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A Twisted Version of Controlled K-theory

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Abstract

There are a number of (co-)homology theories on coarse spaces. Controlled operator K-theory is by far the most popular one of them. Our approach is geometric. We study when does the Roe-algebra of a space restrict to a subspace. Then we show the Roe-algebra is a cosheaf on the coarse topology. A result is a Mayer–Vietoris exact sequence in the presence of a coarse cover. We compute examples as an application.

Keywords Coarse geometry · Operator K-theory · Roe-algebra · Cosheaf · Mayer–Vietoris

Mathematics Subject Classification 19K56 · 51F30

1 Introduction

The K-theory of the Roe-algebra is one of the most popular homological invariants on coarse metric spaces. Meanwhile a new cohomological invariant on coarse spaces recently appeared in [3] which studies sheaf cohomology on coarse spaces.

In this paper,we study the K-theory of the Roe-algebra of a proper metric space X which is introduced in [4, Chapter 6.3]. Note that this theory does not appear as a derived functor as far as we know.

In [5] is studied a coarse excisive property on coarse spaces which we recall now. If $Y \subseteq X$ is a closed subspace then $C^*(Y, X)$ denotes the ideal in $C^*(X)$ which is the norm closure of operators with support near Y. Let $A, B \subseteq X$ be two closed subsets of a proper metric space which are ω-excisive. Then

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$$\begin{array}{ccccc} K_1(C^*(A\cap B,X)) & \longrightarrow & K_1(C^*(A,X))\oplus K_1(C^*(B,X)) & \longrightarrow & K_1(C^*(X)) \\ \uparrow & & & & \downarrow \\ K_0(C^*(X)) & \longleftarrow & K_0(C^*(A,X))\oplus K_0(C^*(B,X)) & \longleftarrow & K_0(C^*(A\cap B,X)) \end{array}$$

is a six-term Mayer–Vietoris exact sequence by [5, Sect. 5].

Note our approach can be compared with [8] where it was shown that a quotient $D(X)/C^*(X)$ is a sheaf on the underlying topological space of X.

The paper [3] introduced a Grothendieck topology X_{ct} associated to a coarse space X. The underlying category of X_{ct} is the poset of subsets of X and the coverings are finite collections of subsets called *coarse covers*.

Theorem 13 shows if X is a proper metric space then the association

$$U \mapsto C^*(\bar{U})/\mathbb{K}(\mathcal{H}_U),$$

for every subset $U \subseteq X$ with restriction maps is a cosheaf on X_{ct}.

Note that in a general setting cosheaves with values in the category of abelian groups `Ab` do not give rise to a derived functor. In [1] is explained that the dual version of sheafification, cosheafification, does not work in general. Moreover, the category of C^*-algebras `CStar` is not abelian.

Our result gives rise to new computational tools one of which is a new Mayer–Vietoris six-term exact sequence which is Corollary 14: If $U_1, U_2 \subseteq X$ are subsets of a proper metric space that coarsely cover a subspace $U \subseteq X$ then

$$\begin{array}{ccccc} K_1(\hat{C}^*(U_1\cap U_2)) & \longrightarrow & K_1(\hat{C}^*(U_1))\oplus K_1(\hat{C}^*(U_2)) & \longrightarrow & K_1(\hat{C}^*(U)) \\ \uparrow & & & & \downarrow \\ K_0(\hat{C}^*(U)) & \longleftarrow & K_0(\hat{C}^*(U_1))\oplus K_0(\hat{C}^*(U_2)) & \longleftarrow & K_0(\hat{C}^*(U_1\cap U_2)) \end{array}$$

is exact. Here $\hat{C}^*(Y) = C^*(\bar{Y})/\mathbb{K}(\mathcal{H}_Y)$ for $Y \subseteq X$ a subset.

The outline of this paper is as follows: The Chapter 2 discusses cosheaves on coarse spaces. The main part of the study is in Chapters 3 and 4 computes examples.

