



On the Curvature of the Bismut Connection: Bismut–Yamabe Problem and Calabi–Yau with Torsion Metrics

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Abstract

We study two natural problems concerning the scalar and the Ricci curvatures of the Bismut connection. Firstly, we study an analog of the Yamabe problem for Hermitian manifolds related to the Bismut scalar curvature, proving that, fixed a conformal Hermitian structure on a compact complex manifold, there exists a metric with constant Bismut scalar curvature in that class when the expected constant scalar curvature is non-negative. A similar result is given in the general case of Gauduchon connections. We then study an Einstein-type condition for the Bismut Ricci curvature tensor on principal bundles over Hermitian manifolds with complex tori as fibers. Thanks to this analysis, we construct explicit examples of Calabi–Yau with torsion Hermitian structures and prove a uniqueness result for them.

Keywords Gauduchon–Yamabe problem · Calabi–Yau with torsion structures · Bismut scalar curvature · Bismut Ricci curvature

Mathematics Subject Classification 53b35 · 53C30 · 53C55 · 32Q25 · 32Q99 · 32M10

1 Introduction

Given a Hermitian manifold (M, J, g) , there are several connections which are compatible with both the metric g and the complex structure J , meaning that they leave

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them parallel. Among these, the Bismut connection is the only one which has skew-symmetric torsion. Thanks to this property, it takes on great interest in String Theory, see the work of Ivanov and Papadopoulos [15]. It also has applications in Differential Geometry, see for example [21], and recently, Garcia-Streets and Streets-Tian showed some interesting links with the Generalized Complex Geometry, see [9] and [24].

In view of having a better understanding of the geometry of the Bismut connection, in this note, we study the classical problems of constant scalar curvature and constant Ricci curvature (in the sense that the Ricci tensor is a multiple of the metric) with a focus on the case Ricci flat curvature. In the first part of this note, we adapt the techniques used in [2] for the Chern–Yamabe problem to the general case of Gauduchon connections ∇^t , which are an affine line of Hermitian connections including the Chern connection ∇^{Ch} (for $t = 1$) and the Bismut connections ∇^+ (for $t = -1$). In this way, in Theorems 3.1 and 3.2, we could solve the *Gauduchon–Yamabe* problem of finding a constant ∇^t -scalar curvature metric in a given conformal class if $\Gamma_M^t(\{\omega\}) \leq 0$ or $\Gamma_M^t(\{\omega\}) \geq 0$ depending on $t > \frac{1}{1-n}$ or $t < \frac{1}{1-n}$, where n is the complex dimension of the manifold. Here, $\Gamma_M^t(\{\omega\})$ is the Gauduchon degree with respect to ∇^t associated to the conformal class $\{\omega\}$, which is defined as

$$\Gamma_M^t(\{\omega\}) := \int_M S^t(\eta) d\mu_\eta,$$

where η is the unique volume-one Gauduchon representative of $\{\omega\}$ and $S^t(\eta)$ is the ∇^t -scalar curvature associated to η . The Gauduchon degree with respect to the Chern connection corresponds to the degree of the anti-canonical line bundle K_M^{-1} (studied in [11]), while for $t = -1$, we get the Gauduchon degree of $\{\omega\}$ with respect to the Bismut connection, which we denote by $\Gamma_M^+(\{\omega\})$. Theorems 3.1 and 3.2 extend the results of [2] about the Chern connection and we have the following theorem as a particular case of them.

Theorem (Corollary 3.5) *Let M be a compact complex manifold with $\dim_{\mathbb{C}} M \geq 3$ and Hermitian structure (ω, J) . If $\Gamma_M^+(\{\omega\}) \geq 0$, then, up to scaling, there exists a unique $\tilde{\omega} \in \{\omega\}$ with constant Bismut scalar curvature. Moreover, its Bismut scalar curvature satisfies $S^+(\tilde{\omega}) = \Gamma_M^+(\{\omega\})$.*

The second part of the note is devoted to the study of metrics with constant Bismut Ricci curvature on the total spaces of rank one toric bundles (T^2 -bundles) over Hermitian manifolds. T^2 -fibrations over Calabi–Yau surfaces were used by Fu and Yau to show explicit solutions to the Hull–Strominger system, see [7, 8], while in [12] the authors studied the so-called Calabi–Yau with torsion (CYT) condition on the total spaces of toric bundles over Kähler manifolds. Here, we focus on the vanishing of the Bismut Ricci tensor, describing the CYT condition for metrics that satisfy a natural ansatz on this class of manifolds. We then briefly analyze the Einstein-type equation $(Ric^+\omega)^{1,1} = \lambda\omega$ with $\lambda \in \mathbb{R}$ in the same setting.

The CYT manifolds play a role in Physics after the works of Strominger [25] and Hull [14]. Moreover, CYT metrics which are also pluriclosed (meaning that the Kähler 2-form associated to the metric is dd^c -closed) are static points of the *pluriclosed flow* of Streets and Tian. Therefore, there is an interest in finding new non-trivial explicit

examples of them. In Sect. 4 we analyze these structures on the Calabi–Eckmann manifolds $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ (with $n, m \geq 0$). Indeed, the simplest examples of *Bismut flat* manifolds are given by the Hopf surface $\mathbb{S}^1 \times \mathbb{S}^3$ and the Calabi–Eckmann threefold $\mathbb{S}^3 \times \mathbb{S}^3$, and in [9], the authors asked if it was possible to construct other special Hermitian structures on the Calabi–Eckmann manifolds of higher dimension. The existence of CYT structures on them can be obtained by applying Theorem 3 of [13], while we prove a result about the uniqueness.

Theorem (Corollary 4.8) *Given a Calabi–Eckmann manifold $M_{n,m} := \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$, the standard Hermitian structure (J, g) gives a CYT structure on it. Moreover, g is the only homogeneous CYT structure on $(M_{n,m}, J)$ up to homothety.*

The above theorem comes as a special case of a more general result that has an analogous statement on Class C manifolds (defined in [19]). These are the total spaces of homogeneous principal T^2 -bundles over the product of two compact irreducible Hermitian symmetric spaces.

Theorem (Theorems 4.6 and 4.7) *Take a class C manifold as in [19], that is a product $M_1 \times M_2$ fibering over two generalized flag manifolds $X_1 = G_1/H_1$ and $X_2 = G_2/H_2$ with \mathbb{S}^1 -fibers, and equip it with a standard complex structure. Then, there exists a CYT metric on it. Moreover, if none of the X_i 's is $SO(k+2)/SO(2) \times SO(k)$ for $k \geq 3$, then this metric is the unique (up to homothety) CYT metric among the homogeneous ones.*

The interest in this kind of result comes from the fact that a compact simply connected homogeneous manifold G/H with an invariant complex structure J (called C-space in [26]) is Kählerian if and only if it is a generalized flag manifold, namely when G is a semisimple Lie group and H is the centralizer of a torus in G , as stated in [5]. In such case, in [18] is proved that they can be endowed with a (unique) invariant Kähler–Einstein metric, while there is a general interest in finding special invariant metrics on the non-Kähler C-spaces G/H . Theorem 3 in [13] proves the existence of a CYT structure on compact simply connected homogeneous manifolds G/H with a G -invariant complex structure of vanishing first Chern class. However, Corollary 4.5 ensures that, in general, it is not unique. On the other hand, the standard Hermitian structures on class C manifolds are the unique invariant CYT structure on them.

2 Preliminaries and Notation

In all the following sections, (M, g, J) will be a Hermitian manifold of complex dimension $n \geq 2$, and $\omega := g(J \cdot, \cdot)$ will denote its associated Kähler $(1, 1)$ -form. In particular, in local holomorphic coordinates $\{z_i\}_i$,

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

where $g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$. Moreover, $(g^{i\bar{j}})_{i,j}$ will henceforth denote the inverse of $(g_{i\bar{j}})_{i,j}$.

