The Journal of Geometric Analysis (2023) 33:105
https://doi.org/10.1007/s12220-022-01143-3

Topology of Asymptotically Conical Calabi–Yau and G_2 Manifolds and Desingularization of Nearly Kähler and Nearly G_2 Conifolds

Lothar Schiemanowski[1]

Received: 27 June 2022 / Accepted: 2 November 2022 / Published online: 30 January 2023
© The Author(s) 2022

Abstract
A natural approach to the construction of nearly G_2 manifolds lies in resolving nearly G_2 spaces with isolated conical singularities by gluing in asymptotically conical G_2 manifolds modelled on the same cone. If such a resolution exits, one expects there to be a family of nearly G_2 manifolds, whose endpoint is the original nearly G_2 conifold and whose parameter is the scale of the glued in asymptotically conical G_2 manifold. We show that in many cases such a curve does not exist. The non-existence result is based on a topological result for asymptotically conical G_2 manifolds: if the rate of the metric is below -3, then the G_2 4-form is exact if and only if the manifold is Euclidean $\mathbb{R}^7$. A similar construction is possible in the nearly Kähler case, which we investigate in the same manner with similar results. In this case, the non-existence results is based on a topological result for asymptotically conical Calabi–Yau 6-manifolds: if the rate of the metric is below -3, then the square of the Kähler form and the complex volume form can only be simultaneously exact, if the manifold is Euclidean $\mathbb{R}^6$.

Keywords Special holonomy · Asymptotically conical manifolds · Einstein manifolds

Mathematics Subject Classification 53C25 · 53C24 · 53C21

1 Introduction

If (L, g_L) is a closed Riemannian manifold of dimension n, then the cone $C(L) = (0, \infty) \times L$ can be equipped with the cone metric $g_C = dr^2 + r^2 g_L$. The manifold L is called the *link* of the cone. If $\mathrm{Ric}[g_L] = (n-1)g_L$, then $\mathrm{Ric}[g_C] = 0$. This

✉ Lothar Schiemanowski
lothar.schiemanowski@math.uni-hannover.de

1 Institut für Differentialgeometrie, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

relationship between positive Einstein manifolds and Ricci flat cones has a very interesting extension to the theory of special holonomy, elucidated by Bär in [1]. This correspondence admits a uniform description in terms of Killing spinors and parallel spinors. It can also be described on the level of each of the holonomy groups whose underlying metric is Ricci flat:

(1) holonomy group $\mathrm{SU}(n)$: the cone over a Sasaki–Einstein manifold is a Calabi–Yau cone,
(2) holonomy group $\mathrm{Sp}(n)$: the cone over a 3-Sasaki manifold is a hyperkähler manifold,
(3) holonomy group G_2: the cone over a nearly Kähler manifold is a G_2 manifold,
(4) holonomy group $\mathrm{Spin}(7)$: the cone over a nearly G_2 manifold is a $\mathrm{Spin}(7)$ manifold.

In the first case, the cone is of dimension $2n$, in the second case, the cone has dimension $4n$, whereas the third case only appears in dimension 7 and the last case in dimension 8. The holonomy groups G_2 and $\mathrm{Spin}(7)$ are also known as the exceptional holonomy groups. Remarkably, the structure group G_2 appears both as a holonomy group and as the structure group of the geometry on the link of a $\mathrm{Spin}(7)$ cone. Similarly, the structure group $\mathrm{SU}(3)$ appears as a holonomy group and as the structure group of the geometry of the link of a G_2 cone. The groups G_2 and $\mathrm{SU}(3)$ are the only groups for which this is true and this observation is foundational for the investigations in this article. A remark concerning terminology is in place: nearly G_2 manifolds are also known as nearly parallel G_2 manifolds in the literature. We prefer the term "nearly G_2" to emphasize the analogy with nearly Kähler manifolds.

The purpose of this article is to study the possibility of constructing nearly G_2 and nearly Kähler manifolds on the basis of this observation.

Given any nearly Kähler manifold, one can construct an incomplete nearly G_2 manifold, which has two singularities, modelled on the G_2 cone associated to the nearly Kähler manifolds [2]. This construction is known as the sine cone construction. Spaces with singularities modelled on cones will be called conifolds in the sequel. It is conceivable that there are more examples of nearly G_2 conifolds.