2 Cosheaves

If X is a metric space, a subset $E \subseteq X \times X$ is called an *entourage* if

$$\sup_{(x,y)\in E} d(x, y) < \infty.$$

A subset B is *bounded* if $\sup_{x,y\in B} \mathrm{d}(x, y) < \infty$. A map $\varphi : X \to Y$ between metric spaces is called *coarse* if $\varphi \times \varphi$ maps entourages to entourages and φ^{-1} maps bounded sets to bounded sets. Two maps $\phi, \psi : X \to Y$ between metric spaces are *close* if $\phi \times \psi$ maps the diagonal to an entourage. The coarse category consists of metric spaces as objects and coarse maps modulo close as morphisms.

We recall [3, Definition 45]:

Definition 1 (*coarse cover*) If X is a metric space and $U \subseteq X$ a subset then a finite family of subsets $U_1, \dots, U_n \subseteq U$ is said to *coarsely cover* U if for every entourage $E \subseteq X^2$ there is a bounded set $B \subseteq X$ such that

$$U^2 \cap \left(\bigcup_i U_i^2 \right)^c \cap E \subseteq B^2.$$

Coarse covers determine a Grothendieck topology X_{ct} associated to a metric space X. If $f : X \to Y$ is a coarse map between metric spaces then there is a morphism of Grothendieck topologies $f^{-1} : Y_{ct} \to X_{ct}$.

Definition 2 (*precosheaf*) A *precosheaf on X_{ct} with values in a category C* is a covariant functor $Cat(X_{ct}) \to C$.

Definition 3 (*cosheaf*) Let C be a category with finite limits and colimits. A precosheaf $\mathcal{F}$ on X_{ct} with values in C is a *cosheaf on X_{ct} with values in C* if for every coarse cover $\{U_i \to U\}_i$ there is a coequalizer diagram:

$$\bigoplus_{ij} \mathcal{F}(U_i \cap U_j) \rightrightarrows \bigoplus_i \mathcal{F}(U_i) \to \mathcal{F}(U). \tag{1}$$

Here the two arrows on the left side relate to the following 2 diagrams:

$$\begin{array}{ccc} \bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) & \longrightarrow & \bigoplus_i \mathcal{F}(U_i) \\ \uparrow & & \uparrow \\ \mathcal{F}(U_i \cap U_j) & \longrightarrow & \mathcal{F}(U_i) \end{array}$$

and

$$\begin{array}{ccc} \bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) & \longrightarrow & \bigoplus_i \mathcal{F}(U_j) \\ \uparrow & & \uparrow \\ \mathcal{F}(U_i \cap U_j) & \longrightarrow & \mathcal{F}(U_j) \end{array}$$

where $\bigoplus$ denotes the coproduct over the index set.

Notation 4 If we write

- $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ then a_i is supposed to be in $\mathcal{F}(U_i)$
- $\sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j)$ then b_{ij} is supposed to be in $\mathcal{F}(U_i \cup U_j)$

Proposition 5 *If $\mathcal{F}$ is a precosheaf on X_{ct} with values in a category C with finite limits and colimits and for every coarse cover $\{U_i \to U\}_i$*

(1) *and every $a \in \mathcal{F}(U)$ there is some $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ such that $\sum_i a_i|_U = a$*
(2) *and for every $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ such that $\sum_i a_i|_U = 0$ there is some $\sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j)$ such that $(\sum_j b_{ij} - b_{ji})|_{U_i} = a_i$ for every i.*

then $\mathcal{F}$ is a cosheaf.

Proof We have to prove that conditions (1) and (2) are equivalent to exactness of the diagram 1. Call the right map β and the left map α. Then exactness at $\mathcal{F}(U)$ means the map α is surjective. That is condition (1).

Now $\text{im}(\beta) \subseteq \ker(\alpha)$ always holds. If $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ then $\sum_i a_i|_U = 0$ is equivalent to $\sum a_i \in \ker \alpha$. If $\ker(\alpha) \subseteq \text{im}(\beta)$ then there exists some $\sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j)$ with $\sum_i (\sum_j b_{ij} - b_{ji})|_{U_i} = \beta(\sum_{ij} b_{ij}) = \sum_i a_i$. This is condition 2). □

Remark 6 Denote by `CStar` the category of C^*-algebras. According to [6], all finite limits and finite colimits exist in `CStar`.

