The Journal of Geometric Analysis (2023) 33:154
https://doi.org/10.1007/s12220-023-01205-0

Horofunction Compactifications and Duality

Bas Lemmens[1] · Kieran Power[1]

Received: 7 October 2022 / Accepted: 9 January 2023 / Published online: 28 February 2023

Abstract

We study the global topology and geometry of the horofunction compactification of classes of symmetric spaces under Finsler distances in three settings: bounded symmetric domains of the form $B = B_1 \times \cdots \times B_r$, where B_i is an open Euclidean ball in $\mathbb{C}^{n_i}$, with the Kobayashi distance, symmetric cones with the Hilbert distance, and Euclidean Jordan algebras with the spectral norm. For these spaces we show, that the horofunction compactification is naturally homeomorphic to the closed unit ball of the dual norm of the Finsler metric in the tangent space at the basepoint. In each case we give an explicit homeomorphism. For finite dimensional normed spaces the link between the geometry of the horofunction compactification and the dual unit ball was suggested by Kapovich and Leeb, which we confirm for Euclidean Jordan algebras with the spectral norm. Our results also show that this duality phenomenon not only occurs in normed spaces, but also in a variety of noncompact type symmetric spaces with invariant Finsler metrics.

Keywords Dual ball · Euclidean Jordan algebras · Hilbert metric · Homeomorphism · Horofunction compactification · Invariant Finsler metrics · Kobayashi metric · Polydiscs · Spectral norm · Symmetric cones · Symmetric spaces

Mathematics Subject Classification Primary 53C60; Secondary 32Q45 · 46B20

B. Lemmens gratefully acknowledges the support of the EPSRC (Grant EP/R044228/1) and Kieran Power gratefully acknowledges the support of the EPSRC (Grant EP/V520093/1).

✉ Bas Lemmens
B.Lemmens@kent.ac.uk

Kieran Power
kcp6@kent.ac.uk

[1] School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury CT2 7NX, UK

1 Introduction

Compactifications of symmetric spaces is a rich subject which has been studied extensively [9, 27]. Recently it was shown that various compactifications of noncompact type symmetric spaces $X = G/K$ can be realised as horofunction compactifications with respect to G-invariant Finsler metrics. For the generalised Satake compactifications this was shown by Haettel et al. [28], and for the Martin compactification this was established by Schilling [53]. The realisation of the maximal Satake compactification as a horofunction compactification was given by Kapovich and Leeb [34].

For symmetric spaces with nonpositive sectional curvature it is well-known that the horofunction compactification with respect to the Riemannian distance is homeomorphic to a Euclidean ball, see [12, 16, 17]. For various finite dimensional normed spaces it was observed that the horofunction compactification is naturally related to the closed dual unit ball. As a matter of fact, Kapovich and Leeb [34, Question 6.18] asked if for finite dimensional normed spaces the horofunction compactification (with its natural stratification) is homeomorphic to the closed unit ball of the dual normed space. This was confirmed by Ji and Schilling [32, 33] for normed spaces with a polyhedral unit ball.

In an analogous manner one can ask for noncompact type symmetric spaces if the horofunction compactification with respect to an invariant Finsler metric is naturally homeomorphic to the closed dual unit ball of the Finsler metric in the tangent space at the basepoint.

The main goal of this paper is to confirm this duality phenomenon for two classes of noncompact type symmetric spaces and a class of normed spaces. More specifically, we will consider bounded symmetric domains of the form $B_1 \times \cdots \times B_r$, where B_i is an open Euclidean ball in $\mathbb{C}^{n_i}$, with the Kobayashi distance, symmetric cones with the Hilbert distance, and Euclidean Jordan algebras with the spectral norm, i.e., finite dimensional JB-algebras.

The bounded symmetric domains $B = B_1 \times \cdots \times B_r$ with the Kobayashi distance are prime examples of noncompact type symmetric spaces with invariant Finsler metric. In this case the open unit ball of the Finsler metric in the tangent space at the origin coincides with B. We will show that the horofunction compactification of B is homeomorphic to the dual unit ball, i.e., the polar of B. In fact, we shall work in a slightly more general domains $D = D_1 \times \cdots \times D_r$, where D_i is the open unit ball of a norm with a strongly convex C^3-boundary, even though these domains no longer correspond to symmetric spaces. The horofunction compactification of these spaces was studied in [40]. It should be noted that for general bounded convex domains $D \subset \mathbb{C}^n$, with the Kobayashi distance, various smoothness conditions on D are known that imply that the identity map extends as a homeomorphism from the horofunction compactification of D onto the norm closure of D, see [5, Theorem 1.2] and [7, 10, 11, 59]. In our setting, however, the domains are not smooth, and the identity does not extend as a homeomorphism.

Symmetric cones with the Hilbert distance are another interesting class of symmetric spaces with invariant Finsler metric. A prime example is the symmetric space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$, which corresponds to the projective symmetric cone consisting of positive definite $n \times n$ Hermitian matrices. More precisely, one can realise this space as $\{A \in \mathrm{Herm}(n, \mathbb{C})\colon\ A$ is positive definite with trace $n\}$. In our analysis we use the cone version of the Hilbert distance, see [42], which provides a convenient way to analyse its Finsler structure [48] and the dual unit ball. The horofunction compactification of symmetric cones with the Hilbert distance was determined in [44, Theorem 5.6] and is naturally described in terms of the Euclidean Jordan algebra associated to the symmetric cone, which will be exploited.

Euclidean Jordan algebras V with the spectral norm, i.e., finite dimensional JB-algebras [4], are an important class of real normed vector spaces. A prime example is the real vector space $\mathrm{Herm}(n, \mathbb{C})$ consisting of $n \times n$ Hermitian matrices with the spectral norm, $\|A\| = \max\{|\lambda|\colon \lambda \text{ eigenvalue of } A\}$. We use the Jordan algebra structure to give a complete description of the horofunctions for these spaces and provide an explicit homeomorphism between the horofunction compactification and the closed dual unit ball. These normed spaces are related to an invariant metric on symmetric cones coming from the Thompson distance. More precisely, for a symmetric cone C in a Euclidean Jordan algebra V, the Finsler metric of the Thompson distance in the tangent space V at the unit is the spectral norm, see [48].

In a sequel [41] to this paper the first author has shown for the Hilbert distance and Thompson distance on symmetric cones that the exponential map at the unit extends as a homeomorphism between the horofunction compactification of the normed space at the unit with the Finsler metric, and the horofunction compactification of the symmetric cone with the Finsler distance. It would be interesting to know if this holds more generally for noncompact type symmetric spaces with invariant Finsler distances.

The origins of the horofunction compactification go back to Gromov [6, 21] who associated a boundary at infinity to any locally compact geodesic metric space. It has found numerous applications in diverse areas of mathematics including, geometric group theory [12], noncommutative geometry [51], complex analysis [1, 5, 7, 10, 11, 59], Teichmüller theory [15, 20, 36, 46, 56], dynamical systems and ergodic theory [8, 19, 35, 44] and in the study of compactifications of noncompact type symmetric spaces [28, 34, 53]. A general set up for metric spaces was discussed by Rieffel [51]. It should, however, be noted that if the metric space is not proper, then the embedding into its horofunction compactification need not be a homeomorphism. So, in that case, the horofunction compactification would not be a compactfication in the usual topological sense.

The horofunction compactification is a particularly powerful tool to study isometry groups of metric spaces and isometric embeddings between metric spaces, see [40, 43, 57, 58]. Especially useful in this context are the so-called Busemann points in the horofunction compactification, which are limits of almost geodesics. They were introduced by Rieffel [51], who asked whether every horofunction is a Busemann point in a finite dimensional normed space. Walsh [54] showed that in general this is not the case and found necessary and sufficient conditions for a finite dimensional normed space to have the property that all its horofunctions are Busemann points.

For the metric spaces considered in this paper, we show that all horofunctions are Busemann points. As a consequence we get that the horofunction boundary has a partition coming from the detour distance. Indeed, on the set of Busemann points the detour distance [2, 43] is a metric, where two Busemann points can lie at infinite distance from each other. This yields a partition of the set of Busemann points into so-called parts where two Busemann points lie in the same part if the detour distance between them is finite. As all horofunctions are Busemann points for our spaces, it follows from [57, Proposition 4.5] that this partition coincides with the partition of the horofunction boundary into subsets, where two horofunctions h and g lie in the same subset if $\sup_x |h(x) - g(x)|$ is bounded.

On the other hand, the closed dual unit ball B_1^* is the disjoint union of the relative interiors of its nonempty faces, see [52, Theorem 18.2]. In each of our settings we will give an explicit homeomorphism that maps the metric space onto the interior of B_1^*, and each part in the horofunction boundary onto the relative interior of a boundary face of B_1^*. It is this property of the homeomorphism that 'naturally' connects the geometry of the horofunction compactification to the closed dual unit ball for our spaces. The homeomorphisms we use are modifications of the maps used by Ji and Schilling [32] in the setting of polyhedral normed spaces. As pointed out there, the homeomorphisms resemble moment maps from symplectic geometry and Lie group actions, but the exact connection is not well understood at present. Similar maps were also used in the study of Satake compactifications of symmetric spaces in [31, 39].

The results are consistent with what is known for the horofunction compactification of the symmetric spaces with nonpositive sectional curvature under the Riemannian distance. In that case all horofunctions are Busemann points and each part is a singleton, which reflects the fact that each point in the boundary of the Euclidean ball is a relatively open face, as it is an extreme point.

In general it is difficult to determine the horofunction compactification of a metric space explicitly and only in relatively few spaces has this been done. For CAT(0) spaces the horofunction compactification is well understood, see [12, Chap. II.8] and coincides with the visual boundary. Gutièrrez [22–24] computed the horofunction compactification of several classes of L_p-spaces. It has also been identified for finite dimensional polyhedral normed spaces, see [13, 28, 33, 37]. For arbitrary (possibly infinite dimensional) normed spaces the Busemann points in the horofunction boundary have been characterised by Walsh [58]. For Hilbert metric spaces there exists a characterisation of the Busemann points [55]. For the Hilbert distance on a symmetric cone in a Euclidean Jordan algebra, the horofunction compactification was obtained in [44], the cone in a (possibly infinite dimensional) spin factor was discussed in [14], and results for the p-metrics, with $1 \le p < \infty$, on the symmetric cone in $\mathrm{Herm}_n(\mathbb{C})$ were obtained in [26].

2 Metric Geometry Preliminaries

We start by recalling the construction of the horofunction compactification and the detour distance.

Let (M, d) be a metric space and let $\mathbb{R}^M$ be the space of all real functions on M equipped with the topology of pointwise convergence. Fix $b \in M$, which is called the *basepoint*, and let $\mathrm{Lip}_b^c(M)$ denote the set of all functions $h \in \mathbb{R}^M$ such that $h(b) = 0$ and h is c-Lipschitz, i.e., $|h(x) - h(y)| \le cd(x, y)$ for all $x, y \in M$.

Then $\mathrm{Lip}_b^c(M)$ is a compact subset of $\mathbb{R}^M$. Indeed, the complement of $\mathrm{Lip}_b^c(M)$ is open, so $\mathrm{Lip}_b^c(M)$ is closed subset of $\mathbb{R}^M$. Moreover, as $|h(x)| = |h(x) -$

$h(b)| \leq cd(x, b)$ for all $h \in \mathrm{Lip}_b^c(M)$ and $x \in M$, we get that $\mathrm{Lip}_b^c(M) \subseteq [-cd(x, b), cd(x, b)]^M$, which is compact by Tychonoff's theorem.

For $y \in M$ define the real valued function,

$$h_y(z) = d(z, y) - d(b, y) \quad \text{with } z \in M. \tag{2.1}$$

Then $h_y(b) = 0$ and $|h_y(z) - h_y(w)| = |d(z, y) - d(w, y)| \leq d(z, w)$. Thus, $h_y \in \mathrm{Lip}_b^1(M)$ for all $y \in M$. Using the previous observation one now defines the *horofunction compactification* of (M, d) to be the closure of $\{h_y : y \in M\}$ in $\mathbb{R}^M$, which is a compact subset of $\mathrm{Lip}_b^1(M)$ and is denoted by $\overline{M}^h$. Its elements are called *metric functionals*, and the boundary $\partial\overline{M}^h = \overline{M}^h \backslash \{h_y : y \in M\}$ is called the *horofunction boundary*. The metric functionals in $\partial\overline{M}^h$ are called *horofunctions*, and all other metric functionals are said to be *internal points*.

The topology of pointwise convergence on $\mathrm{Lip}_b^1(M)$ coincides with the topology of uniform convergence on compact sets, see [47, Sect. 46]. In general the topology of pointwise convergence on $\mathrm{Lip}_b^1(M)$ is not metrisable, and hence horofunctions are limits of nets rather than sequences. If, however, the metric space is separable, then the pointwise convergence topology on $\mathrm{Lip}_b^1(M)$ is metrizable and each horofunction is the limit of a sequence. It should be noted that the embedding $\iota : M \to \mathrm{Lip}_b^1(M)$, where $\iota(y) = h_y$, may not have a continuous inverse on $\iota(M)$, and hence the metric compactification is not always a compactification in the strict topological sense. If, however, (M, d) is proper (i.e. closed balls are compact) and geodesic, then ι is a homeomorphism from M onto $\iota(M)$. Recall that a map γ from a (possibly unbounded) interval $I \subseteq \mathbb{R}$ into a metric space (M, d) is called a *geodesic path* if

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in I.$$

The image, $\gamma(I)$, is called a *geodesic*, and a metric space (M, d) is said to be *geodesic* if for each $x, y \in M$ there exists a geodesic path $\gamma : [a, b] \to M$ connecting x and y, i.e, $\gamma(a) = x$ and $\gamma(b) = y$. We call a geodesic $\gamma([0, \infty))$ a *geodesic ray*.

The following fact, which is slightly weaker than [51, Theorem 4.7], will be useful in the sequel.

Lemma 2.1 *If (M, d) is a proper geodesic metric space, then $h \in \partial\overline{M}^h$ if and only if there exists a sequence (x^n) in M with $d(b, x^n) \to \infty$ such that (h_{x^n}) converges to $h \in \overline{M}^h$ as $n \to \infty$.*

A sequence (x^n) in (M, d) is called an *almost geodesic sequence* if for all $\varepsilon > 0$ there exists a $N \geq 0$ such that

$$d(x^n, x^m) + d(x^m, x^0) - d(x^n, x^0) < \varepsilon \quad \text{for all } n \geq m \geq N.$$

The notion of an almost geodesic sequence goes back to Rieffel [51] and was further developed by Walsh and co-workers in [2, 40, 43, 58]. In particular, every unbounded almost geodesic sequence yields a horofunction in a proper geodesic metric space [58].

Lemma 2.2 *Let (M, d) be a proper geodesic metric space. If (x^n) is an unbounded almost geodesic sequence in M, then*

$$h(z) = \lim_n d(z, x^n) - d(b, x^n)$$

exists for all $z \in M$ and $h \in \partial \overline{M}^h$.

Given a proper geodesic metric space (M, d), a horofunction $h \in \overline{M}^h$ is called a *Busemann point* if there exists an almost geodesic sequence (x^n) in M such that $h(z) = \lim_n d(z, x^n) - d(b, x^n)$ for all $z \in M$. We denote the collection of all Busemann points by $\mathcal{B}_M$.

Suppose that $h, h' \in \partial \overline{M}^h$ be horofunctions and (M, d) is a proper geodesic metric space. Let W_h be the collection of neighbourhoods of h in $\overline{M}^h$. The *detour cost* is given by

$$H(h, h') = \sup_{W \in W_h} \left(\inf_{x \colon \iota(x) \in W} d(b, x) + h'(x) \right).$$

The *detour distance* is given by

$$\delta(h, h') = H(h, h') + H(h', h). \tag{2.2}$$

It is known [58] that if (x^n) is an almost geodesic sequence converging to a horofunction h, then

$$H(h, h') = \lim_n d(b, x^n) + h'(x^n) \tag{2.3}$$

for all horofunctions h'. Moreover, on the set of Busemann points $\mathcal{B}_M$ the detour distance is a metric where points can be at infinite distance from each other, see [58]. The detour distance yields a partition of $\mathcal{B}_M$ into equivalence classes, called *parts*, where h and h' are equivalent if $\delta(h, h') < \infty$. The equivalence class of h is denoted by $\mathcal{P}_h$. So $(\mathcal{P}_h, \delta)$ is a metric space, and $\mathcal{B}_M$ is the disjoint union of metric spaces under the detour distance.

It is known [57, Proposition 4.5] that two Busemann points h and g in the horofunction boundary are in the same part if and only if $\sup_{x \in M} |h(x) - g(x)| < \infty$. Furthermore, any isometry on M extends as an isometry to the set of Busemann points under the detour distance, see [43].

For symmetric spaces with nonpositive sectional curvature, all horofunctions with respect to the Riemannian metric are Busemann points and each part is a singleton. For the spaces under consideration in this paper we show that all horofunctions are Busemann points, but the parts can be nontrivial.

3 Bounded Symmetric Domains

In this section we analyse the geometry and topology of the horofunction compactification of bounded symmetric domains of the form $B = B_1 \times \cdots \times B_r$, where $B_i = \{z \in \mathbb{C}^{n_i} : |z_1|^2 + \cdots + |z_{n_i}|^2 < 1\}$, under the Kobayashi distance. In fact, we shall consider slightly more general product domains where each B_i is the open unit ball of a norm on $\mathbb{C}^{n_i}$ with a strongly convex C^3-boundary. Even though these domains no longer correspond to noncompact type symmetric spaces we shall see that there still exists a homeomorphism between the horofunction compactification and the closed dual unit ball of the Finsler metric at the origin. We will start by recalling some basic concepts.

3.1 Product Domains and Kobayashi Distance

On a convex domain $D \subseteq \mathbb{C}^n$ the *Kobayashi distance* is given by

$$k_D(z, w) = \inf\{\rho(\zeta, \eta) : \exists f : \Delta \to D \text{ holomorphic with } f(\zeta) = z \text{ and } f(\eta) = w\}$$

for all $z, w \in D$, where

$$\rho(z, w) = \log \frac{1 + \left|\frac{w-z}{1-\bar{z}w}\right|}{1 - \left|\frac{w-z}{1-\bar{z}w}\right|} = 2\tanh^{-1}\left(1 - \frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^{1/2}$$

is the *hyperbolic distance* on the open disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

It is known, see [1, Proposition 2.3.10], that if $D \subset \mathbb{C}^n$ is bounded convex domain, then (D, k_D) is a proper metric space, whose topology coincides with the usual topology on $\mathbb{C}^n$. Moreover, (D, k_D) is a geodesic metric space containing geodesic rays, see [1, Theorem 2.6.19] or [38, Theorem 4.8.6].

For the Euclidean ball $B^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 < 1\}$, where $\|z\|^2 = \sum_i |z_i|^2$, the Kobayashi distance satisfies

$$k_{B^n}(z, w) = 2\tanh^{-1}\left(1 - \frac{(1-\|w\|^2)(1-\|z\|^2)}{|1-\langle z, w\rangle|^2}\right)^{1/2}$$

for all $z, w \in B^n$, see [1, Chaps. 2.2 and 2.3].

In our setting we will consider product domains $B = \prod_{i-1}^r B_i$, where each B_i is an open unit ball of a norm in $\mathbb{C}^{n_i}$, and we will use the product property of k_B, which says that

$$k_B(z, w) = \max_{i=1,\ldots,r} k_i(z_i, w_i),$$

where k_i is the Kobayashi distance on B_i, see [38, Theorem 3.1.9]. So for the polydisc $\Delta^r = \{(z_1, \ldots, z_r) \in \mathbb{C}^r :\ \max_i |z_i| < 1\}$, the Kobayashi distance satisfies

$$k_{\Delta^r}(z, w) = \max_i \rho(z_i, w_i) \quad \text{for all } w = (w_1, \ldots, w_r), z = (z_1, \ldots, z_r) \in \Delta^r.$$

For the Euclidean ball, B^n, it is well known that the horofunctions of (B^n, k_{B^n}), with basepoint $b = 0$, are given by

$$h_\xi(z) = \log \frac{|1 - \langle z, \xi\rangle|^2}{1 - \|z\|^2} \quad \text{for all } z \in B^n, \tag{3.1}$$

where $\xi \in \partial B^n$. Moreover, each horofunction h_ξ is a Busemann point, as it is the limit induced by the geodesic ray $t \mapsto \frac{e^t-1}{e^t+1}\xi$, for $0 \leq t < \infty$.

Moreover, if B is a product of Euclidean balls, then the horofunctions are known, see [1, Proposition 2.4.12] and [40, Corollary 3.2]. Indeed, for a product of Euclidean balls $B^{n_1} \times \cdots \times B^{n_r}$ the Kobayashi distance horofunctions with basepoint $b = 0$ are precisely the functions of the form

$$h(z) = \max_{j \in J} \left(h_{\xi_j}(z_j) - \alpha_j\right),$$

where $J \subseteq \{1, \ldots, r\}$ nonempty, $\xi_j \in \partial B^{n_j}$ for $j \in J$, and $\min_{j\in J} \alpha_j = 0$. Moreover, each horofunction is a Busemann point.

The form of the horofunctions of the product of Euclidean balls is essentially due to the product property of the Kobayashi distance and the smoothness and convexity properties of the balls. Indeed, more generally, the following result holds, see [40, Sect. 2 and Lemma 3.3].

Theorem 3.1 *If $D_i \subset \mathbb{C}^{n_i}$ is a bounded strongly convex domain with C^3-boundary, then for each $\xi_i \in \partial D_i$ there exists a unique horofunction h_{ξ_i} which is the limit of a geodesic γ from the basepoint $b_i \in D_i$ to ξ_i. Moreover, these are all the horofunctions. If $D = \prod_{i=1}^r D_i$, where each D_i is a bounded strongly convex domain with C^3-boundary, then each horofunction h of (D, k_D) with respect to the basepoint $b = (b_1, \ldots, b_r)$ is of the form*

$$h(z) = \max_{j \in J} \left(h_{\xi_j}(z_j) - \alpha_j\right), \tag{3.2}$$

where $J \subseteq \{1, \ldots, r\}$ nonempty, $\xi_j \in \partial D_j$ for $j \in J$, and $\min_{j\in J} \alpha_j = 0$. Furthermore, each horofunction is a Busemann point, and the part of h, where h is given by (3.2), consists of those horofunctions h' of the form,

$$h'(z) = \max_{j \in J} \left(h_{\xi_j}(z_j) - \beta_j\right),$$

with $\min_{j\in J} \beta_j = 0$.

