(m_1 + m_2)$ and $m_1 = m_2$ if g is odd.
(vi) For $\varphi^* = \frac{2}{g} \arctan(\sqrt{m_1/m_2})$ the hypersurface M_{φ^*} is minimal in S^n.
(vii) The volume of a regular fibre is given by

$$\operatorname{Vol}(M_\varphi) = c \sin(\frac{g}{2}\varphi)^{m_1} \cos(\frac{g}{2}\varphi)^{m_2} \tag{3}$$

[1] Meaning if a geodesic is perpendicular to a leaf at any time that the geodesic then remains perpendicular to all leaves it intersects, cf. [19] for more details.

where c a positive constant that does not vary in φ (but will be different for different foliations).

(viii) There is a homogenous polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ (called the *Cartan-Münzner polynomial*) of degree g so that $F|_{S^n} = \cos(g\varphi)$ and for which one has:

$$\|\nabla F(x)\|^2 = g^2 \|x\|^{2g-2}, \qquad \Delta F(x) = \frac{g^2}{2}(m_1 - m_2)\|x\|^{g-2}.$$

Example The cases $g \in \{1, 2, 3\}$ were first classified by Cartan. The list of homogenous examples was completed by Takagi and Takahashi [18] based on previous work by Hsiang and Lawson [12], here an example is called homogenous if the fibres of the foliation arise as the orbits of an isometric action on S^n. The homogenous cases always arise as the principal orbit of the isotropy representation of a Riemannian symmetric space of rank 2, see [7] for more detailed remarks and references also for the other cases.

(i) When $g = 1$, the isoparametric foliation is congruent (that is equal up to an isometric transformation) to the latitudes of the sphere S^n. One has $V_1 = \{(1, 0, \ldots, 0)\}$, $V_2 = \{(-1, 0, \ldots, 0)\}$ and $M_\varphi = \{\cos(\varphi)\} \times \sin(\varphi) S^{n-1}$ for $\varphi \in (0, \pi)$. These examples are homogenous and $m_1 = m_2 = n - 1$.
(ii) When $g = 2$, the isoparametric foliation is congruent to the foliation by Clifford tori $S^{m_1} \times S^{m_2}$. One has $V_1 = \{(0, \ldots, 0)\} \times S^{m_2}$, $V_2 = S^{m_1} \times (0, \ldots, 0)$ and $M_\varphi = \sin(\varphi) S^{m_1} \times \cos(\varphi) S^{m_2}$ for $\varphi \in (0, \frac{\pi}{2})$. The integers m_1, m_2 are arbitrary so long as $m_1 + m_2 = n - 1$, in particular $m_1 \neq m_2$ is possible. The fibres are the orbits of an isometric $O(m_1 + 1) \times O(m_2 + 1)$ action on S^n.
(iii) When $g = 3$, one has $m_1 = m_2 \in \{1, 2, 4, 8\}$. The fibres of the foliation arise as the distance tubes of certain embeddings of the projective planes $\mathbb{FP}^2$ $(= V_1)$ in S^{3m_1+1} where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ is one of the real division algebras or the octonions. These examples are homogenous and the fibres are diffeomorphic to $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $SU(2)/\mathbb{T}^2$, $Sp(3)/Sp(1)^3$, $F_4/Spin(8)$, respectively.
(iv) For $g = 4$, there is an infinite family, introduced by Ferus, Karcher, and Münzner in [11], which contains both homogenous as well as inhomogenous examples. Two additional homogenous cases beyond this family exist, else all examples belong to this family. Here $m_1 \neq m_2$ is possible.
(v) For $g = 6$, one has $m_1 = m_2 \in \{1, 2\}$, as was shown by Abresch [1]. For both cases, there exist homogenous examples. If $m_1 = m_2 = 1$, it was shown by Dorfmeister and Neher [8] that the homogenous example is the only one.

2.2 Reduction for Self-shrinkers

Definition 2.2 Let F be the Cartan-Münzner polynomial of an isoparametric foliation of S^n. Define:

$$\mathbf{f} : \mathbb{R}^{n+1} - \mathbb{R}_{\geq 0} \cdot (V_1 \cup V_2) \to (0, \infty) \times (0, \frac{\pi}{g}), \qquad x \mapsto \left(\|x\|, \frac{\arccos(F(x/\|x\|))}{g} \right).$$

A set $X \subset \mathbb{R}^{n+1}$ is called $\mathbf{f}$-*invariant* if there is a set $N \subset (0, \infty) \times (0, \frac{\pi}{g})$ so that $X = \mathbf{f}^{-1}(N)$. Compare with the notion of F-invariant in [20].

Note that the $\mathbf{f}$-invariant sets are precisely those sets that are unions of homothetic copies of the regular fibres of the isoparametric foliation - that is unions of sets of the form $r \cdot M_\varphi$ for $(r, \varphi) \in (0, \infty) \times (0, \frac{\pi}{g})$. (Recall that for $\|x\| = 1$, one has $F(x) = \cos(g\varphi)$, where φ is the distance to the singular fibre V_1. So $\mathbf{f}^{-1}(r, \varphi) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r, \frac{x}{\|x\|} \in M_\varphi\}$.)

Recall (cf. [2, 9]) that a closed submanifold $X \subset \mathbb{R}^{n+1}$ is a *self-shrinker under mean curvature flow* (short: *self-shrinker*) if and only if there is an $\tau > 0$ such that X is a minimal hypersurface in $\mathbb{R}^{n+1}$ equipped with the metric (which we refer to as the *shrinker metric*):

$$g_{\text{sh}} = e^{-\frac{\tau \|x\|^2}{n}} \sum_{i=1}^{n} dx_i^2 = e^{-\frac{\tau \|x\|^2}{n}} g_{\text{Euc}}.$$

The parameter τ is related to the extinction time of X. By rescaling X if necessary we take $\tau = 1$ in what follows.

Proposition 2.3 *Equipping* $\mathbb{R}^{n+1} - \mathbb{R}_{\geq 0} \cdot (V_+ \cup V_-)$ *with the shrinker metric* g_{sh} *(with* $\tau = 1$*) and* $(0, \infty) \times (0, \frac{\pi}{g})$ *with the metric* $g_{Subm} := e^{-\frac{r^2}{n}}(dr^2 + r^2 d\varphi^2)$*, one has:*

(i) $\mathbf{f}$ *is a surjective and proper Riemannian submersion.*
(ii) The mean curvature vector of a fibre $\mathbf{f}^{-1}(r, \varphi)$ *is given by*

$$e^{\frac{r^2}{2n}} \left((\frac{1}{r} - \frac{r}{n}) \nu_r + \frac{H(\varphi)}{r} \nu_\varphi \right),$$

where $H(\varphi)$ *is the mean curvature of* $M_\varphi \subset S^n$, ν_φ *is the unit normal of* $r M_\varphi$ *in* $r S^n$ *equipped with* g_{sh}*, and* ν_r *is the unit normal of* $r S^n$ *in* $\mathbb{R}^{n+1}$ *equipped with* g_{sh}.

The proof is a standard calculation and from (ii) one sees that the mean curvature of the fibres of $\mathbf{f}$ form a basic field of the Riemannian submersion, meaning that it is the horizontal lift of a vector field on the base manifold. Riemannian submersions with this property are the key ingredient in the reduction theory developed by Palais and Terng in [17], recall:

Theorem 2.4 (Palais-Terng, cf. Theorem 4 in [17]) *Let* $\pi : (E, g_E) \to (B, g_B)$ *be a Riemannian submersion for which the mean curvatures of the fibres form a basic field, then for a* k*-dimensional submanifold* $X \subset B$*, one has that* $\pi^{-1}(X)$ *is minimal in* E *if and only if* X *is minimal in* $(B, V^{2/k} g_B)$*. Here* $V^{2/k} g_B$ *is the metric given by*

$$(V^{2/k} g_B)(b) = \text{Vol}_{g_E}(\pi^{-1}(b))^{2/k} \, g_B(b).$$

Using (3) one gets (up to a constant factor):

$$\mathrm{Vol}_{g_{\mathrm{sh}}}(\mathbf{f}^{-1}(r,\varphi))^2 = e^{-\frac{r^2(n-1)}{n}}\,\mathrm{Vol}_{g_{\mathrm{Euc}}}(r\cdot M_\varphi)^2 = r^{2(n-1)}e^{-r^2(1-1/n)}\,\mathrm{Vol}_{S^n}(M_\varphi)^2$$
$$= r^{2(n-1)}e^{-r^2(1-1/n)}\sin^{2m_1}(\frac{g}{2}\varphi)\cos^{2m_2}(\frac{g}{2}\varphi).$$

The problem of finding $\mathbf{f}$-invariant hypersurfaces that are minimal with respect to g_{sh} is then reduced to finding geodesic segments in $(0,\infty)\times(0,\frac{\pi}{g})$ equipped with the metric

$$r^{2n-2}e^{-r^2}\sin(\frac{g}{2}\varphi)^{2m_1}\cos(\frac{g}{2}\varphi)^{2m_2}(dr^2+r^2d\varphi^2). \tag{4}$$

We conclude:

Proposition 2.5 *A $\mathbf{f}$-invariant hypersurface $X\subset\mathbb{R}^{n+1}$ is a closed self-shrinker in $\mathbb{R}^{n+1}$ if and only if $N:=\mathbf{f}(X)$ is a closed geodesic in $(0,\infty)\times(0,\frac{\pi}{g})$ with respect to the metric (4).*

2.3 Geodesic Equation

For the metric (4) one gets the following geodesic equation, where α denotes the angle between $\frac{dr}{dt}$ and $\frac{d\varphi}{dt}$:

$$\frac{dr}{dt} = \cos\alpha\, G(r,\varphi),$$
$$\frac{d\varphi}{dt} = \sin\alpha\,\frac{G(r,\varphi)}{r},$$
$$\frac{d\alpha}{dt} = \sin\alpha\, r\,\partial_r\frac{G(r,\varphi)}{r} - \cos\alpha\,\frac{1}{r}\,\partial_\varphi G(r,\varphi).$$

Here

$$G(r,\varphi) = r^{-n+1}e^{r^2/2}\sin(\frac{g}{2}\varphi)^{-m_1}\cos(\frac{g}{2}\varphi)^{-m_2}.$$

Since we are not directly interested in the parametrisation of the geodesic but rather in its orbit we perform a substitution $\frac{dt^{\mathrm{new}}}{dt^{\mathrm{old}}} = \frac{1}{r}G(r,\varphi)$ to simplify the equation:

$$\frac{dr}{dt} = r\cos\alpha,$$
$$\frac{d\varphi}{dt} = \sin\alpha,$$
$$\frac{d\alpha}{dt} = \sin\alpha\,(r^2-n) + \frac{g}{2}\cos\alpha\,(m_1\cot(\frac{g}{2}\varphi) - m_2\tan(\frac{g}{2}\varphi)).$$

We simplify once more by letting $\theta(t) := \frac{g}{2}\varphi(t)$, substituting $\frac{dt^{\text{new}}}{dt^{\text{old}}} = \frac{g}{2}\frac{1}{\sin(2\theta)}$, letting $\xi(t) := \frac{g}{2}\ln(\sqrt{\frac{2}{g}}r(t))$ and $m := \frac{2}{g}n = m_1 + m_2 + \frac{2}{g}$ to get:

$$\begin{aligned} \frac{d\xi}{dt} &= \cos\alpha\ \sin(2\theta), \\ \frac{d\theta}{dt} &= \sin\alpha\ \sin(2\theta), \\ \frac{d\alpha}{dt} &= \sin\alpha\ \sin(2\theta)(e^{\frac{4}{g}\xi} - m) + 2\cos\alpha\, l(\theta). \end{aligned} \qquad *$$

Here $l(\theta) = m_1\cos^2(\theta) - m_2\sin^2(\theta)$.