2.1 Gauduchon Connections

For $t \in \mathbb{R}$, the *Gauduchon connections* ∇^t associated to (g, J) are Hermitian connections on M with prescribed torsion, where by Hermitian connections we mean connections ∇ which are compatible with both the metric and the complex structure, i.e., $\nabla g = \nabla J = 0$. They are described wrt the Levi-Civita connection ∇^{LC} as,

$$g(\nabla_x^t y, z) = g(\nabla_x^{LC} y, z) + \frac{t-1}{4} Jd\omega(x, y, z) + \frac{1+t}{4} d\omega(Jx, y, z),$$

where J acts as $Jd\omega(\cdot, \cdot, \cdot) = -d\omega(J\cdot, J\cdot, J\cdot)$. Since the Levi-Civita connection is torsion free, the above formula is prescribing the torsion of these connections as

$$T^t(x, y, z) = \frac{1-t}{2} Jd\omega(x, y, z) + \frac{1+t}{4} (d\omega(Jx, y, z) + d\omega(x, Jy, z)).$$

In particular, for $t = 1$, we recover the Chern connection, while for $t = -1$, we get the Bismut connection, which thus has torsion equal to $-Jd\omega$. We remark that the Bismut connection is the unique Hermitian connection with totally skew-symmetric torsion. We will henceforth indicate the Bismut connection as ∇^+ , and any of its curvature tensors with the superscript $+$. The Christoffel symbols of the Gauduchon connection can be easily computed and are

$$\begin{aligned} (\Gamma^t)_{ij}^k &= g^{k\bar{s}} \left(\frac{1+t}{2} \partial_i g_{j\bar{s}} + \frac{1-t}{2} \partial_j g_{i\bar{s}} \right) \\ (\Gamma^t)_{i\bar{j}}^k &= \frac{1-t}{2} g^{k\bar{s}} \left(\bar{\partial}_i g_{j\bar{s}} - \bar{\partial}_s g_{j\bar{i}} \right) \\ (\Gamma^t)_{i\bar{j}}^k &= 0 \end{aligned} \tag{2.1}$$

At a fixed point $p \in M$, we can choose special holomorphic coordinates $\{z_i\}$ such that $g_{i\bar{j}}(p) = \delta_{ij}$ and the Christoffel symbols of the Levi-Civita connection vanish at p , i.e., $(\Gamma^{LC})_{ij}^k(p) = 0$. With these coordinates, we compute the curvature tensors of the Gauduchon connections as

$$\begin{aligned} R_{i\bar{j}k\bar{l}}^t(g) &= -\delta_{p\bar{l}} \left(\frac{\partial}{\partial \bar{z}^j} (\Gamma^t)_{ik}^p - \frac{\partial}{\partial z^i} (\Gamma^t)_{jk}^p + (\Gamma^t)_{ik}^s (\Gamma^t)_{js}^p - (\Gamma^t)_{jk}^s (\Gamma^t)_{is}^p \right) \\ &= \frac{1-t}{2} \left(\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} - \frac{\partial^2 g_{k\bar{j}}}{\partial z_i \partial \bar{z}_l} - \frac{\partial^2 g_{i\bar{l}}}{\partial z_k \partial \bar{z}_j} \right) - \frac{1+t}{2} \frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} \\ &\quad + \sum_q (1-t)^2 \frac{\partial g_{q\bar{l}}}{\partial z_i} \frac{\partial g_{k\bar{j}}}{\partial \bar{z}_q} - t^2 \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{q\bar{l}}}{\partial \bar{z}_j} \end{aligned} \tag{2.2}$$

We define the Ricci tensor associated to the Gauduchon connection ∇^t as the contraction of the endomorphism part of its curvature tensor; hence, in local coordinates,

$$Ric^t(g) = g^{k\bar{l}} R^t_{\cdot k \bar{l}}(g).$$

Contracting again we obtain the ∇^t -scalar curvature

$$S^t(g) = g^{i\bar{j}} Ric^t_{i\bar{j}}(g).$$

From a direct computation using (2.2), we obtain the following useful formula, which is well known for the Bismut connection.

Proposition 2.1 *Let (M, g, J) be a Hermitian manifold of complex dimension n , and let ω be its associated Kähler $(1, 1)$ -form. For $t \in \mathbb{R}$, the Ricci curvature form of the Gauduchon connection ∇^t associated to (g, J) is given by the formula:*

$$Ric^t(g) = \frac{t - 1}{2} dd_g^* \omega - \sqrt{-1} \partial \bar{\partial} \log \omega^n.$$

In particular, the $(1,1)$ -component of the Gauduchon Ricci curvature form satisfies

$$(Ric^t(g))^{1,1} = \frac{t - 1}{2} (\partial \partial_g^* \omega + \bar{\partial} \bar{\partial}_g^* \omega) - \sqrt{-1} \partial \bar{\partial} \log \omega^n. \tag{2.3}$$

In the above proposition, $d_g^* = \partial_g^* + \bar{\partial}_g^*$ where $\partial_g^* : \wedge^{p+1,q} M \rightarrow \wedge^{p,q} M$ and $\bar{\partial}_g^* : \wedge^{p,q+1} M \rightarrow \wedge^{p,q} M$ are the L_g^2 -adjoint operators of ∂ and $\bar{\partial}$, respectively. We recall the local formulas for these operators, which can be found, for example, in [23]. In local holomorphic coordinates,

$$\begin{aligned} (\partial_g^* \omega)_{\bar{k}} &= \sqrt{-1} g^{p\bar{q}} (\partial_{\bar{q}} g_{p\bar{k}} - \partial_{\bar{k}} g_{p\bar{q}}) \\ (\bar{\partial}_g^* \omega)_j &= \sqrt{-1} g^{p\bar{q}} (\partial_p g_{j\bar{q}} - \partial_j g_{p\bar{q}}). \end{aligned}$$

We also fix here the notation for the trace of a two form α with respect to ω :

$$tr_\omega \alpha := \sqrt{-1} g^{i\bar{j}} \alpha_{i\bar{j}},$$

where we used local holomorphic coordinates $\{z_i\}_i$.

Thanks to Proposition 2.1, we can describe the case where two Gauduchon scalar curvatures with different Gauduchon parameters coincide.

Proposition 2.2 *Let (M, g, J) be a compact Hermitian manifold and take two Gauduchon parameters $t_1 \neq t_2$. Then the following conditions are equivalent:*

- i. $Ric^{t_1}(\omega) = Ric^{t_2}(\omega)$;
- ii. $S^{t_1}(\omega) = S^{t_2}(\omega)$;
- iii. g is balanced, namely $d^* \omega = 0$.

Proof Obviously, (i) \Rightarrow (ii). As for (ii) \Rightarrow (iii), taking the trace in (2.3) we have that $s^{t_1} = s^{t_2}$ if and only if

$$\text{tr}_\omega(\partial\bar{\partial}\partial_g^*\omega + \bar{\partial}\partial\bar{\partial}_g^*\omega) = 0.$$

Then, integrating over M we get that

$$0 = \int_M \text{tr}_\omega(\partial\bar{\partial}\partial_g^*\omega) = (\partial\bar{\partial}\partial_g^*\omega, \omega)_g = (\partial_g^*\omega, \partial_g^*\omega)_g = |\partial_g^*\omega|_g^2,$$

and similarly for $\text{tr}_\omega(\bar{\partial}\partial\bar{\partial}_g^*\omega)$. Thus both $\partial_g^*\omega$ and $\bar{\partial}_g^*\omega$ vanish, which means that $\theta = Jd^*\omega = 0$ and g is balanced. Finally, going backward through this argument yields (iii) \Rightarrow (i). \square

We compute the variation of the ∇^t -Ricci curvature under a conformal change, which easily comes from the above formulas:

$$(\text{Ric}^t(e^f g))^{1,1} = (\text{Ric}^t(g))^{1,1} + (t - nt - 1)\sqrt{-1}\partial\bar{\partial}f, \tag{2.4}$$

while the (2, 0) and the (0, 2) components do not change. Taking the trace, we obtain

$$S^t(e^f g) = e^{-f} \left(S^t(g) + (1 + nt - t)\sqrt{-1}\text{tr}_\omega\partial\bar{\partial}f \right).$$

2.2 Chern Laplacian

We recall the definition of the Chern Laplacian Δ_ω^{Ch} associated to the Hermitian metric ω on a smooth function f as

$$\Delta_\omega^{Ch} f = 2\sqrt{-1}\text{tr}_\omega\partial\bar{\partial}f,$$

or, in local holomorphic coordinates $\{z_i\}_i$ as

$$\Delta_\omega^{Ch} \stackrel{\text{loc}}{=} -2g^{i\bar{j}}\partial_i\bar{\partial}_{\bar{j}}.$$

Remark 2.3 With this notation, the variation formula for the ∇^t -scalar curvature under a conformal change becomes:

$$S^t(e^f g) = e^{-f} \left(S^t(g) + \frac{1}{2}(1 + nt - t)\Delta_\omega^{Ch} f \right). \tag{2.5}$$

In [11], Gauduchon made explicit the relation between the Hodge-de Rham Laplacian $\Delta_{d,\omega}$ and the Chern Laplacian Δ_ω^{Ch} on smooth functions through the torsion 1-form, which we recall is defined by the equation

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}.$$

Lemma 2.4 ([11], pp. 502–503) *Let M be a compact complex manifold endowed with a Hermitian metric ω with torsion 1-form θ . The Chern Laplacian on smooth functions f has the form*

$$\Delta_\omega^{Ch} f = \Delta_d f + (df, \theta)_\omega.$$

In particular, the Chern Laplacian is a differential elliptic operator of 2nd order without terms of order 0 and its index agrees with the index of the Hodge-de Rham Laplacian, as is outlined in [10]. Moreover, the Chern Laplacian and the Hodge-de Rham Laplacian on smooth functions coincide when ω is balanced, and $\Delta_\omega^{Ch} f|_p \geq 0$ whenever f is a smooth real function on M which attains a local maximum at $p \in M$.