Now let $D = \prod_{i=1}^{r} D_i$, where each D_i is a bounded strongly convex domain with C^3-boundary. Given $J \subseteq \{1, \dots, r\}$ nonempty, $\xi_j \in \partial D_j$ for $j \in J$, and $\alpha_j \geq 0$ for $j \in J$ with $\min_{j \in J} \alpha_j = 0$, we can find geodesic paths $\gamma_j : [0, \infty) \to D_j$ from b_j to ξ_j, and form the path $\gamma : [0, \infty) \to D$, where

$$\gamma(t)_j = \begin{bmatrix} \gamma_j(t - \alpha_j) & \text{for all } j \in J \text{ and } t \geq \alpha_j \\ b_j & \text{otherwise.} \end{bmatrix} \tag{3.3}$$

Lemma 3.2 *The path $\gamma : [0, \infty) \to D$ in (3.3) is a geodesic path, and $h_{\gamma(t)} \to h$ where h is given by (3.2).*

Proof Let k_i denote the Kobayashi distance on D_i. By the product property we have that

$$k_D(\gamma(s), \gamma(t)) = \max_i k_i(\gamma(s)_i, \gamma(t)_i)$$

for all $s \geq t \geq 0$. By construction $k_i(\gamma(s)_i, \gamma(t)_i) \leq k_i(\gamma_i(s), \gamma_i(t)) = s - t$ for all i and $s \geq t \geq 0$. For $j \in J$ with $\alpha_j = 0$ we have that $k_j(\gamma(s)_j, \gamma(t)_j) = k_j(\gamma_j(s), \gamma_j(t)) = s - t$ for all $s \geq t \geq 0$, and hence

$$k_D(\gamma(s), \gamma(t)) = \max_i k_i(\gamma(s)_i, \gamma(t)_i) = s - t$$

for all $s \geq t \geq 0$.

Note that for $z \in D$ we have

$$\begin{aligned} \lim_{t \to \infty} h_{\gamma(t)}(z) &= \lim_{t \to \infty} k_D(z, \gamma(t)) - k_D(\gamma(t), b) \\ &= \lim_{t \to \infty} \max_i (k_i(z_i, \gamma(t)_i) - t) \\ &= \lim_{t \to \infty} \max_{j \in J} (k_j(z_j, \gamma(t)_j) - t) \\ &= \lim_{t \to \infty} \max_{j \in J} (k_j(z_j, \gamma_j(t - \alpha_j)) - k_j(\gamma_j(t - \alpha_j), b_j) - \alpha_j) \\ &= \max_{j \in J} \left(h_{\xi_j}(z_j) - \alpha_j \right), \end{aligned}$$

which shows that $h_{\gamma(t)} \to h$. $\square$

Consider $B = \prod_{i=1}^{r} B_i \subseteq \mathbb{C}^n$, where each B_i is an open unit ball of a norm in $\mathbb{C}^{n_i}$. Then B is the open unit ball of the norm $\| \cdot \|_B$ on $\mathbb{C}^n$. In fact,

$$\|w\|_B = \max_{i=1,\dots,r} \|w_i\|_{B_i},$$

where $\| \cdot \|_{B_i}$ is the norm on $\mathbb{C}^{n_i}$ with open unit ball B_i.

To analyse the dual norm of $\|\cdot\|_B$ we identify the dual space of $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r}$ with itself using the standard inner-product

$$\langle x, y\rangle = \sum_{i=1}^{r} \langle x_i, y_i\rangle \quad \text{for } x = (x_1, \ldots, x_r),\ y = (y_1, \ldots, y_r) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r}.$$

So, $y \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} \mapsto \langle \cdot, y\rangle \in (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r})^*$. Note that the dual norm $\|\cdot\|_B^*$ satisfies

$$\begin{aligned}\|y\|_B^* &= \sup_{\|x\|_B=1} \mathrm{Re}\langle x, y\rangle = \sup_{\|x\|_B=1} \sum_{i=1}^{r} \mathrm{Re}\langle x_i, y_i\rangle \\ &= \sum_{i=1}^{r} \|y_i\|_{B_i}^* \qquad \text{for } y = (y_1, \ldots, y_r) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r},\end{aligned}$$

as $\|x\|_B = \max_i \|x_i\|_{B_i}$. So we see that the closed dual unit ball is given by

$$\begin{aligned}B_1^* &= \{y \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} : \mathrm{Re}\langle x, y\rangle \leq 1 \text{ for all } x \in \overline{B}\} \\ &= \{y \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} : \sum_{i=1}^{r} \|y_i\|_{B_i}^* \leq 1\}.\end{aligned}$$

Now suppose that each B_i is strictly convex and smooth. The closed ball B_1^* has extreme points $p(\xi_i^*) = (0, \ldots, 0, \xi_i^*, 0, \ldots, 0)$, where $\xi_i^* \in \mathbb{C}^{n_i}$ is the unique supporting functional at $\xi_i \in \partial B_i$, i.e., $\mathrm{Re}\langle \xi_i, \xi_i^*\rangle = 1$ and $\mathrm{Re}\langle w_i, \xi_i^*\rangle < 1$ for $w_i \in \overline{B_i}$ with $w_i \neq \xi_i$.

The relatively open faces of B_1^* are the sets of the form

$$F(\{\xi_j \in \partial B_j : j \in J\}) = \left\{ \sum_{j\in J} \lambda_j p(\xi_j^*) : \sum_{j\in J} \lambda_j = 1 \text{ and } \lambda_j > 0 \text{ for all } j \in J \right\},$$

where $J \subseteq \{1, \ldots, r\}$ is nonempty and $\xi_j \in \partial B_j$ for $j \in J$ are fixed. Here the relative topology is taken with respect to the affine span of $\{p(\xi_j^*) : j \in J\}$.

On B the Kobayashi distance has a Finsler structure in terms of the infinitesimal Kobayashi metric, see e.g., [1, Chap. 2.3]. Indeed, we have that

$$k_B(z, w) = \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all piecewise C^1-smooth paths $\gamma : [0, 1] \to B$ with $\gamma(0) = z$ and $\gamma(1) = w$, and

$$L(\gamma) = \int_0^1 \kappa_B(\gamma(t), \gamma'(t))\mathrm{d}t,$$

with

$$\kappa_B(u, v) = \inf\{|\xi|: \exists\varphi \in \mathrm{Hol}(\Delta, B) \text{ such that } \varphi(0) = u \text{ and } (\mathrm{D}\varphi)_0(\xi) = v\}.$$

Proposition 3.3 [1, Proposition 2.3.24] *If B is the open unit ball of a norm on $\mathbb{C}^n$, then*

$$\kappa_B(0, v) = \|v\|_B \quad \textit{for all } v \in \mathbb{C}^n.$$

For $z \in B$ and $i = 1, \dots, r$, if $z_i \neq 0$, then we let $z_i' = \|z_i\|_{B_i}^{-1} z_i \in \partial B_i$ and we write $p(z_i^*) = (0, \dots, 0, z_i^*, 0, \dots, 0)$, where z_i^* is the unique supporting functional at $z_i' \in \partial B_i$. If $z_i = 0$, we set $p(z_i^*) = 0$.

We now define a map $\varphi_B : \overline{B}^h \to B_1^*$ and show in the remainder of this section that it is a homeomorphism. For $z \in B = B_1 \times \cdots \times B_r$ let

$$\varphi_B(z) = \frac{1}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}} \left(\sum_{i=1}^r (e^{k_i(z_i,0)} - e^{-k_i(z_i,0)}) p(z_i^*) \right),$$

where k_i is the Kobayashi distance on B_i. For a horofunction h given by (3.2) we define

$$\varphi_B(h) = \frac{1}{\sum_{j\in J} e^{-\alpha_j}} \left(\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*) \right).$$

More precisely we prove the following theorem.

Theorem 3.4 *If $B = \prod_{i=1}^r B_i$, where each B_i is the open unit ball of a norm on $\mathbb{C}^{n_i}$ which is strongly convex and has a C^3-boundary, then $\varphi_B : \overline{B}^h \to B_1^*$ is a homeomorphism, which maps each part of $\partial\overline{B}^h$ onto the relative interior of a boundary face of B_1^*.*

3.2 The Map φ_B: Injectivity and Surjectivity

Throughout the remainder of this section we assume that $B = \prod_{i=1}^r B_i$ and each B_i is the open unit ball of a norm on $\mathbb{C}^{n_i}$, which is strongly convex and has a C^3-boundary. So for each $\xi_i \in \partial B_i$ there exists a unique $\xi_i^* \in \mathbb{C}^{n_i}$ such that

$$\mathrm{Re}\langle \xi_i, \xi_i^* \rangle = 1 \text{ and } \mathrm{Re}\langle w_i, \xi_i^* \rangle < 1 \text{ for all } w_i \in \overline{B_i} \text{ with } w_i \neq \xi_i,$$

as $\overline{B_i}$ is strictly convex and smooth.

We start with the following basic observation.

Lemma 3.5 *For each $z \in B$ we have that $\varphi_B(z) \in \mathrm{int}\, B_1^*$, and $\varphi_B(h) \in \partial B_1^*$ for all $h \in \partial\overline{B}^h$.*

Proof Note that for $z \in B$ and $w \in \overline{B}$ we have that

$$\begin{aligned}\operatorname{Re}\langle w, \varphi_B(z)\rangle &= \frac{1}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}}\left(\sum_{i=1}^r (e^{k_i(z_i,0)} - e^{-k_i(z_i,0)})\operatorname{Re}\langle w_i, z_i^*\rangle\right)\\ &\leq \frac{1}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}}\left(\sum_{i=1}^r e^{k_i(z_i,0)} - e^{-k_i(z_i,0)}\right)\\ &< 1-\delta\end{aligned}$$

for some $0 < \delta < 1$, which is independent of w. Thus, $\sup_{w\in\overline{B}} \operatorname{Re}\langle w, \varphi_B(z)\rangle < 1-\delta < 1$, hence $\varphi_B(z) \in \operatorname{int} B_1^*$.

To see that $\varphi_B(h) \in \partial B_1^*$, note that for $w = \sum_{j\in J} p(\xi_j) \in \overline{B}$, where $p(\xi_j) = (0,\dots,0,\xi_j,0,\dots,0)$, we have that $\operatorname{Re}\langle w, \varphi_B(h)\rangle = 1$. ◻

To show that φ_B is injective on B, we need the following basic calculus fact, which can be found in [32, Sect. 4].

Lemma 3.6 *If* $\mu\colon \mathbb{R}^r \to \mathbb{R}$ *is given by* $\mu(x_1,\dots,x_r) = \sum_{i=1}^r e^{x_i} + e^{-x_i}$, *then* $x \mapsto \nabla \log \mu(x)$ *is injective on* $\mathbb{R}^r$.

Note that

$$(\nabla \log \mu(x))_j = \frac{e^{x_j} - e^{-x_j}}{\sum_{i=1}^r e^{x_i} + e^{-x_i}} \quad \text{for all } j.$$

Lemma 3.7 *The map* φ_B *is a continuous bijection from* B *onto* $\operatorname{int} B_1^*$.

Proof Cleary φ_B is continuous on B and $\varphi_B(z) = 0$ if and only if $z = 0$. Suppose that $z, w \in B\setminus\{0\}$ are such that $\varphi_B(z) = \varphi_B(w)$. For simplicity write

$$\alpha_j = \frac{e^{k_j(z_j,0)} - e^{-k_j(z_j,0)}}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}} \geq 0 \quad \text{and} \quad \beta_j = \frac{e^{k_j(w_j,0)} - e^{-k_j(w_j,0)}}{\sum_{i=1}^r e^{k_i(w_i,0)} + e^{-k_i(w_i,0)}} \geq 0.$$

Note that $\alpha_j p(z_j^*) = 0$ if and only if $z_j = 0$, and $\beta_j p(w_j^*) = 0$ if and only if $w_j = 0$. Thus, $z_j = 0$ if and only if $w_j = 0$. Now suppose that $z_j \neq 0$, so $w_j \neq 0$. Then $\langle p(v_j), \varphi_B(z)\rangle = \langle p(v_j), \varphi_B(w)\rangle$ for each $v_j \in B_j$. This implies that

$$\alpha_j\langle v_j, z_j^*\rangle = \beta_j\langle v_j, w_j^*\rangle \quad \text{for all } v_j \in B_j,$$

hence $\alpha_j z_j^* = \beta_j w_j^*$. It follows that $\alpha_j = \beta_j$ and $z_j^* = w_j^*$. Thus $z_j = \mu_j w_j$ for some $\mu_j > 0$. As $\alpha_i = \beta_i$ for all $i \in \{1,\dots,r\}$, we know by Lemma 3.6 that $k_j(z_j,0) = k_j(w_j,0)$, hence $z_j = w_j$ by [1, Proposition 2.3.5]. So $z = w$, which shows that φ_B is injective.

As φ_B is injective and continuous on B, it follows from Brouwer's domain invariance theorem that $\varphi_B(B)$ is an open subset of $\operatorname{int} B_1^*$ by Lemma 3.5. Suppose, by way of contradiction, that $\varphi_B(B) \neq \operatorname{int} B_1^*$. Then $\partial\varphi_B(B) \cap \operatorname{int} B_1^*$ is nonempty, as otherwise $\varphi_B(B)$ is closed and open in $\operatorname{int} B_1^*$, which would imply that $\operatorname{int} B_1^*$ is the disjoint union of the nonempty open sets $\varphi_B(B)$ and its complement contradicting the connectedness of int B_1^*. So let $w \in \partial\varphi_B(B) \cap \text{int } B_1^*$ and (z^n) be a sequence in B such that $\varphi_B(z^n) \to w$. As φ_B is continuous on B, we have that $k_B(z^n, 0) \to \infty$.

Using the product property, $k_B(z^n, 0) = \max_i k_i(z_i^n, 0)$, we may assume after taking subsequences that $\alpha_i^n = k_B(z^n, 0) - k_i(z_i^n, 0) \to \alpha_i \in [0, \infty]$ and $z_i^n \to \zeta_i \in \overline{B_i}$ for all i. Let $I = \{i : \alpha_i < \infty\}$, and note that for each $i \in I$, $\zeta_i \in \partial B_i$, as $k_i(z_i^n, 0) \to \infty$. Then

$$\begin{aligned}\varphi_B(z^n) &= \frac{1}{\sum_{i=1}^r e^{k_i(z_i^n,0)} + e^{-k_i(z_i^n,0)}}\left(\sum_{i=1}^r (e^{k_i(z_i^n,0)} - e^{-k_i(z_i^n,0)})p((z_i^n)^*)\right)\\ &= \frac{1}{\sum_{i=1}^r e^{-\alpha_i^n} + e^{-k_B(z^n,0)-k_i(z_i^n,0)}}\left(\sum_{i=1}^r (e^{-\alpha_i^n} - e^{-k_B(z^n,0)-k_i(z_i^n,0)})p((z_i^n)^*)\right).\end{aligned}$$

Letting $n \to \infty$, the righthand side converges to

$$\frac{1}{\sum_{i\in I} e^{-\alpha_i}}\left(\sum_{i\in I} e^{-\alpha_i} p(\zeta_i^*)\right) = w.$$

But this implies that $w \in \partial B_1^*$, as $\mathrm{Re}\langle \sum_{i\in I} p(\zeta_i), w\rangle = 1$ and $\sum_{i\in I} p(\zeta_i) \in \overline{B}$, where $p(\zeta_i) = (0, \ldots, 0, \zeta_i, 0, \ldots, 0)$. This is impossible and hence $\varphi_B(B) = \text{int } B_1^*$. □

We now analyse φ_B on $\partial\overline{B}^h$.

Lemma 3.8 *The map φ_B maps $\partial\overline{B}^h$ bijectively onto ∂B_1^*. Moreover, the part $\mathcal{P}_h$, where h is given by (3.2), is mapped onto the relative open boundary face*

$$F(\{\xi_j \in \partial B_j : j \in J\}) = \left\{\sum_{j\in J} \lambda_j p(\xi_j^*) \colon \sum_{j\in J} \lambda_j = 1 \textit{ and } \lambda_j > 0 \textit{ for all } j \in J\right\}.$$

Proof We know from Lemma 3.5 that φ_B maps $\partial\overline{B}^h$ into ∂B_1^*. To show that it is onto we let $w \in \partial B_1^*$. As B_1^* is the disjoint union of its relative open faces (see [52, Theorem 18.2]), there exist $J \subseteq \{1, \ldots, r\}$, extreme points $p(\xi_j^*)$ of B_1^*, and $0 < \lambda_j \leq 1$ for $j \in J$ with $\sum_{j\in J} \lambda_j = 1$ such that $w = \sum_{j\in J} \lambda_j p(\xi_j^*)$. Let $\mu_j = -\log \lambda_j$ and $\mu^* = \min_{j\in J} \mu_j$. Now set $\alpha_j = \mu_j - \mu^*$ for $j \in J$. Then $\alpha_j \geq 0$ for $j \in J$ and $\min_{j\in J} \alpha_j = 0$.

Let $h \in \partial\overline{B}^h$ be given by $h(z) = \max_{j\in J}(h_{\xi_j}(z_j) - \alpha_j)$. Then

$$\varphi_B(h) = \frac{\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\alpha_j}} = \frac{\sum_{j\in J} e^{-\mu_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\mu_j}} = \frac{\sum_{j\in J} \lambda_j p(\xi_j^*)}{\sum_{j\in J} \lambda_j} = w.$$

To prove injectivity let $h, h' \in \partial\overline{B}^h$, where h is as in (3.2) and

$$h'(z) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j) \tag{3.4}$$

for $z \in B$. Suppose that $\varphi_B(h) = \varphi_B(h')$, so

$$\varphi_B(h) = \frac{\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\alpha_j}} = \frac{\sum_{j\in J'} e^{-\beta_j} p(\eta_j^*)}{\sum_{j\in J'} e^{-\beta_j}} = \varphi_B(h').$$

We have that $J = J'$. Indeed, if $k \in J$ and $k \notin J'$, then

$$0 = \mathrm{Re}\langle p(\xi_k), \varphi_B(h')\rangle = \mathrm{Re}\langle p(\xi_k), \varphi_B(h)\rangle > 0,$$

which is impossible. For the other case a contradiction can be derived in the same way.

Now suppose there exists $k \in J$ such that $\xi_k \neq \eta_k$. If

$$\frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}} \leq \frac{e^{-\beta_k}}{\sum_{j\in J} e^{-\beta_j}},$$

then

$$\begin{aligned}\mathrm{Re}\langle p(\eta_k), \varphi_B(h)\rangle &= \frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}}\mathrm{Re}\langle \eta_k, \xi_k^*\rangle < \frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}} \\ &\leq \frac{e^{-\beta_k}}{\sum_{j\in J} e^{-\beta_j}} = \mathrm{Re}\langle p(\eta_k), \varphi_B(h')\rangle,\end{aligned}$$

as $\overline{B_k}$ is smooth and strictly convex, which contradicts $\varphi_B(h) = \varphi_B(h')$. The other case goes in the same way. Thus, $J = J'$ and $\xi_j = \eta_j$ for all $j \in J$.

It follows that

$$\frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}} = \mathrm{Re}\langle p(\xi_k), \varphi_B(h)\rangle = \mathrm{Re}\langle p(\eta_k), \varphi_B(h')\rangle = \frac{e^{-\beta_k}}{\sum_{j\in J} e^{-\beta_j}}$$

for all $k \in J$. To show that $\alpha_k = \beta_k$ for all $k \in J$ let $\nu\colon \mathbb{R}^J \to \mathbb{R}$ be given by $\nu(x) = \sum_{j\in J} e^{-x_j}$. Then for $x, y \in \mathbb{R}^J$ and $0 < t < 1$ we have that

$$\nu(tx + (1-t)y) \leq \nu(x)^t \nu(y)^{1-t},$$

and we have equality if and only if there exists a constant c such that $x_k = y_k + c$ for all $k \in J$. So, if $x \neq y + (c, \ldots, c)$ for all c, then $-\nabla \log \nu(x) \neq -\nabla \log \nu(y)$.

As $\min_{j\in J} \alpha_j = 0 = \min_{j\in J} \beta_j$, we can conclude that $\alpha_k = \beta_k$ for all $k \in J$. This shows that $h = h'$ and hence φ_B is injective on $\partial\overline{B}^h$.

To complete the proof, note that $\varphi_B(h)$ is in the relative open boundary face $F(\{\xi_j \in \partial B_j : j \in J\})$ of B_1^*. Moreover, h' given by (3.4) is in the same part as h if, and only if, $J = J'$ and $\xi_j = \eta_j$ for all $j \in J$ by [40, Propositions 2.8 and 2.9]. So, $\varphi_B(h')$ lies in $F(\{\xi_j \in \partial B_j : j \in J\})$ if and only if h' lies in the same part as h. $\square$

3.3 Continuity and the Proof of Theorem 3.4

We now show that φ_B is continuous on $\overline{B}^h$.

Proposition 3.9 *The map* $\varphi_B : \overline{B}^h \to B_1^*$ *is continuous.*

Proof Clearly φ_B is continuous on B. Suppose that (z^n) is sequence in B converging to $h \in \partial\overline{B}^h$, where h is given by (3.2). To show that $\varphi_B(z^n) \to \varphi_B(h)$ we show that every subsequence of $(\varphi_B(z^n))$ has a subsequence converging to $\varphi_B(h)$. So, let $(\varphi_B(z^{n_k}))$ be a subsequence. We can take a further subsequence $(z^{n_{k,m}})$ such that

(1)

$$\beta_j^m = k_B(z^{n_{k,m}}, 0) - k_j(z_j^{n_{k,m}}, 0) \to \beta_j \in [0, \infty] \quad \text{for all } j \in \{1, \dots, r\}.$$

(2) There exists j_0 such that $\beta_{j_0}^m = 0$ for all $m \geq 1$.
(3) $(z_j^{n_{k,m}})$ converges to $\eta_j \in \overline{B_j}$ and $h_{z_j^{n_{k,m}}} \to h_{\eta_j}$ for all $j \in \{1, \dots, r\}$.

Let $J' = \{j : \beta_j < \infty\}$. Then $h_{z^{n_{k,m}}} \to h'$, where $h'(z) = \max_{j \in J'}(h_{\eta_j}(z_j) - \beta_j)$ for $z \in B$, as

$$\begin{aligned}
&\lim_{m\to\infty} k_B(z, z^{n_{k,m}}) - k_B(z^{n_{k,m}}, 0) \\
&\quad = \lim_{m\to\infty} \max_j (k_j(z_j, z_j^{n_{k,m}}) - k_j(z_j^{n_{k,m}}, 0) - \beta_j^m) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j),
\end{aligned}$$

by the product property of k_B.