On the other hand, for some of the known nearly Kähler manifolds, there exist complete, non-compact G_2 manifolds, which are asymptotic at infinity to the cone over the nearly Kähler manifold. These will be called asymptotically conical G_2 manifolds.

A G_2 structure on a manifold M is given by certain 3-forms $\varphi \in \Omega^3(M)$. This will be explained in more detail in Sect. 2.

If (N, φ_{AC}) is an asymptotically conical G_2 manifold, then the rescalings $(N, t^3\varphi_{AC})$ are also asymptotically conical G_2 manifolds for every $t > 0$. Moreover, $(N, t^3\varphi_{AC})$ converges to a G_2 cone as $t \to 0$. Given a nearly G_2 conifold $(\overline{X}, \varphi_{CS})$, one can "chop off" the singularity and instead glue on a piece of $(N, t^3\varphi_{AC})$. If t is sufficiently small, the error made is small to zeroth order. The equations satisfied by the two pieces are quite different, however. Therefore it is a priori unclear if there exists a nearly G_2 manifold close to this resolved nearly G_2 conifold.

Suppose for the moment, there exists a nearby nearly G_2 structure for every sufficiently small $t > 0$. Then we obtain a curve $(\overline{M}, \varphi(t))$ of nearly G_2 conifolds, such that $(\overline{M}, \varphi(t))$ converges to $(\overline{X}, \varphi_{CS})$ as $t \to 0$ and $(\overline{M}, t^{-3}\varphi(t))$ converges to

(N, φ_{AC}). Such a family of nearly G_2 structures will be called a *smooth desingularization*, if it also satisfies certain regularity assumptions. (See Definition 4.1 for the precise meaning.)

The purpose of this article is to investigate whether such families can exist. It will turn out that this question is closely related to the topology of asymptotically conical G_2 manifolds.

Theorem A *Let $(\overline{X}, \varphi_{CS})$ be a nearly G_2 conifold with a singularity at x_0 modelled on the G_2 cone $(C = C(L), \varphi_C)$. Suppose (N, φ_{AC}) is an asymptotically conical G_2 manifold, asymptotic to (C, φ_C).*

If there exists a smooth desingularization of $(\overline{X}, \varphi_{CS})$ by (N, φ_{AC}) at x_0, then $*\varphi_{AC}$ is exact.

A precise definition of the notions of cones, conifolds and asymptotically conical manifolds will be given in Sect. 3. Smooth desingularizations will be defined in Sect. 4 and this theorem and the next will also be proven there.

The equation for nearly G_2 manifolds implies that the dual of the G_2 form $*\varphi_{CS}$ is exact. Therefore the theorem essentially says that the asymptotically conical G_2 manifold inherits this property if it arises from a smooth desingularization.

In six dimensions, a very similar construction is possible with nearly Kähler conifolds and asymptotically conical Calabi–Yau 6-manifolds. Both classes of manifolds are described by SU(3) structures. These will be parametrized by a pair (ω, Ω), where ω is a 2-form and Ω is a complex valued 3-form.

Theorem B *Let $(\overline{X}, \omega_{CS}, \Omega_{CS})$ be a nearly Kähler conifold with a singularity at x_0 modelled on the Calabi–Yau cone $(C = C(L), \omega_C, \Omega_C)$. Suppose $(N, \omega_{AC}, \Omega_{AC})$ is an asymptotically conical Calabi–Yau manifold asymptotic to (C, ω_C, Ω_C).*

If there exists a smooth desingularization of $(\overline{X}, \omega_{CS}, \Omega_{CS})$ by $(N, \omega_{AC}, \Omega_{AC})$ at x_0, then ω_{AC}^2 and $\operatorname{Re}\Omega_{AC}$ are exact.

Similarly to the nearly G_2 case, the nearly Kähler equations imply that the forms ω_{CS}^2 and $\operatorname{Re}\Omega_{CS}$ are exact and this theorem shows that a smoothly desingularizing asymptotically conical Calabi–Yau manifold inherits this property.

It turns out that below a certain rate threshold asymptotically conical G_2 or Calabi–Yau 6-manifolds can only satisfy the conclusions of the preceding theorems if they are already Euclidean spaces. Therefore, no asymptotically conical G_2 manifold with rate less than -3 can desingularize a nearly G_2 conifold in the sense of Definition 4.1. Likewise, no asymptotically conical Calabi–Yau 6-manifold with rate less than -3 can desingularize a nearly Kähler manifold in the sense of Definition 4.2.