If $\theta'(t) \neq 0$ then ξ may be (locally) given the form of a graph over θ. This graph obeys the following ODE:

$$\frac{d^2\xi}{d\theta^2}(\theta) = \frac{\xi''(t)}{\theta'(t)^2} - \frac{\xi'(t)}{\theta'(t)}\frac{\theta''(t)}{\theta'(t)^2} = -\left(1 + (\frac{d\xi}{d\theta})^2\right)\left(e^{\frac{4}{g}\xi} - m + 2H(\theta)\frac{d\xi}{d\theta}\right). \qquad **$$

Here

$$H(\theta) = \frac{l(\theta)}{\sin(2\theta)} = \frac{m_1}{2}\cot(\theta) - \frac{m_2}{2}\tan(\theta).$$

The ODE $(*)$ can be formulated for all initial conditions $(\xi, \theta, \alpha) \in \mathbb{R}^3$. But in coordinates (ξ, θ), the domain $(0, \infty) \times (0, \frac{\pi}{g})$ has been transformed to $\mathbb{R} \times (0, \frac{\pi}{2})$; so we are only interested in solutions where the ξ and θ components remain in that domain. We set $\mathcal{D} := \mathbb{R} \times (0, \frac{\pi}{2}) \times \mathbb{R}$.

The ODE $(*)$ admits two trivial families of solutions in $\mathcal{D}$, namely for any $k \in \mathbb{Z}$:

$$\begin{aligned} (\xi, \theta, \alpha)(t) &= (\xi(0) + 2(-1)^k\frac{\sqrt{m_1 m_2}}{m_1 + m_2}t,\ \arctan\left(\sqrt{\frac{m_1}{m_2}}\right), \pi k), \\ (\xi, \theta, \alpha)(t) &= (\frac{g}{4}\ln m,\ \operatorname{arccot}\left(e^{(-1)^k 2t}\cot(\theta(0))\right), \frac{\pi}{2} + \pi k). \end{aligned}$$

The first of these solutions lifts to the cone $\mathbb{R}_{>0} \cdot M_{\varphi^*}$ over the minimal hypersurface of the isoparametric foliation, which is a minimal submanifold of $\mathbb{R}^{n+1}$. The second lifts to the round sphere, albeit with the singular fibres V_1, V_2 removed.

2.4 Elementary Properties of $(*)$ and Symmetry

We briefly note some elementary properties of solutions of $(*)$, proofs are standard and are thus omitted.

Proposition 2.6 (i) *For any* $(\xi_0, \theta_0, \alpha_0) \in \mathbb{R}^3$,*there is a unique solution* γ *of* $(*)$ *with initial condition* $\gamma(0) = (\xi_0, \theta_0, \alpha_0)$. *This solution is smooth and has domain of definition all of* $\mathbb{R}$, *i.e. solutions exist for all times.*

(ii) *Suppose* $(\xi_s, \theta_s, \alpha_s)$ *converges in* $\mathbb{R}^3$ *to a point* $(\xi_\infty, \theta_\infty, \alpha_\infty)$. *Denote by* γ_s *the solution of* $(*)$ *with initial condition* $(\xi_s, \theta_s, \alpha_s)$ *and* γ_∞ *the solution of* $(*)$ *with initial condition* $(\xi_\infty, \theta_\infty, \alpha_\infty)$. *Then* γ_s *converges uniformly on compacta to* γ_∞.

(iii) *Solutions of* $(*)$ *with initial condition in* $\mathcal{D}$ *remain in* $\mathcal{D}$ *for all times.*

For our investigation, we are interested in periodic solutions of $(*)$. These will be found with the help of a discrete symmetry of the ODE $(*)$.

Definition 2.7 Define $\theta^* := \arctan(\sqrt{m_1/m_2})$ and let

$$S : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (\xi, \theta, \alpha) \mapsto (\xi, 2\theta^* - \theta, \pi - \alpha).$$

Remark 2.8 Note that θ^* is the solution in $(0, \frac{\pi}{2})$ of $l(\theta) = 0$. Additionally the map S is an involution that reflects θ at θ^* while sending $\cos\alpha \to -\cos\alpha$ and $\sin\alpha \to \sin\alpha$. In the event that $m_1 = m_2$, one has $\theta^* = \frac{\pi}{4}$ and $l(2\theta^* - \theta) = -l(\theta)$ as well as $\sin(2(2\theta^* - \theta)) = \sin(2\theta)$. For $m_1 = m_2$, one has $S(\mathcal{D}) = \mathcal{D}$.

Proposition 2.9 *If* $m_1 = m_2$ *then for any* $x \in \mathcal{D}$, *one has* $S(\gamma_1(-t)) = \gamma_2(t)$ *for all* $t \in \mathbb{R}$, *where* γ_1, γ_2 *are the solutions to* $(*)$ *with initial conditions* x *and* $S(x)$, *respectively.*

If $x = S(x)$ then $\gamma_1 = \gamma_2$ in the above proposition and one gets $S(\gamma_1(t)) = \gamma_1(-t)$. Noting that $(*)$ is further invariant under transformations of the form $\alpha \mapsto \alpha + 2\pi k$ for $k \in \mathbb{Z}$ then immediately gives a criterium for finding periodic solutions:

Corollary 2.10 *Let* $m_1 = m_2$ *and* $x \in \mathcal{D}$ *with* $S(x) = x$, *let* γ *be solution of* $(*)$ *with initial condition* x. *If there are* $T \neq 0$ *and* $k \in \mathbb{Z}$ *so that*

$$S(\gamma(T)) = \gamma(T) + \begin{pmatrix} 0 \\ 0 \\ 2\pi k \end{pmatrix}$$

then the ξ *and* θ *components of* γ *are periodic and* $2T$ *is a period.*

Note that one has $S(\xi, \theta, \alpha) = (\xi, \theta, \alpha + 2\pi k)$ for some $k \in \mathbb{Z}$ if and only if $\theta = \theta^*$ and $\alpha = \frac{\pi}{2} + \pi j$ for some $j \in \mathbb{Z}$.

3 Existence of Periodic Curves

In light of Corollary 2.10 we wish to find geodesic segments that begin and end on the line $\theta = \theta^*$, with both intersections being orthogonal. We begin with the following definition:

Definition 3.1 For $\xi_0 \in \mathbb{R}$ let $(\theta_{\xi_0}(t), \xi_{\xi_0}(t), \alpha_{\xi_0}(t))$ denote the solution of $(*)$ with initial condition $\xi(0) = \xi_0$, $\theta(0) = \theta^*$ and $\alpha(0) = \frac{\pi}{2}$. Then:

(i) ξ_0 is said to be of *type 1* if there is a $T > 0$ so that $\theta_{\xi_0}(T) = \theta^*$ and $\xi'_{\xi_0}(t) \neq 0$ for all $t \in (0, T)$.

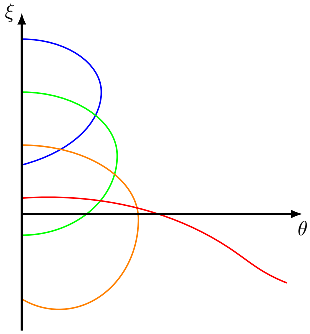

Fig. 1 Examples of curves of different type. The blue curve is type 1 but not type 2, the green curve is both type 1 and type 2, the orange curve is type 2 but not type 1, the red curve is type 3

(ii) ξ_0 is said to be of *type 2* if there is a $T > 0$ so that $\xi'_{\xi_0}(T) = 0$ and $\theta_{\xi_0}(t) \neq \theta^*$ for all $t \in (0, T)$.
(iii) ξ_0 is said to be of *type 3* if $\xi'_{\xi_0}(t) \neq 0$ and $\theta_{\xi_0}(t) \neq \theta^*$ for all $t > 0$.

Note that type 1 and type 2 are not exclusive, whereas a point is type 3 precisely if it is not type 1 or type 2. In fact, a point that is both of type 1 and type 2 corresponds to a curve segment that orthogonally meets the $\theta = \theta^*$ line at its start and its end. If $m_1 = m_2$, this leads to a solution for which the ξ and θ components are periodic, which corresponds to a closed embedded self-shrinker in $\mathbb{R}^{n+1}$ of topological type $S^1 \times M$, here M is diffeomorphic to the leaves of the isoparametric foliation. The following argument then finds a value ξ_0^* that is both type 1 and type 2.

Remark 3.2 For $m_1 = m_2$, one can see that $\xi_0 = \frac{g}{4} \ln m$ is the only type 3 point, as in this case, type 3 points correspond to embedded mean curvature convex self-shrinkers that are topologically a sphere. By [13], the only closed embedded mean curvature convex self-shrinkers are round spheres, which in this setting are given by the line $\xi = \frac{g}{4} \ln m$.