As $h = h'$, we know by [40, Propositions 2.8 and 2.9] that $J = J'$, $\xi_j = \eta_j$ and $\alpha_j = \beta_j$ for all $j \in J$. We also know by Lemma 2.1 that $k_B(z^{n_{k,m}}, 0) \to \infty$, as h is a horofunction. So,

$$\begin{aligned}
\varphi_B(z^{n_{k,m}}) &= \frac{\sum_{i=1}^r (e^{-\beta_i^m} - e^{-k_B(z^{n_{k,m}},0) - k_i(z_i^{n_{k,m}},0)}) p((z_i^{n_{k,m}})^*)}{\sum_{i=1}^r e^{-\beta_i^m} - e^{-k_B(z^{n_{k,m}},0) - k_i(z_i^{n_{k,m}},0)}} \\
&\to \frac{\sum_{j\in J} e^{-\beta_j} p(\eta_j^*)}{\sum_{j\in J} e^{-\beta_j}} = \varphi_B(h),
\end{aligned}$$

which shows that $\varphi_B(z^n) \to \varphi_B(h)$.

We know from Lemma 3.5 that $\varphi_B(B) \subseteq \operatorname{int} B_1^*$ and $\varphi_B(\partial\overline{B}^h) \subseteq \partial B_1^*$. So, to complete the proof it remains to show that if (h_n) in $\partial\overline{B}^h$ converges to $h \in \partial\overline{B}^h$, where h is as in (3.2), then $\varphi_B(h_n) \to \varphi_B(h)$. For $n \geq 1$ let h_n be given by

$$h_n(z) = \max_{j\in J_n}(h_{\eta_j^n}(z_j) - \beta_j^n) \qquad \text{for } z \in B.$$

Again we show that every subsequence of $(\varphi_B(h_n))$ has a convergent subsequence with limit $\varphi_B(h)$.

Let $(\varphi_B(h_{n_k}))$ be a subsequence. Taking a further subsequence we may assume that

(1) There exists $J_0 \subseteq \{1, \ldots, r\}$ such that $J_{n_k} = J_0$ for all k.
(2) There exists $j_0 \in J_0$ such that $\beta_{j_0}^{n_k} = 0$ for all k.
(3) $\beta_j^{n_k} \to \beta_j \in [0, \infty]$ for all $j \in J_0$.
(4) $\eta_j^{n_k} \to \eta_j$ for all $j \in J_0$.

Note that for each $j \in J_0$ we have that $h_{\eta_j^{n_k}} \to h_{\eta_j}$ in $\overline{B}_j^h$, as the identity map on $\overline{B_j}$, that is $\xi_j \in \overline{B_j} \to h_{\xi_j} \in \overline{B}_j^h$, is a homeomorphism by [5, Theorem 1.2].

Let $J' = \{j \in J_0 \colon \beta_j < \infty\}$ and note that $j_0 \in J'$. Then for each $z \in B$ we have that

$$\begin{aligned}\lim_{m\to\infty} h_{n_k}(z) &= \lim_{k\to\infty}\max_{j\in J_0}(h_{\eta_j^{n_k}}(z_j) - \beta_j^{n_k}) = \lim_{k\to\infty}\max_{j\in J'}(h_{\eta_j^{n_k}}(z_j) - \beta_j^{n_k})\\ &= \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j).\end{aligned}$$

So, if we let $h'(z) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j)$ for $z \in B$, then h' is a horofunction by Theorem 3.1 and $h_{n_k} \to h'$ in $\overline{B}^h$. As $h_n \to h$, we conclude that $h' = h$. This implies that $J' = J$ and $\eta_j = \xi_j$ and $\beta_j = \alpha_j$ for all $j \in J$, as otherwise $\delta(h, h') \neq 0$ by [40, Proposition 2.9 and Lemma 3.3]. This implies that $\beta_j^{k_m} \to \alpha_j$ and $\eta_j^{k_m} \to \xi_j$ for all $j \in J'$. Moreover, by definition $\beta_j^{n_k} \to \infty$ for all $j \in J_0 \backslash J'$. Thus,

$$\varphi_B(h_{n_k}) = \frac{\sum_{j\in J_0} e^{-\beta_j^{n_k}} p((\eta_j^{n_k})^*)}{\sum_{j\in J_0} e^{-\beta_j^{n_k}}} \to \frac{\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\alpha_j}} = \varphi_B(h),$$

which completes the proof. □

The proof of Theorem 3.4 is now straightforward.

Proof of Theorem 3.4 It follows from Lemmas 3.7 and 3.8 and Proposition 3.9 that $\varphi_B \colon \overline{B}^h \to B_1^*$ is a continuous bijection. As $\overline{B}^h$ is compact and B_1^* is Hausdorff, we conclude that φ_B is a homeomorphism. Moreover, φ_B maps each part of $\partial\overline{B}^h$ onto the relative interior of a boundary face of B_1^* by Lemma 3.8. □

4 Euclidean Jordan Algebras with Spectral Norm

Every finite dimensional normed space $(V, \|\cdot\|)$ has a Finsler structure. Indeed, if we let

$$L(\gamma) = \int_0^1 \|\gamma'(t)\| \mathrm{d}t$$

be the length of a piecewise C^1-smooth path $\gamma\colon [0,1] \to V$, then

$$\|x - y\| = \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all C^1-smooth paths $\gamma\colon [0,1] \to V$ with $\gamma(0) = x$ and $\gamma(1) = y$. So, for normed spaces V the unit ball in the tangent space $T_b V$ is the same for all $b \in V$.

In this section we analyse the problem posed by Kapovich and Leeb [34, Question 6.18] concerning the existence of a natural homeomorphism between the horofunction compactification of a finite dimensional normed space V and the closed dual unit ball of V in the setting of Euclidean Jordan algebras equipped with the spectral norm. So we consider the Euclidean Jordan algebra not as inner-product space, but as an order-unit space, which makes it a finite dimensional (formally real) JB-algebra, see [4, Theorem 1.11]. We will give an explicit description of the horofunctions of these normed spaces and identify the parts and the detour distance. In our analysis we make frequent use of the theory of Jordan algebras and order-unit spaces. For the reader's convenience we will recall some of the basic concepts. Throughout the paper we will follow the terminology used in [3, 4, 25].

4.1 Preliminaries

Order-unit spaces A *cone* V_+ in a real vector space V is a convex subset of V with $\lambda V_+ \subseteq V_+$ for all $\lambda \geq 0$ and $V_+ \cap -V_+ = \{0\}$. The cone V_+ induces a partial ordering $\leq$ on V by $x \leq y$ if $y - x \in V_+$. We write $x < y$ if $x \leq y$ and $x \neq y$. The cone V_+ is said to be *Archimedean* if for each $x \in V$ and $y \in V_+$ with $nx \leq y$ for all $n \geq 1$ we have that $x \leq 0$. An element u of V_+ is called an *order-unit* if for each $x \in V$ there exists $\lambda \geq 0$ such that $-\lambda u \leq x \leq \lambda u$. The triple (V, V_+, u), where V_+ is an Archimedean cone and u is an order-unit, is called an *order-unit space*. An order-unit space admits a norm

$$\|x\|_u = \inf\{\lambda \geq 0\colon -\lambda u \leq x \leq \lambda u\},$$

which is called the *order-unit norm*, and we have that $-\|x\|_u u \leq x \leq \|x\|_u u$ for all $x \in V$. The cone V_+ is closed under the order-unit norm and $u \in \operatorname{int} V_+$.

A linear functional φ on an order-unit space is said to be *positive* if $\varphi(x) \geq 0$ for all $x \in V_+$. It is called a *state* if it is positive and $\varphi(u) = 1$. The set of all states is denoted by $S(V)$ and is called the *state space*, which is a convex set. In our case, the order-unit space is finite dimensional, hence $S(V)$ is compact. The extreme points of $S(V)$ are called the *pure states*.

The dual space V^* of an order-unit space V is a *base norm space*, see [3, Theorem 1.19]. More specifically, V^* is an ordered normed vector space with cone $V_+^* = \{\varphi \in V^* \colon \varphi \text{ is positive}\}$, $V_+^* - V_+^* = V^*$, and the unit ball of the norm of V^* is given by

$$B_1^* = \operatorname{conv}(S(V) \cup -S(V)).$$

Jordan algebras Important examples of order-unit spaces come from Jordan algebras. A *Jordan algebra* (over $\mathbb{R}$) is a real vector space V equipped with a commutative bilinear product $\bullet$ that satisfies the identity

$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x \quad \text{for all } x, y \in V.$$

A basic example is the space $\operatorname{Herm}(n, \mathbb{C})$ consisting of $n \times n$ Hermitian matrices with Jordan product $A \bullet B = (AB + BA)/2$.

Throughout the paper we will assume that V has a *unit*, denoted u. For $x \in V$ we let L_x be the linear map on V given by $L_x y = x \bullet y$. A finite dimensional Jordan algebra is said to be *Euclidean* if there exists an inner-product $(\cdot|\cdot)$ on V such that

$$(L_x y|z) = (y|L_x z) \quad \text{for all } x, y, z \in V.$$

A Euclidean Jordan algebra has a cone $V_+ = \{x^2 \colon x \in V\}$. The interior of V_+ is a *symmetric cone*, i.e., it is self-dual and $\operatorname{Aut}(V_+) = \{A \in \operatorname{GL}(V) \colon A(V_+) = V_+\}$ acts transitively on the interior of V_+. In fact, the Euclidean Jordan algebras are in one-to-one correspondence with the symmetric cones by the Koecher-Vinberg theorem, see for example [25].

The algebraic unit u of a Euclidean Jordan algebra is an order-unit for the cone V_+, so the triple (V, V_+, u) is an order-unit space. We will consider the Euclidean Jordan algebras as an order-unit space equipped with the order-unit norm. These are precisely the finite dimensional formally real JB-algebras, see [4, Theorem 1.11]. In the analysis, however, the inner-product structure on V will be exploited to identify V^* with V.

Throughout we will fix the rank of the Euclidean Jordan algebra V to be r. In a Euclidean Jordan algebra each x can be written in a unique way as $x = x^+ - x^-$, where x^+ and x^- are orthogonal element x^+ and x^- in V_+, see [4, Proposition 1.28]. This is called the *orthogonal decomposition of* x.

Given x in a Euclidean Jordan algebra V, the *spectrum* of x is given by $\sigma(x) = \{\lambda \in \mathbb{R} \colon \lambda u - x \text{ is not invertible}\}$, and we have that $V_+ = \{x \in V : \sigma(x) \subset [0, \infty)\}$. We write $\Lambda(x) = \inf\{\lambda \colon x \leq \lambda u\}$ and note that $\Lambda(x) = \max\{\lambda \colon \lambda \in \sigma(x)\}$, so that

$$\|x\|_u = \max\{\Lambda(x), \Lambda(-x)\} = \max\{|\lambda| \colon \lambda \in \sigma(x)\}$$

for all $x \in V$. We also note that

$$\Lambda(x + \mu u) = \Lambda(x) + \mu$$

for all $x \in V$ and $\mu \in \mathbb{R}$. Moreover, if $x \leq y$, then $\Lambda(x) \leq \Lambda(y)$.

Recall that $p \in V$ is an *idempotent* if $p^2 = p$. If, in addition, p is non-zero and cannot be written as the sum of two non-zero idempotents, then it is said to be a *primitive* idempotent. The set of all primitive idempotent is denoted $\mathcal{J}_1(V)$ and is known to be a compact set [30]. Two idempotents p and q are said to be orthogonal if $p \bullet q = 0$, which is equivalent to $(p|q) = 0$. According to the spectral theorem [25, Theorem III.1.2], each x has a *spectral decomposition*, $x = \sum_{i=1}^{r} \lambda_i p_i$, where each p_i is a primitive idempotent, the λ_i's are the eigenvalues of x (including multiplicities), and $p_1, \ldots, p_r$ is a Jordan frame, i.e., the p_i's are mutually orthogonal and $p_1 + \cdots + p_r = u$.

Throughout the paper we will fix the inner-product on V to be

$$(x|y) = \mathrm{tr}(x \bullet y),$$

where $\mathrm{tr}(x) = \sum_{i=1}^{r} \lambda_i$ and $x = \sum_{i=1}^{r} \lambda_i p_i$ is the spectral decomposition of x.

For $x \in V$ we denote the *quadratic representation* by $U_x \colon V \to V$, which is the linear map,

$$U_x y = 2x \bullet (x \bullet y) - x^2 \bullet y = 2L_x(L_x y) - L_{x^2} y.$$

In case of a Euclidean Jordan algebra U_x is self-adjoint, i.e. $(U_x y|z) = (y|U_x z)$.

We identify V^* with V using the inner-product. So, $S(V) = \{w \in V_+ \colon (u|w) = 1\}$, which is a compact convex set, as V is finite dimensional. Moreover, the extreme points of $S(V)$ are the primitive idempotents, see [25, Proposition IV.3.2]. The dual space $(V, \|\cdot\|_u^*)$ is a base norm space with norm,

$$\|z\|_u^* = \sup\{(x|z) \colon x \in V \text{ with } \|x\|_u = 1\}.$$

If V is a Euclidean Jordan algebra, it is known that the (closed) boundary faces of the dual ball $B_1^* = \mathrm{conv}(S(V) \cup -S(V))$ are precisely the sets of the form,

$$\mathrm{conv}\left((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V))\right), \tag{4.1}$$

where p and q are orthogonal idempotents not both zero, see [18, Theorem 4.4].

4.2 Summary of Results

To conveniently describe the horofunction compactification $\overline{V}^h$ of $(V, \|\cdot\|_u)$, where V is a Euclidean Jordan algebra, we need some additional notation. Throughout this section we will fix the basepoint $b \in V$ to be 0.

Let $p_1, \ldots, p_r$ be a Jordan frame in V. Given $I \subseteq \{1, \ldots, r\}$ nonempty, we write $p_I = \sum_{i \in I} p_i$ and we let $V(p_I) = U_{p_I}(V)$. For convenience we set $p_\emptyset = 0$, so $V(p_\emptyset) = U_0(V) = \{0\}$.

Recall that $V(p_I)$ is the Peirce 1-space of the idempotent p_I:

$$V(p_I) = \{x \in V \colon p_I \bullet x = x\},$$

which is a subalgebra, see [25, Theorem IV.1.1]. Given $z \in V(p_I)$, we write $\Lambda_{V(p_I)}(z)$ to denote the maximal eigenvalue of z in the subalgebra $V(p_I)$.

The following theorem characterises the horofunctions in $\overline{V}^h$.

Theorem 4.1 *Let $p_1, \dots, p_r$ be a Jordan frame, $I, J \subseteq \{1, \dots, r\}$, with $I \cap J = \emptyset$ and $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I \cup J}$ such that* $\min\{\alpha_i : i \in I \cup J\} = 0$. *The function $h \colon V \to \mathbb{R}$, given by*

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i \in I}\alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j \in J}\alpha_j p_j\right)\right\} \quad \text{for } x \in V, \tag{4.2}$$

is a horofunction, where we use the convention that if I or J is empty, the corresponding term is omitted from the maximum. Each horofunction in $\overline{V}^h$ is of the form (4.2) and a Busemann point.

To conveniently describe the parts and the detour distance (2.2) we introduce the following notation. Given orthogonal idempotents p_I and p_J we let $V(p_I, p_J) = V(p_I) + V(p_J)$, which is a subalgebra of V with unit $p_{IJ} = p_I + p_J$. The subspace $V(p_I, p_J)$ can be equipped with the *variation norm*,

$$\|x\|_{\mathrm{var}} = \Lambda_{V(p_I,p_J)}(x) + \Lambda_{V(p_I,p_J)}(-x) = \operatorname{diam} \sigma_{V(p_I,p_J)}(x),$$

which is a semi-norm on $V(p_I, p_J)$. The variation norm is, however, a norm on the quotient space $V(p_I, p_J)/\mathbb{R}p_{IJ}$.

Theorem 4.2 *Given horofunctions h and h', where*

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i \in I}\alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j \in J}\alpha_j p_j\right)\right\} \tag{4.3}$$

and

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i \in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j \in J'}\beta_j q_j\right)\right\}, \tag{4.4}$$

we have that

(i) *h and h' are in the same part if and only if $p_I = q_{I'}$ and $p_J = q_{J'}$.*
(ii) *If h and h' are in the same part, then $\delta(h, h') = \|a - b\|_{\mathrm{var}}$, where $a = \sum_{i \in I}\alpha_i p_i + \sum_{j \in J}\alpha_j p_j$ and $b = \sum_{i \in I'}\beta_i q_i + \sum_{j \in J'}\beta_j q_j$ in $V(p_I, p_J)$.*

(iii) *The part* $(\mathcal{P}_h, \delta)$ *is isometric to* $(V(p_I, p_J)/\mathbb{R}p_{IJ}, \|\cdot\|_{\mathrm{var}})$.

Remark 4.3 A basic example is $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|z\|_\infty = \max_i |z_i|$, which is an associative Euclidean Jordan algebra. In that case every horofunction is a Busemann points and of the form

$$h(x) = \max\{\max_{i\in I}(-x_i - \alpha_i), \max_{j\in J}(x_j - \alpha_i)\},$$

where $I, J \subseteq \{1, \ldots, n\}$ are disjoint, $I \cup J$ is nonempty and $\alpha \in \mathbb{R}^{I\cup J}$ with $\min_{k\in I\cup J} \alpha_k = 0$, (see [22, Theorem 5.2] and [40]). Moreover, $(\mathcal{P}_h, \delta)$ is isometric to $(\mathbb{R}^{I\cup J}/\mathbb{R}\mathbf{1}, \|\cdot\|_{\mathrm{var}})$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{I\cup J}$.

We will show that the following map is a homeomorphism from $\overline{V}^h$ onto B_1^*. Let $\varphi\colon \overline{V}^h \to B_1^*$ be given by

$$\varphi(x) = \frac{e^x - e^{-x}}{(e^x + e^{-x}|u)} = \frac{1}{\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i}} \left(\sum_{i=1}^r (e^{\lambda_i} - e^{-\lambda_i}) p_i\right) \tag{4.5}$$

for $x = \sum_{i=1}^r \lambda_i p_i \in V$, and

$$\varphi(h) = \frac{1}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} \left(\sum_{i\in I} e^{-\alpha_i} p_i - \sum_{j\in J} e^{-\alpha_j} p_j\right) \tag{4.6}$$

for $h \in \partial\overline{V}^h$ given by (4.2).

We should note that φ is well defined. To verify this assume that the horofunction h given by (4.2) is represented as

$$h(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'} \beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'} \beta_j q_j\right)\right\}$$

for $x \in V$. Then it follows from Theorem 4.2 that $p_I = q_{I'}$ and $p_J = q_{J'}$. Moreover, as $\delta(h, h) = 0$, we have that $a = \sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j = \sum_{i\in I'} \beta_i q_i + \sum_{j\in J'} \beta_j q_j = b$, as $\min\{\alpha_i\colon I \cup J\} = 0 = \min\{\beta_i\colon I \cup J\}$. This implies that $U_{p_I}a = U_{q_{I'}}b$ and $U_{p_J}a = U_{q_{J'}}b$, so that

$$\sum_{i\in I} \alpha_i p_i = \sum_{i\in I'} \beta_i q_i \quad \text{and} \quad \sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j.$$

Using the map $v \in V \mapsto e^{-v}$ we deduce that $\sum_{i\in I} e^{-\alpha_i} p_i + (u - p_I) = \sum_{i\in I'} e^{-\beta_i} q_i + (u - q_{I'})$, and hence $\sum_{i\in I} e^{-\alpha_i} p_i = \sum_{i\in I'} e^{-\beta_i} q_i$. Likewise $\sum_{j\in J} e^{-\alpha_j} p_j = \sum_{j\in J'} e^{-\beta_j} q_j$. We also find that

$$\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j} = \left(\sum_{i\in I} e^{-\alpha_i} p_i + \sum_{j\in J} e^{-\alpha_j} p_j \,\middle|\, u \right)$$
$$= \left(\sum_{i\in I'} e^{-\beta_i} q_i + \sum_{j\in J'} e^{-\beta_j} q_j \,\middle|\, u \right) = \sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j},$$

so $\varphi(h)$ is well defined.

We will also show that φ maps each part of the horofunction boundary onto the relative interior of a boundary face of the dual unit ball. Recall that the relative interior of a face F of B_1^* is the interior of F as a subset of the affine span of F.

Theorem 4.4 *Given a Euclidean Jordan algebra $(V, \|\cdot\|_u)$, the map $\varphi\colon \overline{V}^h \to B_1^*$ is a homeomorphism. Moreover, the part $\mathcal{P}_h$, with h given by (4.2), is mapped onto the relative interior of the closed boundary face*

$$\operatorname{conv}\,(U_{p_I}(V) \cap S(V)) \cup (U_{p_J}(V) \cap -S(V))).$$

4.3 Horofunctions

In this subsection we will prove Theorem 4.1. We first make some preliminary observations. Note that $x \le \lambda u$ if and only if $0 \le \lambda u - x$, which by the Hahn–Banach separation theorem is equivalent to $(\lambda u - x|w) \ge 0$ for all $w \in S(V)$. As the state space is compact, we have for each $x \in V$ that

$$\Lambda(x) = \max_{w\in S(V)} (x|w). \tag{4.7}$$

As $\|\cdot\|_u$ is the JB-algebra norm, $\|x \bullet y\|_u \le \|x\|_u\|y\|_u$, see [4, Theorem 1.11]. It follows that if $x^n \to x$ and $y^n \to y$ in $(V, \|\cdot\|_u)$, then $x^n \bullet y^n \to x \bullet y$. Thus, we have the following lemma.

Lemma 4.5 *If $x^n \to x$ and $y^n \to y$ in $(V, \|\cdot\|_u)$, then $U_{x^n} y^n \to U_x y$.*

We will also use the following technical lemma several times.

Lemma 4.6 *For $n \ge 1$, let $p_1^n, \dots, p_r^n$ be a Jordan frame in V and $I \subseteq \{1, \dots, r\}$ nonempty. Suppose that*

(i) *$p_i^n \to p_i$ for all $i \in I$.*
(ii) *$x^n \in V(p_I^n)$ with $x^n \to x \in V(p_I)$.*
(iii) *$\beta_i^n \ge 0$ with $\beta_i^n \to \beta_i \in [0, \infty]$ for all $i \in I$.*

If $I' = \{i \in I : \beta_i < \infty\}$ is nonempty, then

$$\lim_{n\to\infty} \Lambda_{V(p_I^n)} \left(x^n - \sum_{i\in I} \beta_i^n p_i^n \right) = \Lambda_{V(p_{I'})} \left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i \right).$$

Proof We will show that every subsequence of $(\Lambda_{V(p_I^n)}(x^n - \sum_{i\in I}\beta_i^n p_i^n))$ has a convergent subsequence with limit $\Lambda_{V(p_{I'})}(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i)$. So let $(\Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}))$ be a subsequence. By (4.7) there exists $d^{n_k} \in S(V(p_I^{n_k}))$ with

$$\Lambda_{V(p_I^{n_k})}\left(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}\right) = \left(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}\,\middle|\, d^{n_k}\right).$$

By taking subsequences we may assume that $d^{n_k} \to d \in S(V(p_I))$.