2.3 Hermitian Conformal Structures

Given a metric ω on M , the Hermitian conformal class of ω will be denoted by

$$\{\omega\} := \left\{ \exp(f) \omega \mid f \in C^\infty(M; \mathbb{R}) \right\}.$$

The following fundamental result by Gauduchon ensures the existence of a Gauduchon metric (i.e., a metric with $d^*\theta = 0$) in any Hermitian conformal class.

Theorem 2.5 ([10], Théorème 1) *Let M be a compact complex manifold of complex dimension $\dim_{\mathbb{C}} M \geq 2$, and fix a Hermitian conformal structure $\{\omega\}$. Then there exists a unique Gauduchon metric η in $\{\omega\}$ such that $\int_M d\mu_\eta = 1$.*

Using this result, we can consider the following normalized conformal class

$$\{\omega\}_1 := \left\{ \exp(f)\eta \in \{\omega\} \mid \int_M \exp(f) d\mu_\eta = 1 \right\} \subset \{\omega\},$$

where we denoted by $\eta \in \{\omega\}$ the unique Gauduchon representative of volume 1. With this choice of η , we can also introduce a natural *invariant* of the conformal class $\{\omega\}$, namely, the *Gauduchon degree*

$$\Gamma_M(\{\omega\}) \in \mathbb{R},$$

defined as

$$\Gamma_M(\{\omega\}) := \frac{1}{(n-1)!} \int_M c_1^{BC}(K_M^{-1}) \wedge \eta^{n-1} = \int_M S^{Ch}(\eta) d\mu_\eta.$$

We extend it to any Gauduchon parameter $t \in \mathbb{R}$,

$$\Gamma_M^t(\{\omega\}) := \int_M S^t(\eta) d\mu_\eta.$$

This value is related to the expected constant ∇^t -scalar curvature as follows.

Proposition 2.6 *Let (M, g, J) be a compact Hermitian manifold. Assume that $\omega' \in \{\omega\}$ has constant ∇^t -scalar curvature equal to $\lambda \in \mathbb{R}$. Then $\omega' \in \{\omega\}_1$ if and only if*

$$\Gamma_M^t(\{\omega\}) = \lambda.$$

In particular, the sign of the ∇^t -scalar curvature of a potential constant ∇^t -scalar curvature metric in $\{\omega\}$ agrees with the sign of $\Gamma_M^t(\{\omega\})$.

Proof Suppose that $e^f \omega \in \{\omega\}_1$ has constant ∇^t -scalar curvature λ . As representative in $\{\omega\}$, fix the unique Gauduchon metric $\eta \in \{\omega\}$ of volume 1 and denote by θ its torsion 1-form. Equation (2.5) yields

$$\frac{1}{2}(1 + nt - t) \int_M \Delta_\eta^{Ch} f d\mu_\eta + \int_M S^t(\eta) d\mu_\eta = \lambda \int_M \exp(f) d\mu_\eta,$$

where

$$\begin{aligned} \int_M \Delta_\eta^{Ch} f d\mu_\eta &= \int_M \Delta_d f d\mu_\eta + \int_M (df, \theta) d\mu_\eta \\ &= \int_M \Delta_d f d\mu_\eta + \int_M (f, d^*\theta) d\mu_\eta = 0, \end{aligned}$$

since η is Gauduchon. Therefore,

$$\Gamma_M^t(\{\omega\}) = \int_M S^t(\eta) d\mu_\eta = \lambda \int_M \exp(f) d\mu_\eta = \lambda,$$

yielding the first implication.

On the other hand, if we have a metric $e^f \omega \in \{\omega\}$ with constant ∇^t -scalar curvature equal to $\Gamma_M^t(\{\omega\})$ we can scale it by a constant e^c so that $e^{f+c} \omega$ stays in the normalized conformal class $\{\omega\}_1$ and its Gauduchon scalar curvature becomes $e^{-c} \Gamma_M^t(\{\omega\})$. Here c is such that

$$e^{-c} = \int_M e^f d\mu_\eta.$$

Note that $e^{f+c} \omega$ is a constant ∇^t -scalar curvature metric in $\{\omega\}_1$, hence it has scalar curvature equal to $\Gamma_M^t(\{\omega\})$. Thus, finally, $c = 0$. □

3 Gauduchon–Yamabe Problem

The Yamabe problem, consisting in finding a constant scalar curvature metric in the conformal class of a given Riemannian metric, is well understood in the Riemannian setting, while in the Hermitian setting, the Chern–Yamabe problem was introduced and studied in [2]. In this note, we study it for all the Gauduchon connections, i.e., given

a Hermitian manifold (M, g, J) we look for a constant ∇^t -scalar curvature metric $\tilde{\omega}$ in the conformal class $\{\omega\}$. Thanks to the conformal changing equation for the scalar curvature of ∇^t (2.5), this problem reduces to solve a semi-linear elliptic equation of 2nd order:

$$C_t \Delta_\omega^{Ch} f + S^t(\omega) = \lambda \exp(2f), \tag{3.1}$$

where $C_t = 1 + nt - t$ and λ is the expected constant scalar curvature value, equal to $\Gamma_M^t(\{\omega\})$ by Proposition 2.6.

Note that for $t = 1$ we recover the Chern–Yamabe problem which has been studied in [2] by Angella, Calamai and Spotti. In what follows, we extend their arguments to the more general Gauduchon–Yamabe problem, obtaining their results on the Chern connection and our results on the Bismut connection as particular choice of Gauduchon connections in Theorems 3.1 and 3.2.

3.1 Linear Case

In case of $\Gamma_M^t(\{\omega\}) = 0$, the semi-linear elliptic differential Eq. (3.1) becomes just linear since we shall take $\lambda = 0$, and so we get a solution for the corresponding Gauduchon–Yamabe problem whenever $C_t \neq 0$.

Theorem 3.1 *Let M be a compact complex manifold with Hermitian structure (ω, J) . If the Gauduchon parameter t is such that $C_t \neq 0$ and $\Gamma_M^t(\{\omega\}) = 0$, then there exists a unique metric $\tilde{\omega} \in \{\omega\}_1$ such that it has constant scalar curvature with respect to the ∇^t Gauduchon connection. Moreover, $S^t(\tilde{\omega}) = \Gamma_M^t(\{\omega\}) = 0$.*

Proof Fix $\eta \in \{\omega\}$ the unique Gauduchon representative in $\{\omega\}$ with volume 1. We should solve (3.1) with $\lambda = 0$, that is

$$C_t \Delta_\eta^{Ch} f = -S^t(\eta).$$

Using the relation in Lemma 2.4, it can be shown (see [11]) that the Kernel of the Chern Laplacian consists of just the constant functions. Indeed, we recall that

$$\Delta_\eta^{Ch} f = \Delta_d f + (df, \theta)_\eta,$$

where θ denotes the torsion 1-form of η . Thus if we take a function u in $\ker(\Delta^{Ch})$ we have

$$0 = \int_M u \Delta_\eta^{Ch} u \, d\mu_\eta = \int_M \left(|\nabla u|^2 + \frac{1}{2} (du^2, \theta) \right) d\mu_\eta = \int_M |\nabla u|^2 \, d\mu_\eta,$$

since $d^*\theta = 0$ because η is Gauduchon.