Define:

$$\xi_0^* = \inf\{r \in \mathbb{R} \mid \xi_0 \text{ is type 1 for all } \xi_0 > r\}.$$

Proposition 3.3 *We have:*

(i) $\xi_0^* < \infty$.
(ii) $\xi_0^* > \frac{g}{4} \ln m$.
(iii) ξ_0^* *is not type 3.*

This proposition will be proven in Sect. 4. For now, we make use of the following elementary lemma:

Lemma 3.4 *For* $(\xi, \theta)(t) \in \mathcal{D}$*, one has the following characterisation of extrema:*

(i) *If* $\xi'(t) = 0$*, then* $\operatorname{sign}(\xi''(t)) = \operatorname{sign}(\frac{g}{4} \ln m - \xi(t))$.
(ii) *If* $\theta'(t) = 0$*, then* $\operatorname{sign}(\theta''(t)) = \operatorname{sign}(\theta^* - \theta)$.

Proof $\xi'(t) = 0$ if and only if $\cos(\alpha(t)) = 0$, so

$$\xi''(t) = -\sin\alpha\sin(2\theta)\alpha'(t) = -\sin^2\alpha\sin^2(2\theta)(e^{\frac{4}{g}\xi} - m).$$

In the same way $\theta'(t) = 0$ if and only if $\sin(\alpha(t)) = 0$, so

$$\theta''(t) = \cos\alpha\sin(2\theta)\alpha'(t) = 2\cos^2\alpha\sin(2\theta)l(\theta).$$

Now $l(\theta) > 0$ for $\theta < \theta^*$ and $l(\theta) > 0$ for $\theta > \theta^*$. □

This now gives:

Proposition 3.5 *ξ_0^* is type 1 and type 2.*

Proof Since ξ_0^* is not type 3 by Proposition 3.3 (iii), it must be at least one of type 1 or type 2. We first assume that ξ_0^* is type 1 but not type 2, then we have a $T > 0$ so that for all $\epsilon > 0$ small enough, one gets:

$$\cos(\alpha_{\xi_0^*}(t)) \neq 0 \quad \forall t \in [\epsilon, T+\epsilon], \qquad \theta_{\xi_0^*}(T+\epsilon) < \theta^*.$$

Since solutions of the ODE $(*)$ vary continuously in the initial conditions (with respect to the topology of uniform convergence on compacta), one finds a neighbourhood $U(\epsilon)$ of ξ_0^* so that for all $\xi_0 \in U(\epsilon)$, one has

$$\cos(\alpha_{\xi_0}(t)) \neq 0 \quad \forall t \in [\epsilon, T+\epsilon], \qquad \theta_{\xi_0}(T+\epsilon) < \theta^*.$$

Additionally, one has $|\frac{d}{dt}\xi_{\xi_0}(t)| \leq 1$, whence if one chooses ϵ small enough, there is a neighbourhood V of $\xi_0^* > \frac{g}{4}\ln m$ with $\xi_{\xi_0}(t) > \frac{g}{4}\ln m$ for all $t \in [0, \epsilon]$ and $\xi_0 \in V$. By Lemma 3.4, $\xi(t)$ can only have maxima when $\xi(t) > \frac{g}{4}\ln m$, which implies that $\cos(\alpha_{\xi_0}(t)) \neq 0$ for all $t \in (0, \epsilon]$ and $\xi_0 \in V$.

The above shows that for $\xi_0 \in V \cap U(\epsilon)$, one has that ξ_0 is type 1, contradicting the definition of ξ_0^*.

The assumption that ξ_0^* is type 2 but not type 1 leads to a contradiction via similar argument, we carry this out:

Note first that $\xi''_{\xi_0^*}(0) < 0$, so the next extremum must be a minimum (Lemma 3.4 implies that whenever $\xi'(t) = 0$, one has either a maximum, a minimum, or ξ is the trivial solution $\xi = \frac{g}{4}m$). This gives a $T > 0$ such that for all $\epsilon > 0$ small enough:

$$\theta_{\xi_0^*}(t) \neq \theta^* \quad \forall t \in [\epsilon, T+\epsilon], \qquad \cos(\alpha_{\xi_0^*}(T+\epsilon)) > 0$$

As before one gets a neighbourhood $U(\epsilon)$ of ξ_0^* so that this extends to all initial conditions $\xi_0 \in U(\epsilon)$, i.e. for all $\xi_0 \in U(\epsilon)$:

$$\theta_{\xi_0}(t) \neq \theta^* \quad \forall t \in [\epsilon, T+\epsilon], \qquad \cos(\alpha_{\xi_0}(T+\epsilon)) > 0$$

Since $\theta'_{\xi_0}(t) = 1$ one again gets for ϵ small enough a neighbourhood V of ξ_0^* with $\theta_{\xi_0}(t) \neq \theta^*$ for all $t \in (0, \epsilon]$ and $\xi_0 \in V$. This shows that all points $\xi_0 \in V \cap U(\epsilon)$ are of type 2 but not type 1, again contradicting the definition of ξ_0^*. □

As discussed, this yields a periodic geodesic via Corollary 2.10 in the case $m_1 = m_2$.

Lemma 3.6 *When $m_1 = m_2$, one has that the periodic geodesic $t \mapsto (\xi_{\xi_0^*}, \theta_{\xi_0^*})(t)$ is simple.*

Proof Let $2T > 0$ denote the period of the geodesic, then $\theta(0) = \theta(T) = \theta^*$. For $t \in (0, T)$ this gives $\xi'_{\xi_0^*}(t) < 0$ and $\theta_{\xi_0^*}(t) > \theta^*$, similarly if $t \in (T, 2T)$ then $\xi'_{\xi_0^*}(t) > 0$ and $\theta_{\xi_0^*}(t) < \theta^*$. This immediately implies that the geodesic is simple. □

Together with Proposition 2.5, this proves the main theorem of the paper:

Theorem A *For any isoparametric hypersurface M in S^n, $n \geq 2$, for which the multiplicities m_1 and m_2 of the principal curvatures agree, there is a closed embedded self-shrinker of topological type $S^1 \times M$ in $\mathbb{R}^{n+1}$. This hypersurface is a union of homothetic copies of the leaves of the isoparametric foliation of S^n associated to M.*

In the case $m_1 \neq m_2$, Proposition 3.5 remains true and yields a simple geodesic segment that starts and ends on the $\theta = \theta^*$, meeting this line orthogonally in both places. In between the two ends, one has $\theta > \theta^*$ and the same arguments give another geodesic arc with the same properties, except now $\theta < \theta^*$.

It may be useful to connect the end-points of these two arcs by line segments $\theta = \theta^*$ (which are geodesics). Doing so gives a simple closed curve consisting piecewise geodesic segments and having external angle sum equal to 0. By the Theorem of Gauß-Bonnet, this curve then encloses a total Gauß curvature of 2π. Such a curve is an essential ingredient in [10], where an adapted curve shortening flow is used to generate closed geodesics.

3.1 The Case Studied by Angenent

The case $g = 1$ yields an embedded self-shrinker in $\mathbb{R}^{n+1}$ of topological type $S^1 \times S^{n-1}$, which is invariant under an isometric $O(n)$ action on $\mathbb{R}^{n+1}$. This is the case investigated by Angenent in [2]. In this subsection, we relate Angenent's construction to ours and show that they give the same self-shrinker.

Let $\omega := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1, x_0 = 0\}$, $e_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. Then the construction of [2] yields a self-shrinker of the form

$$\{x(t)\, e_0 + \tilde{r}(t)\, \omega \mid t \in (a, b)\}, \tag{5}$$

where $x : (a, b) \to \mathbb{R}$, $\tilde{r} : (a, b) \to \mathbb{R}_{>0}$ are smooth functions and $a, b \in \mathbb{R}$. Note that sets the form (5) are precisely the $\mathbf{f}$-invariant sets from Definition 2.2, where $\mathbf{f}$ arises from the isoparametric foliation of S^n with $V_1 = e_0$. To be more precise, if $r : (a, b) \to \mathbb{R}_{>0}$, $\varphi : (a, b) \to (0, \pi)$, one has:

$$\mathbf{f}^{-1}(\{(r(t), \varphi(t)) \mid t \in (a, b)\}) = \{r(t) \cos(\varphi(t))\, e_0 + r(t) \sin(\varphi(t))\, \omega \mid t \in (a, b)\}. \tag{6}$$

In [2], the variational condition that a set of the form (5) is a self-shrinker is reduced to $\{(x(t), r(t)) \mid t \in (a, b)\}$ being a geodesic segment in

$$(\mathbb{R} \times \mathbb{R}_{>0}, \tilde{r}^{2n-2} e^{-\tau(x^2+\tilde{r}^2)}(dx^2 + d\tilde{r}^2)). \tag{7}$$

Here $\tau > 0$ is related to the extinction time and in [2], one has $\tau = \frac{1}{4}$. For ease of comparison, we take $\tau = 1$ and then up to a constant conformal factor, the transformation implicit in (6) gives an isometry to the metric (4) on $\mathbb{R}_{>0} \times (0, \pi)$, as is easy to see (recall $g = 1$). It follows that any geodesic of (7) is a reparametrisation of a geodesic in (4). Carrying out the additional coordinate changes of SubSect. 2.3, one sees that the following map sends the orbits of solutions of $(*)$ to the orbits of geodesics of (7):

$$\Psi : \mathbb{R} \times (0, \frac{\pi}{2}) \to \mathbb{R} \times \mathbb{R}_{>0}, \qquad (\xi, \theta) \mapsto (e^{\frac{2}{g}\xi} \cos(2\theta), e^{\frac{2}{g}\xi} \sin(2\theta)).$$

In particular, $x(t) = 0$ if and only if $\theta = \frac{\pi}{4} = \theta^*$.

Denote with $(\tilde{r}_R, x_R)(t)$ the evolution of a geodesic in (7) with initial conditions $(\tilde{r}(0), x(0)) = (R, 0)$ and $(\tilde{r}'(0), x'(0)) = (0, 1)$. Define:

$$R_* := \inf\{\tilde{R} > 0 \mid \forall R > \tilde{R} : \exists t_1 > 0 \text{ so that } x_R(t_1) = 0 \text{ and } \tilde{r}'_R(t) < 0 \; \forall t \in (0, t_1)\}.$$

Angenent then shows that the geodesic $(\tilde{r}_{R_*}, x_{R_*})$ meets the line $x = 0$ orthogonally after a finite time. A symmetry argument as in Corollary 2.10 then shows that this yields a simple periodic geodesic.

In order to show Proposition 1, we start with the following lemma. It follows from elementary arguments using continuity of solutions of the relevant ODEs in initial conditions, similar to Proposition 3.5.