Using the Peirce decomposition with respect to the Jordan frame $p_i^{n_k}$, $i \in I$, in $V(p_I^{n_k})$, we can write

$$d^{n_k} = \sum_{i\in I}\mu_i^{n_k} p_i^{n_k} + \sum_{i<j\in I} d_{ij}^{n_k}.$$

Note that as $d^{n_k} \geq 0$, we have that $\mu_i^{n_k} = (d^{n_k}|p_i^{n_k}) \geq 0$ for all $i \in I$.

We claim that for each $i \in I \setminus I'$ we have that $\mu_i^{n_k} \to 0$. Indeed, as I' is nonempty, there exist $l \in I'$ and a constant $C > 0$ such that

$$\begin{aligned}\left(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}\,\middle|\, d^{n_k}\right) &\geq \left(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}\,\middle|\, p_l^{n_k}\right)\\ &= \left(x^{n_k}\,\middle|\, p_l^{n_k}\right) - \beta_l^{n_k} \geq -\|x^{n_k}\|_u - \beta_l^{n_k} > -C\end{aligned}$$

for all k, since $(x^{n_k}|p_l^{n_k}) \leq \|x^{n_k}\|_u$. Moreover,

$$\begin{aligned}\left(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}\,\middle|\, d^{n_k}\right) &= \left(x^{n_k}|d^{m_k}\right) - \sum_{i\in I}\beta_i^{n_k}\mu_i^{n_k}\\ &\leq \|x^{n_k}\|_u - \sum_{i\in I'}\beta_i^{n_k}\mu_i^{n_k} - \sum_{i\in I\setminus I'}\beta_i^{n_k}\mu_i^{n_k}.\end{aligned}$$

As $\beta_i^{n_k}, \mu_i^{n_k} \geq 0$ for all $i \in I$ and $\beta_i^{n_k} \to \infty$ for all $i \in I\setminus I'$, we conclude from the previous two inequalities that $\mu_i^{n_k} \to 0$ for all $i \in I \setminus I'$.

Using the Peirce decomposition with respect to the Jordan frame p_i, $i \in I$, we write

$$d = \sum_{i\in I}\mu_i p_i + \sum_{i<j\in I} d_{ij}.$$

We now show that

$$d = \sum_{i\in I'}\mu_i p_i + \sum_{i<j\in I'} d_{ij}, \tag{4.8}$$

and hence $d \in V(p_{I'})$. Note that

$$\mu_i - \mu_i^{n_k} = (d|p_i) - (d^{n_k}|p_i^{n_k}) = (d - d^{n_k}|p_i) + (d^{n_k}|p_i - p_i^{n_k}) \to 0.$$

We conclude that $\mu_i^{n_k} \to \mu_i$ for all $i \in I$, and hence $(d|p_j) = \mu_j = 0$ for all $j \in I\backslash I'$. This implies by [25, III, Exercise 3] that $d \bullet p_j = 0$ for all $j \in I\backslash I'$. So,

$$0 = d \bullet p_j = \frac{1}{2}\left(\sum_{l<j} d_{lj} + \sum_{j<m} d_{jm}\right),$$

which shows that $d_{lj} = 0 = d_{jm}$ for all $l < j < m$, as they are all orthogonal. This gives (4.8).

Next we show that $\lim_{k\to\infty} \Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k}) = (U_{p_{I'}}x - \sum_{i\in I'} \beta_i p_i|d)$. First note that

$$\begin{aligned}
\Lambda_{V\left(p_I^{n_k}\right)}\left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k}\right) &= \left(x^{n_k} - \sum_{i\in I'} \beta_i^{n_k} p_i^{n_k} \,\middle|\, d^{n_k}\right) - \sum_{i\in I\backslash I'} \left(\beta_i^{n_k} p_i^{n_k}|d^{n_k}\right) \\
&= \left(x^{n_k} - \sum_{i\in I'} \beta_i^{n_k} p_i^{n_k} \,\middle|\, d^{n_k}\right) - \sum_{i\in I\backslash I'} \beta_i^{n_k} \mu_i^{n_k} \\
&\leq \left(x^{n_k} - \sum_{i\in I'} \beta_i^{n_k} p_i^{n_k} \,\middle|\, d^{n_k}\right)
\end{aligned}$$

as $\beta_i^{n_k}, \mu_i^{n_k} \geq 0$ for all i and k. This implies that

$$\begin{aligned}
\limsup_{k\to\infty} \Lambda_{V\left(p_I^{n_k}\right)}\left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k}\right) &\leq \lim_{k\to\infty}\left(x^{n_k} - \sum_{i\in I'} \beta_i^{n_k} p_i^{n_k} \,\middle|\, d^{n_k}\right) \\
&= \left(x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right)
\end{aligned}$$

As $U_{p_{I'}}d = d$ and $U_{p_{I'}}$ is self-adjoint, we find that

$$\left(x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right) = \left(x - \sum_{i\in I'} \beta_i p_i \,\middle|\, U_{p_{I'}}d\right) = \left(U_{p_{I'}}x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right),$$

so that

$$\limsup_{k\to\infty} \Lambda_{V\left(p_I^{n_k}\right)}\left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k}\right) \leq \left(U_{p_{I'}}x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right). \tag{4.9}$$

Now let $p_{I'}^{n_k} = \sum_{i\in I'} p_i^{n_k}$. As $p_{I'}^{n_k} \to p_{I'}$, it follows from Lemma 4.5 that $U_{p_{I'}^{n_k}} d \to U_{p_{I'}} d = d$. This implies that

$$\left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k} \,\middle|\, U_{p_{I'}^{n_k}} d\right) \left(U_{p_{I'}^{n_k}} d | p_I^{n_k}\right)^{-1} \leq \Lambda_{V\left(p_I^{n_k}\right)} \left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k}\right)$$

for all k large, as $(U_{p_{I'}^{n_k}} d | p_I^{n_k}) \to (U_{p_{I'}} d | p_I) = (d | U_{p_{I'}} p_I) = (d | p_{I'}) = (d | p_I) = 1$. Moreover,

$$\begin{aligned}
&\lim_{k\to\infty} \left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k} \,\middle|\, U_{p_{I'}^{n_k}} d\right) \left(U_{p_{I'}^{n_k}} d | p_I^{n_k}\right)^{-1} \\
&\quad = \lim_{k\to\infty} \left(U_{p_{I'}^{n_k}} x^{n_k} - \sum_{i\in I'} \beta_i^{n_k} p_i^{n_k} \,\middle|\, d\right) \left(U_{p_{I'}^{n_k}} d | p_I^{n_k}\right)^{-1} \\
&\quad = \left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right).
\end{aligned}$$

This shows that $(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i | d) \leq \liminf_{k\to\infty} \Lambda_{V(p_I^{n_k})} (x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k})$. From (4.9) we conclude that

$$\left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right) = \lim_{k\to\infty} \Lambda_{V\left(p_I^{n_k}\right)} \left(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k}\right). \tag{4.10}$$

To complete the proof we show that

$$\left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right) = \Lambda_{V(p_{I'})} \left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i\right). \tag{4.11}$$

As $(d | p_{I'}) = (d | p_I) = 1$, we know that $d \in S(V_{p_{I'}})$. So, we get from (4.7) that

$$\begin{aligned}
\left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i \,\middle|\, d\right) &\leq \sup_{z\in S(V(p_{I'}))} \left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i \,\middle|\, z\right) \\
&= \Lambda_{V(p_{I'})} \left(U_{p_{I'}} x - \sum_{i\in I'} \beta_i p_i\right).
\end{aligned}$$

On the other hand, if $w \in S(V(p_{I'}))$ is such that

$$\left(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i \,\middle|\, w\right) = \sup_{z\in S(V(p_{I'}))}\left(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i \,\middle|\, z\right)$$
$$= \Lambda_{V(p_{I'})}\left(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i\right),$$

then by definition of d^{n_k} we get for all k large that

$$\left(x - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k} \,\middle|\, d^{n_k}\right) \geq \left(x - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k} \,\middle|\, U_{p_{I'}^{n_k}} w\right)\left(U_{p_{I'}^{n_k}} w \,\middle|\, p_I^{n_k}\right)^{-1}$$
$$= \left(U_{p_{I'}^{n_k}}x - \sum_{i\in I'}\beta_i^{n_k} p_i^{n_k} \,\middle|\, w\right)\left(U_{p_{I'}^{n_k}} w \,\middle|\, p_I^{n_k}\right)^{-1},$$

as $(U_{p_{I'}^{n_k}} w | p_I^{n_k}) \to (U_{p_{I'}} w | p_I) = (w | p_{I'}) = 1$. This implies that

$$\lim_{k\to\infty} \Lambda_{V\left(p_I^{n_k}\right)}\left(x^{n_k} - \sum_{i\in I}\beta_i^{n_k} p_i^{n_k}\right)$$
$$\geq \lim_{k\to\infty}\left(U_{p_{I'}^{n_k}}x - \sum_{i\in I'}\beta_i^{n_k} p_i^{n_k} \,\middle|\, w\right)\left(U_{p_{I'}^{n_k}} w \,\middle|\, p_I^{n_k}\right)^{-1} = \left(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i \,\middle|\, w\right),$$

and hence (4.11) holds by (4.10). □

To prove that all horofunctions in $\overline{V}^h$ are of the form (4.2), we first establish the following proposition using the previous lemma.

Proposition 4.7 *Let (y^n) be a sequence in V, with $y^n = \sum_{i=1}^r \lambda_i^n p_i^n$. Suppose that $h_{y^n} \to h \in \partial\overline{V}^h$ and (y^n) satisfies the following properties:*

(1) *There exists $1 \leq s \leq r$ such that $|\lambda_s^n| = r^n$ for all n, where $r^n = \|y^n\|_u$.*
(2) *$p_k^n \to p_k$ for all $1 \leq k \leq r$.*
(3) *There exist $I, J \subseteq \{1, \ldots, r\}$ disjoint with $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I\cup J}$ with $\min\{\alpha_i : i \in I \cup J\} = 0$ such that $r^n - \lambda_i^n \to \alpha_i$ for all $i \in I$, $r^n + \lambda_j^n \to \alpha_j$ for all $j \in J$, and $r^n - |\lambda_k^n| \to \infty$ for all $k \notin I \cup J$.*

Then h satisfies (4.2).

Proof Take $x \in V$ fixed. Note that for all $n \geq 1$,

$$\|x - y^n\|_u - \|y^n\|_u = \max\{\Lambda(x - y^n), \Lambda(-x + y^n)\} - r^n$$
$$= \max\{\Lambda(x - y^n - r^n u), \Lambda(-x + y^n - r^n u)\}.$$

As h is a horofunction, $\|y^n\|_u = r^n \to \infty$ by Lemma 2.1. Thus, $\lambda_i^n \to \infty$ for all $i \in I$ and $\lambda_j^n \to -\infty$ for all $j \in J$. Now suppose that J is nonempty. Then $r^n + \lambda_k^n \geq r^n - |\lambda_k^n| \to \infty$ for all $k \notin J$. As

$$\Lambda(x - y^n - r^n u) = \Lambda(x - \sum_{j\in J}(r^n + \lambda_j^n)p_j^n - \sum_{k\notin J}(r^n + \lambda_k^n)p_k^n),$$

it follows that

$$\lim_{n\to\infty} \Lambda(x - y^n - r^n u) = \Lambda_{V(p_J)}(U_{p_J} x - \sum_{j\in J}\alpha_j p_j)$$

by Lemma 4.6. Likewise, if I is nonempty, then

$$\lim_{n\to\infty} \Lambda(-x + y^n - r^n u) = \Lambda_{V(p_I)}(-U_{p_I} x - \sum_{i\in I}\alpha_i p_i)$$

by Lemma 4.6. We conclude that if I and J are both nonempty, then

$$\begin{aligned} h(x) &= \lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u \\ &= \lim_{n\to\infty} \max\{\Lambda(-x + y^n - r^n u), \Lambda(x - y^n - r^n u)\} \\ &= \max\{\Lambda_{V(p_I)}(-U_{p_I} x - \sum_{i\in I}\alpha_i p_i), \Lambda_{V(p_J)}(U_{p_J} x - \sum_{j\in J}\alpha_j p_j)\}. \end{aligned}$$

To complete the proof it remains to show that $\lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u = \lim_{n\to\infty} \Lambda(-x + y^n - r^n u)$ if J is empty, and $\lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u = \lim_{n\to\infty} \Lambda(x - y^n - r^n u)$ if I is empty. Suppose that I is empty, so J is nonempty. Then for each $i \in \{1, \dots, r\}$ we have that $r^n - \lambda_i^n \to \infty$. Note that

$$\begin{aligned} -x + y^n - r^n u = -x - \sum_i (r^n - \lambda_i^n)p_i^n &\le -x - \min_i (r^n - \lambda_i^n)u \\ &\le (\|x\|_u - \min_i (r^n - \lambda_i^n))u. \end{aligned}$$

Thus, $\Lambda(-x + y^n - r^n u) \le \Lambda((\|x\|_u - \min_i (r^n - \lambda_i^n))u) = \|x\|_u - \min_i (r^n - \lambda_i^n)$ for all n, hence $\Lambda(-x + y^n - r^n u) \to -\infty$. As

$$\max\{\Lambda(x - y^n - r^n u), \Lambda(-x + y^n - r^n u)\} = \left\|x - y^n\right\|_u - \left\|y^n\right\|_u \ge -\|x\|_u,$$

we conclude that $\|x - y^n\|_u - |y^n\|_u = \Lambda(x - y^n - r^n u)$ for all n sufficiently large, hence

$$h(x) = \lim_{n\to\infty} \Lambda(x - y^n - r^n u) = \Lambda_{V(p_J)}(U_{p_J} x - \sum_{j\in J}\alpha_j p_j).$$

The argument for the case where J is empty goes in the same way. $\square$

The following corollary shows that each horofunction is of the form (4.2).

Corollary 4.8 *If h is a horofunction in $\overline{V}^h$, then there exist a Jordan frame $p_1, \ldots, p_r$ in V, disjoint subsets $I, J \subseteq \{1, \ldots, r\}$, with $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I\cup J}$ with* $\min\{\alpha_i : i \in I \cup J\} = 0$, *such that $h\colon V \to \mathbb{R}$ satisfies (4.2) for all $x \in V$.*

Proof Suppose that (y^n) is a sequence in V with $h_{y^n} \to h$ in $\overline{V}^h$. Then for each $x \in V$ we have that

$$\lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u = h(x)$$

and $\|y^n\|_u \to \infty$ by Lemma 2.1.

To show that the limit is equal to (4.2) it suffices to show that we can take a subsequences of (y^n) that satisfies the conditions in Proposition 4.7. First we note that by the spectral theorem [25, Theorem III.1.2], there exist for each $n \geq 1$ a Jordan frame $p_1^n, \ldots, p_r^n$ in V and $\lambda_1^n, \ldots, \lambda_r^n \in \mathbb{R}$ such that

$$y^n = \lambda_1^n p_1^n + \cdots + \lambda_r^n p_r^n,$$

where r is the rank of V. Denote $r^n = \|y^n\|_u = \max_i |\lambda_i^n|$.

Now by taking subsequences we may assume that there exist $I_+ \subseteq \{1, \ldots, r\}$ and $1 \leq s \leq r$ such that for each $n \geq 1$ we have $r^n = |\lambda_s^n|$ and

$$\lambda_i^n > 0 \text{ for all } i \in I_+ \quad \text{and } \lambda_i^n \leq 0 \text{ for all } i \notin I_+.$$

Now for each $i \in \{1, \ldots, r\}$ and $n \geq 1$ define

$$\alpha_i^n = \begin{bmatrix} r^n - \lambda_i^n & \text{for } i \in I_+ \\ r^n + \lambda_i^n & \text{for } i \notin I_+. \end{bmatrix}$$

Note that $\alpha_i^n \in [0, \infty)$ for all i. Again by taking subsequences we may assume that $\alpha_i^n \to \alpha_i \in [0, \infty]$ as $n \to \infty$, for all i. Recall that $\alpha_s^n = 0$ for all n, so $\alpha_s = 0$. Furthermore, we may assume that $p_i^n \to p_i$ in $\mathcal{J}_1(V)$ for all i, as it is a compact set [30]. Note that $p_1, \ldots, p_r$ is a Jordan frame in V.

Now let

$$I = \{i : \alpha_i < \infty \text{ and } i \in I_+\} \quad \text{and} \quad J = \{j : \alpha_j < \infty \text{ and } j \notin I_+\}.$$

So, $I \cap J$ is empty, $s \in I \cup J$ and $\min\{\alpha_i : i \in I \cup J\} = \alpha_s = 0$. Then the subsequence of (y^n) satisfies the conditions in Proposition 4.7, hence h is a horofunction of the form (4.2). ◻

The next proposition shows that each function of the form (4.2) can be realised as a horofunction, and is a Busemann point.

Proposition 4.9 *Let $p_1, \ldots, p_r$ be a Jordan frame in V. Suppose that $I, J \subseteq \{1, \ldots, r\}$ with $I \cap J = \emptyset$ and $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I \cup J}$ with* $\min\{\alpha_i : i \in I \cup J\} = 0$. *If for $n \geq 1$ we let $y^n = \lambda_1^n p_1 + \cdots + \lambda_r^n p_r$, where*

$$\lambda_i^n = \begin{bmatrix} n - \alpha_i & \text{if } i \in I \\ -n + \alpha_i & \text{if } i \in J \\ 0 & \text{otherwise,} \end{bmatrix}$$

then (y^n) is an almost geodesic sequence and $h_{y^n} \to h$, where h satisfies (4.2) *for all $x \in V$. In particular, h is a Busemann point in $\overline{V}^h$.*

Proof Let $k \geq \max\{\alpha_i : i \in I \cup J\}$ and note that for $n \geq k$ we have that $r^n = \|y^n\|_u = n$, as $\min\{\alpha_i : i \in I \cup J\} = 0$. The sequence (y^n), where $n \geq k$, satisfies the conditions in Proposition 4.7. Indeed, for $n \geq k$ we have that $r^n - \lambda_i^n = \alpha_i$ for all $i \in I$, $r^n + \lambda_i^n = \alpha_i$ for all $i \in J$, and $r^n - \lambda_i^n = n$ otherwise. Also for s with $\alpha_s = 0$, we have that $|\lambda_s^n| = n = \|y^n\|_u$.

Finally to see that (h_{y^n}) converges, we note that if we define $z = \sum_{i \in I} -\alpha_i p_i + \sum_{j \in J} \alpha_j p_j$ and $w = \sum_{i \in I} p_i - \sum_{j \in J} p_j$, then $y^n = nw + z$, which lies on the straight-line $t \mapsto tw + z$. Hence (y^n) is an almost geodesic sequence, so

$$h(x) = \lim_{n \to \infty} \|x - y^n\|_u - \|y^n\|_u$$

exists for all $x \in V$. Thus, we can apply Proposition 4.7 and conclude that h satisfies (4.2), and h is a Busemann point in the horofunction boundary. $\square$

Combining the results so far we now prove Theorem 4.1.

Proof of Theorem 4.1 Corollary 4.8 shows that each horofunction in $\overline{V}^h$ is of the form (4.2). It follows from Proposition 4.9 that any function of the form (4.2) is a horofunction and by the second part of that proposition each horofunction is a Busemann point. $\square$

4.4 Parts and the Detour Metric

In this subsection we will identify the parts in the horofunction boundary of $\overline{V}^h$, derive a formula for the detour distance (2.2), and establish Theorem 4.2. We begin by proving the following proposition.

Proposition 4.10 *If*

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i \in I} \alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j \in J} \alpha_j p_j\right)\right\}, \tag{4.12}$$

and

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right)\right\} \tag{4.13}$$

are horofunctions with $p_I = q_{I'}$ *and* $p_J = q_{J'}$*, then* h *and* h' *are in the same part and*

$$\delta(h,h') = \|a-b\|_{\mathrm{var}} = \Lambda_{V(p_I,p_J)}(a-b) + \Lambda_{V(p_I,p_J)}(b-a),$$

where $a = \sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j$ *and* $b = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j$ *in* $V(p_I,p_J) = V(p_I) + V(p_J)$*.*

Proof As in Proposition 4.9, for $n \geq 1$ let $y^n = \lambda_1^n p_1 + \cdots + \lambda_r^n p_r$, where

$$\lambda_i^n = \left[\begin{array}{rl} n - \alpha_i & \text{if } i \in I \\ -n + \alpha_i & \text{if } i \in J \\ 0 & \text{otherwise,} \end{array}\right.$$

and let $w^n = \mu_1^n q_1 + \cdots + \mu_r^n q_r$, where

$$\mu_i^n = \left[\begin{array}{rl} n - \beta_i & \text{if } i \in I' \\ -n + \beta_i & \text{if } i \in J' \\ 0 & \text{otherwise.} \end{array}\right.$$

By Proposition 4.9 we know that (y^n) and (w^n) are almost geodesic sequences with $h_{y^n} \to h$ and $h_{w^n} \to h'$. Note that

$$U_{p_I}w^m = U_{q_{I'}}w^m = \sum_{i\in I'}\mu_i^m U_{q_{I'}}q_i = \sum_{i\in I'}\mu_i^m q_i$$

for all m, so

$$\begin{aligned} &\Lambda_{V(p_I)}\left(-U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u p_I\right) \\ &\quad= \Lambda_{V(p_I)}\left(-U_{q_{I'}}w^m - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u q_{I'}\right) \\ &\quad= \Lambda_{V(p_I)}\left(\sum_{i\in I'}\left(\|w^m\|_u - \mu_i^m\right) q_i - \sum_{i\in I}\alpha_i p_i\right). \end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{m\to\infty} \Lambda_{V(p_I)}\left(-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i + \|w^m\|_u p_I\right)\\
&= \lim_{m\to\infty} \Lambda_{V(p_I)}\left(\sum_{i\in I'} \left(\|w^m\|_u - \mu_i^m\right) q_i - \sum_{i\in I} \alpha_i p_i\right)\\
&= \Lambda_{V(p_I)}\left(\sum_{i\in I'} \beta_i q_i - \sum_{i\in I} \alpha_i p_i\right)\\
&= \Lambda_{V(p_I)}(b-a)\,.
\end{aligned}$$

In the same way it can be shown that

$$\begin{aligned}
&\lim_{m\to\infty} \Lambda_{V(p_J)}\left(U_{p_J} w^m - \sum_{j\in J} \alpha_j p_i + \|w^m\|_u p_J\right)\\
&= \Lambda_{V(p_J)}\left(\sum_{j\in J'} \beta_j q_j - \sum_{j\in J} \alpha_j p_j\right) = \Lambda_{V(p_J)}(b-a)\,.
\end{aligned}$$

So, it follows from (2.3) that

$$\begin{aligned}
H\left(h, h'\right) &= \lim_{m\to\infty} \|w^m\|_u\\
&\quad + \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J} w^m - \sum_{j\in J} \alpha_j p_j\right)\right\}\\
&= \lim_{m\to\infty} \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i + \|w^m\|_u p_I\right), \Lambda_{V(p_J)}\left(U_{p_J} w^m\right.\right.\\
&\quad \left.\left. - \sum_{j\in J} \alpha_j p_j + \|w^m\|_u p_J\right)\right\}\\
&= \max\left\{\Lambda_{V(p_I)}\left(\sum_{i\in I'} \beta_i q_i - \sum_{i\in I} \alpha_i p_i\right), \Lambda_{V(p_J)}\left(\sum_{j\in J'} \beta_j q_j - \sum_{j\in J} \alpha_j p_j\right)\right\}\\
&= \Lambda_{V(p_I, p_J)}(b-a)\,.
\end{aligned}$$

Interchanging the roles of h and h' gives $H(h', h) = \Lambda_{V(p_I, p_J)}(a-b)$, hence $\delta(h, h') = \|a-b\|_{\text{var}}$. □

To show that h and h' are in different part if $p_I \neq q_{I'}$ or $p_J \neq q_{J'}$, we need the following lemma.