It follows that two conformal metrics with zero Gauduchon scalar curvature differ by a multiplicative constant, which in turn must be one if they are both in $\{\omega\}_1$, and so we get the uniqueness.

From the above equality, it is also possible to compute (see [11]) the adjoint of Δ_η^{Ch} on smooth functions u as

$$(\Delta_\eta^{Ch})^*u = \Delta_d u - (du, \theta)_\eta.$$

Thus the same computation applies and hence also the Kernel of the adjoint of the Chern Laplacian of a Gauduchon metric consists of just the constants. Since the integral of $-S^t(\eta)$ is zero by hypothesis, $-C_t^{-1}S^t(\eta) \in (\ker(\Delta_\eta^{Ch})^*)^\perp = \text{im } \Delta_\eta^{Ch}$. We thus achieve the existence of a metric of zero Gauduchon scalar curvature. \square

3.2 Non-linear Case

Here we provide a positive answer for the Gauduchon–Yamabe problem when $C_t \Gamma_M^t(\{\omega\}) < 0$. As a particular case, we will obtain the solution of the Bismut–Yamabe problem when the Gauduchon degree $\Gamma_M^{-1}(\{\omega\})$ is strictly positive and the complex dimension of the manifold is greater than three.

Theorem 3.2 *Let M be a compact complex manifold with Hermitian structure (ω, J) . Fix a Gauduchon parameter t for which $C_t \Gamma_M^t(\{\omega\}) < 0$. Then there exists a unique $\tilde{\omega} \in \{\omega\}_1$ with constant ∇^t -scalar curvature. Moreover, its Gauduchon scalar curvature satisfies $S^t(\tilde{\omega}) = \Gamma_M^t(\{\omega\})$.*

Proof Fix $\eta \in \{\omega\}$ the unique Gauduchon representative in $\{\omega\}$ with volume 1. By hypothesis, we have

$$C_t \Gamma_M^t(\{\omega\}) = C_t \int_M S^t(\eta) d\mu_\eta < 0.$$

The proof of the existence of a constant ∇^t -scalar curvature metric consists of two steps. We apply a continuity method to prove the existence of a constant ∇^t -scalar curvature metric in $\{\omega\}$ of class $\mathcal{C}^{2,\alpha}$; then we exploit the structure of the elliptic equation by a standard bootstrap argument to prove that it is smooth.

Before starting with the continuity method, we need a preliminary step. Namely, we prove that in the normalized conformal class $\{\omega\}_1$, there is a metric which has ∇^t -scalar curvature of constant sign $-sign(C_t)$. By this, we can assume that $C_t S^t(\omega) < 0$ at every point.

Consider the equation

$$\Delta_\eta^{Ch} f = -S^t(\eta) + \int_M S^t(\eta) d\mu_\eta. \tag{3.2}$$

Since η is Gauduchon, arguing as in the proof of Theorem 3.1, the above equation has a solution $f \in \mathcal{C}^\infty(M; \mathbb{R})$, which is unique once we require $\int_M \exp(2f/C_t) d\mu_\eta = 1$. Then $\exp(2f/C_t)\eta \in \{\omega\}_1$ satisfies

$$\begin{aligned}
 C_t S^t(\exp(2f/C_t)\eta) &= \exp(-2f/C_t)C_t \left(S^t(\eta) + \Delta_\eta^{Ch} f \right) \\
 &= \exp(-2f/C_t)C_t \int_M S^t(\eta) d\mu_\eta \\
 &= \exp(-2f/C_t)C_t \Gamma_M^t(\{\omega\}) < 0 .
 \end{aligned}$$

Now we can set up the following continuity path using as reference metric in the conformal class of η the above metric ω with $C_t S^t(\omega) < 0$. Consider the map, for $\alpha \in (0, 1)$,

$$\text{GaYa} : [0, 1] \times C^{2,\alpha}(M; \mathbb{R}) \rightarrow C^{0,\alpha}(M; \mathbb{R}) ,$$

such that

$$\text{GaYa}(s, f) := \Delta_\omega^{Ch} f + s S^t(\omega) - \lambda \exp(2f/C_t) + \lambda(1 - s).$$

Let us define the set

$$S := \left\{ s \in [0, 1] \mid \exists f_s \in C^{2,\alpha}(M; \mathbb{R}) \text{ such that } \text{GaYa}(s, f_s) = 0 \right\} ,$$

which trivially is non-empty since $\text{GaYa}(0, 0) = 0$. Thus, we should prove that it is also open and closed since the expected solution is achieved when $s = 1$.

We start with the open condition. The implicit function theorem for Hilbert spaces guarantees that S is open as long as the linearization of GaYa with respect to the second variable is bijective. Hence, we prove that, for a fixed solution $\text{GaYa}(s_0, f_{s_0}) = 0$, the linearized operator of GaYa ,

$$D : C^{2,\alpha}(M; \mathbb{R}) \rightarrow C^{0,\alpha}(M; \mathbb{R}),$$

defined by

$$v \mapsto Dv := \Delta_\omega^{Ch} v - \lambda \exp(2f_{s_0}/C_t) \cdot 2v/C_t,$$

is bijective. Let us remark that D differs from the Chern Laplacian by a compact operator, thus they have the same index, zero. This means that injectivity directly implies surjectivity, hence we are reduced to prove the former.

If v belongs to $\ker D$, then at a maximum point p for v there holds

$$-\lambda \exp(2f_{t_0}(p)/C_t) \cdot 2v(p)/C_t \leq 0, \tag{3.3}$$

and hence $v(p) \leq 0$, since $-\lambda/C_t > 0$. Similarly, at a minimum point q for v , there holds $v(q) \geq 0$. Thus, $\ker D = \{0\}$.

To show that S is also closed we argue as follows. Take $\{s_n\} \subset S$ a sequence converging to s_∞ and $f_{s_n} \in C^{2,\alpha}(M; \mathbb{R})$ such that $\text{GaYa}(s_n, f_{s_n}) = 0$ for any n ; we will use the Ascoli–Arzelà theorem to prove that the f_{s_n} converge in $C^{2,\alpha}(M; \mathbb{R})$ to

a function f_∞ such that $\text{GaYa}(s_\infty, f_{s_\infty}) = 0$. To use that theorem, we firstly need uniform L^∞ estimates of the solutions f_{s_n} .

Lemma 3.3 *There exists a positive constant K , depending only on M, ω, λ and t such that, for any n , we have*

$$\|f_{s_n}\|_{L^\infty} \leq K . \tag{3.4}$$

Proof By hypothesis the functions f_{s_n} satisfy $\text{GaYa}(s_n, f_{s_n}) = 0$, which means that the following equality holds:

$$\Delta_\omega^{Ch} f_{s_n} + s_n S^t(\omega) - \lambda \exp(2f_{s_n}/C_t) + \lambda(1 - s_n) = 0 . \tag{3.5}$$

We distinguish the two cases $C_t > 0$ and $C_t < 0$ which correspond to $\lambda < 0$ or $\lambda > 0$ respectively. We also recall that, by the preliminary step in the proof, $S^t(\omega)$ can be supposed to be a negative function when $C_t > 0$ and positive when $C_t < 0$.