Lemma 3.7 *For any neighbourhood U of R_*, there are $R \in U$ and $t_1(R) > 0$ so that*

$$x_R(t_1(R)) = 0, \quad \tilde{r}'_R(t_1(R)) > 0, \quad \textit{and } x(t) > 0 \textit{ for all } t \in (0, t_1(R)).$$

Similarly for any neighbourhood V of ξ_0^*, there are $\xi_0 \in V$ and $T(\xi_0) > 0$ so that

$$\theta_{\xi_0}(T(\xi_0)) = \theta^*, \quad \xi'_{\xi_0}(T(\xi_0)) > 0, \quad \textit{and } \theta_{\xi_0}(t) > \theta^* \textit{ for all } t \in (0, T(\xi_0)).$$

Proposition B *In the case $g = 1$, the construction of Theorem A gives Angenent's shrinking doughnut* [2].

Proof Let $L : \mathbb{R} \to \mathbb{R}$ be the reparametrisation of a geodesic so that $\Psi((\xi, \theta)(L(t))) = (x, \tilde{r})(t)$. We take $L'(t) > 0$ for all t. Then if $x(t) = 0$, one has by an elementary calculation:

$$\operatorname{sign}(\xi'(L(t))) = \operatorname{sign}(\tilde{r}'(t)) \tag{8}$$

Assume first $R_* > e^{\frac{2}{g}\xi_0^*}$. Then apply the first part of Lemma 3.7 to R_* to get initial conditions R arbitrarily close to R_* and times $t_1(R)$ for which

$$x_R(t_1(R)) = 0, \quad \tilde{r}'_R(t_1) > 0, \quad \text{and} \quad x_R(t) > 0 \text{ for all } t \in (0, t_1(R)).$$

This then transforms under Ψ^{-1} to values ξ_0 close to $\frac{g}{2}\ln(R_*)$ (which is larger than ξ_0^*), by (8) one then gets $\xi'_{\xi_0}(L(t_1)) > 0$, while $\theta_{\xi_0}(L(t)) > \theta^*$ for all $t \in (0, t_1)$.

These points are not type 1, contradicting the definition of ξ_0^*. The contradiction for $R_* < e^{\frac{2}{g}\xi_0^*}$ is similar. □

4 Proof of Proposition 3.3

For the proof of Proposition 3.3, each of the points (i), (ii), and (iii) is considered separately in Subsects. 4.2, 4.3, and 4.4, respectively. For the proof of these statements, we also use two lemmas about the general dynamics of $(*)$, which are proven in Subsect. 4.1.

For a rough overview of the proof of points (i) and (ii), which are the more technical parts, see the beginnings of Subsects. 4.2 and 4.3 as well as Figs. 2 and 3.

4.1 Crossings in Finite Time

In this subsection, we prove two useful lemmas that expand on the analysis of extrema in Lemma 3.4. The lemmas state that if $\xi'(t)$ points towards the $\frac{g}{4}\ln m$ line, then we reach this line in finite time, the same holding true for θ if $\theta'(t)$ points towards θ^*. The proof of Proposition 3.3 (i) uses Lemma 4.2 below, and Proposition 3.3 (iii) and Lemma 4.2 use Lemma 4.1.

Lemma 4.1 *If for some $t_0 \in \mathbb{R}$ one has $\xi(t_0) > \frac{g}{4}\ln m$ ($\xi(t_0) < \frac{g}{4}\ln m$) and $\xi'(t_0) < 0$ ($\xi'(t_0) > 0$), then there exists a time $T \in (0, \infty)$ so that $\xi(t_0 + T) = \frac{g}{4}\ln m$.*

Proof If this were not true then $\xi(t) > \frac{g}{4}\ln m$ for all $t > t_0$. By Lemma 3.4, we would then have that any extremum of ξ is a maximum when $t > t_0$. Since $\xi'(t_0) < 0$ it follows that ξ has no extrema for times $> t_0$ and then $\xi(t)$ is monotonically decreasing in t and bounded below by $\frac{g}{4}\ln m$ by assumption, hence it must converge.

Since $\xi(t)$ is bounded we find that $\alpha'(t)$ is bounded, whence $\xi'(t) = \cos\alpha\ \sin(2\theta)$ must converge to 0 (since $\theta'(t)$ is also automatically bounded). So either $\lim_{t\to\infty}\alpha(t) \in \frac{\pi}{2} + \pi\mathbb{Z}$ or $\lim_{t\to\infty}\theta(t) \in \{0, \frac{\pi}{2}\}$. We briefly show that the first condition implies the second and then continue working only with the second.

The condition $\alpha(t) \to \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$ gives $\sin\alpha \to (-1)^k$. For large times, the dynamics of $\theta(t)$ are then given by

$$\theta'(t) = (-1)^k \sin(2\theta) + O(\cos\alpha).$$

This gives that $\theta(t)$ converges to either 0 or $\frac{\pi}{2}$ as $t \to \infty$, depending on whether k is even or odd.

Assuming now $\theta(t) \to 0$ as $t \to \infty$ one gets from Lemma 3.4 that $\theta(t)$ admits no extrema for t large enough, so $\sin\alpha < 0$ for t large enough. This gives $\alpha(t) \in 2\pi\mathbb{Z} + (\pi, \frac{3\pi}{2})$ for t large enough. However, for such t

$$\alpha'(t) = \sin\alpha \sin(2\theta)(e^{\frac{4}{g}\xi} - m) + 2\cos\alpha\, l(\theta)$$

is a sum of two strictly negative terms. α then decreases for large times, so there is an $\epsilon > 0$ with $|\cos\alpha| > \epsilon$ for large enough t. In particular $\alpha'(t) < -2\epsilon l(\theta)$ for large t, where $l(\theta)$ converges to m_1. This means that in finite time, α exits the interval $2\pi k + (\pi, \frac{3\pi}{2})$ from the bottom, contradicting that $\sin\alpha < 0$ for all t large enough.

The case $\theta(t) \to \frac{\pi}{2}$ can be treated in the same way. This contradiction then implies the statement for $\xi(t_0) > \frac{g}{4}\ln m$ and $\xi'(t_0) < 0$. The case $\xi(t_0) < \frac{g}{4}\ln m$ and $\xi'(t_0) > 0$ is also completely analogous. □

Lemma 4.2 *If for some t_0 one has $\theta(t_0) < \theta^*$ ($\theta(t_0) > \theta^*$) and $\theta'(t_0) > 0$ ($\theta'(t_0) < 0$) then there exists a time $T \in (0, \infty)$ so that $\theta(t_0 + T) = \theta^*$.*

Proof As before assuming that $\theta(t) < \theta^*$ for all $t > t_0$ and $\theta'(t_0) > 0$ implies that $\theta(t)$ has only minima as extrema, hence $\theta'(t) > 0$ for all t and $\theta(t)$ converges (being bounded from above by θ^*).

We first assume that $e^{\frac{4}{g}\xi(t)}$ remains bounded as $t \to \infty$, this implies that $\alpha'(t)$ remains bounded and then from convergence of θ one gets that $\theta'(t) = \sin\alpha\ \sin(2\theta)$ converges to 0. Since $\sin(2\theta)$ remains bounded away from 0 one gets that α converges to some element of $\pi\mathbb{Z}$. Since $\xi(t)$ is not allowed to go to $+\infty$, one gets that $\lim_{t\to\infty}\alpha(t) \in \pi + 2\pi\mathbb{Z}$.

Performing a coordinate transform $R = \ln\xi$, the system of ODEs $(*)$ becomes:

$$\begin{aligned}
\theta'(t) &= \sin\alpha\ \sin(2\theta),\\
R'(t) &= \cos\alpha\ \sin(2\theta)R,\\
\alpha'(t) &= \sin\alpha\ \sin(2\theta)(R^{\frac{4}{g}} - m) + 2\cos\alpha\, l(\theta).
\end{aligned}$$

From $\alpha(t) \to \pi + 2\pi k$ for some $k \in \mathbb{Z}$, one gets $R(t) \to 0$ and $\theta(t) \to \theta^*$. Note, however, that the fixpoint $(\theta, R, \alpha) = (\theta^*, 0, \pi + 2\pi k)$ is hyperbolic and at this point the above ODE has as linearisation:

$$\frac{d}{dt}\begin{pmatrix}\theta\\ R\\ \alpha\end{pmatrix} \approx \begin{pmatrix} 0 & 0 & -\sin(2\theta^*)\\ 0 & -\sin(2\theta^*) & 0\\ -2l'(\theta^*) & 0 & \sin(2\theta^*)m\end{pmatrix}\begin{pmatrix}\theta - \theta^*\\ R\\ \alpha - (\pi + 2\pi k)\end{pmatrix}.$$

The system then has a one-dimensional stable manifold - this is the line

$$(\theta(t), R(t), \alpha(t)) = (\theta^*, R(0)\exp(-\sin(2\theta^*)t), \pi + 2\pi k).$$

Since we are assuming $\theta(t) < \theta^*$ for all $t > t_0$ the solution cannot lie on the stable manifold, yielding a contradiction.

To complete the proof of the lemma, we must show that $e^{\frac{4}{g}\xi(t)}$ cannot be unbounded under the hypothesis $\theta(t) < \theta^*$ for all $t > t_0$ and $\theta'(t_0) > 0$. First recall the graph form $(**)$:

$$\frac{d^2\xi}{d\theta^2}(\theta) = -\left(1 + (\frac{d\xi}{d\theta})^2\right)\left(e^{\frac{4}{g}\xi} - m + 2H(\theta)\frac{d\xi}{d\theta}\right).$$

Whence if $\theta < \theta^*$, $\xi > \frac{g}{4}\ln m$ and $\frac{d\xi}{d\theta} > 0$, one has $\frac{d^2\xi}{d\theta^2} < 0$, even becoming arbitrarily negative if ξ becomes arbitrarily large. So if $\xi(t)$ is unbounded from above, it cannot eventually be monotonic in θ (and hence in t by $\theta'(t) > 0$) and must admit maxima, in fact infinitely many such maxima. Between two maxima, there must be a minimum, which can only happen for values of $\xi(t)$ less than $\frac{g}{4}\ln m$.

However, at each minimum, one has $\theta'(t) = \sin(2\theta)$, which may be bounded from below since θ stays away from $\{0, \frac{\pi}{2}\}$. Since the system $(*)$ admits a Lipschitz constant on $\{(\theta, \xi, \alpha) \mid \xi < \frac{g}{4}\ln(m) + 1\}$, one finds that at each minimum of ξ, the parameter θ increases by some positive number admitting a bound from below. This contradicts the assumption that $\theta(t) < \theta^*$ for all $t > t_0$.