Lemma 4.11 *If p and q are idempotents in V with $p \not\leq q$, then $U_p q < p$.*

Proof We have that $U_p q \leq U_p u = p$. In fact, $U_p q < p$. Indeed, if $U_p q = p$, then

$$p = U_p u = U_p(u-q) + U_p q = U_p(u-q) + p,$$

and hence $U_p(u-q) = 0$. This implies that $p + (u-q) \leq u$ by [29, Lemma 4.2.2], so that $p \leq q$. This is impossible, as $p \not\leq q$, and hence $U_p q < p$. □

Proposition 4.12 *If h and h' are horofunctions given by (4.12) and (4.13), respectively, and $p_I \neq q_{I'}$ or $p_J \neq q_{J'}$, then*

$$\delta(h, h') = \infty.$$

Proof Suppose that $p_I \neq q_{I'}$. Then $p_I \not\leq q_{I'}$ or $q_{I'} \not\leq p_I$. Without loss of generality assume that $p_I \not\leq q_{I'}$. Let (y^n) in $V(p_I)$ and (w^n) in $V(q_{I'})$ be as in Proposition 4.9, so $h_{y^n} \to h$ and $h_{w^m} \to h'$. To prove the statement in this case, we use (2.3) and show that

$$H(h', h) = \lim_{m\to\infty} \|w^m\|_u + h(w^m) = \infty. \tag{4.14}$$

Note that

$$\begin{aligned}\|w^m\|_u + h\left(w^m\right) &\geq \|w^m\|_u + \Lambda_{V(p_I)}\left(-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i\right)\\ &= \Lambda_{V(p_I)}\left(-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i + \|w^m\|_u p_I\right).\end{aligned}$$

As $w^m \leq \|w^m\|_u q_{I'}$ for all m, we have that $U_{p_I} w^m \leq \|w^m\|_u U_{p_I} q_{I'}$ for all m. Thus,

$$\begin{aligned}-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i + \|w^m\|_u p_I &\geq -\|w^m\|_u U_{p_I} q_{I'} - \sum_{i\in I} \alpha_i p_i + \|w^m\|_u p_I\\ &= \|w^m\|_u (p_I - U_{p_I} q_{I'}) - \sum_{i\in I} \alpha_i p_i\end{aligned}$$

for all m.

We know from Lemma 4.11 that $p_I - U_{p_I} q_{I'} > 0$. As $p_I - U_{p_I} q_{I'} \in V(p_I)$ we also have that $p_I - U_{p_I} q_{I'} = \sum_{j=1}^s \gamma_j r_j$, where $\gamma_j > 0$ for all j and the r_j's are orthogonal idempotents in $V(p_I)$. It now follows that for all m,

$$\begin{aligned}&\Lambda_{V(p_I)}\left(-U_{p_I} w^m - \sum_{i\in I} \alpha_i p_i + \|w^m\|_u p_I\right)\\ &\quad\geq \left(\|w^m\|_u \sum_{j=1}^s \gamma_j r_j - \sum_{i\in I} \alpha_i p_i \middle| r_1\right) (p_I \mid r_1)^{-1}\end{aligned}$$

$$= \left(\|w^m\|_u \gamma_1 - \left(\sum_{i \in I} \alpha_i p_i \middle| r_1 \right) \right) (p_I \mid r_1)^{-1} .$$

The right-hand side goes to ∞ as $m \to \infty$, and hence (4.14) holds.

For the case $p_J \neq q_{J'}$ a similar argument can be used. □

We now prove Theorem 4.2.

Proof Parts (i) and (ii) follow directly from Propositions 4.10 and 4.12. Clearly the map $\rho\colon \mathcal{P}_h \to V(p_I, p_J)/\mathbb{R}p_{IJ}$ given by $\rho(h') = [b]$, where

$$h'(x) = \max \left\{ \Lambda_{V(q_{I'})} \left(-U_{q_{I'}} x - \sum_{i \in I'} \beta_i q_i \right), \Lambda_{V(q_{J'})} \left(U_{q_{J'}} x - \sum_{j \in J'} \beta_j q_j \right) \right\},$$

and $b = \sum_{i \in I'} \beta_i q_i + \sum_{j \in J'} \beta_j q_j \in V(p_I, p_J)$ with $\min_{i \in I \cup J} \beta_i = 0$, is a bijection. So, by Proposition 4.10, ρ is an isometry from $(\mathcal{P}_h, \delta)$ onto $(V(p_I, p_J)/\mathbb{R}p_{IJ}, \|\cdot\|_{\text{var}})$. □

4.5 The Homeomorphism onto the Dual Unit Ball

In this subsection we prove Theorem 4.4. To start we prove a lemma that will be useful in the sequel.

Lemma 4.13 *If $q \leq p$ are idempotents in V and $z \in V(p)$, then $\Lambda_{V(q)}(U_q z) \leq \Lambda_{V(p)}(z)$.*

Proof If $\lambda = \Lambda_{V(p)}(z)$, then $0 \leq \lambda p - z$, so that $0 \leq \lambda U_q p - U_q z$. As $q = U_q q \leq U_q p \leq U_q u = q^2 = q$, we find that $0 \leq \lambda U_q p - U_q z = \lambda q - U_q z$, hence $\Lambda_{V(q)}(U_q z) \leq \lambda$. □

We will show that φ given by (4.5) and (4.6) is a continuous bijection from $\overline{V}^h$ onto B_1^*. As $\overline{V}^h$ is compact and B_1^* is Hausdorff, we can then conclude that φ is a homeomorphism. We begin by showing that φ maps V into the interior of B_1^*.

Lemma 4.14 *For each $x \in V$ we have that $\varphi(x) \in \operatorname{int} B_1^*$.*

Proof For $x \in V$ there exists $y \in V$ with $\|y\|_u = 1$, such that

$$\|\varphi(x)\|_u^* = \sup_{w \in V\colon \|w\|_u \leq 1} |(w|\varphi(x))| = (y|\varphi(x)),$$

where $(v|w) = \operatorname{tr}(v \bullet w)$. So, if x has spectral decomposition $x = \sum_{i=1}^r \lambda_i p_i$, then we can consider the Peirce decomposition of y,

$$y = \sum_{i=1}^r \mu_i p_i + \sum_{i<j} y_{ij},$$

to find that

$$\|\varphi(x)\|_u^* = (\varphi(x)\,|y) = \frac{1}{\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i}} \left(\sum_{i=1}^r \left(e^{\lambda_i} - e^{-\lambda_i}\right) p_i \,\middle|\, y \right)$$
$$\leq \frac{\sum_{i=1}^r \left(e^{\lambda_i} - e^{-\lambda_i}\right) |\mu_i|}{\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i}} < 1,$$

as $\mu_i = (y|p_i) \leq (u|p_i) = 1$ and $\mu_i = (y|p_i) \geq (-u|p_i) = -1$. □

Lemma 4.15 *The map φ is injective on V.*

Proof Suppose that $x, y \in V$ with $x = \sum_{i=1}^r \sigma_i p_i$ and $y = \sum_{i=1}^r \tau_i q_i$, where $\sigma_1 \leq \ldots \leq \sigma_r$ and $\tau_1 \leq \ldots \leq \tau_r$, satisfy $\varphi(x) = \varphi(y)$. Then $\varphi(x) = \sum_{i=1}^r \alpha_i p_i = \sum_{i=1}^r \beta_i q_i = \varphi(y)$. where

$$\alpha_j = \frac{e^{\sigma_j} - e^{-\sigma_j}}{\sum_{i=1}^r e^{\sigma_i} + e^{-\sigma_i}} \quad \text{and} \quad \beta_j = \frac{e^{\tau_j} - e^{-\tau_j}}{\sum_{i=1}^r e^{\tau_i} + e^{-\tau_i}} \qquad \text{for all } j.$$

As $\alpha_1 \leq \ldots \leq \alpha_r$ and $\beta_1 \leq \ldots \leq \beta_r$, it follows from the spectral theorem (version 2) [25, Theorem III.1.2] that $\alpha_j = \beta_j$ for all j. Lemma 3.6 now implies that $\sigma = (\sigma_1, \ldots, \sigma_r) = (\tau_1, \ldots, \tau_r) = \tau$, as

$$(\alpha_1, \ldots, \alpha_r) = \nabla \log \mu(\sigma) \quad \text{and} \quad (\beta_1, \ldots, \beta_r) = \nabla \log \mu(\tau).$$

Note that $\alpha_i = \alpha_j$ if and only if $\sigma_i = \sigma_j$, and $\beta_i = \beta_j$ if and only if $\tau_i = \tau_j$, as $\nabla \log \mu(x)$ is injective. It now follows from the spectral theorem (version 1) [25, Theorem III.1.1] that $x = y$. □

Lemma 4.16 *The map φ maps V onto* int B_1^*.

Proof As φ is continuous on V and $\varphi(V) \subseteq \operatorname{int} B_1^*$, it follows from Brouwer's domain invariance theorem that $\varphi(V)$ is open in $\operatorname{int} B_1^*$. Suppose, for the sake of contradiction, that $\varphi(V) \neq \operatorname{int} B_1^*$. So, we can find a $z \in \partial\varphi(V) \cap \operatorname{int} B_1^*$. Let (y^n) in V be such that $\varphi(y^n) \to z$ and write $y^n = \sum_{i=1}^r \lambda_i^n p_i^n$. As φ is continuous on V, we may assume that $r^n = \|y^n\|_u \to \infty$. Furthermore, after taking a subsequence, we may assume that (y^n) satisfies the conditions in Proposition 4.7. So, using the notation as in Proposition 4.7, we get that

$$\varphi(y^n) = \frac{\sum_{i=1}^r (e^{\lambda_i^n} - e^{-\lambda_i^n}) p_i^n}{\sum_{i=1}^r e^{\lambda_i^n} + e^{-\lambda_i^n}} = \frac{\sum_{i=1}^r (e^{-r^n+\lambda_i^n} - e^{-r^n-\lambda_i^n}) p_i^n}{\sum_{i=1}^r e^{-r^n+\lambda_i^n} + e^{-r^n-\lambda_i^n}}.$$

The right-hand side converges to

$$\frac{1}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} \left(\sum_{i\in I} e^{-\alpha_i} p_i - \sum_{j\in J} e^{-\alpha_j} p_j \right) = z.$$

But this implies that $z \in \partial B_1^*$, which is impossible. Indeed, if we let $p_I = \sum_{i \in I} p_i$ and $p_J = \sum_{j \in J} p_j$, then $1 \geq \|z\|_u^* \geq (z|p_I - p_J) = 1$, as $-u \leq p_I - p_J \leq u$. □

For simplicity we denote the (closed) boundary faces of B_1^* by

$$F_{p,q} = \text{conv}\,((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V)))$$

where p and q are orthogonal idempotents in V not both zero, see [18, Theorem 4.4].

Lemma 4.17 *If h is a horofunction given by* (4.2)*, then φ maps $\mathcal{P}_h$ into* relint F_{p_I,p_J}.

Proof Clearly, $\varphi(h) \in F_{p_I,p_J}$ if h is given by (4.2). So, φ maps $\mathcal{P}_h$ into F_{p_I,p_J} by Theorem 4.2(i). To show that φ maps $\mathcal{P}_h$ into relint F_{p_I,p_J}, it suffices to show that $\varphi(h) \in \text{relint}\, F_{p_I,p_J}$.

To do this we first consider $w = (|I| + |J|)^{-1}(p_I - p_J) \in F_{p_I,q_J}$ and show that $w \in \text{relint} F_{p_i,q_J}$. Let $c \in F_{p_I,p_J}$ be arbitrary. Note that we can write $c = \sum_{i \in I'} \lambda_i q_i - \sum_{j \in J'} \lambda_j q_j$, where $\sum_{i \in I'} q_i = p_I$, $\sum_{j \in J'} q_j = p_J$, and $\sum_{i \in I'} \lambda_i + \sum_{j \in J'} \lambda_j = 1$ with $0 \leq \lambda_i, \lambda_j \leq 1$ for all i and j. We see that $w + \varepsilon(w - c) = (1+\varepsilon)w - \varepsilon c \in F_{p_I,p_J}$ for all $\varepsilon > 0$ small, so $w \in \text{relint}\, F_{p_I,p_J}$ by [52, Theorem 6.4].

To complete the proof we argue by contradiction. So suppose that $\varphi(h) \notin \text{relint} F_{p_I,p_J}$. Then $\varphi(h)$ is in the (relative) boundary of F_{p_I,p_J}, hence

$$z_\varepsilon = (1 + \varepsilon)\varphi(h) - \varepsilon w \notin F_{p_I,p_J}$$

for all $\varepsilon > 0$, as $w \in \text{relint} F_{p_I,p_J}$ and F_{p_I,p_J} is convex. However, for each $i \in I$ we have that the coefficient of p_i in z_ε,

$$\frac{(1+\varepsilon)e^{-\alpha_i}}{\sum_{i \in I} e^{-\alpha_i} + \sum_{j \in J} e^{-\alpha_j}} - \frac{\varepsilon}{|I| + |J|},$$

is strictly positive for all $\varepsilon > 0$ sufficiently small. Likewise, for each $j \in J$ we have that the coefficient of $-p_j$ in z_ε,

$$\frac{(1+\varepsilon)e^{-\alpha_j}}{\sum_{i \in I} e^{-\alpha_i} + \sum_{j \in J} e^{-\alpha_j}} - \frac{\varepsilon}{|I| + |J|},$$

is strictly positive for all $\varepsilon > 0$ sufficiently small. This implies that $z_\varepsilon \in F_{p_I,p_J}$ for all $\varepsilon > 0$ small, which is impossible. This completes the proof. □

Using the previous results we now show that φ is injective on $\overline{V}^h$.

Corollary 4.18 *The map $\varphi \colon \overline{V}^h \to B_1^*$ is injective.*

Proof We already saw in Lemmas 4.14 and 4.15 that φ maps V into int B_1^* and is injective on V. So by the previous lemma, it suffices to show that if $\varphi(h) = \varphi(h')$

for horofunctions $h \sim h'$, then $h = h'$. Let h be given by (4.2) and suppose that h' is given by

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right)\right\}.$$

Then

$$\frac{\sum_{i\in I} e^{-\alpha_i} p_i - \sum_{j\in J} e^{-\alpha_j} p_j}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} = \frac{\sum_{i\in I'} e^{-\beta_i} q_i - \sum_{j\in J'} e^{-\beta_j} q_j}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}}.$$

As $\min_k \alpha_k = 0 = \min_k \beta_k$, it follows from the spectral theorem [25, Theorem III.1.2] that

$$\frac{1}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} = \|\varphi(h)\|_u = \|\varphi(h')\|_u = \frac{1}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}},$$

so that

$$\sum_{i\in I} e^{-\alpha_i} p_i - \sum_{j\in J} e^{-\alpha_j} p_i = \sum_{i\in I'} e^{-\beta_i} q_i - \sum_{j\in J'} e^{-\beta_j} q_j.$$

As each $x \in V$ can be written in a unique way as $x = x^+ - x^-$, where x^+ and x^- are orthogonal element x^+ and x^- in V_+, see [4, Proposition 1.28], we find that $\sum_{i\in I} e^{-\alpha_i} p_i = \sum_{i\in I'} e^{-\beta_i} q_i$ and $\sum_{j\in J} e^{-\alpha_j} p_i = \sum_{j\in J'} e^{-\beta_j} q_j$. This implies that

$$\begin{aligned}\sum_{i\in I} \alpha_i p_i &= -\log\left(\sum_{i\in I} e^{-\alpha_i} p_i + (u - p_I)\right) = -\log\left(\sum_{i\in I'} e^{-\beta_i} q_i + (u - q_{I'})\right)\\ &= \sum_{i\in I'} \beta_i q_i\end{aligned}$$

and

$$\begin{aligned}\sum_{j\in J} \alpha_j p_j &= -\log\left(\sum_{j\in J} e^{-\alpha_j} p_i + (u - p_J)\right) = -\log\left(\sum_{j\in J'} e^{-\beta_j} q_j + (u - q_{J'})\right)\\ &= \sum_{j\in J'} \beta_j q_j,\end{aligned}$$

and hence $h = h'$. □

The next result shows that φ is continuous on $\partial \overline{V}^h$.

Theorem 4.19 *The map* $\varphi\colon \overline{V}^h \to B_1^*$ *is continuous.*

Proof Clearly φ is continuous on V. Suppose (y^n) is a sequence in V with $h_{y^n} \to h \in \partial\overline{V}^h$. We wish to show that $\varphi(y^n) \to \varphi(h)$. Let $(\varphi(y^{n_k}))$ be a subsequence. We will show that it has a subsequence which converges to $\varphi(h)$.

As h is a horofunction, we know that $r^n = \|y^{n_k}\|_u \to \infty$ by Lemma 2.1. For each k there exists a Jordan frame $q_1^{n_k}, \dots, q_r^{n_k}$ in V and $\lambda_1^{n_k}, \dots, \lambda_r^{n_k} \in \mathbb{R}$ such that

$$y^{n_k} = \sum_{i=1}^{r} \lambda_i^{n_k} q_i^{n_k}.$$

By taking a subsequence we may assume that there exist $I_+ \subseteq \{1, \dots, r\}$ and $1 \leq s \leq r$ such that for each k, $r^{n_k} = \|y^{n_k}\|_u = |\lambda_s^{n_k}|$, and $\lambda_i^{n_k} > 0$ if and only if $i \in I_+$.

For each k, let $\beta_i^{n_k} = r^{n_k} - \lambda_i^{n_k}$ for $i \in I_+$, and $\beta_i^{n_k} = r^{n_k} + \lambda_i^{n_k}$ for $i \notin I_+$. Note that $\beta^{n_k} \geq 0$ for all i and k, and $\beta_s^{n_k} = 0$ for all k. By taking a further subsequence we may assume that $\beta_i^{n_k} \to \beta_i \in [0, \infty]$ and $q_i^{n_k} \to q_i$ for all i. Let $I' = \{i \in I_+ \colon \beta_i < \infty\}$ and $J' = \{j \notin I_+ \colon \beta_j < \infty\}$. Note that $s \in I' \cup J'$ and we can apply Proposition 4.7 to conclude that $h_{y^{n_k}} \to h' \in \partial\overline{V}^h$, where

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i \in I'} \beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j \in J'} \beta_j q_j\right)\right\}.$$

As $h_{y^{n_k}} \to h$, we know that $h = h'$ and hence $\delta(h, h') = 0$. This implies that $p_I = q_{I'}$ and $p_J = q_{J'}$ by Theorem 4.2. Moreover,

$$\sum_{i \in I} \alpha_i p_i + \sum_{j \in J} \alpha_j p_j = \sum_{i \in I'} \beta_i q_i + \sum_{j \in J'} \beta_j q_j.$$

It follows that

$$\begin{aligned}\sum_{i \in I} \alpha_i p_i &= U_{p_I}\left(\sum_{i \in I} \alpha_i p_i + \sum_{j \in J} \alpha_j p_j\right) = U_{q_{I'}}\left(\sum_{i \in I'} \beta_i q_i + \sum_{j \in J'} \beta_j q_j\right)\\ &= \sum_{i \in I'} \beta_i q_i\end{aligned}$$

and

$$\sum_{j \in J} \alpha_j p_j = U_{p_J}\left(\sum_{i \in I} \alpha_i p_i + \sum_{j \in J} \alpha_j p_j\right) = U_{q_{J'}}\left(\sum_{i \in I'} \beta_i q_i + \sum_{j \in J'} \beta_j q_j\right) = \sum_{j \in J'} \beta_j q_j,$$

so that $\sum_{i\in I} e^{\alpha_i} p_i = \sum_{i\in I'} e^{\beta_i} q_i$ and $\sum_{j\in J} e^{\alpha_j} p_j = \sum_{j\in J'} e^{\beta_j} q_j$. We conclude that

$$\lim_{k\to\infty} \varphi\left(y^{n_k}\right) = \lim_{k\to\infty} \frac{\sum_{i=1}^{r}\left(e^{-r^{n_k}+\lambda_i^{n_k}} - e^{-r^{n_k}-\lambda_i^{n_k}}\right) q_i^{n_k}}{\sum_{i=1}^{r}\left(e^{-r^{n_k}+\lambda_i^{n_k}} + e^{-r^{n_k}-\lambda_i^{n_k}}\right)}$$
$$= \frac{\sum_{i\in I'} e^{-\beta_i} q_i - \sum_{j\in J'} e^{-\beta_j} q_j}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}} = \varphi\left(h\right).$$

From Lemmas 4.14 and 4.17 we know that φ maps V into int B_1^* and $\partial\overline{V}^h$ into ∂B_1^*. So to complete the proof it remains to show that if (h_n) in $\partial\overline{V}^h$ converges to $h \in \partial\overline{V}^h$, then $\varphi(h_n) \to \varphi(h)$. Suppose h is given by (4.2) and for each n the horofunction h_n is given by

$$h_n(x) = \max\left\{\Lambda_{V(q_{I_n}^n)}\left(-U_{q_{I_n}^n} x - \sum_{i\in I_n}\beta_i^n q_i^n\right), \Lambda_{V(q_{J_n}^n)}\left(U_{q_{J_n}^n} x - \sum_{j\in J_n}\beta_j^n q_j^n\right)\right\} \quad \text{for } x \in V, \tag{4.15}$$

where $I_n, J_n \subseteq \{1,\ldots,r\}$ are disjoint, $I_n \cup J_n$ is nonempty, and $\min\{\beta_k^n : k \in I_n \cup J_n\} = 0$.