Suppose $C_t > 0$ and take a maximum point p for f_{s_n} . Then, at p , there holds

$$-\lambda \exp(2f_{s_n}(p)/C_t) \leq -s_n S^t(\omega)(p) - \lambda(1 - s_n) \leq -\left(\min_M S^t(\omega)\right) - \lambda . \tag{3.6}$$

On the other hand, at a minimum point for f_{s_n} , say q , there holds

$$\begin{aligned} -\lambda \exp(2f_{s_n}(q)/C_t) &\geq -s_n S^t(\omega)(q) - \lambda(1 - s_n) \geq s_n(-S^t(\omega)(q) + \lambda) - \lambda \\ &\geq \min\left\{\min_M(-S^t(\omega)), -\lambda\right\} > 0. \end{aligned}$$

The above estimates provide the claimed uniform constant K_0 . The same argument holds also for $C_t < 0$, indeed, in this case, at a maximum point p for f_{s_n} , we have

$$\begin{aligned} \lambda \exp(2f_{s_n}(p)/C_t) &\geq s_n S^t(\omega)(p) + \lambda(1 - s_n) \geq s_n(S^t(\omega)(p) - \lambda) + \lambda \\ &\geq \min\left\{\min_M(S^t(\omega)), \lambda\right\}, \end{aligned}$$

while at a minimum point q for f_{s_n} , there holds

$$\lambda \exp(2f_{s_n}(q)/C_t) \leq s_n S^t(\omega)(q) + \lambda(1 - s_n) \leq \max_M S^t(\omega) + \lambda . \tag{3.7}$$

Hence the lemma is proved. □

Now it remains to prove the uniform equicontinuity of the functions $\{f_{s_n}\}$ in $C^{2,\alpha}(M; \mathbb{R})$. We define the elliptic operators

$$L_n f := \Delta_\omega^{Ch} f + s_n S^t(\omega) + \lambda(1 - s_n).$$

For the functions f_{s_n} , we get the equalities

$$L_n f_{s_n} = \lambda \exp(2f_{s_n}/C_t).$$

The estimate of Lemma 3.3 gives a uniform L^∞ control of the right-hand side $\lambda \exp(2f_{s_n}/C_t)$ of the equation and hence a uniform L^p control of $L_n f_{s_n}$ for any $p \in (1, \infty)$. Then, by the Calderon–Zygmund inequality, we can control the p -norm of the second-order derivatives by the p -norms of the function and its Laplacian; hence, iterating it twice, we get that $f_{s_n} \in W^{4,p}(M; \mathbb{R})$ with uniform bound on the norms. Finally, we can use the Sobolev embedding taking p large enough so that we find an a-priori C^3 uniform bound on the solutions. Thus now we can apply the Ascoli–Arzelà theorem so that we get a subsequence (which we still call $\{f_{s_n}\}$) converging in $C^{2,\alpha}(M; \mathbb{R})$ to a function f_{s_∞} . We can take the limit in the Eq. (3.5); in this way, we see that f_{s_∞} is a solution of $\text{GaYa}(s_\infty, f_{s_\infty}) = 0$ as needed.

So far we achieved the existence of a $C^{2,\alpha}$ solution f to the Gauduchon–Yamabe equation, $\text{GaYa}(1, f) = 0$. Hence we have $f \in C^{2,\alpha}$ such that

$$\Delta_\omega^{Ch} f = \lambda e^{2f/C_t} - S^t(\omega).$$

Notice that the right-hand side has the same regularity of f , hence the smooth regularity of the solution follows by the usual bootstrap argument via Schauder’s estimates for elliptic operators.

Now we have a smooth function f solving $\Delta_\omega^{Ch} f = \lambda e^{2f/C_t} - S^t(\omega)$ and we want to prove its uniqueness.

Notice that by Proposition 2.6 since we have $\lambda = \Gamma_M^t(\{\omega\})$, $e^f \omega$ must be in $\{\omega\}_1$; moreover, any other metric in $\{\omega\}_1$ with constant ∇^t -scalar curvature must solve the same equation.

Now suppose we have two conformal metrics $\omega_1 = \exp(2f_1/C_t)\omega$ and $\omega_2 = \exp(2f_2/C_t)\omega$ in $\{\omega\}_1$ with constant ∇^t -scalar curvatures equal to λ . Hence we have the equations

$$\Delta_\omega^{Ch} f_1 + S^t(\omega) = \lambda \exp(2f_1/C_t) \quad \text{and} \quad \Delta_\omega^{Ch} f_2 + S^t(\omega) = \lambda \exp(2f_2/C_t).$$

Taking the difference of these, we get the equation

$$\Delta_\omega^{Ch}(f_1 - f_2) = \lambda(\exp(2f_1/C_t) - \exp(2f_2/C_t)).$$

At a first glance, we should distinguish the cases $C_t > 0$ or $C_t < 0$ for which we, respectively, have $\lambda < 0$ and $\lambda > 0$; however, in both cases at a maximum point p for $f_1 - f_2$, we find $f_1(p) - f_2(p) \leq 0$, while at a minimum point q , we have $f_1(q) - f_2(q) \geq 0$, proving that f_1 and f_2 coincide. \square

Remark 3.4 In case $C_t \Gamma_M^t(\{\omega\}) > 0$, the maximum principle does not apply and the Gauduchon–Yamabe equation loses its good analytical properties. For the Chern connection, this case corresponds to have positive Gauduchon degree and it is investigated

in Section 5 of [2] where some sufficient criterion for the existence of positive constant Chern scalar curvature metrics is found. Moreover, non-homogeneous examples of Hermitian metrics of positive constant Chern scalar curvature have been constructed in [3, 16].

Similarly, it would be interesting to find new explicit examples of constant Bismut scalar curvature metrics as well as some sufficient (and, possibly, necessary) conditions which ensure the existence of metrics with negative constant scalar curvature for the Bismut connection.

We also remark that the “critical” Gauduchon connection for which the constant C_t vanishes are left out by these theorems. In particular, in complex dimension 2, it happens for the Bismut connection, since $C_t = 1 + nt - t = 1 + t$.

We then have the following result.

Corollary 3.5 *Let M be a compact complex manifold with $\dim_{\mathbb{C}} M \geq 3$ and Hermitian structure (ω, J) . If $\Gamma_M^+(\{\omega\}) \geq 0$, then there exists a unique $\tilde{\omega} \in \{\omega\}_1$ with constant Bismut scalar curvature. Moreover, $S^+(\tilde{\omega}) = \Gamma_M^+(\{\omega\})$.*

Remark 3.6 In [17], the authors extended the results of Angella, Calamai, and Spotti on the Chern–Yamabe problem to the non-integrable case. M. Lejmi suggested that it should also be possible to extend our results (Theorems 3.1 and 3.2) to the non-integrable case.

4 Constant Bismut Ricci curvature

We study the CYT equation on rank one toric bundles over Hermitian manifolds imitating the setting of [12]. In that article, the authors derive useful formulas for the Bismut Ricci curvature of special metrics on principal toric bundles over compact Kähler manifolds. Using them they construct CYT structures on the manifolds $(k - 1)(\mathbb{S}^2 \times \mathbb{S}^4) \# k(\mathbb{S}^3 \times \mathbb{S}^3)$ for all $k \geq 1$.

Given a Hermitian manifold (X, ω_X) , consider a principal toric bundle

$$\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow M \xrightarrow{\pi} X,$$

with characteristic classes of type $(1, 1)$. We take a connection one form with values in the Lie algebra of $\mathbb{S}^1 \times \mathbb{S}^1$ given by (θ_1, θ_2) such that $d\theta_i = \pi^* \omega_i$, with ω_i $(1, 1)$ -forms on X . Once we fix a complex structure on the torus, M inherits a complex structure from that of X , such that the projection map π from M to X is holomorphic, see Lemma 1 of [12] for details on this. We consider the Hermitian metrics on M for which π becomes a Riemannian submersion. These are all of the forms

$$\omega = \pi^*(\omega_X) + f\theta_1 \wedge \theta_2,$$

where f is a positive function on M , which is constant along the fibers (thus we will usually think of it as a function on X). The Bismut Ricci form of ω is given by

$$Ric^+(\omega) = \pi^*(Ric^+(\omega_X)) - dd^*(f\theta_1 \wedge \theta_2). \tag{4.1}$$

Indeed, from Proposition 2.1,

$$\begin{aligned} Ric^+\omega &= Ric^{Ch}\omega - dd^*\omega = \pi^* Ric^{Ch}\omega_X - dd^*\omega \\ &= \pi^*(Ric^+\omega_X + dd^*\omega_X) - dd^*\omega \\ &= \pi^*(Ric^+\omega_X) + dd^*(\pi^*\omega_X) - dd^*\omega = \pi^* Ric^+(\omega_X) - dd^*(f\theta_1 \wedge \theta_2) \end{aligned}$$

where the second equality comes from the following lemma.

Lemma (Lemma 3 of [12]) *Let $Ric^{Ch}\omega$ and $Ric^{Ch}\omega_X$ be the Ricci forms of the Chern connections on (M, ω) and (X, ω_X) respectively. Then $Ric^{Ch}\omega = \pi^*(Ric^{Ch}\omega_X)$.*

We work on a Hermitian frame $\{e_1, \dots, e_{2n}, t_1, t_2\}$ on an open subset of M which comes from a local Hermitian frame $\{e_1, \dots, e_{2n}\}$ on an open subset of X extended so that the vector fields t_1, t_2 are dual to the 1-forms θ_1, θ_2 .