The case $\theta(t_0) > \theta^*$, $\theta'(t_0) < 0$ can be treated analogously. □

4.2 Proof of Proposition 3.3 (i)

The proof of Proposition 3.3 (i) is divided into two parts. First we show that for ξ_0 large enough, there is a $T_2 > 0$ so that the solution to $(*)$ with initial value (ξ, θ, α) $(t = 0) = (\xi_0, \theta^*, \frac{\pi}{2})$ has the property:

$$\theta'(T_2) = 0, \quad 0 < \theta(T_2) - \theta^* < \frac{1}{\xi_0}, \quad \xi_0 - \xi(T_2) < \frac{1}{\xi_0}, \quad \text{while } \xi'(t) < 0 \text{ for all } t \in (0, T_2).$$

So θ has an extremum at T_2, which by Lemma 3.4 is a maximum and $\theta'(T_2+\epsilon) < 0$ for small $\epsilon > 0$. Then by Lemma 4.2, one has that θ reaches θ^* in finite time and so ξ_0 cannot be of type 3. The proof then continues by contradiction, assuming that ξ_0 is not of type 1 means it must be of type 2. Being of type 2 means that ξ must travel all the way to some value $< \frac{g}{4}\ln(m)$ where we have an extremum of ξ - all the while θ is not allowed to cross the line θ^*.

The proof by contradiction is carried out in Lemma 4.6, here one assumes that conditions of this scenario have been set: there is some time T_3 at which $\xi(T_3) = \frac{4}{g}\ln(m)$ all the while $\theta(t) > \theta^*$ and $\xi'(t) < 0$ for $t \in (0, T_3]$. Using bounds for the value of $\theta(T_3)$ one is, however, able to show that even in this worst-case-scenario θ crosses the value θ^* before any extremum of ξ is possible, contradicting the assumption that ξ_0 is type 2. Hence, since it cannot be type 3, it must have been type 1.

In what follows ξ, θ and α will denote the components of the solution of $(*)$ with initial condition (ξ, θ, α) $(0) = (\xi_0, \theta^*, \frac{\pi}{2})$. The proof begins by establishing an auxilliary time T_1, at which $\frac{\xi'(T_1)}{\theta'(T_1)} = -1$.

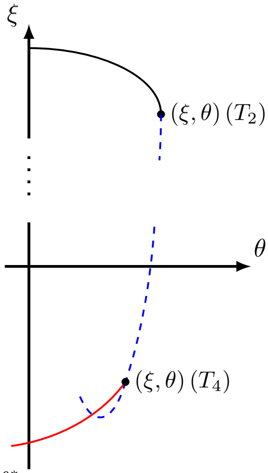

Fig. 2 A sketch of the argument for Proposition 3.3(i). The black curve gives the evolution of (ξ, θ) up until the extremum of θ. The dashed blue line describes the worst-case scenario for the evolution of $(\xi, \theta)\,(t)$ after this extremum. The red line, which crosses the line $\theta = \theta^*$ without any extrema of ξ, is an estimate of actual evolution starting on a certain point of the worst-case scenario

Lemma 4.3 *If ξ_0 is large enough then there is a time $T_1 > 0$ so that $\frac{\xi'(T_1)}{\theta'(T_1)} = -1$ while $\theta'(t) > 0, \xi'(t) < 0$ for all $t \in (0, T_1]$.*

Proof Note that one initially has $\frac{d}{dt} \cos \alpha|_{t=0} < 0$ whence one gets $\xi'(t) < 0$ for small t. By Lemma 4.1 $\xi(t)$ then descends to $\frac{g}{4} \ln m$ and does not have any extrema until after this value is reached, meaning $\cos \alpha(t) < 0$ for all $t \in (0, t_m]$ and some $t_m \in \mathbb{R}$ at which $\xi(t_m) = \frac{g}{4} \ln m$. For ξ_0 large enough there will then be some intermediate time $T_1 < t_m$ for which $\frac{d}{d\theta}\xi(\theta) = \frac{\xi'(t)}{\theta'(t)} = -1$ holds, since either $\theta'(t) = 0$ for some $t \in [0, t_m]$ or the graph $\xi(\theta)$ must descend from ξ_0 at θ^* to $\frac{g}{4} \ln m$ at some value $\theta < \frac{\pi}{2}$. In the second case, the mean value theorem implies that the graph achieves slope -1 at some point. □

Lemma 4.4 *There are constants $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ so that if ξ_0 is large enough, one has*

$$c_1 e^{-\frac{4}{g}\xi_0} \leq \theta(T_1) - \theta^* \leq c_2 e^{-\frac{4}{g}\xi_0}, \qquad \xi(T_1) \geq \xi_0 + c_3 e^{-\frac{4}{g}\xi_0}$$

Proof Before beginning with the proper analysis, one notes that by well definedness, one has $\theta(T_1) \leq \frac{\pi}{2}$, whence by the mean value theorem $\xi_0 - \xi(T_1) \leq \frac{\pi}{2} - \theta^*$, some finite value bounded from above.

For the proof of this lemma, it is more convenient to work with the following system of ODEs:

$$\begin{aligned} \xi'(t) &= \cos \alpha, \\ \theta'(t) &= \sin \alpha, \\ \alpha'(t) &= \sin \alpha \,(e^{\frac{4}{g}\xi} - m) + 2 \cos \alpha \, H(\theta). \end{aligned}$$

which one can get from $(*)$ by rescaling time by the law $\frac{dt_{\text{new}}}{dt_{\text{old}}} = \sin(2\theta)$. To keep the number of superfluous parameters at a minimum, we still use T_1 to denote the time at which $\frac{\xi'(T_1)}{\theta'(T_1)} = -1$ in this new ODE. One then gets for all $t \in [0, T_1]$:

$$\cos\alpha(t) \in [-\frac{1}{\sqrt{2}}, 0], \qquad \sin\alpha(t) \in [\frac{1}{\sqrt{2}}, 1].$$

Which implies:

$$\xi_0 - \xi(T_1) \in [0, \frac{1}{\sqrt{2}} T_1], \qquad \theta(T_1) - \theta^* \in [\frac{1}{\sqrt{2}} T_1, T_1].$$

We then proceed by bounding T_1 from above and below. Noting that for $t \in [0, T_1]$:

$$\frac{1}{\sqrt{2}}(e^{\frac{4}{g}\xi(T_1)} - m) \leq \alpha'(t) \leq e^{\frac{4}{g}\xi_0} - \sqrt{2} H(\theta(T_1)).$$

Integrating the left inequality from 0 to T_1 yields:

$$\frac{T_1}{\sqrt{2}}(e^{\frac{4}{g}\xi(T_1)} - m) \leq \alpha(T_1) - \alpha(0) = \frac{\pi}{4}.$$

Recalling that $\xi_0 - \xi(T_1) \leq \frac{\pi}{2} - \theta^*$, one gets $T_1 \leq d_1 e^{-\frac{4}{g}\xi_0}$ for an appropriate constant d_1. This implies $\theta(T_1) - \theta^* \leq c_2 e^{-\frac{4}{g}\xi_0}$ and $\xi_0 - \xi(T_1) \leq c_3 e^{-\frac{4}{g}\xi_0}$ for appropriate c_2, c_3.

Combining $\theta(T_1) - \theta^* \leq c_2 e^{-\frac{4}{g}\xi_0}$ with $H(\theta^*) = 0$ gives for ξ_0 large enough that $-\sqrt{2}\, H(\theta(T_1)) \leq d_2 e^{-\frac{4}{g}\xi_0}$ for some other constant d_2. Integrating the other inequality for $\alpha'(t)$ from 0 to T_1 then gives:

$$\frac{\pi}{4} \leq T_1(e^{\frac{4}{g}\xi_0} + d_2 e^{-\frac{4}{g}\xi_0}) \quad \Longrightarrow \quad T_1 \geq d_3 e^{-\frac{4}{g}\xi_0}$$

for another constant d_3, provided ξ_0 is large enough. This yields the final bound of the lemma, namely: $c_1 e^{-\frac{4}{g}\xi_0} \leq \theta(T_1) - \theta^*$. □

Lemma 4.5 *For ξ_0 large enough, there is a time $T_2 > T_1$ so that $\theta'(T_2) = 0$, $\xi(T_2) > \xi_0 - \frac{1}{\xi_0}$ while $\theta(t) \in \theta^* + (0, \frac{1}{\xi_0})$ and $\xi'(t) < 0$ for all $t \in (0, T_2]$.*

Proof For all $t \in (0, \frac{1}{\xi_0})$, one has from $|\xi'(t)| \leq 1$ that $\xi(t) > \xi_0 - \frac{1}{\xi_0}$, so for ξ_0 large enough $\xi'(t) < 0$ for such t. Further as long as $\theta > \theta^*$ and $\alpha \in [\frac{3\pi}{4}, \pi)$, one has for such t that

$$\alpha'(t) = \sin\alpha\, \sin(2\theta)(e^{\frac{4}{g}\xi} - m) + 2\cos\alpha\, l(\theta)$$

is a sum of two positive terms and so α is increasing. Recalling that by definition $\alpha(T_1) = \frac{3\pi}{4}$ and assuming that $\alpha(t) < \pi$ (i.e. $\theta'(t) > 0$) for all $t \in (T_1, \frac{1}{\xi_0})$ yields:

$$\alpha'(t) > Ae^{\frac{4}{g}(\xi_0 - \frac{1}{\xi_0})}(\pi - \alpha).$$

Here $A > 0$ is some constant. For this estimate, we used that $\theta(t)$ is bounded away from $\{0, \frac{\pi}{2}\}$ for $t \in (0, \frac{1}{\xi_0})$, which follows from $|\theta'(t)| \leq 1$. From the intermediate value theorem, one then gets a $\widetilde{t} \in (T_1, \frac{1}{2\xi_0})$ so that

$$\alpha(\widetilde{t}) = \pi - A \exp\left(-e^{\frac{4}{g}(\xi_0 - \frac{1}{\xi_0})}\frac{1}{2\xi_0}\right).$$

For ξ_0 large enough, one then finds $\alpha(\widetilde{t}) > \pi - e^{-\xi_0^2}$. Now $\theta(T_1) > \theta^* + c_1 e^{-\frac{4}{g}\xi_0}$ from Lemma 4.4, so $l(\theta(T_1)) \leq c_1 l'(\theta^*)e^{-\frac{4}{g}\xi_0}$ for ξ_0 large enough. By assumption $\theta(\widetilde{t}) > \theta(T_1)$, so another estimate yields for $t \in (\widetilde{t}, \frac{1}{\xi_0})$:

$$\alpha'(t) > B\, e^{-\frac{4}{g}\xi_0}$$

where $B > 0$ is some constant incorporating the $2\cos(\alpha)$ term (which is close to -2) and $c_1 l'(\theta^*)$ (which is bounded away from 0). This then yields

$$\alpha(\widetilde{t} + \frac{1}{2\xi_0}) > \pi - e^{-\xi_0^2} + B\frac{e^{-\frac{4}{g}\xi_0}}{2\xi_0}$$

which is larger than π, contradicting our assumption that $\alpha(t) < \pi$ for all $t \in (T_1, \frac{1}{\xi_0})$. The lemma then follows. □

We will now prove that ξ_0 is of type 1 by contradiction.