To prove the assertion we show that each subsequence of $(\varphi(h_n))$ has a convergent subsequence with limit $\varphi(h)$. Let $(\varphi(h_{n_k}))$ be a subsequence. By taking subsequences we may assume that

(1) There exist $I_0, J_0 \subseteq \{1,\ldots,r\}$ disjoint with $I_0 \cup J_0$ nonempty, such that $I_{n_k} = I_0$ and $J_{n_k} = J_0$ for all k.
(2) $\beta_i^{n_k} \to \beta_i \in [0,\infty]$ and $q_i^{n_k} \to q_i$ for all $i \in I_0 \cup J_0$.
(3) There exists $i^* \in I_0 \cup J_0$ such that $\beta_{i^*}^{n_k} = 0$ for all k.

Let $I' = \{i \in I_0 : \beta_i < \infty\}$ and $J' = \{j \in J_0 : \beta_j < \infty\}$, and note that $i^* \in I' \cup J'$.

Using Lemma 4.6 we now show that $h_{n_k} \to h'$, where

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}} x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}} x - \sum_{j\in J'}\beta_j q_j\right)\right\}. \tag{4.16}$$

Note that if I' is nonempty, then by Lemma 4.6 we have that

$$\lim_{k\to\infty} \Lambda_{V(q_{I_0}^{n_k})}\left(-U_{q_{I_0}^{n_k}} x - \sum_{i\in I_0}\beta_i^{n_k} q_i^{n_k}\right) = \Lambda_{V(q_{I'})}\left(-U_{q_{I'}} x - \sum_{i\in I'}\beta_i q_i\right),$$

as $U_{q_{I_0}^{n_k}} x \to U_{q_{I_0}} x$ by Lemma 4.5 and $U_{q_{I'}}(U_{q_{I_0}} x) = U_{q_{I'}} x$ by [4, Proposition 2.26]. Likewise if J' is nonempty, we have that

$$\lim_{k\to\infty} \Lambda_{V(q_{J_0}^{n_k})} \left(U_{q_{J_0}^{n_k}} x - \sum_{j\in J_0} \beta_j^{n_k} q_j^{n_k} \right) = \Lambda_{V(q_{J'})} \left(U_{q_{J'}} x - \sum_{j\in J'} \beta_j q_j \right).$$

Thus, if I' and J' are both nonempty (4.16) holds.

Now suppose that I' is empty, so J' is nonempty. As $-x \leq \|x\|_u u$, we get that

$$-U_{q_{I_0}^{n_k}} x \leq \|x\|_u U_{q_{I_0}^{n_k}} u = \|x\|_u q_{I_0}^{n_k}.$$

This implies that $-U_{q_{I_0}^{n_k}} x - \sum_{i\in I_0} \beta_i^{n_k} q_i^{n_k} \leq \sum_{i\in I_0} (\|x\|_u - \beta_i^{n_k}) q_i^{n_k}$, hence

$$\Lambda_{V(q_{I_0}^{n_k})} \left(-U_{q_{I_0}^{n_k}} x - \sum_{i\in I_0} \beta_i^{n_k} q_i^{n_k} \right) \leq \max_{i\in I_0} (\|x\|_u - \beta_i^{n_k}) \to -\infty.$$

On the other hand, $h_{n_k}(x) \geq -\|x\|_u$ for all k. Thus, for all k sufficiently large, we have that

$$h_{n_k}(x) = \Lambda_{V(q_{J_0}^{n_k})} \left(U_{q_{J_0}^{n_k}} x - \sum_{j\in J_0} \beta_j^{n_k} q_j^{n_k} \right),$$

which implies that (4.16) holds if I' is empty. In the same way it can be shown that (4.16) holds if J' is empty.

As $h_n \to h$, we know that $h' = h$, so $\delta(h, h') = 0$. It follows from Theorem 4.2 that $p_I = q_{I'}$, $p_J = q_{J'}$, and $\sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j = \sum_{i\in I'} \beta_i q_i + \sum_{j\in J'} \beta_j q_j$. This implies that

$$\sum_{i\in I} \alpha_i p_i = \sum_{i\in I'} \beta_i q_i \quad \text{and} \quad \sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j,$$

so that $\sum_{i\in I} e^{-\alpha_i} p_i = \sum_{i\in I'} e^{-\beta_i} q_i$ and $\sum_{j\in J} e^{-\alpha_j} p_j = \sum_{j\in J'} e^{-\beta_j} q_j$. Thus,

$$\begin{aligned} \lim_{k\to\infty} \varphi(h_{n_k}) &= \lim_{k\to\infty} \frac{\sum_{i\in I_0} e^{-\beta_i^{n_k}} q_i^{n_k} - \sum_{j\in J_0} e^{-\beta_j^{n_k}} q_j^{n_k}}{\sum_{i\in I_0} e^{-\beta_i^{n_k}} + \sum_{j\in J_0} e^{-\beta_j^{n_k}}} \\ &= \frac{\sum_{i\in I'} e^{-\beta_i} q_i - \sum_{j\in J'} e^{-\beta_j} q_j}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}} = \varphi(h), \end{aligned}$$

which completes the proof. □

Theorem 4.20 *The map* $\varphi\colon \overline{V}^h \to B_1^*$ *is onto.*

Proof From Lemma 4.16 we know that $\varphi(V) = \operatorname{int} B_1^*$. Let $z \in \partial B_1^*$. As B_1^* is the disjoint union of the relative interiors of its faces, see [52, Theorem 18.2], we know that there exist orthogonal idempotents p_I and p_J such that $z \in \operatorname{relint} F_{p_I,p_J}$. Thus, we can write

$$z = \sum_{i\in I} \lambda_i p_i - \sum_{j\in J} \lambda_j p_j,$$

where $p_I = \sum_{i\in I} p_i, q_J = \sum_{j\in J} q_j, 0 < \lambda_k \leq 1$ for all $k \in I \cup J$, and $\sum_{k\in I\cup J} \lambda_k = 1$.

Define $\mu_k = -\log \lambda_k$ for $k \in I \cup J$. So, $\mu_k \geq 0$. Let $\mu^* = \min\{\mu_k : k \in I \cup J\}$ and set $\alpha_k = \mu_k - \mu^* \geq 0$. Note that $\min\{\alpha_k : k \in I \cup J\} = 0$.

Then h, given by

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i\in I}\alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j\in J}\alpha_j p_j\right)\right\}$$

for $x \in V$, is a horofunction by Proposition 4.9. Moreover,

$$\begin{aligned}&\frac{1}{\sum_{i\in I} e^{-\mu_i} + \sum_{j\in J} e^{-\mu_j}}\left(\sum_{i\in I} e^{-\mu_i} p_i - \sum_{j\in J} e^{-\mu_j} p_j\right)\\ &= \frac{1}{\sum_{i\in I} \lambda_i + \sum_{j\in J}\lambda_j}\left(\sum_{i\in I}\lambda_i p_i - \sum_{j\in J}\lambda_j p_j\right),\end{aligned}$$

hence $\varphi(h) = z$, which completes the proof. $\square$

The proof of Theorem 4.4 is now straightforward.

Proof of Theorem 4.4 It follows from Theorems 4.19 and 4.20 and Corollary 4.18 that $\varphi\colon \overline{V}^h \to B_1^*$ is a continuous bijection. As $\overline{V}^h$ is compact and B_1^* is Hausdorff, we conclude that φ is a homeomorphism. It follows from Lemma 4.17 that φ maps each part onto the relative interior of a boundary face of B_1^*. $\square$

Remark 4.21 It is interesting to note that a similar idea can be used to show that the horofunction compactification of a finite dimensional normed space $(V, \|\cdot\|)$ with a smooth and strictly convex norm is homeomorphic to the closed dual unit ball. Indeed, in that case the horofunctions are given by $h\colon z \mapsto -x^*(z)$, where $x^* \in V^*$ has norm 1, see for example [23, Lemma 5.3]. Moreover, for (y^n) in V we have that $h_{y^n} \to h$ if and only if $y^n/\|y^n\| \to x$ and $\|y^n\| \to \infty$.

In this case we define a map $\psi\colon \overline{V}^h \to B_1^*$ as follows. For $x \in V$ with $x \neq 0$, let

$$\psi(x) = -\left(\frac{e^{\|x\|} - e^{-\|x\|}}{e^{\|x\|} + e^{-\|x\|}}\right) x^*,$$

where $x^* \in V^*$ is the unique functional with $x^*(x) = \|x\|$ and $\|x^*\| = 1$, and let $\psi(0) = 0$. For $h \in \partial\overline{V}^h$ with $h\colon z \mapsto -x^*(z)$ let

$$\psi(h) = -x^*.$$

It is straightforward to check that ψ is a bijection from $\overline{V}^h$ onto B_1^*, and ψ is continuous on int B_1^*. To show continuity on $\partial\overline{V}^h$, we assume, by way of contradiction, that (h_n) is a sequence of horofunctions with $h_n \to h$ and $h_n(z) = -x_n^*(z)$ for all $z \in V$, and there exists a neighbourhood U of $\psi(h)$ in B_1^* such that $\psi(h_n) \notin U$ for all n. Then, for each $z^* \in \partial B_1^*$ with $z^* \notin U$ we have that $z^*(x) < 1$. So, by compactness, $\delta = \max\{1 - z^*(x)\colon z^* \in \partial B_1^* \backslash U\} > 0$. It now follows that

$$h_n(x) - h(x) = -x_n^*(x) + x^*(x) = 1 - x_n^*(x) \geq \delta > 0$$

for all n, which contradicts $h_n \to h$. This shows that ψ is a continuous bijection, and hence a homeomorphism, as $\overline{V}^h$ is compact and B_1^* is Hausdorff.

More generally, one can consider product spaces $V = \prod_{i=1}^r V_i$ with norm $\|x\|_V = \max_{i=1}^r \|v_i\|_i$, where each $(V_i, \|\cdot\|_i)$ is a finite dimensional normed space with a smooth and strictly convex norm. In that case we have by [40, Theorem 2.10] that the horofunctions of V are given by

$$h(v) = \max_{j\in J}(h_{\xi_j^*}(v_j) - \alpha_j), \tag{4.17}$$

where $J \subseteq \{1,\dots,r\}$ nonempty, $\min_{j\in J}\alpha_j = 0$, $\xi_j^* \in V_j^*$ with $\|\xi_j^*\| = 1$, and $h_{\xi_j^*}(v_j) = -\xi_j^*(v_j)$. One can use similar ideas as the ones in Sect. 3 to show that the horofunction compactification is homeomorphic to the closed unit dual ball of V. Indeed, one can define a map $\varphi_V : \overline{V}^h \to B_1^*$ by

$$\varphi_V(v) = \frac{1}{\sum_{i=1}^r e^{\|v_i\|_i} + e^{-\|v_i\|_i}}\left(\sum_{i=1}^r (e^{\|v_i\|_i} - e^{-\|v_i\|_i})p(v_i^*)\right) \qquad \text{for } v \in V \setminus \{0\}$$

and $\varphi_V(0) = 0$. Here $p(v_i^*) = (0,\dots,0,v_i^*,0,\dots,0)$ and v_i^* is the unique functional such that $v_i^*(v_i) = \|v_i\|_i$ and $\|v_i^*\|_i = 1$ if $v_i \neq 0$, and we set $p(v_i^*) = 0$, if $v_i = 0$. For a horofunction h given by (4.17) we define

$$\varphi_V(h) = \frac{1}{\sum_{j\in J} e^{-\alpha_j}}\left(\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)\right).$$

Following the same line of reasoning as in Sect. 3 one can prove that φ_V is a homeomorphism.

Remark 4.22 The connection between the geometry of the horofunction compactification and the dual unit ball seems hard to establish for general finite dimensional normed spaces, and might not even hold. For the normed spaces discussed in this paper and in [32, 33] all horofunctions are Busemann points, but there are normed spaces with horofunctions that are not Busemann, see [54]. It could well be the case that the horofunction compactification of these spaces is not naturally homeomorphic to the closed dual unit ball, but no counter example is known at present.

5 Symmetric Cones with the Hilbert Distance

In this section we study the global topology and geometry of the horofunction compactification of symmetric cones under the Hilbert distance. Recall that the Hilbert distance is defined as follows. Let A be a real finite dimensional affine space. Consider a bounded, open, convex set $\Omega \subseteq A$. For $x, y \in \Omega$, let ℓ_{xy} be the straight-line through x and y in A, and denote the points of intersection of ℓ_{xy} and $\partial\Omega$ by x' and y', where x is between x' and y, and y is between x and y'. On Ω the *Hilbert distance* is then defined by

$$\rho_H(x, y) = \log\left(\frac{|x'-y|}{|x'-x|}\frac{|y'-x|}{|y'-y|}\right) \tag{5.1}$$

for all $x \neq y$ in Ω, and $\rho_H(x, x) = 0$ for all $x \in \Omega$. The metric space (Ω, ρ_H) is called the *Hilbert geometry* on Ω.

These metric spaces generalise Klein's model of hyperbolic space and have a Finsler structure, see [48, 49]. In our analysis we will work with Birkhoff's version of the Hilbert metric, which is defined on a cone in an order-unit space in terms of its partial ordering. This provides a convenient way to work with the Hilbert distance and its Finsler structure. In the next subsection we will recall the basic concepts involved in our analysis. Throughout we will follow the terminology used in [42, Chap. 2], which contains a detailed discussion of Hilbert geometries and some of their applications. We refer the reader to [49] for a comprehensive account of the theory of Hilbert geometries.

5.1 Preliminaries and Finsler Structure

Let (V, V_+, u) be a finite dimensional order-unit space. So, V_+ is a closed cone in V with $u \in \operatorname{int} V_+$. Recall that the cone V_+ induces a partial ordering on V by $x \leq y$ if $y - x \in V_+$, see Sect. 4.1. For $x \in V$ and $y \in V_+$, we say that y *dominates* x if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \leq x \leq \beta y$. In that case, we write

$$M(x/y) = \inf\{\beta \in \mathbb{R} \colon x \leq \beta y\} \quad \text{and} \quad m(x/y) = \sup\{\alpha \in \mathbb{R} \colon \alpha y \leq x\}.$$

By the Hahn–Banach theorem, $x \leq y$ if and only if $\psi(x) \leq \psi(y)$ for all $\psi \in V_+^* = \{\varphi \in V^* \colon \varphi \text{ positive}\}$, which is equivalent to $\psi(x) \leq \psi(y)$ for all $\psi \in S(V)$. Using this fact we see that for each $x \in V$ and $y \in \operatorname{int} V_+$,

$$M(x/y) = \sup_{\psi \in S(V)} \frac{\psi(x)}{\psi(y)} \quad \text{and} \quad m(x/y) = \inf_{\psi \in S(V)} \frac{\psi(x)}{\psi(y)}.$$

We also note that if $A \in \mathrm{GL}(V)$ is a linear automorphism of V_+, i.e., $A(V_+) = V_+$, then $x \leq \beta y$ if, and only if, $Ax \leq \beta Ay$. It follows that $M(Ax/Ay) = M(x/y)$ and $m(x/y) = m(Ax/Ay)$.

If $w \in \operatorname{int} V_+$, then w dominates each $x \in V$, and we define

$$|x|_w = M(x/w) - m(x/w).$$

One can verify that $|\cdot|_w$ is a semi-norm on V, see [42, Lemma A.1.1], and a genuine norm on the quotient space $V/\mathbb{R}w$, as $|x|_w = 0$ if and only if $x = \lambda w$ for some $\lambda \in \mathbb{R}$.

Clearly, if $x, y \in V$ are such that $y = 0$ and y dominates x, then $x = 0$, as V_+ is a cone. On the other hand, if $y \in V_+ \backslash \{0\}$, and y dominates x, then $M(x/y) \geq m(x/y)$. The domination relation yields an equivalence relation on V_+ by $x \sim y$ if y dominates x and x dominates y. The equivalence classes are called the *parts* of V_+. As V_+ is closed, one can check that $\{0\}$ and $\operatorname{int} V_+$ are parts of V_+. The parts of a finite dimensional cone are closely related to its faces. Indeed, if V_+ is the cone of a finite dimensional order-unit space, then it can be shown that the parts correspond to the relative interiors of the faces of V_+, see [42, Lemma 1.2.2]. Recall that a *face* of a convex set $S \subseteq V$ is a subset F of S with the property that if $x, y \in S$ and $\lambda x + (1 - \lambda) y \in F$ for some $0 < \lambda < 1$, then $x, y \in F$.

It is easy to verify that if $x, y \in V_+ \backslash \{0\}$, then $x \sim y$ if, and only if, there exist $0 < \alpha \leq \beta$ such that $\alpha y \leq x \leq \beta y$. Furthermore, if $x \sim y$, then

$$m(x/y) = \sup\{\alpha > 0 \colon y \leq \alpha^{-1} x\} = M(y/x)^{-1}. \tag{5.2}$$

Birkhoff's version of the Hilbert distance on V_+ is defined as follows:

$$d_H(x, y) = \log\left(\frac{M(x/y)}{m(x/y)}\right) = \log M(x/y) + \log M(y/x) \tag{5.3}$$

for all $x \sim y$ with $y \neq 0$, $d_H(0, 0) = 0$, and $d_H(x, y) = \infty$ otherwise.

Note that $d_H(\lambda x, \mu y) = d_H(x, y)$ for all $x, y \in V_+$ and $\lambda, \mu > 0$, so d_H is not a distance on V_+. It is, however, a distance between pairs of rays in each part of V_+. In particular, if $\varphi \colon V \to \mathbb{R}$ is a linear functional such that $\varphi(x) > 0$ for all $x \in V_+ \backslash \{0\}$, then d_H is a distance on

$$\Omega_V = \{x \in \operatorname{int} V_+ \colon \varphi(x) = 1\},$$

which is a (relatively) open, bounded, convex set, see [42, Lemma 1.2.4]. Moreover, the following holds, see [42, Proposition 2.1.1 and Theorem 2.1.2].

Theorem 5.1 *(Ω_V, d_H) is a metric space and $d_H = \rho_H$ on Ω_V.*

It is worth noting that any Hilbert geometry can be realised as (Ω_V, d_H) for some order-unit space V and strictly positive linear functional φ.

A Hilbert geometry (Ω_V, d_H) has a Finsler structure, see [48]. Indeed, if one defines the length of a piecewise C^1-smooth path $\gamma\colon [0,1] \to \Omega_V$ by

$$L(\gamma) = \int_0^1 |\gamma'(t)|_{\gamma(t)} \mathrm{d}t,$$

then $d_H(x, y) = \inf_\gamma L(\gamma)$, where the infimum is taken over all piecewise C^1-smooth paths in Ω_V with $\gamma(0) = x$ and $\gamma(1) = y$.

It should be noted that in the case of Hilbert geometries the unit ball $\{x \in V/\mathbb{R}w\colon |x|_w \le 1\}$ in the tangent space at $w \in \Omega_V$ may have a different facial structure for different w. This phenomenon appears frequently in the case where Ω_V is a polytope, but does not appear in the Hilbert geometries considered here.

Let (V, V_+, u) be an order-unit space, where V is a Euclidean Jordan algebra of rank r, V_+ is the cone of squares, and u is the algebraic unit. So, int V_+ is a symmetric cone and $\mathrm{Isom}(\Omega_V)$ acts transitively on Ω_V.

Throughout we will take $\varphi\colon V \to \mathbb{R}$ with $\varphi(x) = \frac{1}{r}\mathrm{tr}(x)$, which is a state, and

$$\Omega_V = \{x \in \operatorname{int} V_+\colon \varphi(x) = 1\} = \{x \in \operatorname{int} V_+\colon \mathrm{tr}(x) = r\}.$$

We shall call (Ω_V, d_H) a *symmetric* Hilbert geometry. A prime example is

$$\Omega_V = \{A \in \mathrm{Herm}(n, \mathbb{C})\colon \mathrm{tr}(A) = n \text{ and } A \text{ positive definite}\}.$$

These spaces are important examples of noncompact type symmetric spaces with an invariant Finsler metric, see [50]. In particular, the example above corresponds to the symmetric space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$.

In a symmetric Hilbert geometry the distance can be expressed in terms of the spectrum. Indeed, we know that for $x \in V$ invertible, the quadratic representation $U_x\colon V \to V$ is a linear automorphism of V_+, see [25, Proposition III.2.2]. Moreover, $U_x^{-1} = U_{x^{-1}}$ and $U_{x^{-1/2}}x = u$. Furthermore, for $x \in V$ we have that

$$M(x/u) = \inf\{\lambda\colon x \le \lambda u\} = \max \sigma(x) \quad \text{and} \quad m(x/u) = \sup\{\lambda\colon \lambda u \le x\} = \min \sigma(x),$$

so that $|x|_u = \max \sigma(x) - \min \sigma(x)$. Also for $x, y \in \operatorname{int} V_+$ we have that

$$\log M(x/y) = \max \sigma(\log U_{y^{-1/2}}x) \quad \text{and} \quad \log M(y/x) = -\min \sigma(\log U_{y^{-1/2}}x).$$

It follows that

$$\begin{aligned} d_H(x, y) &= \log M(x/y) + \log M(y/x) = |\log U_{y^{-1/2}}x|_u \\ &= \operatorname{diam} \sigma(\log U_{y^{-1/2}}x) \quad \text{for all } x, y \in \operatorname{int} V_+. \end{aligned}$$

Moreover, for each $w \in \Omega_V$ we have that

$$
\begin{aligned}
|x|_w &= M(x/w) - m(x/w) = M(U_{w^{-1/2}}x/u) - m(U_{w^{-1/2}}x/u) \\
&= |U_{w^{-1/2}}x|_u \quad \text{for all } x \in V,
\end{aligned}
$$

which shows that the facial structure of the unit ball in each tangent space is identical, as $U_{w^{-1/2}}$ is an invertible linear map.