Theorem C *Let (M, g, φ) be an asymptotically conical G_2 manifold of rate $\nu < -3$.*

Then $*\varphi \in \Omega^4(M)$ is exact if and only if (M, g) is isometric to $(\mathbb{R}^7, g_{\text{eucl}})$.

In particular, if (M, g) is not isometric to $(\mathbb{R}^7, g_{\text{eucl}})$ the cohomology group $H^4(M, \mathbb{R})$ is non-trivial.

In 1989, Bryant and Salamon constructed the first three asymptotically conical Riemannian manifolds with holonomy group G_2, which were also the first complete examples with holonomy group G_2. [8] The underlying manifolds of these Bryant–Salamon spaces are $\Lambda^2_- S^4$, $\Lambda^2_- \mathbb{CP}^2$ and $S^3 \times \mathbb{R}^4$. The rate of $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{CP}^2$ is -4

and to these spaces the theorem applies. On the other hand, the metric on $S^3 \times \mathbb{R}^4$ has rate -3 and therefore the theorem does not apply, which is also evident from the fact that $H^4(S^3 \times \mathbb{R}^4) = 0$. This example also shows the sharpness of the rate threshold. Recently, Foscolo, Haskins and Nordström constructed infinitely many more examples of asymptotically conical G_2 manifolds, see [17]. These examples are of rate -3 and the theorem therefore does not apply.

Theorem D *Let* (M, g, ω, Ω) *be an asymptotically conical* SU(3) *manifold of rate* $\nu < -3$.

Then $\omega^2 \in \Omega^2(M)$ *and* $\operatorname{Re}\Omega \in \Omega^3(M)$ *are simultaneously exact if and only if* (M, g) *is isometric to* $(\mathbb{R}^6, g_{\text{eucl}})$.

In particular, if (M, g) *is not isometric to* $(\mathbb{R}^6, g_{\text{eucl}})$*, the cohomology group* $H^2(M, \mathbb{R})$ *or* $H^3(M, \mathbb{R})$ *is non-trivial.*

There exists an extensive literature for asymptotically conical Calabi–Yau manifolds, which includes existence results, see for example [13] and references included there. A class of examples to which the theorem can be applied are crepant resolution of Calabi–Yau cones with compactly supported Kähler class. These have rate $-6+\delta$ for every $\delta > 0$, see [31]. See also the discussion in Sect. 4 of [13] for an extension to almost compactly supported Kähler classes. An explicit example in this class is given by the Calabi metric on the total space of the line bundle $\mathcal{O}(-3) \to \mathbb{CP}^2$, which has rate -6. This space is asymptotic to $\mathbb{C}^3/\mathbb{Z}_3$.

Two examples, in which the Theorem D does not apply, are given by the Stenzel metric on T^*S^3, which has rate -3, and the Candelas-DeLaOssa metric on the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$, which has rate -2. Both spaces are asymptotic to the cone over the Sasaki–Einstein manifold $S^2 \times S^3$. Since the third and fourth cohomology group vanish in the second example, the forms $\operatorname{Re}\Omega$ and ω^2 are necessarily exact. While the topology of T^*S^3 permits non-exact 3-forms, after a multiplication by a $\lambda \in S^1 \subset \mathbb{C}$ the form $\operatorname{Re}\Omega$ of the Stenzel Calabi–Yau structure can be shown to be exact, as pointed out to the author by the anonymous referee. Since the rate of this Calabi–Yau structure is -3, this also shows that the rate threshold in the theorem is sharp.

Let us mention that the rate of asymptotically conical Ricci flat manifolds has been studied in [11] and can be estimated using analytical properties of the link, see the recent preprint [21]. In particular, asymptotically locally Euclidean Ricci flat spaces of dimension n have rate $\leq -n$.

The proofs of Theorems C and D can be found in Sect. 8. Both proofs are very similar and they rely on a study of approximate potential functions. These are functions on asymptotically conical manifolds, which are asymptotic to $r^2/2$ and whose Laplacian is constant. Such an approximate potential can be used to define certain forms using the forms that are given by the G_2 or SU(3) structure. Appealing to a Hodge theoretical result shows that these forms must in fact vanish. Application of this result first requires a closer examination of the asymptotics of the approximate potential beyond the leading term. This is the technical core of the theorem and this is also where the condition on the rate comes in, see also Theorem 6.2. Representation theoretic properties of G_2 and SU(3) structures then imply that the vanishing of the forms implies that the trace free part of the Hessian of the approximate potential vanishes –

a theorem of Tashiro will then yield the desired statement, i.e. that the asymptotically conical manifold must be isometric to a Euclidean space. Approximate potentials have been used earlier in [5] to define an invariant of ALE gravitational instantons, which vanishes if and only if the instanton is Euclidean $\mathbb{R}^4$.