Lemma 4.1 *The following equations hold:*

- $[t_i, e_j] = 0$ for any $i = 1, 2$ and $j = 1, \dots, 2n$;
- $\theta_i(\sum_j [e_{2j-1}, e_{2j}]) = -tr_{\omega_X}(\omega_i)$ for $i = 1, 2$.

Proof We derive these equations from the conditions $d\theta_i = \pi^*\omega_i$ ($i = 1, 2$). First of all, since the Lie brackets are π -related, i.e., $\pi_*[u, v] = [\pi_*u, \pi_*v]$ for any smooth vector fields u, v , we have that $[t_i, e_j]$ must be vertical. However,

$$\theta_k([t_i, e_j]) = -d\theta_k(t_i, e_j) = -\pi^*\omega_k(t_i, e_j) = 0,$$

We similarly obtain the second equation, indeed

$$\theta_i([e_{2j-1}, e_{2j}]) = -d\theta_i(e_{2j-1}, e_{2j}) = -\omega_i(e_{2j-1}, e_{2j}),$$

thus

$$\theta_i\left(\sum_j [e_{2j-1}, e_{2j}]\right) = -\sum_j \omega_i(e_{2j-1}, e_{2j}) = -tr_{\omega_X} \omega_i.$$

□

We now compute $dd^*\widehat{\omega}$ where $\widehat{\omega} := f\theta_1 \wedge \theta_2$. First of all, recall that the co-differential of a tensor could be expressed in terms of the contraction of Levi-Civita connection as

$$d^*\widehat{\omega} = -\sum_{j=1}^{2n} \nabla_{e_j}^{LC} \widehat{\omega}(e_j, \cdot) - \sum_{i=1,2} \nabla_{t_i}^{LC} \widehat{\omega}(t_i, \cdot),$$

moreover for any smooth vector fields u, v, w on a Hermitian manifold

$$-2\left(\nabla_u^{LC} \widehat{\omega}(v, w)\right) = d\widehat{\omega}(u, Jv, Jw) - d\widehat{\omega}(u, v, w),$$

hence we have

$$\begin{aligned}
 dd^*\widehat{\omega} &= d \left(\sum_j d\widehat{\omega}(J\cdot, e_{2j-1}, e_{2j}) + d\widehat{\omega}(J\cdot, t_1, t_2) \right) \\
 &= d \left(\sum_j d\widehat{\omega}(t_2, e_{2j-1}, e_{2j}) \theta_1 - \sum_j d\widehat{\omega}(t_1, e_{2j-1}, e_{2j}) \theta_2 + d\widehat{\omega}(J\cdot, t_1, t_2) \right) \\
 &= d \left(- \sum_j \widehat{\omega}([e_{2j-1}, e_{2j}], t_2) \theta_1 + \sum_j \widehat{\omega}([e_{2j-1}, e_{2j}], t_1) \theta_2 + d\widehat{\omega}(J\cdot, t_1, t_2) \right) \\
 &= d \left(-f \left(\theta_1 \left(\sum_j [e_{2j-1}, e_{2j}] \right) \theta_1 + \theta_2 \left(\sum_j [e_{2j-1}, e_{2j}] \right) \theta_2 \right) + (((Je_j) f) e^j \right) \\
 &= d \left(f (tr \omega_1 \theta_1 + tr \omega_2 \theta_2) + ((Je_j) f) e^j \right) \\
 &= df \wedge (tr \omega_1 \theta_1 + tr \omega_2 \theta_2) + fd(tr \omega_1 \theta_1 + tr \omega_2 \theta_2) + d \left(((Je_j) f) e^j \right) \\
 &= (e_i (Je_j) f - e_j (Je_i) f) e^i \wedge e^j + f(tr \omega_1 \pi^* \omega_1 + tr \omega_2 \pi^* \omega_2) \\
 &\quad + [e_j (f tr \omega_i) - t_i (Je_j) f] e^j \wedge \theta^i + [t_1 (f tr \omega_2) - t_2 (f tr \omega_1)] \theta_1 \wedge \theta_2 \tag{4.2}
 \end{aligned}$$

where we used the Einstein notation and dropped the subscript ω_X on the traces $tr_{\omega_X} \omega_i$ for convenience. Since f is constant along the fibers, we obtain

$$dd^*\widehat{\omega} = \pi^* dd^c f + f(tr \omega_1 \pi^* \omega_1 + tr \omega_2 \pi^* \omega_2) + [e_j (f tr \omega_i)] e^j \wedge \theta^i . \tag{4.3}$$

From this identity and Eq. (4.1), we get the following result.

Proposition 4.2 *On the total space M of a principal toric bundle of rank one $\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow M \xrightarrow{\pi} X$ over a Hermitian manifold (X, ω_X) with connection one forms (θ_1, θ_2) , the metric $\omega = \pi^*(\omega_X) + f\theta_1 \wedge \theta_2$, defines a Calabi–Yau with torsion structure if*

$$\begin{cases} Ric^+(\omega_X) = dd^c f + (c_1\omega_1 + c_2\omega_2) \\ f tr_{\omega_X} \omega_i = c_i \end{cases}$$

Notice that if $tr_{\omega_X} \omega_i$ vanishes at some point then it must vanish everywhere since $f tr_{\omega_X} \omega_i$ is supposed to be a constant function and $f > 0$. We now analyze the case of $tr_{\omega_X} \omega_1$ and $tr_{\omega_X} \omega_2$ simultaneously zero, which is completely understood from the following Lemma.

Lemma 4.3 (Lemma 6 of [12]) *Suppose that the Ricci form of the Bismut connection of a Hermitian metric g_M is $\partial\bar{\partial}$ -exact on a manifold M of dimension greater than two. Then the metric g_M is conformally a CYT structure. In other words, there exists a conformal change of g_M such that the Ricci form of the induced Bismut connection vanishes.*

Proof We recall the argument of the proof in [12]. The result directly comes from the formula for the conformal change of the Ricci curvature form of the Gauduchon connections. Indeed, setting $t = -1$ in (2.4), we obtain

$$\left(Ric^+(e^f \omega) \right)^{1,1} = \left(Ric^+(\omega) \right)^{1,1} + (n - 2) dd^c f .$$

Then it is sufficient to notice that the $(2, 0)$ and the $(0, 2)$ components of the Bismut Ricci tensor are invariant for conformal changes and are zero by hypothesis. \square

We may consider a trivial $\mathbb{S}^1 \times \mathbb{S}^1$ -principal bundle over (X, ω_X) since the computations are the same. In this case, we have the following result.

Proposition 4.4 *Given a compact Hermitian manifold (X, ω_X) such that the Bismut Ricci curvature is dd^c -exact, i.e., $Ric^+(\omega_X) = \sqrt{-1} \partial \bar{\partial} f$, then $M := \mathbb{S}^1 \times \mathbb{S}^1 \times X$, equipped with the induced complex structure, admits a CYT metric.*

Proof We can suppose that f is positive since it is defined up to constants on a compact manifold. Thus we can define the metric on M as usual: $\omega = \pi^*(\omega_X) + f\theta_1 \wedge \theta_2$, where π is the trivial projection of M onto X and θ_1, θ_2 are dual to the coordinates t_1, t_2 on the fibers. We hence have that

$$Ric^+(\pi^*(\omega_X) + f\theta_1 \wedge \theta_2) = \pi^*(Ric^+(\omega_X)) - dd^*(f\theta_1 \wedge \theta_2) = 0,$$

since from (4.3) we get

$$dd^*(f\theta_1 \wedge \theta_2) = \pi^*(dd^c f) .$$

\square

From this result, using the transformation law of the Bismut Ricci form under a conformal change (2.4), we directly get the following corollary.