3 Roe–Calkin Algebra

This exposition uses notation from [4, Chapter 6] which is a standard reference for K-theory of the Roe-algebra. We recall a few of the definitions:

Let X be a proper metric space. A presentation $\rho : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$ of $C_0(X)$ on a separable Hilbert space $\mathcal{H}_X$ is called an X-*module* if it is non-degenerate and ample. The *support* of a vector $v \in \mathcal{H}_X$ is the complement in X of the union of all open subsets $U \subseteq X$ such that $\rho(f)v = 0$ for all $f \in C_0(X)$. An operator $T \in \mathbb{B}(\mathcal{H}_X)$ is called *locally compact on* X if $\rho(f)T$ and $T\rho(f)$ are compact operators for all $f \in C_0(X)$. The *support* of an operator $T \in \mathbb{B}(\mathcal{H}_X, \mathcal{H}_Y)$ is the complement in $Y \times X$ of the union of all open subsets $U \times V \subseteq Y \times X$ such that $\rho(f)T\rho(g) = 0$ for every $f \in C_0(U)$ and $g \in C_0(V)$. An operator $T \in \mathbb{B}(\mathcal{H}_X)$ is said to be *controlled* if $\text{supp}(T)$ is an entourage. The *Roe algebra $C^*(X)$* is the norm closure of the algebra of locally compact, controlled operators on $\mathcal{H}_X$. If $C_0(X)$ is represented by an X-module then the K-theory of the Roe-algebra is a functor on coarse proper metric spaces. If $\varphi : X \to Y$ is a coarse map between metric spaces then a bounded operator $V : \mathcal{H}_X \to \mathcal{H}_Y$ *covers* φ if the two maps π_1 and $\varphi \circ \pi_2$ from $\text{supp}(V) \subseteq Y \times X$ are close.

Lemma 7 *If X is a proper metric space and $Y \subseteq X$ is a closed subspace then*

- *The subset $I(Y) = \{f \in C_0(X) : f|_Y = 0\}$ is an ideal of $C_0(X)$ and we have*

$$C_0(Y) = C_0(X)/I(Y)$$

- *There exists a sub-Hilbert space $\mathcal{H}_Y \subseteq \mathcal{H}_X$ and a non-degenerate representation $\rho_Y : C_0(Y) \to \mathbb{B}(\mathcal{H}_Y)$ that is a natural restriction of a non-degenerate representation $\rho_X : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$.*

- *The inclusion* $i_Y : \mathcal{H}_Y \to \mathcal{H}_X$ *covers the inclusion* $i : Y \to X$.

Proof • This one follows by Gelfand duality.

- We define $\mathcal{H}_{I(Y)} = \overline{\rho_X(I(Y))\mathcal{H}_X}$. Then

$$\mathcal{H}_X = \mathcal{H}_{I(Y)} \oplus \mathcal{H}_{I(Y)}^{\perp}$$

is the direct sum of reducing subspaces for $\rho_X(C_0(X))$. We define

$$\mathcal{H}_Y = \mathcal{H}_{I(Y)}^{\perp}$$

and a representation of $C_0(Y)$ on $\mathcal{H}_Y$ by

$$\rho_Y([a]) = \rho_X(a)|_{\mathcal{H}_Y}$$

for every $[a] \in C_0(Y)$. Note that $\rho_X(\cdot)|_{\mathcal{H}_Y}$ annihilates $I(Y)$ so this is well defined.

- Note that the support of i_Y is

$$\begin{aligned} \operatorname{supp}(i_Y) &= \Delta_Y \\ &\subseteq X \times Y \end{aligned}$$

□

Remark 8 Note that we can not conclude the following: If the representation $\rho_X : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$ is ample and $Y \subseteq X$ is a closed subspace then the induced representation $\rho_Y : C_0(Y) \to \mathbb{B}(\mathcal{H}_Y)$ is ample. Thus if we want to restrict representations to subspaces, we have to check the ample property each time.