Lemma 4.6 *If ξ_0 is large enough, then it is of type 1.*

Proof If ξ_0 is large enough by Lemma 4.5, there is a $T_2 > 0$ for which $\theta'(T_2) = 0$, which corresponds to a local maximum of θ. This means θ reaches θ^* in finite time by Lemma 4.2 and so ξ_0 is not type 3. We now assume it is not type 1, so it must be type 2.

Hence $\xi(t)$ has an extremum before $\theta(t)$ reaches θ^*. We let $T_3 > T_2$ denote the time of this extremum and note first that $\theta(t) \in (\theta^*, \theta^* + \frac{1}{\xi_0})$ for all $t \in [T_2, T_3)$ (since $\theta(T_2) \leq \theta^* + \frac{1}{\xi_0}$ is a maximum and any further extremum of θ must take place behind the line $\theta = \theta^*$).

Next one sees that $\xi(T_3) < \frac{g}{4}\ln m$, since the extremum must be a minimum. With $\xi(T_2) \geq \xi_0 + \frac{1}{\xi_0}$ and $|\xi'(t)| \leq 1$, one gets that $T_3 - T_2$ will become arbitrarily large as ξ_0 grows. And so, for ξ_0 large enough, one sees that $\xi(t) \leq \frac{g}{4}\ln m + 1$, $|\theta(t) - \theta^*| \leq 1$ for all $t \in [T_3 - 1, T_3]$. Note that $\alpha'(t)$ admits a bound if ξ, θ are in this region.

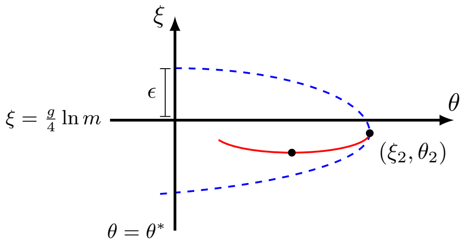

Fig. 3 The figure sketches the argument for Proposition 3.3(ii). The dashed blue line denotes the form that $(\xi, \theta)(t)$ must be if the initial condition were type 1. The red line, which has an extremum of ξ, is an estimate of the actual evolution starting at (ξ_2, θ_2)

Now $\xi'(T_3) = 0$ implies $\cos\alpha(T_3) = 0$ and so $\sin\alpha(T_3) = -1$, which in turn gives $\theta'(T_3) = -\sin(2\theta) \leq -1 + O(\frac{1}{\xi_0^2})$. From $(*)$, one sees directly that $|\theta''(t)| \leq |\alpha'(t)| + 1$, and so the bound on $\alpha'(t)$ give a finite $b > 0$ (independent of ξ_0) so that $\theta'(t) < -\frac{1}{2}$ for all $t \in [T_3 - b, T_3]$.

But if ξ_0 is large enough, one notes that $\theta(t) \in [\theta^*, \theta^* + \frac{1}{\xi_0})$ and $\theta'(t) \leq -\frac{1}{2}$ cannot both simultaneously hold for all $t \in [T_3 - b, T_3]$. This contradiction shows that ξ_0 cannot be type 2, hence (since it is also not type 3) it is type 2. □

4.3 Proof of Proposition 3.3 (ii)

We consider the solution curve with initial condition $\xi_0 = \frac{g}{4}\ln m + \epsilon$ and show that this is not of type 1 for ϵ sufficiently small. To do this, we assume that it is of type 1 - only to later arrive at a contradiction. If it were of type 1, then there is a $T > 0$ with $\theta(T) = \theta^*$ and $\xi'(t) < 0$ for all $t \in (0, T)$. Since θ can only have maxima when $\theta > \theta^*$, we find that the trajectory $\{(\theta, \xi)(t) \mid t \in [0, T]\}$ must be the union of two graphs of ξ over θ. The maximum of θ occurs at the point denoted by (ξ_2, θ_2) in Fig. 3.

In the upper graph, one has that the slope $\frac{d\xi}{d\theta}$ starts at 0 and must go to $-\infty$ (which occurs when $\theta'(t) = 0$). Along the way ξ' has been negative and one verifies that ξ_2 has decreased to a value far enough below $\frac{g}{4}\ln m$ (c.f. Lemma 4.10). When we then switch to the lower graph the $e^{\frac{4}{g}\xi} - m$ term in the ODE for $\frac{d^2\xi}{d\theta^2}(\theta)$ will be large enough to push $\frac{d\xi}{d\theta}$ over the value 0 before θ reaches θ^*, contradicting the assumption that ξ_0 was type 1.

Note that this does not prove that $\frac{g}{4}\ln m + \epsilon$ is of type 2, because the proof by contradiction assumes that $\theta(t)$ has a maximum.

In what follows ξ, θ and α will denote the components of the solution of $(*)$ with initial condition $(\xi, \theta, \alpha)(0) = (\frac{4}{g}\ln m + \epsilon, \theta^*, \frac{\pi}{2})$. The proof begins by investigating an auxilliary value θ_1, which is defined to be the least (and for small ϵ only) value of θ for which one has $\frac{d\xi}{d\theta}(\theta_1) = -1$ in the upper graph.

Lemma 4.7 *If $\xi_0 = \frac{g}{4}\ln m + \epsilon$ is type 1, then there are $T_2(\epsilon) > T_1(\epsilon) > 0$ so that $\theta'(T_2) = 0$ and $\frac{\xi'(T_1)}{\theta'(T_1)} = -1$, while $\xi'(t) < 0$ and $\theta'(t) > 0$ for all $t \in (0, T_2)$.*

Proof Assuming that ξ_0 is of type 1 means that there is a time $T > 0$ for which $\theta(T) = \theta^*$ and $\xi'(t) \neq 0, \theta(t) \neq \theta^*$ for all $t \in (0, T)$. Since $\theta'(0) = \sin(2\theta^*) > 0$, one finds that $\theta'(t) > 0$ for small t, whence $\theta(t)$ must go through an extremum before it can go back to θ^* and there is a $T_2 < T$ so that $\theta'(T_2) = 0$. On the other hand, one has $\xi''(0) = -\sin(2\theta^*)m(e^{\frac{4}{g}\epsilon} - 1) < 0$, whence $\xi'(t) < 0$ for all $t \in (0, T]$, in particular for all $t \in (0, T_2)$.

This means that $\frac{\xi'(t)}{\theta'(t)}$ is 0 at $t = 0$ and diverges to $-\infty$ at $t = T_2$. There must then be a T_1 so that $\frac{\xi'(T_1)}{\theta'(T_1)} = -1$. □

Lemma 4.8 *For ϵ small enough, there is only one pair (T_1, T_2) satisfying the conditions of Lemma 4.7 and $\theta(T_1) \to \frac{\pi}{2}$ as $\epsilon \to 0$.*

Proof The time T_2 is obviously unique.

On the other hand, the initial condition $\epsilon = 0$ has as solution the line $(\theta, \xi, \alpha) = (\arctan(\tan(\theta^*)e^{2t}), \frac{g}{4}\ln m, \frac{\pi}{2})$. So one finds that as $\epsilon \to 0$ the solution and its derivatives converge uniformly on compacta to the above curve, in particular for on any finite interval $[0, T]$, one can make $\frac{\xi'(t)}{\theta'(t)}$ arbitrarily small for all $t \in [0, T]$ (by taking ϵ small), while $\theta(T)$ is arbitrarily close to $\frac{\pi}{2}$ (by taking T large and then ϵ small). This means that as $\epsilon \to 0$ one must have $\frac{\pi}{2} - \theta(T_1(\epsilon)) \to 0$, where T_1 is any of the times satisfying Lemma 4.7.

Looking, however, at the graph ODE (∗∗)

$$\frac{d^2\xi}{d\theta^2} = -(1 + (\frac{d\xi}{d\theta})^2)(e^{\frac{4}{g}\xi} - m + 2H(\theta)\frac{d\xi}{d\theta})$$

it follows if $\frac{d\xi}{d\theta} \leq -1$ and θ is close enough to $\frac{\pi}{2}$ while ξ is not too large that then $\frac{d^2\xi}{d\theta^2} < 0$, since $H(\theta) \to -\infty$ as $\theta \to \frac{\pi}{2}$. For ϵ small enough one gets that for any pair (T_1, T_2) satisfying Lemma 4.7, there is no pair (T_1', T_2) satisfying the lemma with $T_1' > T_1$. □

This allows us to introduce the following notation:

$$\theta_1 := \theta(T_1), \qquad \theta_2 := \theta(T_2), \qquad \xi_2 := \xi(T_2).$$

Lemma 4.9 *For ϵ small enough, one has that $\xi(T_1) \leq \frac{g}{4}\ln m$.*

Proof We assume that $\xi(T_1) > \frac{g}{4}\ln m$ and get a contradiction. The first step is to show that this assumption leads to constants $c_1, c_2 > 0$ (independent of ϵ) so that for ϵ small enough one has:

$$c_1\epsilon < (\frac{\pi}{2} - \theta_1)^{m_2} < c_2\epsilon. \tag{9}$$

Since $\xi'(t) \leq 0$ for all $t \in (0, T_2)$ we may assume $\xi(t) > \frac{g}{4}\ln m$ for such t. Then from the graph ODE

$$\frac{d^2\xi}{d\theta^2} = -(1 + (\frac{d\xi}{d\theta})^2)(e^{\frac{4}{g}\xi} - m + 2H(\theta)\frac{d\xi}{d\theta})$$

one gets that $\frac{d^2\xi}{d\theta^2} < 0$ for all $\theta \in (\theta^*, \theta_1)$ (recall that $H(\theta) < 0$ for $\theta > \theta^*$) and then $\frac{d\xi}{d\theta}$ is strictly decreasing in this interval. Monotonicity of $\frac{d\xi}{d\theta}$ and mean value theorem then imply

$$\frac{d\xi}{d\theta}\left(\frac{\theta_1-\theta^*}{2}\right) > -\epsilon\,\frac{\theta_1-\theta^*}{2}, \qquad \xi\left(\frac{\theta_1-\theta^*}{2}\right) > \frac{g}{4}\ln m + \frac{\epsilon}{2}. \tag{10}$$