5.2 Horofunctions of Symmetric Hilbert Geometries

The main objective is to show for symmetric Hilbert geometries (Ω_V, d_H) that there exists a natural homeomorphism between $\overline{\Omega}_V^h$ and the closed dual unit ball of the Finsler metric $|\cdot|_u$ in the tangent space $V/\mathbb{R}u$ at the unit u. To describe the homeomorphism, we recall the description of the horofunction compactification of symmetric Hilbert geometries given in [44, Theorem 5.6].

Theorem 5.2 *The horofunctions of a symmetric Hilbert geometry (Ω_V, d_H) are precisely the functions $h\colon \Omega_V \to \mathbb{R}$ of the form*

$$
h(x) = \log M(y/x) + \log M(z/x^{-1}) \quad \textit{for } x \in \Omega_V, \tag{5.4}
$$

where $y, z \in \partial V_+$ are such that $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$.

It follows from the proof of [44, Theorem 5.6] that all horofunctions are in fact Busemann points. Indeed, if y and z have spectral decompositions

$$
y = \sum_{i \in I} \lambda_i p_i \quad \text{and} \quad z = \sum_{j \in J} \mu_j p_j,
$$

where $I, J \subset \{1, \dots, r\}$ are nonempty and disjoint, and $p_1, \dots, p_r$ is a Jordan frame, then the sequence $(y_n) \in \operatorname{int} V_+$ given by

$$
y_n = \sum_{i \in I} \lambda_i p_i + \sum_{j \in J} \frac{1}{n^2 \mu_j} p_j + \sum_{k \notin I \cup J} \frac{1}{n} p_k
$$

has the property that $y_n \to y$, $y_n^{-1}/\|y_n^{-1}\|_u \to z$ and $h_{y_n} \to h$, where h is as in (5.4). Note that if we let $v_n = y_n/\varphi(y_n) \in \Omega_V$, then $h_{v_n}(z) = h_{y_n}(z)$ for all $z \in \Omega_V$, so $h_{v_n} \to h$.

Also note that for $n, m \geq 1$,

$$
U_{y_n^{-1/2}} y_m = \sum_{i \in I} p_i + \sum_{j \in J} \frac{n^2}{m^2} p_j + \sum_{k \notin I \cup J} \frac{n}{m} p_k.
$$

This implies that for each $n \geq m \geq 1$,

$$
M(y_m/y_n) = M(U_{y_n^{-1/2}} y_m/u) = \|U_{y_n^{-1/2}} y_m\|_u = n^2/m^2,
$$

so that $\log M(y_m/y_n) = 2\log n - 2\log m$. Moreover, $\log M(y_n/y_m) = \log 1 = 0$ for all $n \geq m \geq 1$. It follows that

$$d_H(v_n, v_m) + d_H(v_m, v_1) = d_H(y_n, y_m) + d_H(y_m, y_1) = d_H(y_n, y_1) = d_H(v_n, v_1)$$

for all $n \geq m \geq 1$. Thus, (v_n) is an almost geodesic sequence in Ω_V, and hence each horofunction in $\overline{\Omega}_V^h$ is a Busemann point.

To identify the parts and describe the detour distance (2.2) we need the following general lemma.

Lemma 5.3 *Let (V, V_+, u) be a finite dimensional order-unit space. If $v \in \partial V_+ \backslash \{0\}$ and $w_n \in \operatorname{int} V_+$ with $w_{n+1} \leq w_n$ for all $n \geq 1$ and $w_n \to w \in \partial V_+ \backslash \{0\}$, then*

$$\lim_{n\to\infty} M(v/w_n) = \begin{cases} M(v/w) & \textit{if } w \textit{ dominates } v \\ \infty & \textit{otherwise.} \end{cases}$$

Proof Set $\lambda_n = M(v/w_n)$ for $n \geq 1$. Then for $n \geq m \geq 1$ we have that $0 \leq \lambda_n w_n - v \leq \lambda_n w_m - v$. This implies that $\lambda_m \leq \lambda_n$ for all $m \leq n$, hence (λ_n) is monotonically increasing.

Now suppose that $\lambda = M(v/w) < \infty$, i.e., w dominates v. Then $0 \leq \lambda w - v \leq \lambda w_n - v$, hence $\lambda_n \leq \lambda$ for all n. This implies that $\lambda_n \to \lambda^* \leq \lambda < \infty$. As $0 \leq \lambda_n w_n - v$ for all n and V_+ is closed, we know that $\lim_{n\to\infty} \lambda_n w_n - v = \lambda^* w - v \in V_+$. So $\lambda^* \geq \lambda$, hence $\lambda^* = \lambda$. We conclude that if w dominates v, then $\lim_{n\to\infty} M(v/w_n) = M(v/w)$.

On the other hand, if w does not dominate v, then

$$\lambda w - v \notin V_+ \quad \text{for all } \lambda \geq 0. \tag{5.5}$$

Assume, by way of contradiction, that (λ_n) is bounded. Then $\lambda_n \to \lambda^* < \infty$, since (λ_n) is increasing, and $\lambda_n w_n - v \to \lambda^* w - v \in V_+$, as V_+ is closed. This contradicts (5.5), and hence $\lambda_n = M(v/w_n) \to \infty$, if w does not dominate v. □

Before we identify the parts in $\partial\overline{\Omega}_V^h$ and the detour distance, it is useful to recall the following fact:

$$M(x/y) = M(y^{-1}/x^{-1}) \quad \text{for all } x, y \in \operatorname{int} V_+,$$

if $\operatorname{int} V_+$ is a symmetric cone, see [45, Sect. 2.4].

Proposition 5.4 *Let (Ω_V, d_H) be a symmetric Hilbert geometry and $h, h' \in \partial\overline{\Omega}_V^h$ with*

$$h(x) = \log M(y/x) + \log M(z/x^{-1}) \quad \textit{and} \quad h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$$

for $x \in \Omega_V$. The following assertions hold:

(i) *h and h' are in the same part if and only if $y \sim y'$ and $z \sim z'$.*
(ii) *If h and h' are in the same part, then $\delta(h, h') = d_H(y, y') + d_H(z, z')$.*

Proof Consider the spectral decompositions: $y = \sum_{i\in I}\lambda_i p_i$, $z = \sum_{j\in J}\mu_j p_j$, $y' = \sum_{i\in I'}\alpha_i q_i$, and $z' = \sum_{j\in J'}\beta_j q_j$. Set

$$y_n = \sum_{i\in I}\lambda_i p_i + \sum_{j\in J}\frac{1}{n^2\mu_j}p_j + \sum_{k\notin I\cup J}\frac{1}{n}p_k \quad \text{and}$$
$$w_n = \sum_{i\in I'}\alpha_i q_i + \sum_{j\in J'}\frac{1}{n^2\beta_j}q_j + \sum_{k\notin I'\cup J'}\frac{1}{n}q_k.$$

Then $h_{y_n}\to h$ and $h_{w_n}\to h'$ by the proof of [44, Theorem 5.6].

For all $n\geq 1$ large we have that $\|w_n\|_u = \|y'\|_u = 1$, so that

$$d_H(w_n,u) = \log M(w_n/u) + \log M(u/w_n) = \log\|w_n\|_u + \log M(w_n^{-1}/u) = \log\|w_n^{-1}\|_u.$$

Now set $v_n = w_n^{-1}/\|w_n^{-1}\|_u$ and note that by (2.3),

$$\begin{aligned} H(h',h) &= \lim_{n\to\infty} d_H(w_n,u) + h(w_n) \\ &= \lim_{n\to\infty} \log\|w_n^{-1}\|_u + \log M(y/w_n) + \log M(z/w_n^{-1}) \\ &= \lim_{n\to\infty} \log M(y/w_n) + \log M(z/v_n^{-1}). \end{aligned}$$

Clearly $w_{n+1}\leq w_n$ and $w_n\to y'$. Also,

$$w_n^{-1} = \sum_{i\in I'}\alpha_i^{-1}q_i + \sum_{j\in J'}n^2\beta_j q_j + \sum_{k\notin I'\cup J'}nq_k.$$

So, for all $n\geq 1$ large, we have that $\|w_n^{-1}\|_u = n^2$, as $\max_{j\in J}\beta_j = \|z'\|_u = 1$. It follows that

$$v_n = \sum_{i\in I'}\frac{1}{n^2\alpha_i}q_i + \sum_{j\in J'}\beta_j q_j + \sum_{k\notin I'\cup J'}\frac{1}{n}q_k$$

for all $n\geq 1$ large. So, $v_{n+1}\leq v_n$ for all $n\geq 1$ large and $v_n\to z'$. It now follows from Lemma 5.3 that $H(h',h)=\infty$ if y' does not dominate y, or, z' does not dominate z. Moreover, if y' dominates y, and, z' dominates z, then $H(h',h) = \log M(y/y') + \log M(z/z')$.

Interchanging the roles between h and h' we find that $H(h,h')=\infty$ if y does not dominate y', or, z does not dominate z', and $H(h,h') = \log M(y'/y) + \log M(z'/z)$, otherwise. Thus, $\delta(h,h') = d_H(y,y') + d_H(z,z')$ if and only if $y\sim y'$ and $z\sim z'$, and $\delta(h,h')=\infty$ otherwise. $\square$

The characterisation of the parts and the detour distance is a more explicit description of the general one one given in [43, Theorem 4.9] in the case of symmetric Hilbert geometries.

5.3 The Homeomorphism

Let us now define a map $\varphi_H : \overline{\Omega}_V^h \to B_1^*$, where B_1^* is the unit ball of the dual norm of $|\cdot|_u$ on $V/\mathbb{R}u$. For $x \in \Omega_V$ let

$$\varphi_H(x) = \frac{x}{\mathrm{tr}(x)} - \frac{x^{-1}}{\mathrm{tr}(x^{-1})},$$

and for $h \in \partial\overline{\Omega}_V^h$ given by (5.4) let

$$\varphi_H(h) = \frac{y}{\mathrm{tr}(y)} - \frac{z}{\mathrm{tr}(z)}.$$

We note that $\varphi_H(h)$ is well-defined by Proposition 5.4.

We will prove the following theorem in the sequel.

Theorem 5.5 *If (Ω_V, d_H) is a symmetric Hilbert geometry, then the map $\varphi_H : \overline{\Omega}_V^h \to B_1^*$ is a homeomorphism which maps each part of $\partial\overline{\Omega}_V^h$ onto the relative interior of a boundary face of B_1^*.*

We first analyse the dual unit ball B_1^* of $|\cdot|_u$ and its facial structure. The following fact, which can be found in [45, Sect. 2.2], will be useful.

Lemma 5.6 *Given an order-unit space (V, V_+, u), the norm $|\cdot|_u$ on $V/\mathbb{R}u$ coincides with the quotient norm of $2\|\cdot\|_u$ on $V/\mathbb{R}u$.*

Recall that in a Euclidean Jordan algebra V each x has a unique orthogonal decomposition $x = x^+ - x^-$, where x^+ and x^- are orthogonal elements in V_+, see [4, Proposition 1.28]. Let

$$\mathbb{R}u^\perp = \{x \in V : (u|x) = 0\} = \{x \in V : \mathrm{tr}(x^+) = \mathrm{tr}(x^-)\}.$$

It follows from Lemma 5.6 that

$$(V/\mathbb{R}u, |\cdot|_u)^* = (\mathbb{R}u^\perp, \frac{1}{2}\|\cdot\|_u^*).$$

So the dual unit ball B_1^* in $\mathbb{R}u^\perp$ is given by

$$B_1^* = 2\mathrm{conv}(S(V) \cup -S(V)) \cap \mathbb{R}u^\perp,$$

see [3, Theorem 1.19], and its (closed) boundary faces are precisely the nonempty sets of the form

$$A_{p,q} = 2\mathrm{conv}\left((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V))\right) \cap \mathbb{R}u^\perp,$$

where p and q are orthogonal idempotents, see [18, Theorem 4.4].

To prove Theorem 5.5 we collect a number of preliminary results.

Lemma 5.7 *For each $x \in \Omega_V$ we have that $\varphi_H(x) \in \operatorname{int} B_1^*$, and for each $h \in \partial\overline{\Omega}_V^h$ we have that $\varphi_H(h) \in \partial B_1^*$.*

Proof Let $x = \sum_{i=1}^r \lambda_i p_i \in \Omega_V$, so $\lambda_i > 0$ for all i. Note that $(u|\varphi_H(x)) = 1 - 1 = 0$ and hence $\varphi_H(x) \in \mathbb{R}u^\perp$. Given $-u \leq z \leq u$, we have the Peirce decomposition of z with respect to the frame $p_1, \ldots, p_r$,

$$z = \sum_{i=1}^r \sigma_i p_i + \sum_{i<j} z_{ij},$$

with $-1 = -(u|p_i) \leq \sigma_i = (z|p_i) \leq (u|p_i) = 1$. As this is an orthogonal decomposition we have that

$$(z|\varphi_H(x)) = \sum_{i=1}^r \sigma_i \left(\frac{\lambda_i}{\sum_{j=1}^r \lambda_j} - \frac{\lambda_i^{-1}}{\sum_{j=1}^r \lambda_j^{-1}} \right) < \sum_{i=1}^r \left(\frac{\lambda_i}{\sum_{j=1}^r \lambda_j} \right) + \sum_{i=1}^r \left(\frac{\lambda_i^{-1}}{\sum_{j=1}^r \lambda_j^{-1}} \right) = 2.$$

This implies that $\frac{1}{2}\|\varphi_H(x)\|_u^* = \frac{1}{2}\sup_{-u\leq z\leq u}(z|\varphi_H(x)) < 1$, hence $\varphi_H(x) \in \operatorname{int} B_1^*$.

To prove the second assertion let h be a horofunction given by $h(x) = \log M(y/x) + \log M(z/x^{-1})$, where $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$. Write $y = \sum_{i\in I}\alpha_i q_i$ and $z = \sum_{j\in J}\beta_j q_j$. If we now let $q_I = \sum_{i\in I} q_i$ and $q_J = \sum_{j\in J} q_j$, then $-u \leq q_I - q_J \leq u$ and

$$\|\varphi_H(h)\|_u^* \geq \frac{1}{2}(q_I - q_J|\varphi_H(h)) = (1+1)/2 = 1.$$

Moreover, for each $-u \leq w \leq u$ we have that

$$|(w|\varphi_H(h))| \leq |(w|y/\operatorname{tr}(y))| + |(w|z/\operatorname{tr}(z))| \leq (u|y/\operatorname{tr}(y)) + (u|z/\operatorname{tr}(z)) = 2.$$

Combining the inequalities shows that $\varphi_H(h) \in \partial B_1^*$. □

To prove injectivity of φ_H on Ω_V we need the following lemma, which has a proof similar to the one of Lemma 3.6 given in [32, Sect. 4].

Lemma 5.8 *Let $\mu_i\colon \mathbb{R}^r \to \mathbb{R}$, for $i = 1, 2$, be given by $\mu_1(x) = \sum_{i=1}^r e^{x_i}$ and $\mu_2(x) = \sum_{i=1}^r e^{-x_i}$ for $x \in \mathbb{R}^r$, and let $g\colon x \mapsto \log\mu_1(x) + \log\mu_2(x)$. If $x, y \in \mathbb{R}^r$ are such that $y \neq x + c(1, \ldots, 1)$ for all $c \in \mathbb{R}$, then $\nabla g(x) \neq \nabla g(y)$.*

Lemma 5.9 *The map φ_H is injective on Ω_V.*

Proof Suppose that $\varphi_H(x) = \varphi_H(y)$, where $x = \sum_{i=1}^r \lambda_i p_i$ and $y = \sum_{i=1}^r \mu_i q_i$ in Ω_V. Note that $0 < \lambda_i, \mu_i$ for all i and $(x|u) = \operatorname{tr}(x) = r = \operatorname{tr}(y) = (y|u)$. After possibly relabelling we can write

$$\varphi_H(x) = \sum_{i=1}^r \left(\frac{\lambda_i}{\sum_{j=1}^r \lambda_j} - \frac{\lambda_i^{-1}}{\sum_{j=1}^r \lambda_j^{-1}} \right) p_i = \sum_{i=1}^r \alpha_i p_i$$

and

$$\varphi_H(y) = \sum_{i=1}^r \left(\frac{\mu_i}{\sum_{j=1}^r \mu_j} - \frac{\mu_i^{-1}}{\sum_{j=1}^r \mu_j^{-1}} \right) q_i = \sum_{i=1}^r \beta_i q_i,$$

where $\alpha_1 \leq \ldots \leq \alpha_r$ and $\beta_1 \leq \ldots \leq \beta_r$. By the spectral theorem (version 2) [25] we conclude that $\alpha_i = \beta_i$ for all i.

Consider the injective map $\mathrm{Log}\colon \operatorname{int} \mathbb{R}^r_+ \to \mathbb{R}^r$ given by $\mathrm{Log}(\gamma) = (\log \gamma_1, \ldots, \log \gamma_r)$. Let $\Delta = \{\gamma \in \operatorname{int} \mathbb{R}^r_+ \colon \sum_{i=1}^r \gamma_i = r\}$. The map $(\nabla g) \circ \mathrm{Log}$ is injective on Δ by Lemma 5.8 and

$$\nabla g(\mathrm{Log}(\gamma)) = \left(\frac{\gamma_1}{\sum_{i=1}^r \gamma_i} - \frac{\gamma_1^{-1}}{\sum_{i=1}^r \gamma_i^{-1}}, \ldots, \frac{\gamma_r}{\sum_{i=1}^r \gamma_i} - \frac{\gamma_r^{-1}}{\sum_{i=1}^r \gamma_i^{-1}} \right).$$

Writing $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$, we have that $\lambda, \mu \in \Delta$ and

$$\nabla g(\mathrm{Log}(\lambda)) = (\alpha_1, \ldots, \alpha_r) = (\beta_1, \ldots, \beta_r) = \nabla g(\mathrm{Log}(\mu)),$$

so that $\lambda = \mu$.

As $(\nabla g) \circ \mathrm{Log}$ is injective on Δ, we also know that $\alpha_k = \alpha_{k+1}$ if and only if $\lambda_k = \lambda_{k+1}$. Likewise, $\beta_k = \beta_{k+1}$ if and only if $\mu_k = \mu_{k+1}$. From the spectral theorem (version 1) [25] we now conclude that $x = y$. □

In the next couple of lemmas we show that φ_H is onto.

Lemma 5.10 *The map φ_H maps Ω_V onto* $\operatorname{int} B_1^*$.

Proof Note that Ω_V is an open set of the affine space $\{x \in V \colon \operatorname{tr}(x) = r\}$, which has dimension $\dim V - 1$. Also $B_1^* \subset \mathbb{R}u^\perp$ has dimension $\dim V - 1$. As φ_H is a continuous injection from Ω_V into $\operatorname{int} B_1^*$ by Lemmas 5.7 and 5.9, we know that $\varphi_H(\Omega_V)$ is a open subset of $\operatorname{int} B_1^*$ by Brouwer's invariance of domain theorem. We now argue by contradiction. So, suppose that $\varphi_V(\Omega_V) \neq \operatorname{int} B_1^*$. There then exists a $w \in \partial \varphi_H(\Omega_V) \cap \operatorname{int} B_1^*$. Let (v_n) in Ω_V be such that $\varphi_H(v_n) \to w$.

As φ_H is continuous on Ω_V, we may assume that $d_H(v_n, u) \to \infty$. After taking a subsequence, we may also assume that $v_n \to v \in \partial \Omega_V$. Now let $y_n = v_n / \|v_n\|_u$ and set $y = v/\|v\|_u$. Furthermore, let $z_n = y_n^{-1} / \|y_n^{-1}\|_u$. After taking subsequences we may assume that $z_n \to z \in \partial V_+$ and $y_n \to y \in \partial V_+$, so $\|y\|_u = \|z\|_u = 1$. As $y_n \bullet z_n = u / \|y_n^{-1}\|_u \to 0$, we find that $y \bullet z = 0$, which implies that $(y|z) = 0$.

Using the spectral decomposition we write $y_n = \sum_{i=1}^r \lambda_i^n p_i^n$ and $y = \sum_{i \in I} \lambda_i p_i$, where $\lambda_i > 0$ for all $i \in I$. Likewise, we let $z_n = \sum_{i=1}^r \mu_i^n p_i^n$ and $z = \sum_{j \in J} \mu_j p_j$ with $\mu_j > 0$ for all $j \in J$. Note that $\mu_i^n = (\lambda_i^n)^{-1} / \|y_n^{-1}\|_u$.

Then

$$\varphi_h(v_n) = \frac{\sum_{i=1}^r \lambda_i^n p_i^n}{\sum_{k=1}^r \lambda_k^n} - \frac{\sum_{i=1}^r (\lambda_i^n)^{-1} p_i^n}{\sum_{k=1}^r (\lambda_k^n)^{-1}} = \frac{\sum_{i=1}^r \lambda_i^n p_i^n}{\sum_{k=1}^r \lambda_k^n} - \frac{\sum_{i=1}^r \mu_i^n p_i^n}{\sum_{k=1}^r \mu_k^n} \to \frac{\sum_{i\in I} \lambda_i p_i}{\sum_{k\in I} \lambda_k}$$
$$- \frac{\sum_{j\in J} \mu_j p_j}{\sum_{k\in J} \mu_j} = w.$$

Now let $w^* = \sum_{i\in I} p_i - \sum_{j\in J} p_j$ and note that $-u \le w^* \le u$, as $(y|z) = 0$. We find that

$$\frac{1}{2}\|w\|_u^* \ge \frac{1}{2}(w|w^*) = (1+1)/2 = 1,$$

hence $w \in \partial B_1^*$, which is a contradiction. ◻

Lemma 5.11 *The map φ_H maps $\partial\overline{\Omega}_V^h$ onto ∂B_1^*.*

Proof We know from Lemma 5.7 that φ_H maps $\partial\overline{\Omega}_V^h$ into ∂B_1^*. To prove that it is onto let $w \in \partial B_1^*$. Then there exists a face, say

$$A_{p,q} = 2\mathrm{conv}\,((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V))) \cap \mathbb{R}u^\perp$$

where p and q are orthogonal idempotents, such that w is in the relative interior of $A_{p,q}$, as B_1^* is the disjoint union of the relative interiors of its faces [52, Theorem 18.2]. So,

$$w = \sum_{i\in I} \alpha_i p_i - \sum_{j\in J} \beta_j q_j,$$

where $\alpha_i > 0$ for all $i \in I$, $\beta_j > 0$ for all $j \in J$, and $\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j = 2$. Moreover, $\sum_{i\in I} p_i = p$ and $\sum_{j\in J} q_j = q$.