Lemma 9 *If* X *is a proper metric space,*

- $B \subseteq X$ *is a compact subset and* $T \in C^*(X)$ *is an operator with*

$$\operatorname{supp} T \subseteq B^2$$

then T *is a compact operator.*

- *The converse does not hold. If* $T \in C^*(X)$ *is a compact operator then there does not necessarily exist a bounded set* $B \subseteq X^2$ *such that* $\operatorname{supp} T \subseteq B^2$
- *The* C^**-algebra of compact operators* $\mathbb{K}(\mathcal{H}_X)$ *is an ideal in* $C^*(X)$.

Proof • Suppose there is a non-degenerate representation $\rho : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$. For every $f \in C_0(B^c)$, $g \in C_0(X)$ the equations $\rho(f)T\rho(g) = 0$ and $\rho(g)T\rho(f) = 0$ hold. This implies $T(I(B)) = 0$ and $\operatorname{im} T \cap I(B) = 0$. Thus $T : \mathcal{H}_B \to \mathcal{H}_B$ is the same map. Thus $T \in C^*(B)$ already. Now T is locally compact, B is compact thus T is a compact operator.

- Note the set of ghost operator as defined in [9, Definition 1.2] contains the compact operators. The space X has property A if and only if every ghost operators is compact by [9, Theorem 1.3]. Not all of them have bounded support. The paper

[2] shows that there exists a metric space $\mathcal{O}$ with non-compact ghost projections in $\mathbb{B}(L_2(\mathcal{O}))$. This implies that $\mathcal{O}$ does not coarsely embed into Hilbert space. More precisely if X is the space $\mathbb{Z}$, the Hilbert space $\mathcal{H}_X := \ell^2(X) \otimes \ell^2(\mathbb{N})$ the standard ample X-module, D is a diagonal operator with coefficients in $C_0(X)$ and e is a finite rank projection then $T := D \otimes e \in C^*(X, \mathcal{H}_X)$ is compact and has support the full diagonal which cannot be contained in $B \times B$ with B bounded.

- This is already [7, Lemma 4.12]. For the convenience of the reader, we recall the proof: Let $K \subseteq X$ be a set and $v \in \mathcal{H}_X$ be a vector with $\operatorname{supp} v \subseteq K$. Then for every $f \in I(K)$, we obtain $\rho(f)v = 0$. Now a vector in $\mathcal{H}_{I(K)}$ can be written as $\rho(f)w$ where $f \in I(K)$, $w \in \mathcal{H}_X$. Then

$$\begin{aligned}\langle v, \rho(f)w\rangle &= \langle \rho(f^*)v, w\rangle \\ &= 0\end{aligned}$$

Thus $v \in \mathcal{H}_K$. If on the other hand $v' \in \mathcal{H}_K$ is any vector then

$$\begin{aligned}0 &= \langle \rho(f)w, v'\rangle \\ &= \langle w, \rho(f^*)v'\rangle\end{aligned}$$

for every $f \in I(K)$, $w \in \mathcal{H}_X$. This implies $\rho(f^*)v' = 0$ for every $f \in I(K)$. Thus $\operatorname{supp} v' \subseteq K$. Since X can be written as a union of bounded sets, $X = \bigcup B_i$ with B_i bounded for every i, the vectors with compact support form an orthonormal basis of $\mathcal{H}_X$.

A finite rank operator T with respect to this basis belongs to $C^*(X)$: First of all, T is locally compact since it is compact. We can write

$$T : h \mapsto \sum_{i=1}^{n} \alpha_i \langle h, v_i\rangle u_i$$

here $\alpha_i \geq 0$ and v_i, u_i are vectors with compact support $\operatorname{supp} v_i \subseteq B_i$, $\operatorname{supp} u_i \subseteq A_i$ for $1 \leq i \leq n$. Let $v \in \mathcal{H}_X$ be a vector. If $g \in I(B_i)$ and $f \in C_0(X)$ are two functions then

$$\begin{aligned}\alpha_i \langle \rho(g)v, v_i\rangle \rho(f)u_i &= \alpha_i \langle v, \rho(g^*)v_i\rangle \rho(f)u_i \\ &= 0\end{aligned}$$

Now let $g \in C_0(X)$, $f \in I(A_i)$ be functions. Then $\alpha_i \langle \rho(g)v, v_i\rangle \rho(f)u_i = 0$ since $\operatorname{supp} u_i \subseteq A_i$. Thus $\operatorname{supp} T \subseteq \bigcup_{i=1}^n A_i \times B_i$ is controlled.