Corollary 4.5 *Given a complex manifold (X, J) with a CYT metric g_X , for any positive function $f > 0$, we can construct a CYT structure on $\mathbb{S}^1 \times \mathbb{S}^1 \times X$ by taking the submersion metric $\omega = \pi^*(e^f \omega_X) + (n - 2)f \theta_1 \wedge \theta_2$.*

4.1 Class \mathcal{C} manifolds

In [19] the author defined Class \mathcal{C} manifolds as homogeneous manifolds $M = G/L$, where $G = G_1 \times G_2$ are compact simply connected simple Lie groups G_1, G_2 and L is a connected closed subgroup of G . There should also exist two irreducible compact Hermitian symmetric spaces $G_1/H_1, G_2/H_2$ so that the subgroups H_i are of the form $H_i = \langle Z_i \rangle \cdot L_i$ for $i = 1, 2$ and $L = L_1 \times L_2$. Therefore, we have the following setting

$$\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow (G_1/L_1) \times (G_2/L_2) \xrightarrow{\phi_1 \times \phi_2} (G_1/H_1) \times (G_2/H_2)$$

\nearrow^{π_1}

G_1/H_1

\searrow_{π_2}

G_2/H_2

where the ϕ_i are the Tits fibrations given by

$$\phi_i : G_i/L_i \rightarrow G_i/H_i : g \cdot L_i \mapsto g \cdot H_i.$$

M is then the product of two manifolds $M = (G_1/L_1) \times (G_2/L_2)$ where G_1/L_1 and G_2/L_2 are M -manifolds as defined in [19], meaning that L_1 and L_2 are the semisimple part of the centralizer of some torus, $H_i = C_{G_i}(\langle Z_i \rangle)$. Then by Theorem C in [26], the manifolds G_i/H_i are also simply connected, hence they are generalized flag manifolds. In [18], it is proved that any generalized flag manifold can be endowed with an invariant Kähler–Einstein Fano metric which is unique (up to homothety) once we fix the invariant complex structure on it. Moreover, the left-invariant complex structures on M are all given by choosing left-invariant complex structures on the symmetric spaces G_1/H_1 , G_2/H_2 and on the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Once we fix the complex structures on the symmetric spaces, the standard complex structure on M is that given by choosing $I(Z_1) = Z_2$. By exploiting the structure of class \mathcal{C} manifolds, we can explicitly construct submersion metrics on them which are CYT, namely, we prove the following theorem.

Theorem 4.6 *Take a class \mathcal{C} manifold as in [19], that is a product $M = M_1 \times M_2$ of M -manifolds which fibers through the Tits fibrations ϕ_i over two generalized flag manifolds $X_1 = G_1/H_1$ and $X_2 = G_2/H_2$ with \mathbb{S}^1 -fibers, and equip it with a standard complex structure. Set ω_i the unique invariant Kähler–Einstein metrics on X_i with Einstein constants $n_i = \dim(X_i)$. Then the metric on M given by*

$$\omega = \phi_1^*(\omega_1) + \phi_2^*(\omega_2) + \theta_1 \wedge \theta_2,$$

where θ_1 and θ_2 are the connections one forms on the fiber bundles such that $d\theta_i = \phi_i^* \omega_i$, defines a CYT structure on M .

Proof The metric on the base space $X = X_1 \times X_2$ is $\omega_X = \omega_1 + \omega_2$. Then the metric ω satisfies

$$\begin{cases} Ric^+(\omega_X) = dd^c(1) + n_1\omega_1 + n_2\omega_2 \\ tr_{\omega_X} \omega_i = n_i \quad \text{for } i = 1, 2 \end{cases}$$

hence it is Bismut Ricci flat by Proposition 4.2. Thus, we only need to check that the Tits fibrations represent the $U(1)$ -principal bundles over G_i/H_i with curvature $\omega_i \in c_1(G_i/H_i)$ chosen to be the unique Kähler–Einstein metrics on X_i . We know that the isomorphism classes of principal $U(1)$ -bundles over a manifold X are parametrized by its cohomology group $H^2(X)$; moreover, we can extract the following piece from

the exact sequence in the cohomology of the Tits fibration:

$$\mathbb{R} \cong H^1(\mathbb{S}^1) \xrightarrow{\delta} H^2(G/H) \xrightarrow{\phi^*} H^2(G/L) = 0,$$

where the last term vanishes since the M-manifolds have zero second Betti number by Theorem D in [26]. Thus, on the $U(1)$ -principal bundles on the G_i/H_i given by the Tits fibrations, we can always find connections one forms θ_i with curvature in $c_1(G_i/H_i)$. □

The existence of CYT Hermitian structures on the class \mathcal{C} manifolds can also be derived by Theorem 3 in [13]. Indeed, the metric $-B(\cdot, \cdot)$ given by the negative of the Killing form of G is Hermitian with respect to the standard complex structures on M . To see this, consider the decomposition of the Lie algebra \mathfrak{g} of G as

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 = (\mathfrak{m}_1 + \mathfrak{l}_1) + (\mathfrak{m}_2 + \mathfrak{l}_2) = (\mathfrak{n}_1 + \mathbb{R} Z_1 + \mathfrak{l}_1) + (\mathfrak{n}_2 + \mathbb{R} Z_2 + \mathfrak{l}_2).$$

Here, \mathfrak{g}_1 and \mathfrak{g}_2 are the Lie algebras of G_1 and G_2 , respectively, while the \mathfrak{l}_i 's are the Lie algebras of the L_i 's; moreover, the Lie algebras \mathfrak{h}_i 's of the H_i 's satisfy $\mathfrak{h}_i = \mathfrak{l}_i + \mathbb{R} Z_i$ for $i = 1, 2$. The Killing form B is Hermitian on \mathfrak{n} , moreover, the tori $\langle Z_1 \rangle$ and $\langle Z_2 \rangle$ are orthogonal to the \mathfrak{n}_i 's as well as one to each other. It only remains to verify that B is Hermitian on \mathfrak{t} , that is $B(Z_1, Z_1) = B(Z_2, Z_2)$.

The CYT metrics constructed above can be characterized as the unique CYT metrics among the homogeneous ones. Namely, we prove the following result.

Theorem 4.7 *Take a class \mathcal{C} manifold M as in Theorem 4.6. Suppose that none of the X_i 's is $SO(k + 2)/SO(2) \times SO(k)$ for $k \geq 3$, then the metric*

$$\omega = \phi_1^*(\omega_1) + \phi_2^*(\omega_2) + \theta_1 \wedge \theta_2,$$

constructed in Theorem 4.6, is the unique (up to homothety) homogeneous CYT metric on M .

Proof First of all, we verify that the homogeneous metrics on M make $\phi_1 \times \phi_2$ a Riemannian submersion. Indeed, with the same notations as above, a G -invariant Hermitian metric g' on M , can be seen as an $\text{ad}(\mathfrak{l}_1 + \mathfrak{l}_2)$ -invariant Hermitian inner product on $\mathfrak{m}_1 + \mathfrak{m}_2$. As the \mathfrak{l}_i 's are not trivial, $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ acts non-trivially on $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ and trivially on \mathfrak{t} , therefore $g'(\mathfrak{t}, \mathfrak{n}) = 0$. Moreover, the $\text{ad}(\mathfrak{l})$ -modules \mathfrak{n}_i are mutually non-equivalent, hence $g'(\mathfrak{n}_1, \mathfrak{n}_2) = 0$. Since we are avoiding the special case of $\mathfrak{g}_i = \mathfrak{so}(n + 2)$ and $\mathfrak{h}_i = \mathfrak{so}(2) + \mathfrak{so}(n)$, for $n \geq 3$ the \mathfrak{n}_i 's are \mathfrak{l}_i -irreducible. Hence, the Schur Lemma implies that g' on $\mathfrak{n}_i \times \mathfrak{n}_i$ restricts to a multiple $(\lambda_i \in \mathbb{R}_+)$ of the Killing form on G_i , i.e.,

$$g'_{|\mathfrak{n}_i \times \mathfrak{n}_i} = -\lambda_i(B_i)|.$$

In other words, the homogeneous metrics on M are all of the types

$$\omega' = \lambda_1 \phi_1^*(\omega_1) + \lambda_2 \phi_2^*(\omega_2) + \lambda \theta_1 \wedge \theta_2,$$

and by Proposition 4.2 any homogeneous CYT metric g' have to satisfy

$$\begin{cases} Ric^+(\lambda_1\omega_1 + \lambda_2\omega_2) = n_1\omega_1 + n_2\omega_2 \\ \lambda tr(\lambda_1\omega_1 + \lambda_2\omega_2) \omega_i = n_i \quad \text{for } i = 1, 2 \end{cases}$$

However, for $i = 1, 2$,

$$n_i = \lambda tr(\lambda_1\omega_1 + \lambda_2\omega_2) \omega_i = \frac{\lambda}{\lambda_i} tr(\omega_1 + \omega_2) \omega_i = \frac{\lambda}{\lambda_i} n_i,$$

proving that $\lambda = \lambda_1 = \lambda_2$, and hence g' is a positive multiple of g . □