As $w \in \mathbb{R}u^\perp$, we have that $0 = (u|w) = \sum_{i\in I} \alpha_i - \sum_{j\in J} \beta_j$, hence $\sum_{i\in I} \alpha_i = \sum_{j\in J} \beta_j = 1$.

Put $\alpha^* = \max_{i\in I} \alpha_i$ and $\beta^* = \max_{j\in J} \beta_j$. Furthermore, for $i \in I$ set $\lambda_i = \alpha_i/\alpha^*$ and for $j \in J$ set $\mu_j = \beta_j/\beta^*$. Then

$$w = \left(\frac{\sum_{i\in I} \alpha_i p_i}{\sum_{k\in I} \alpha_k}\right) - \left(\frac{\sum_{j\in J} \beta_j q_j}{\sum_{k\in J} \beta_k}\right) = \left(\frac{\sum_{i\in I} \lambda_i p_i}{\sum_{k\in I} \lambda_k}\right) - \left(\frac{\sum_{j\in J} \mu_j q_j}{\sum_{k\in J} \mu_k}\right).$$

Note that $0 < \lambda_i \le 1$ for all $i \in I$ and $\max_{i\in I} \lambda_i = 1$. Likewise, $0 < \mu_j \le 1$ for all $j \in J$ and $\max_{j\in J} \beta_j = 1$.

Now let $y = \sum_{i\in I} \lambda_i p_i$ and $z = \sum_{j\in J} \mu_j q_j$. Then $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$. Furthermore, if we let $h\colon \Omega_V \to \mathbb{R}$ be given by

$$h(x) = \log M(y/x) + \log M(z/x^{-1})$$

for $x \in \Omega_V$, then h is a horofunction by Theorem 5.2 and $\varphi_H(h) = w$, which completes the proof. □

We already saw in Lemma 5.10 that φ_H is injective on Ω_V. The next lemma shows that φ_H is injective on $\overline{\Omega}_V^h$.

Lemma 5.12 *The map* $\varphi_H : \overline{\Omega}_V^h \to B_1^*$ *is injective.*

Proof We know from Lemma 5.10 that φ_H is injective on Ω_V. So, it remains to show that if $h, h' \in \partial\overline{\Omega}_V^h$ and $\varphi_H(h) = \varphi_H(h')$, then $h = h'$.

Suppose $h(x) = \log M(y/x) + \log M(z/x^{-1})$ and $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$ for all $x \in \Omega_V$. Then

$$\varphi_H(h) = \frac{y}{\operatorname{tr}(y)} - \frac{z}{\operatorname{tr}(z)} = \frac{y'}{\operatorname{tr}(y')} - \frac{z'}{\operatorname{tr}(z')} = \varphi_H(h').$$

Using the fact that the orthogonal decomposition of an element in V is unique, see [4, Proposition 1.26], we conclude that

$$\frac{y}{\operatorname{tr}(y)} = \frac{y'}{\operatorname{tr}(y')} \quad \text{and} \quad \frac{z}{\operatorname{tr}(z)} = \frac{z'}{\operatorname{tr}(z')}.$$

As $\|y\|_u = \|y'\|_u = 1$, we get that $\operatorname{tr}(y) = \operatorname{tr}(y')$, and hence $y = y'$. Likewise, $\|z\|_u = \|z'\|_u = 1$ implies that $z = z'$, hence $h = h'$. □

5.4 Proof of Theorem 5.5

Before we prove Theorem 5.5, we recall a fact from Jordan theory. For $x, z \in V$ let $[x, z] = \{y \in V : x \le y \le z\}$ be the order-interval. Given $y \in V_+$ we write

$$\operatorname{face}(y) = \{x \in V_+ : x \le \lambda y \text{ for some } \lambda \ge 0\}.$$

In a Euclidean Jordan algebra V every idempotent p satisfies

$$\operatorname{face}(p) \cap [0, u] = [0, p],$$

see [4, Lemma 1.39]. Also note that $y \sim y'$ if and only if $\operatorname{face}(y) = \operatorname{face}(y')$.

Proof of Theorem 5.5 We know from the results in the previous subsection that $\varphi_H : \overline{\Omega}_V^h \to B_1^*$ is a bijection, which is continuous on Ω_V.

To prove continuity of φ_H on the whole of $\overline{\Omega}_V^h$ we first show that if (v_n) in Ω_V is such that $h_{v_n} \to h \in \partial\overline{\Omega}_V^h$, then $\varphi_H(v_n) \to \varphi_H(h)$. Let $h(x) = \log M(y/x) + \log M(z/x^{-1})$ for $x \in \Omega_V$, where $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$. For $n \ge 1$ let $y_n = v_n/\|v_n\|_u$ and note that $\varphi_H(v_n) = \varphi_H(y_n)$ for all n. Let $w_k = \varphi_H(v_{n_k})$, $k \ge 1$ be a subsequence of $(\varphi_H(v_n))$. We need to show that (w_k) has a subsequence that converges to $\varphi_H(h)$.

As h is a horofunction and (Ω_V, d_H) is a proper metric space, $d_H(v_n, u) = d_H(y_n, u) \to \infty$ by Lemma 2.1. It follows that (y_{n_k}) has a subsequence (y_{k_m}) with $y_{k_m} \to y' \in \partial V_+$ and $z_{k_m} = y_{k_m}^{-1}/\|y_{k_m}^{-1}\|_u \to z' \in V_+$. Note that as $y \in \partial V_+$, we have that $\|y_{k_m}^{-1}\|_u \to \infty$. This implies that

$$y' \bullet z' = \lim_{m\to\infty} y_{k_m} \bullet \frac{y_{k_m}^{-1}}{\|y_{k_m}^{-1}\|_u} = \lim_{m\to\infty} \frac{u}{\|y_{k_m}^{-1}\|_u} = 0,$$

hence $(y'|z') = 0$ (see [25, III, Exercise 3.3]) and $z' \in \partial V_+$. For $x \in \Omega_V$,

$$\begin{aligned}
\lim_{m\to\infty} h_{y_{k_m}}(x) &= \lim_{m\to\infty} \log M(y_{k_m}/x) + \log M(x/y_{k_m}) - \log M(y_{k_m}/u) - \log M(u/y_{k_m}) \\
&= \lim_{m\to\infty} \log M(y_{k_m}/x) + \log M(y_{k_m}^{-1}/x^{-1}) - \log \|y_{k_m}^{-1}\|_u \\
&= \lim_{m\to\infty} \log M(y_{k_m}/x) + \log M(z_{k_m}/x^{-1}) \\
&= \log M(y'/x) + \log M(z'/x^{-1}).
\end{aligned}$$

So, if we let $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$, then h' is a horofunction by Theorem 5.2 and $h_{y_{k_m}} \to h'$. As $h = h'$, we know that $\delta(h, h') = d_H(y, y') + d_H(z, z') = 0$, hence $y = y'$ and $z = z'$. It follows that

$$\varphi_H(v_{k_m}) = \varphi_H(y_{k_m}) = \frac{y_{k_m}}{\operatorname{tr}(y_{k_m})} - \frac{y_{k_m}^{-1}}{\operatorname{tr}(y_{k_m}^{-1})} \to \frac{y}{\operatorname{tr}(y)} - \frac{z}{\operatorname{tr}(z)} = \varphi_H(h).$$

Recall that φ_H maps Ω_V into int B_1^* and φ_H maps $\partial\overline{\Omega_V}^h$ into ∂B_1^* by Lemma 5.7. So, to prove the continuity of φ_H it remains to show that if (h_n) is a sequence in $\partial\overline{\Omega_V}^h$ converging to $h \in \partial\overline{\Omega_V}^h$, then $\varphi_H(h_n) \to \varphi_H(h)$.

Let $(\varphi_H(h_{n_k}))$ be a subsequence of $(\varphi_H(h_n))$. We show that it has a subsequence $(\varphi_H(h_{k_m}))$ converging to $\varphi_H(h)$. We know there exists $v_m, w_m \in \partial V_+$, with $\|v_m\|_u = \|w_m\|_u = 1$ and $(v_m|w_m) = 0$ such that

$$h_{k_m}(x) = \log M(v_m/x) + \log M(w_m/x^{-1})$$

for $x \in \Omega_V$. By taking a further subsequence we may assume that $v_m \to v \in \partial V_+$ and $w_m \to w \in \partial V_+$. Then $\|v\|_u = \|w\|_u = 1$ and $(v|w) = 0$. Moreover,

$$\log M(v_m/x) \to \log M(v/x) \quad \text{and} \quad \log M(w_m/x^{-1}) \to \log M(w/x^{-1})$$

for each $x \in \Omega_V$, as $y \mapsto M(y/x)$ is continuous on V, see [44, Lemma 2.2]. Thus, $h_{k_m} \to h^* \in \partial\overline{\Omega_V}^h$, where

$$h^*(x) = \log M(v/x) + \log M(w/x^{-1}),$$

by Theorem 5.2. As $h_n \to h$, we have that $h = h^*$. This implies that $y = v$ and $z = w$ by Proposition 5.4. Thus, $v_m \to y$ and $w_m \to z$, hence

$$\varphi_H(h_{k_m}) = \frac{v_m}{\mathrm{tr}(v_m)} - \frac{w_m}{\mathrm{tr}(w_m)} \to \frac{y}{\mathrm{tr}(y)} - \frac{z}{\mathrm{tr}(z)} = \varphi_H(h).$$

This completes the proof of the continuity of φ_H.

As $\varphi_H \colon \overline{\Omega}_V^h \to B_1^*$ is a continuous bijection, $\overline{\Omega}_V^h$ is compact, and B_1^* is Hausdorff, we conclude that φ_H is a homeomorphism.

To prove the second assertion let $h(x) = \log M(y/x) + \log M(z/x^{-1})$ be a horofunction, where $y = \sum_{i \in I} \lambda_i p_i$ and $z = \sum_{j \in J} \mu_j p_j$ with $\lambda_i, \mu_j > 0$ for all $i \in I$ and $j \in J$. Let $p_I = \sum_{i \in I} p_i$ and $p_J = \sum_{j \in J} p_j$. As φ_H is surjective, it suffices to show that φ_H maps $\mathcal{P}_h$ into the relative interior of

$$A_{p_I,p_J} = 2\mathrm{conv}\,((U_{p_I}(V) \cap S(V)) \cup (U_{p_J}(V) \cap -S(V))) \cap \mathbb{R}u^{\perp}.$$

So, let $h' \in \mathcal{P}_h$ where $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$ for $x \in \Omega_V$. Then $p_I \sim y \sim y'$ and $p_J \sim z \sim z'$. Using the spectral decomposition write $y' = \sum_{i \in I'} \alpha_i q_i$ and $z' = \sum_{j \in J'} \beta_j q_j$, where $\alpha_i > 0$ for all $i \in I'$ and $\beta_j > 0$ for all $j \in J'$. Now let $q_{I'} = \sum_{i \in I'} q_i$ and $q_{J'} = \sum_{j \in J'} q_j$. It follows that $p_I \sim q_{I'}$ and $p_J \sim q_{J'}$. So, $\mathrm{face}(p_I) = \mathrm{face}(q_{I'})$ and $\mathrm{face}(p_J) = \mathrm{face}(q_{J'})$. As $\mathrm{face}(p_I) \cap [0, u] = [0, p_I]$ and $\mathrm{face}(q_{I'}) \cap [0, u] = [0, q_{I'}]$ by [4, Lemma 1.39], we conclude that $p_I = q_{I'}$. In the same way we get that $p_J = q_{J'}$. As $\alpha_i > 0$ for all $i \in I'$ and $\beta_j > 0$ for all $j \in J'$, we have that

$$\varphi_H(h') = \frac{y'}{\mathrm{tr}(y')} - \frac{z'}{\mathrm{tr}(z')}$$

is in the relative interior of $A_{q_{I'},q_{J'}} = A_{p_I,p_J}$. □

6 Final Remarks

Besides the problem posed by Kapovich and Leeb [34, Question 6.18] for finite dimensional normed spaces the results in this paper show that there should be milage in analysing the following problem.

Problem 6.1 *Suppose $X = G/K$ is a noncompact type symmetric space with a G-invariant Finsler metric. When does there exist a homeomorphism between the horofunction compactification of X with basepoint b under the Finsler distance, and the closed dual unit ball B_1^* of the Finsler metric in the tangent space at b, which maps each part in the horofunction boundary onto the relative interior of a boundary face of B_1^*?*

In a sequel to this paper [41] the first author has shown for various classes of noncompact type symmetric spaces X with invariant Finsler distances coming from symmetric cones that the exponential map $\exp_b$ from the tangent space T_b at the basepoint b onto

X extends as a homeomorphism between the horofunction compactification of T_b as a normed space under the Finsler metric and the horofunction compactification of X under the Finsler distance. Moreover, the extension of the exponential map preserves the parts in the horofunction boundaries. In particular, this is true for symmetric Hilbert geometries (Ω_V, d_H) and the normed spaces $(V/\mathbb{R}u, |\cdot|_u)$. It would be interesting to know if this is true for all noncompact type symmetric spaces with invariant Finsler metrics.

References

1. Abate, M.: Iteration Theory of Holomorphic Maps on Taut Manifolds. Mediterranean Press, Cosenza (1989)
2. Akian, M., Gaubert, S., Walsh, C.: The max-plus Martin boundary. Doc. Math. **14**, 195–240 (2009)
3. Alfsen, E.M., Shultz, F.W.: State Spaces of Operator Algebras. Basic Theory, Orientations, and C^*-Products. Mathematics: Theory & Applications. Birkhäuser, Boston (2001)
4. Alfsen, E.M., Shultz, F.W.: Geometry of State Spaces of Operator Algebras. Mathematics: Theory & Applications. Birkhäuser, Boston (2003)
5. Arosio, L., Fiacchi, M., Gontard, S., Guerini, L.: The horofunction boundary of a Gromov hyperbolic space. Math. Ann. (2020, to appear). arXiv:2012.09848v1
6. Ballmann, W.: Lectures on Spaces of Nonpositive Curvature. DMV-Seminar, Bd. 25. Birkhäuser, Boston (1995)
7. Balogh, Z.M., Bonk, M.: Gromov hyperbolicity and the Kobayashi metric on strictly pseudo convex domains. Comment. Math. Helv. **75**, 504–533 (2000)
8. Beardon, A.F.: The dynamics of contractions. Ergod. Theory Dyn. Syst. **17**, 1257–1266 (1997)
9. Borel, A., Ji, L.: Compactifications of Symmetric and Locally Symmetric Spaces. Birkhäuser, Boston (2006)
10. Bracci, F., Gaussier, H.: Horosphere topology. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **20**(1), 239–289 (2020)
11. Bracci, F., Gaussier, H., Zimmer, A.: Homeomorphic extension of quasi-isometries for convex domains in $\mathbb{C}^d$ and iteration theory. Math. Ann. **379**(1–2), 691–718 (2021)
12. Bridson, M.R., Haefliger, A.: Metric Spaces of Non-Positive Curvature, Grundlehren Math. Wiss. 319. Springer, Berlin (1999)
13. Ciobotaru, C., Kramer, L., Schwer, P.: Polyhedral compactifications, I (2020). arXiv:2002.12422
14. Claassens, F.: The horofunction boundary of the infinite dimensional hyperbolic space. Geom. Dedicata **207**, 255–263 (2020)
15. Duchin, M., Fisher, N.: Stars at infinity in Teichmüller space. Geom. Dedicata **213**, 531–545 (2021)
16. Eberlein, P.B.: Geometry of Nonpositively Curved Manifolds. Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1996)
17. Eberlein, P., O'Neill, B.: Visibility manifolds. Pac. J. Math. **46**, 45–109 (1973)
18. Edwards, C.M., Rüttimann, G.T.: On the facial structure of the unit balls in a GL-space and its dual. Math. Proc. Camb. Philos. Soc. **98**(2), 305–322 (1985)
19. Gouëzel, S., Karlsson, A.: Subadditive and multiplicative ergodic theorems. J. Eur. Math. Soc. (JEMS) **22**(6), 1893–1915 (2020)

20. Greenfield, M., Ji, L.: Metrics and compactifications of Teichmüller spaces of flat tori. Asian J. Math. **25**(4), 477–504 (2021)
21. Gromov, M.: Hyperbolic manifolds, groups and actions. In: Riemann Surfaces and Related Topics (Stony Brook, NY, 1978). Annals of Mathematical Studies, vol. 97. Princeton University Press, Princeton (1981)
22. Gutiérrez, A.W.: The horofunction boundary of finite-dimensional ℓ_p spaces. Colloq. Math. **155**(1), 51–65 (2019)
23. Gutiérrez, A.W.: On the metric compactification of infinite dimensional ℓ_p spaces. Can. Math. Bull. **62**(3), 491–507 (2019)
24. Gutiérrez, A.W.: Characterizing the metric compactification of L_p spaces by random measures. Ann. Funct. Anal. **11**(2), 227–243 (2020)
25. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford Mathematical Monographs. Oxford University Press, New York (1994)
26. Friedland, S., Freitas, P.J.: p-metrics on $\mathrm{GL}(n, \mathbb{C})/U_n$ and their Busemann compactifications. Linear Algebra Appl. **376**, 1–18 (2004)
27. Guivarc'h, Y., Ji, L., Taylor, J.C.: Compactifications of Symmetric Spaces. Birkhäuser, Boston (1998)
28. Haettel, T., Schilling, A., Walsh, C., Wienhard, A.: Horofunction compactifications of symmetric spaces (2018). arXiv:1705.05026v2
29. Hanche-Olsen, H., Stormer, E.: Jordan Operator Algebras. Monographs and Studies in Mathematics, vol. 21. Pitman, Boston (1984)
30. Hirzebruch, U.: Über Jordan-Algebren und kompakte Riemannsche symmetrische Räume vom Rang 1. Math. Z. **90**, 339–354 (1965)
31. Ji, L.: Satake and Martin compactifications of symmetric spaces are topological balls. Math. Res. Lett. **4**, 79–89 (1997)
32. Ji, L., Schilling, A.: Polyhedral horofunction compactification as polyhedral ball (2016). arXiv:1607.00564v2
33. Ji, L., Schilling, A.: Toric varieties vs. horofunction compactifications of polyhedral norms. Enseign. Math. **63**(3?4), 375–401 (2017)
34. Kapovich, M., Leeb, B.: Finsler bordifications of symmetric and certain locally symmetric spaces. Geom. Topol. **22**(5), 2533–2646 (2018)
35. Karlsson, A.: Non-expanding maps and Busemann functions. Ergod. Theory Dyn. Syst. **21**, 1447–1457 (2001)
36. Karlsson, A.: Two extensions of Thurston's spectral theorem for surface diffeomorphisms. Bull. Lond. Math. Soc. **46**(2), 217–226 (2014)
37. Karlsson, A., Metz, V., Noskov, G.A.: Horoballs in simplices and Minkowski spaces. Int. J. Math. Math. Sci. **2006**, Art. ID 23656 (2006)
38. Kobayashi, S.: Hyperbolic Complex Spaces. Grundlehren der Mathematischen Wissenschaften, 318. Springer, Berlin (1998)
39. Korányi, A.: Remarks on the Satake compactifications. Pure Appl. Math. Q. **1**, 851–866 (2005)
40. Lemmens, B.: A metric version of Poincaré's theorem concerning biholomorphic inequivalence of domains. J. Geom. Anal. **32**(5), Paper No. 160 (2022)
41. Lemmens, B.: Horofunction compactifications of symmetric cones under Finsler distances (2021). arXiv:2111.12468
42. Lemmens, B., Nussbaum, R.: Nonlinear Perron–Frobenius Theory. Cambridge Tracts in Mathematics, vol. 189. Cambridge University Press, Cambridge (2012)
43. Lemmens, B., Walsh, C.: Isometries of polyhedral Hilbert geometries. J. Topol. Anal. **3**(2), 213–241 (2011)
44. Lemmens, B., Lins, B., Nussbaum, R., Wortel, M.: Denjoy–Wolff theorems for Hilbert's and Thompson's metric spaces. J. Anal. Math. **134**(2), 671–718 (2018)
45. Lemmens, B., Roelands, M., Wortel, M.: Hilbert and Thompson isometries on cones in JB-algebras. Math. Z. **292**(3–4), 1511–1547 (2019)
46. Miyachi, H.: Extremal length boundary of the Teichmüller space contains non-Busemann points. Trans. Am. Math. Soc. **366**(10), 5409–5430 (2014)
47. Munkres, J.R.: Topology, 2nd edn. Prentice-Hall, Upper Saddle River (2000)
48. Nussbaum, R.D.: Finsler structures for the part metric and Hilbert's projective metric and applications to ordinary differential equations. Differ. Integral Equ. **7**(5–6), 1649–1707 (1994)

49. Papadopoulos, P., Troyanov, M.: Handbook of Hilbert Geometry. IRMA Lectures in Mathematics and Theoretical Physics, vol. 22. European Mathematical Society (EMS), Zürich (2014)
50. Planche, P.: Géométrie de Finsler sur les Espaces Symétriques. PhD thesis, Université de Genève (1995)
51. Rieffel, M.A.: Group C^*-algebras as compact quantum metric spaces. Doc. Math. **7**, 605–651 (2002)
52. Rockafellar, R.T.: Convex Analysis. Princeton Landmarks in Mathematics. Princeton University Press, Princeton (1997)
53. Schilling, A.-S.: The Horofunction Compactification of Finite-Dimensional Normed Vector Spaces and of Symmetric Spaces. PhD thesis, Ruprecht-Karls-Universität Heidelberg (2021)
54. Walsh, C.: The horofunction boundary of finite-dimensional normed spaces. Math. Proc. Camb. Philos. Soc. **142**(3), 497–507 (2007)
55. Walsh, C.: The horofunction boundary of the Hilbert geometry. Adv. Geom. **8**(4), 503–529 (2008)
56. Walsh, C.: The horoboundary and isometry group of Thurston's Lipschitz metric. In: Papadopoulos, A. (ed.) Handbook of Teichmüller Theory, vol. IV. IRMA Lecture Notes in Mathematics and Theoretical Physics 19. European Mathematical Society, Helsinki (2013)
57. Walsh, C.: Gauge-reversing maps on cones, and Hilbert and Thompson isometries. Geom. Topol. **22**(1), 55–104 (2018)
58. Walsh, C.: Hilbert and Thompson geometries isometric to infinite-dimensional Banach spaces. Ann. Inst. Fourier (Grenoble) **68**(5), 1831–1877 (2018)
59. Zimmer, A.M.: Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type. Math. Ann. **365**(3–4), 1425–1498 (2016)