Since the finite rank operators are dense in the compact operators $\mathbb{K}(\mathcal{H}_X)$, we obtain the inclusion $\mathbb{K}(\mathcal{H}_X) \subseteq C^*(X)$. Since the composition with a compact operator yields a compact operator,the subset $\mathbb{K}(\mathcal{H}_X)$ is an ideal in $C^*(X)$.

□

Definition 10 Let X be a proper metric space then

$$\hat{C}^*(X) = C^*(X)/\mathbb{K}(\mathcal{H}_X)$$

where $\mathbb{K}(\mathcal{H}_X)$ denotes the compact operators of $\mathbb{B}(\mathcal{H}_X)$ is called the *Roe–Calkin algebra* of X.

We want to assign a C^*-algebra to every subset $U \subseteq X$.

Remark 11 If $U \subseteq X$ is a subset of a proper metric space then the inclusion $U \to \bar{U}$ is coarsely surjective which means that there is some $R \geq 0$ such that every point of $\bar{U}$ lies in an R-neighborhood of U. We define

$$\hat{C}^*(U) := \hat{C}^*(\bar{U})$$

This way we can use Lemma 7 to restrict representations and elements of the Roe–Calkin algebra to subspaces.

Lemma 12 *If $Y \subseteq X$ is a closed subspace and $i_Y : \mathcal{H}_Y \to \mathcal{H}_X$ the inclusion operator of Lemma 7 then*

- *the operator*

$$\begin{aligned} Ad(i_Y) : C^*(Y) &\to C^*(X) \\ T &\mapsto i_Y T i_Y^* \end{aligned}$$

  is well defined and maps compact operators to compact operators.
- *Then the induced operator on quotients*

$$\hat{Ad}(i_Y) : \hat{C}^*(Y) \to \hat{C}^*(X)$$

  is the dual version of a restriction map, which means $U \mapsto \hat{C}^*(U)$ is a precosheaf on X.

Proof • i_Y covers the inclusion the other statement follows since composition with compact operators gives a compact operator.

- The assignment is a covariant functor.

□

Theorem 13 *If X is a proper metric space, then the assignment*

$$U \mapsto \hat{C}^*(U)$$

for every subspace $U \subseteq X$ is a cosheaf with values in `CStar`.

Proof Let $U_1, \ldots, U_n \subseteq U$ be subsets that coarsely cover $U \subseteq X$ and $V_i : \mathcal{H}_{U_i} \to \mathcal{H}_U$ and $V_{ij} : \mathcal{H}_{U_{ij}} \to \mathcal{H}_{U_i}$ the corresponding inclusion operators for $i, j = 1, \ldots, n$.

Let $T \in C^*(U)$ be a locally compact controlled operator. We need to construct $T_i \in C^*(U_i)$ such that