Combining the second inequality with the graph ODE yields (recall that $H(\theta) < 0$ if $\theta > \theta^*$) additionally the bound $\frac{d\xi}{d\theta}\left(\frac{\theta_1-\theta^*}{2}\right) < -\frac{m\epsilon}{g}(\theta_1-\theta^*) + O(\epsilon^2)$. Using that $\theta_1 \to \frac{\pi}{2}$ as $\epsilon \to 0$ then gives constants $\widetilde{c}_1, \widetilde{c}_2$ so that

$$\widetilde{c}_1\epsilon < \left|\frac{d\xi}{d\theta}\left(\frac{\theta_1-\theta^*}{2}\right)\right| < \widetilde{c}_2\epsilon. \tag{11}$$

Rewriting the graph ODE as

$$\frac{\frac{d^2\xi}{d\theta^2}}{\frac{d\xi}{d\theta}(1+(\frac{d\xi}{d\theta})^2)} = -\frac{e^{\frac{4}{g}\xi}-m}{\frac{d\xi}{d\theta}} - 2H(\theta), \tag{12}$$

one notes that the first term on the right-hand side is $O(1)$ in the interval $(\frac{\theta_1-\theta^*}{2}, \theta_1)$ by (11) and monotonicity of $\frac{d\xi}{d\theta}$, and further that the left-hand side has $-\ln(\frac{|\frac{d\xi}{d\theta}|}{\sqrt{1+(\frac{d\xi}{d\theta})^2}})$ as an anti-derivative. Integrating (12) over $(\frac{\theta_1-\theta^*}{2}, \theta_1)$ then gives

$$\ln(\left|\frac{d\xi}{d\theta}\left(\frac{\theta_1-\theta^*}{2}\right)\right|) + O(1) = m_2\ln(\cos(\theta_1)) + O(1).$$

(For the left-hand side: From $-\epsilon\frac{\theta_1-\theta^*}{2} < \frac{d\xi}{d\theta} \leq -1$ for $\theta \in (\frac{\theta_1-\theta^*}{2}, \theta_1)$ the $1+(\frac{d\xi}{d\theta})^2$ term in the anti-derivative is absorbed into the $O(1)$, similarly from $\frac{d\xi}{d\theta}(\theta_1) = -1$ only the lower boundary of the integral has a contribution.)

The approximation $\cos(\theta) = \frac{\pi}{2} - \theta + O((\frac{\pi}{2}-\theta)^3)$ then gives the bounds (9) from (11).

For the next step, note that $H(\theta) - (\frac{d\xi}{d\theta}(\theta))^{-1}$ becomes $+\infty$ as $\theta \to \theta^*$ and $H(\theta_1)+1$ as $\theta \to \theta_1$, which for ϵ small enough will be negative. Hence for small enough ϵ, there is a $\theta_0 \in (\theta^*, \theta_1)$ so that $H(\theta_0) = (\frac{d\xi}{d\theta}(\theta_0))^{-1}$. In the same way as Lemma 4.8 one shows that $\theta_0 \to \frac{\pi}{2}$ as $\epsilon \to 0$.

We let

$$q := \frac{\pi/2-\theta_0}{\pi/2-\theta_1}$$

and show next that there is a constant c_3 so that $q^{m_2} \geq \frac{c_3}{\pi/2-\theta_0}$, and hence that q grows unboundedly as $\epsilon \to 0$.

To show this we again integrate (12), this time from θ_0 to $\theta \leq \theta_1$. The result is:

$$\ln(|\frac{d\xi}{d\theta}(\theta_0)|) - \ln(|\frac{d\xi}{d\theta}(\theta)|) + O(1) = m_2 \ln\left(\frac{\cos(\theta)}{\cos(\theta_0)}\right) + O(1).$$

(As before the $1 + (\frac{d\xi}{d\theta})^2$ term on the left-hand side is absorbed into the $O(1)$, while the $\frac{e^{\frac{4}{g}\xi}-m}{\frac{d\xi}{d\theta}}$ term on the right is also $O(1)$ by (11) and monotonicity of $\frac{d\xi}{d\theta}$.)

Noting that $\frac{d\xi}{d\theta}(\theta_0) = \frac{1}{H(\theta_0)} \approx -m_2(\frac{\pi}{2} - \theta_0) + O((\frac{\pi}{2} - \theta_0)^2)$ and again using the expansion of cosine close to $\frac{\pi}{2}$ implies the existence of $c_3, c_4 > 0$ so that:

$$c_4 \left|\frac{d\xi}{d\theta}(\theta)\right| \geq \left(\frac{\pi/2 - \theta_0}{\pi/2 - \theta}\right)^{m_2} (\frac{\pi}{2} - \theta_0) \geq c_3 \left|\frac{d\xi}{d\theta}(\theta)\right|. \tag{13}$$

Taking $\theta = \theta_1$ shows $q^{m_2} \geq \frac{c_3}{\pi/2-\theta_0}$, in particular $q \to \infty$ as $\epsilon \to 0$. On the other hand, if one integrates $\frac{d\xi}{d\theta}(\theta)$ over (θ_0, θ_1) one finds by (13) that

$$c_4|\xi(\theta_1) - \xi(\theta_0)| \geq \begin{cases} (\frac{\pi}{2} - \theta_0)^2 \ln q & m_2 = 1 \\ \frac{1}{m_2-1}(\frac{\pi}{2} - \theta_0)^2(q^{m_2-1} - 1) & m_2 > 1 \end{cases}.$$

Noting that $(\frac{\pi}{2} - \theta_0)^2 = \frac{(\pi/2-\theta_1)^{m_2}}{(\pi/2-\theta_0)^{m_2}} q^{m_2} (\frac{\pi}{2} - \theta_0)^2 \geq \epsilon\, c_1 c_3 (\frac{\pi}{2} - \theta_0)^{1-m_2}$ and using unboundedness of q one sees that $|\xi(\theta_1) - \xi(\theta_0)|$ will be larger than ϵ, contradicting our assumption that $\xi(\theta_1) \geq \frac{g}{4} \ln(m)$. $\square$

Lemma 4.10 *For θ_1 sufficiently close to $\frac{\pi}{2}$, one has that*

$$\xi(T_2) \leq \frac{g}{4} \ln(m) - \left(2^{\frac{1}{2m_2}} - 1\right)(\frac{\pi}{2} - \theta_2) + O((\frac{\pi}{2} - \theta_2)^2).$$

Proof By Lemma 4.9, one has that $\xi(T_1) \leq \frac{g}{4} \ln(m)$. Additionally for ϵ small enough $\frac{d^2\xi}{d\theta^2}(\theta) < 0$ for all $\theta \in (\theta_1, \theta_2)$, whence $\xi(T_2) \leq \xi(T_1) - (\theta_2 - \theta_1)$ and the statement reduces to checking $\theta_2 - \theta_1 = \left(2^{\frac{1}{2m_2}} - 1\right)(\frac{\pi}{2} - \theta_2) + O((\frac{\pi}{2} - \theta_2)^2)$.

From the graph ODE (∗∗), one recovers:

$$\frac{\frac{d^2\xi}{d\theta^2}}{\frac{d\xi}{d\theta}(1 + (\frac{d\xi}{d\theta})^2)} = -\frac{e^{\frac{4}{g}\xi} - m}{\frac{d\xi}{d\theta}} - 2H(\theta).$$

The absolute value of the first summand on the right-hand side is bounded by m over the interval (θ_1, θ_2), so integrating the equation from θ_1 to θ_2 gives:

$$\frac{1}{2}\ln(2) = O(\frac{\pi}{2} - \theta_1) - m_2 \ln\left(\frac{\cos(\theta_2)}{\cos(\theta_1)}\right) - m_1 \ln\left(\frac{\sin(\theta_2)}{\sin(\theta_1)}\right).$$

As θ becomes arbitrarily close to $\frac{\pi}{2}$, we use that $\cos(\theta) = \frac{\pi}{2} - \theta + O((\frac{\pi}{2} - \theta)^3)$, $\sin(\theta) = 1 + O((\frac{\pi}{2} - \theta)^2)$ to get:

$$\frac{1}{2}\ln(2) = -m_2 \ln\left(\frac{\frac{\pi}{2} - \theta_2}{\frac{\pi}{2} - \theta_1}\right) + O(\frac{\pi}{2} - \theta_1).$$

Together with some arithmetic this implies the statement about $\theta_2 - \theta_1$. □

Lemma 4.11 *If ϵ is small enough then $\xi_0 = \frac{g}{4}\ln m + \epsilon$ is not type 1.*

Proof As noted before the assumption that $\xi_0 = \frac{g}{4}\ln(m) + \epsilon$ is type 1 leads to (ξ, θ) being the union of two graphs of ξ over θ. In the previous lemmas we investigated the upper graph and found that it ends at the turning point (ξ_2, θ_2).

The lower graph is then determined by the graph ODE and the initial conditions $\xi(\theta_2) = \xi_2$, $\lim_{\theta \to \theta_2^-} \frac{d\xi}{d\theta}(\theta) = +\infty$. The assumption that ξ_0 is type 1 necessitates that for all $\theta \in (\theta^*, \theta_2)$, one has $\frac{d\xi}{d\theta}(\theta) > 0$ for the lower graph. In particular if we integrate

$$\frac{\frac{d^2\xi}{d\theta^2}}{\frac{d\xi}{d\theta}(1 + (\frac{d\xi}{d\theta})^2)} = -\frac{e^{\frac{4}{g}\xi} - m}{\frac{d\xi}{d\theta}} - 2H(\theta) \tag{14}$$

from $\frac{\theta^* + \theta_2}{2}$ to θ_2 one gets

$$-\ln\left(\frac{\frac{d\xi}{d\theta}(\frac{\theta^*+\theta_2}{2})}{\sqrt{1 + (\frac{d\xi}{d\theta})^2}}\right) \geq -m_2 \ln(\frac{\pi}{2} - \theta_2) + O(1),$$

where we used $e^{\frac{4}{g}\xi(\theta)} - m \leq e^{\frac{4}{g}\xi_2} - m \leq 0$. Inverting the above inequality gives the existence of a $c_1 > 0$ so that:

$$\frac{d\xi}{d\theta}\left(\frac{\theta^* + \theta_2}{2}\right) \leq \frac{c_1(\frac{\pi}{2} - \theta_2)^{m_2}}{\sqrt{1 - c_1^2(\frac{\pi}{2} - \theta_2)^{2m_2}}} \leq 2c_1(\frac{\pi}{2} - \theta_2)^{m_2}. \tag{15}$$

For $m_2 > 1$ this implies the lemma, since for all $\theta \in (\theta^*, \theta_2)$, one has

$$\frac{d^2\xi}{d\theta^2}(\theta) > -(e^{\frac{4}{g}\xi(\theta)} - m) \geq c_2(\frac{\pi}{2} - \theta_2)$$

with $c_2 > 0$ some constant by Lemma 4.10, and then

$$\frac{d\xi}{d\theta}(\theta) \leq 2c_1(\frac{\pi}{2} - \theta_2)^{m_2} - c_2(\frac{\pi}{2} - \theta_2)\,(\frac{\theta^* + \theta_2}{2} - \theta)$$

for $\theta \in (\theta^*, \frac{\theta^*+\theta_2}{2})$ and one gets $\frac{d\xi}{d\theta}(\theta) = 0$ for one such θ.