Theorem 4.6 and Theorem 4.7 apply to give unique homogeneous CYT metrics on the Calabi–Eckmann manifolds when they are equipped with their standard complex structures. Indeed, these are the total spaces $M_{n_1, n_2} \cong \mathbb{S}^{2n_1+1} \times \mathbb{S}^{2n_2+1}$ of rank one toric bundles over the product of complex projective spaces $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$. As class C manifolds, they are given by taking $G_i = SU(n_i + 1)$, $L_i = SU(n_i)$, and $H_i = SU(n_i) \times U(1)$. The Tits fibrations agree with the Hopf fibrations $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n_i+1} \xrightarrow{\phi_i} \mathbb{C}P^{n_i}$. Then the standard Calabi–Eckmann structure J on

$$TM_{n_1, n_2} = H_{\mathbb{C}P^{n_1}} \oplus \langle Z_1, Z_2 \rangle \oplus H_{\mathbb{C}P^{n_2}},$$

is $J = J_{\mathbb{C}P^{n_1}} \oplus I \oplus J_{\mathbb{C}P^{n_2}}$ where $J_{\mathbb{C}P^{n_1}}$ and $J_{\mathbb{C}P^{n_2}}$ are the complex structures of $\mathbb{C}P^{n_1}$ and $\mathbb{C}P^{n_2}$, respectively, pulled-back on the horizontal spaces $H_{\mathbb{C}P^{n_1}}$ and $H_{\mathbb{C}P^{n_2}}$, and $I(Z_1) = Z_2$.

Corollary 4.8 *Given a Calabi–Eckmann manifold M_{n_1, n_2} equipped with its standard complex structure. Consider the Fubini–Study metrics ω_i ’s on the complex projective spaces $\mathbb{C}P^{n_i}$ ’s with Einstein constants n_i ’s, and set θ_1 and θ_2 the connections one forms on the fiber bundles such that $d\theta_i = \phi_i^* \omega_i$ for $i = 1, 2$. Then, the metric*

$$\omega = \phi_1^*(\omega_1) + \phi_2^*(\omega_2) + \theta_1 \wedge \theta_2,$$

is a CYT metric on M_{n_1, n_2} ; moreover, it is the unique homogeneous CYT metric on it.

We remark that in [4] a complete description of the Bismut curvature tensor for the homogeneous metrics on the Hopf manifolds was given. In particular, given a homogeneous metrics $g(\alpha, \beta)$ on an n -dimensional Hopf manifold, it holds (in the standard local holomorphic coordinates $\{z_i\}$),

$$Ric^+(g(\alpha, \beta)) = \left(2 - n + \frac{\beta}{\alpha}(1 - n)\right) \left(\frac{\delta_{ij}}{|z|^2} - \frac{\bar{z}_i z_j}{|z|^4}\right) dz^i \wedge d\bar{z}^j.$$

This identically vanishes if and only if the ratio $\frac{\beta}{\alpha}$ equals $\frac{2-n}{n-1}$, giving explicit CYT metrics on any Hopf manifold $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$, unique among the homogeneous ones.

4.2 Bismut Hermitian Einstein

The *pluriclosed flow* is a parabolic flow of metrics in the family of Hermitian curvature flows introduced by Streets and Tian in [22]. It has the property of preserving the pluriclosed condition, that is $\partial\bar{\partial}\omega = 0$. As a matter of fact, given a pluriclosed metric ω_0 , it evolves as

$$\begin{cases} \frac{\partial}{\partial t}\omega = - (Ric^+(\omega))^{1,1} \\ \omega(0) = \omega_0. \end{cases}$$

The static points of the pluriclosed flow are pluriclosed metrics ω which satisfy

$$(Ric^+(\omega))^{1,1} = \lambda\omega, \quad \lambda \in \mathbb{R}, \tag{4.4}$$

and are called in the literature *Bismut Hermitian Einstein metrics*. We distinguish the case of $\lambda = 0$ from that of $\lambda \neq 0$. The only known examples of non-Kähler pluriclosed metrics which also satisfy $(Ric^+\omega)^{1,1} = 0$ are that given by the *Bismut flat* structures. These are Hermitian structures such that the whole Bismut curvature tensor R^+ vanishes and Theorem 9 in [1] ensures that their Hermitian metrics are pluriclosed. Particular examples are given by the standard Calabi–Eckmann structures on the Hopf surface $\mathbb{S}^1 \times \mathbb{S}^3$ and the Calabi–Eckmann threefold $\mathbb{S}^3 \times \mathbb{S}^3$ which are known to be Bismut flat. In [9], the authors asked if the other Calabi–Eckmann manifolds admit such special Hermitian structures. The answer is negative and it comes from the fact that, for cohomological reasons, $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^3 \times \mathbb{S}^3$ are the only Calabi–Eckmann manifolds which can admit a pluriclosed structure, see Example 5.17 in [6]. On the other hand, when equipped with the standard Calabi–Eckmann complex structure, these manifolds can be equipped with metrics such that $(Ric^+\omega)^{1,1}$ vanishes. Hence we have the following picture:

- $\mathbb{S}^1 \times \mathbb{S}^1$ has a flat Kähler metric;
- $\mathbb{S}^3 \times \mathbb{S}^1$ has a Bismut flat, hence pluriclosed, metric;
- $\mathbb{S}^3 \times \mathbb{S}^3$ has a Bismut flat, hence pluriclosed, metric;
- $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ with $n \geq 2, m \geq 0$ have Bismut Ricci flat metrics (by Corollary 4.8) which are not pluriclosed.

On the other hand, when $\lambda \neq 0$, there are restrictions that suggest that the Eq. (4.4) should imply that the Hermitian structure is Kähler (see [9], p. 172). Some of these can be found in [20] (e.g., Proposition 3.5), where the author classifies solitons of the Pluriclosed Flow.

Proposition (Proposition 3.5 of [20]) *Let (M^{2n}, J) be a compact Kähler manifold, and suppose (g, f) is a pluriclosed steady or shrinking soliton on M . Then (g, f) is a Kähler-Ricci soliton.*

Here we show that on $\mathbb{S}^1 \times \mathbb{S}^1$ -principal bundles over Hermitian manifolds there are no metrics satisfying (4.4) with $\lambda \neq 0$. We take a slightly more general setting than that of the previous section, namely, we equip the total space M of the rank one

toric fibration over (X, ω_X) with a metric $\omega = \pi^*(\omega_X) + f\theta_1 \wedge \theta_2$ where f is any positive function on M . Using the computation of (4.2) we prove the following result.

Proposition 4.9 *Given a principal toric bundle, $\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow M \xrightarrow{\pi} X$, over a Hermitian manifold (X, ω_X) , with connection one forms θ_1, θ_2 such that $d\theta_i = \pi^*\omega_i$ for $(1, 1)$ -forms ω_i , there are no Hermitian metric of type $\omega = \pi^*(\omega_X) + f\theta_1 \wedge \theta_2$, where f is a positive function on M , which satisfy the equation*

$$(Ric^+(\omega))^{1,1} = \lambda\omega,$$

for $\lambda \neq 0$.

Proof Thanks to (4.2) the Bismut Einstein problem (4.4) in this setting reduces to solve

$$\begin{cases} \pi^*(Ric^+(\omega_X))^{1,1} = \lambda\pi^*(\omega_X) + (e_i(Je_j)f - e_j(Je_i)f) e^i \wedge e^j + f(tr_{\omega_X}\omega_1\pi^*\omega_1 \\ + tr_{\omega_X}\omega_2\pi^*\omega_2) \\ tr_{\omega_X}\omega_1 t_2 f - tr_{\omega_X}\omega_2 t_1 f = \lambda f \\ ((e_j(f tr_{\omega_X}\omega_i) - t_i(Je_j)f)\theta^i \wedge e^j)^{1,1} = 0 \end{cases}$$

In particular, f has to verify

$$tr_{\omega_X}\omega_1 t_2 f - tr_{\omega_X}\omega_2 t_1 f = \lambda f .$$

Since the fibers are compact, if we fix one of them, there should be a critical point for f on it. At this point, both $t_1 f$ and $t_2 f$ vanish giving a contradiction with the above equality, since $f > 0$. □

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