$$\sum_i V_i T_1 V_i^* = T$$

modulo compacts. Denote by $E = \mathrm{supp}(T)$ the support of T in U. Define

$$T_i := V_i^* T V_i - \sum_{j=1}^{i-1} V_{ij} V_{ji}^* T_j V_{ji} V_{ij}^*$$

then the T_i are locally compact controlled operators, thus elements in $C^*(U_i)$. We now show that $T - \sum_i V_i T_i V_i^* = 0$ on $U_1 \times U_1 \cup \cdots \cup U_n \times U_n$ which shows the result since $E \cap (U_1^2 \times \cdots U_n^2)^c$ is bounded. Let $(x, y) \in (U_1^2 \times U_n^2) \cap \mathrm{supp}\, T$ be a point and choose k minimal with $(x, y) \in U_k \times U_k$. Then $T_k(x) = T(x) = y$. We now show $(x, y) \notin \mathrm{supp}\, T_i$ for $i \neq k$. For $i = 1, \ldots, k-1$, this is clear since k is minimal. We now use induction on $i = k+1, \ldots, n$: If $i = k+1$ then $(x, y) \in \mathrm{supp}(V_{k+1}^* T V_{k+1})$ exactly when $(x, y) \in \mathrm{supp}(V_{k+1,k} V_{k,k+1}^* T_k V_{k,k+1} V_{k+1,k}^*)$. Thus $(x, y) \notin \mathrm{supp}\, T_{k+1}$. If the statement holds for $k+1, \ldots, i$ then $(x, y) \in \mathrm{supp}(V_{i+1}^* T V_{i+1})$ exactly when $(x, y) \in \mathrm{supp}(V_{i+1,k} V_{k,i+1}^* T_k V_{k,i+1} V_{i+1,k}^*)$. Thus $(x, y) \notin \mathrm{supp}\, T_{i+1}$. This implies $\sum_i T_i|_U = T$, axiom 1).

Suppose $T_i \in C^*(U_i)$ are elements with

$$\sum_i V_i T_i V_i^* = 0$$

modulo compacts. Denote by $V_{ijk} : \mathcal{H}_{U_i \cap U_j \cap U_k} \to \mathcal{H}_{U_i \cap U_j}$ the covering isometry operator associated to the inclusion $U_i \cap U_j \cap U_k \to U_i \cap U_j$. Define for $1 \leq i < j \leq n$:

$$T_{ij} := V_{ij}^* T_i V_{ij} + \sum_{k<i} V_{ijk} V_{kij}^* T_{ki} V_{kij} V_{ijk}^* - \sum_{i<k<j} V_{ijk} V_{ikj}^* T_{ik} V_{ikj} V_{ijk}^*.$$

Using $V_i V_{ij} = V_j V_{ji}$ and combinatorical information,we can show

$$\sum_j (T_{ij} - T_{ji})|_{U_i} = \sum_{i<j} T_{ij}|_{U_i} - \sum_{j<i} T_{ji}|_{U_i} = T_i$$

modulo compacts. Thus axiom 2) of a cosheaf holds. □

4 Computing Examples

Corollary 14 *If U_1, U_2 coarsely cover a subset U of a proper metric space X then there is a six-term Mayer–Vietoris exact sequence*

$$\begin{array}{ccccc} K_1(\hat{C}^*(U_1 \cap U_2)) & \longrightarrow & K_1(\hat{C}^*(U_1)) \oplus K_1(\hat{C}^*(U_2)) & \longrightarrow & K_1(\hat{C}^*(U)) \\ \uparrow & & & & \downarrow \\ K_0(\hat{C}^*(U)) & \longleftarrow & K_0(\hat{C}^*(U_1)) \oplus K_0(\hat{C}^*(U_2)) & \longleftarrow & K_0(\hat{C}^*(U_1 \cap U_2)) \end{array}$$

Proof If $A \subseteq X$ is a subset define $C^*(A, X)$ to be the C^*-algebra generated by all locally compact operators with finite propagation on $\mathcal{H}_X$ whose support is contained in $E[A] \times E[A]$ for some entourage $E \subseteq X \times X$. It can be observed that $C^*(A, X)$ forms an ideal in $C^*(X)$. The inclusion $C^*(A) \to C^*(A, X)$ induces an isomorphism on the K-theory of the algebras obtained by modding out $\mathbb{K}(\mathcal{H}_X)$. That is because we have a commuting diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K}(\mathcal{H}_X) & \longrightarrow & C^*(A) & \longrightarrow & \hat{C}^*(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{K}(\mathcal{H}_X) & \longrightarrow & C^*(A, X) & \longrightarrow & C^*(A, X)/\mathbb{K}(\mathcal{H}_X) & \longrightarrow & 0 \end{array}$$

where the left vertical arrow is an isomorphism, the middle vertical arrow induces an isomorphism in K-theory by [4, Proposition 6.4.7]. By the five lemma, the right vertical arrow induces an isomorphism in K-theory.