For $m_2 = 1$, one gets first from (15) that $\frac{d\xi}{d\theta}$ takes on all values in $(\frac{c_1}{2}(\frac{\pi}{2} - \theta_2), \infty)$ as θ varies from $\frac{\theta^*+\theta_2}{2}$ to θ_2. In particular for θ_2 close enough to $\frac{\pi}{2}$ there exists a $\theta_3 < \theta_2$ so that $\frac{d\xi}{d\theta}(\theta_3) = -\frac{1}{H(\theta_3)}$.

By integrating (14), one shows that θ_3 gets arbitrarily close to $\frac{\pi}{2}$ if $\frac{\pi}{2} - \theta_2$ is small enough. We carry this out explicitly:

Integrate (14) from θ_3 to θ_2, the left-hand side evaluates to $\frac{-1}{2}\ln(1 + H(\theta_3)^2)$, which is negative and remains bounded as $\theta_2 \to \frac{\pi}{2}$ unless θ_3 also gets close to $\frac{\pi}{2}$ (since otherwise $H(\theta_3)$ is bounded). For the right-hand side, one first notes that the integral over $-\frac{e^{\frac{4}{g}\xi} - m}{\frac{d\xi}{d\theta}}$ from θ_3 to θ_2 is positive. The other term on the right-hand side, however, yields $-\ln(\frac{\pi/2-\theta_2}{\pi/2-\theta_3}) + O(1)$ which is, crucially, negative and unbounded unless $\theta_3 \to \frac{\pi}{2}$ together with θ_2. So a situation where $\theta_2 \to \frac{\pi}{2}$ but $\theta_3 \not\to \frac{\pi}{2}$ is impossible.

In fact, this integral shows that if $q := \frac{\pi/2-\theta_3}{\pi/2-\theta_2}$ that q grows unboundedly as θ_2 approaches $\frac{\pi}{2}$. Finally if we integrate (14) from $\theta \geq \theta_3$ to θ_2 one gets:

$$-\ln\left(\frac{\frac{d\xi}{d\theta}(\theta)}{\sqrt{1 + (\frac{d\xi}{d\theta}(\theta))^2}}\right) + O(1) = -\ln(\frac{\pi/2 - \theta_3}{\pi/2 - \theta}) + O(\frac{\pi}{2} - \theta).$$

(Here the $-\frac{e^{\frac{4}{g}\xi} - m}{\frac{d\xi}{d\theta}}$ term is bounded by $-H(\theta_3)m = \frac{m}{\pi/2-\theta_3} + O(1)$. Its contribution to the integral then an $O(1)$ term, since $\frac{\theta_2-\theta}{\pi/2-\theta_3} \leq 1$.)

Inverting this expression gives a constant c_3 so that $\frac{d\xi}{d\theta}(\theta) \geq c_3\frac{\pi/2-\theta_2}{\pi/2-\theta}$, and then integrating this from θ_3 to θ_2 gives:

$$\xi(\theta_2) - \xi(\theta_3) \geq c_3(\frac{\pi}{2} - \theta_2)\ln q$$

Plugging this into $(**)$ then implies for any $\theta \in (\theta^*, \frac{\theta^*+\theta_2}{2})$ that:

$$\frac{d^2\xi}{d\theta^2}(\theta) \geq m - e^{\frac{4}{g}\xi(\theta_3)} \geq m(1 - e^{-\frac{4}{g}c_3(\frac{\pi}{2}-\theta_2)\ln q}).$$

Together with (15), which states $\frac{d\xi}{d\theta}(\frac{\theta^*+\theta_2}{2}) \leq 2c_1(\frac{\pi}{2} - \theta_2)$, and unboundedness of q as $\theta_2 \to \frac{\pi}{2}$, one recovers $\frac{d\xi}{d\theta}(\theta) = 0$ for some $\theta \in (\theta^*, \frac{\theta^*+\theta_2}{2})$, provided $\frac{\pi}{2} - \theta_2$ is small enough.

So also in the case $m_2 = 1$, we get a contradiction to the assumption that $\frac{g}{4}\ln m + \epsilon$ was type 1 for ϵ small enough. □

4.4 Proof of Proposition 3.3 (iii)

Lemma 4.12 *If ξ_0^* is of type 3 then:*

(i) *There is a* $\delta > 0$ *and a* $T > 0$ *so that* $\xi_{\xi_0^*}(t) < \frac{g}{4}\ln(m) - \delta$ *for all* $t > T$.
(ii) $\theta'_{\xi_0^*}(t) > 0$ *for all* $t > 0$.
(iii) $\lim_{t\to\infty}\theta_{\xi_0^*}(t) = \frac{\pi}{2}$.

Proof Since $\xi_0^* > \frac{g}{4}\ln m$ and $\xi''_{\xi_0^*}(0) = -\sin(2\theta^*)(e^{\frac{4}{g}\xi_0^*} - m) < 0$, $\xi'_{\xi_0^*}(0) = 0$ one has that $\xi'_{\xi_0^*}(t)$ becomes negative for small times t. Then by Lemma 4.1, it must reach $\frac{g}{4}\ln(m)$ in finite time. Since $\xi'_{\xi_0^*}(t) < 0$ for all $t > 0$ this yields part (i).

One also has that $\theta'_{\xi_0^*}(t) \neq 0$ for all $t > 0$, as otherwise Lemma 4.2 implies that $\theta_{\xi_0^*} = \theta^*$ in finite time. This gives part (ii).

$\theta_{\xi_0^*}$ is then monotonous in t and bounded by $\frac{\pi}{2}$, whence it converges. Since ξ_0^* is bounded above, one has that all derivatives of the parameters are bounded and hence $\theta'_{\xi_0^*}(t) = \sin\alpha\ \sin(2\theta) \to 0$. It is clear that $\sin\alpha$ cannot converge to 0 as $\alpha'(t)$ would then be asymptotically equal to $-2l(\theta)$, which does not converge to 0. Hence $\theta \to \frac{\pi}{2}$, giving part (iii). □

Lemma 4.13 ξ_0^* *is not of type 3.*

Proof Let $(\xi_\epsilon(t), \theta_\epsilon(t))$ denote the solution to $(*)$ with initial condition (ξ, θ, α) $(t = 0) = (\xi_0^* + \epsilon, \theta^*, \frac{\pi}{2})$ where $\epsilon > 0$. Note that $\xi_0^* + \epsilon$ is type 1 by the definition of ξ_0^*, in particular there is a $T_1(\epsilon)$ so that $\theta_\epsilon(t)$ has a local maximum and $\xi'_\epsilon(T_1) < 0$ for all $t \in (0, T_1]$.

Assuming that ξ_0^* is of type 3 and using that ξ_ϵ and θ_ϵ converge uniformly on compacta to $\xi_{\epsilon=0}$ and $\theta_{\epsilon=0}$ as $\epsilon \to 0$ one finds that $\theta_\epsilon(T_1) \to \frac{\pi}{2}$ as $\epsilon \to 0$. This also implies $T_1(\epsilon) \to \infty$ for $\epsilon \to 0$, giving for ϵ small enough that one has $\xi_\epsilon(T_1) < \frac{g}{4}\ln(m) - \delta$.

After the extremum at $\theta_\epsilon(T_1)$, one has that ξ_ϵ becomes a graph over θ because no more extrema of θ are possible until after $\theta = \theta^*$. As in Lemma 4.9, the graph ODE for ξ yields:

$$\frac{\frac{d^2\xi_\epsilon}{d\theta^2}}{\frac{d\xi_\epsilon}{d\theta}(1 + (\frac{d\xi}{d\theta})^2)} = -\frac{e^{\frac{4}{g}\xi_\epsilon} - m}{\frac{d\xi_\epsilon}{d\theta}} - 2H(\theta).$$

Integrating this for the lower graph from $x{:=}\frac{\theta^* + \pi/2}{2}$ to $\theta_\epsilon(T_1)$, one gets

$$-\ln\left(\frac{d\xi_\epsilon/d\theta}{\sqrt{1 + (\frac{d\xi_\epsilon}{d\theta})^2}}(x)\right) \geq -m_2\ln(\frac{\pi}{2} - \theta_\epsilon(T_1)) + O(1)$$

where $-\frac{e^{\frac{4}{g}\xi} - m}{\frac{d\xi}{d\theta}} \geq 0$ and $\frac{d\xi}{d\theta}(\theta(T_1)) = +\infty$ were used.

As $\epsilon \to 0$ this implies that $\frac{d\xi_\epsilon}{d\theta}(x)$ becomes arbitrarily small, in particular we may assume it to be smaller than $\frac{4m}{g}\frac{1}{2}(x - \theta^*)\delta$. Remarking, however, that if $\xi < \frac{g}{4}\ln(m) - \delta$

then $-(e^{\frac{4}{g}\xi} - m) > \frac{4}{g}m\delta$, one gets:

$$\frac{d^2\xi_\epsilon}{d^2\theta}(\theta) > \frac{4}{g}m\delta$$

for all $\theta \in (\theta^*, x)$. With $\frac{d\xi_\epsilon}{d\theta}(x) < \frac{4m}{g}\frac{1}{2}(x-\theta^*)\delta$, one immediately gets $\frac{d\xi_\epsilon}{d\theta}(\frac{\theta^*+x}{2}) < 0$, contradicting that ξ_ϵ is type 1. □

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