We define $I_1 := C^*(U_1, X)/\mathbb{K}(\mathcal{H}_X)$, $I_2 := C^*(U_2, X)/\mathbb{K}(\mathcal{H}_X)$ and $I_{12} = C^*(U_1 \cap U_2, X)/\mathbb{K}(\mathcal{H}_X)$. They are the smallest ideals containing $\hat{C}^*(U_1)$, $\hat{C}^*(U_2)$ and $C^*(U_1 \cap U_2)$, respectively. Now Theorem 13 and the first cosheaf axiom imply $\hat{C}^*(U) = \hat{C}^*(U_1) + \hat{C}^*(U_2)$ and the second cosheaf axiom implies $\hat{C}^*(U_1) \cap \hat{C}^*(U_2) = \hat{C}^*(U_1 \cap U_2)$. This implies $I_1 + I_2 = C^*(U, X)/\mathbb{K}(\mathcal{H}_X)$ and $I_1 \cap I_2 = I_{12}$. With those properties and [4, Exercise 4.10.21], the six-term exact sequence in K-theory is obtained. □

Remark 15 Now for every proper metric space there is a short exact sequence

$$0 \to \mathbb{K}(\mathcal{H}_X) \to C^*(X) \to \hat{C}^*(X) \to 0$$

which induces a 6-term sequence in K-theory:

$$\begin{array}{ccccc} K_0(\mathbb{K}(\mathcal{H}_X)) & \longrightarrow & K_0(C^*(X)) & \longrightarrow & K_0(\hat{C}^*(X)) \\ \uparrow & & & & \downarrow \\ K_1(\hat{C}^*(X)) & \longleftarrow & K_1(C^*(X)) & \longleftarrow & K_1(\mathbb{K}(\mathcal{H}_X)) \end{array}$$

If X is flasque [10] then

$$K_i(\hat{C}^*(X)) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \end{cases}$$

Remark 16 Note that the result of Corollary 14 is applicable when computing controlled K-theory if the property ample is preserved by restricting the representation of U to the representations of U_1, U_2.

If X is a Riemannian manifold then fixing a volume form ν the Hilbert space $\mathcal{H}_X = L^2(X, \nu) \otimes \ell^2$ is an ample X-module with ℓ^2 the standard separable Hilbert space and

$\rho_X : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$ trivial on the second factor. In Example 17, Example 18, we will use this canonical representation on $\mathbb{R}$, $\mathbb{R}^2$ and certain subspaces of them without mentioning it. In those cases, the property ample is preserved by restricting $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$, the space R^2 to V_1, V_2 and $V_1 \cap V_2$ to U_1, U_2, respectively.

Example 17 ($\mathbb{R}$) Now $\mathbb{R}$ is the coarse disjoint union of two copies of $\mathbb{R}_{\geq 0}$ which is a flasque space. By Corollary 14, there is an isomorphism

$$K_i(\hat{C}^*(\mathbb{R})) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \end{cases}$$

Applying Remark 15,we can compare

$$K_i(C^*(\mathbb{R})) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \end{cases},$$

in [4, Theorem 6.4.10], if it matches our computation. And indeed it does.

Example 18 ($\mathbb{R}^2$) We coarsely cover $\mathbb{R}^2$ with $V_1 = \mathbb{R}_{\geq 0} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_{\geq 0}$ and $V_2 = \mathbb{R}_{<0} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_{<0}$. The space $V_1 \cap V_2$ is coarsely covered by $U_1 = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $U_2 = \mathbb{R}_{<0} \times \mathbb{R}_{<0}$. The second cover and Corollary 14 gives

$$K_i(\hat{C}^*(V_1 \cap V_2)) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \end{cases}$$

Since V_1, V_2 are coarsely equivalent to flasque spaces, the first cover and Corollary 14 imply that $K_0(\hat{C}^*(\mathbb{R}^2))$ and $K_1(\hat{C}^*(\mathbb{R}^2))$ have the same (free abelian) rank. Translating back using Remark 15, the groups

$$K_i(C^*(\mathbb{R}^2)) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \end{cases}$$

of [4, Theorem 6.4.10] also fit in the exact sequence.

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