Theorems A and B assume that there are smooth curves desingularizing the nearly G_2 or nearly Kähler conifolds. Note that in general deformations are obstructed in both the nearly Kähler and the nearly G_2 setting [14, 15, 25]. For this reason the assumption of a smooth curve of nearly G_2 resp. nearly Kähler structures is fairly restrictive. Examples of Einstein desingularizations of sine cones due to Böhm suggest that it can happen that there is only a discrete sequence approaching the singular metric rather than a smooth curve. By analogy with recent results of Ozuch [26, 27, 29], we believe that the obstructions we found also exclude such desingularizations and we therefore propose the following two conjectures. See also the discussion in Sect. 4.

Conjecture A *Let $(\overline{X}, \varphi_{CS})$ be a nearly G_2 conifold with a singularity at x_0 modelled on the G_2 cone $(C = C(L), \varphi_C)$.*

If every asymptotically conical G_2 manifold asymptotic to (C, φ_C) has rate $\nu < -3$, then there is no smooth nearly G_2 manifold Gromov–Hausdorff close to $(\overline{X}, \omega_{CS}, \rho_{CS})$.

Conjecture B *Let $(\overline{X}, \omega_{CS}, \rho_{CS})$ be a nearly Kähler conifold with a singularity at x_0 modelled on the Calabi–Yau cone $(C = C(L), \omega_C, \rho_C)$.*

If every asymptotically conical Calabi–Yau manifold asymptotic to (C, φ_C) has rate $\nu < -3$, then there is no smooth nearly Kähler manifold Gromov–Hausdorff close to $(\overline{X}, \omega_{CS}, \rho_{CS})$.

There are formidable analytical challenges to transferring Ozuch's approach to the weak holonomy setting.

As pointed out by the anonymous reviewer, the G_2 cones over the nearly Kähler manifolds $\mathbb{CP}^3 = \mathrm{Sp}(2)/(U(1) \times \mathrm{Sp}(1))$ and $F_{1,2} = \mathrm{SU}(3)/T^2$ have the property in Conjecture A. Indeed, by corollary 6.10 in [20] an asymptotically conical G_2 manifold with cone at infinity $C(\mathbb{CP}^3)$ or $C(F_{1,2})$ must be the corresponding Bryant–Salamon manifold. As indicated earlier, any Ricci flat ALE manifold of (real) dimension n has rate $\leq -n$. In particular, any asymptotically conical Calabi–Yau manifold asymptotic to $\mathbb{C}^3/\Gamma$ with $\Gamma \subset \mathrm{SU}(3)$ a finite subgroup acting freely on S^5 will have rate ≤ -6 and therefore these cones have the property posed in Conjecture B.

It should be noted that the theorems and the conjectures are statements only about metrics near the conifold metrics. Indeed, the only two known inhomogeneous examples of nearly Kähler metrics, constructed by Foscolo and Haskins in [16], *do* arise as desingularizations of sine cones in a certain sense, but they do not come in families that converge back to the sine cone. Indeed, Foscolo and Haskins conjecture that the examples they construct are the only cohomogeneity one nearly Kähler metrics. From the point of view of the present article, this means that the construction should only work for one specific size of the exceptional divisor of the asymptotically conical Calabi–Yau manifold.

Previously, similar questions have been studied in the context of Einstein manifolds. Biquard investigated the possibility of desingularizing $\mathbb{Z}_2$ orbifold singularities in 4-manifolds in the article [3] by gluing in an Eguchi–Hanson space and more general

isolated orbifold singularities on 4-manifolds in [4]. Biquard found a first obstruction in terms of the curvature tensor at the orbifold singularity in [3] and secondary obstructions in [4]. Morteza and Viaclovsky extended this analysis to $2n$-dimensional Einstein manifolds with $\mathbb{Z}_n$-orbifold singularities. Ozuch deepened the understanding of the Biquard obstructions in the papers [26, 27], investigated higher order obstructions in [28] and gave a proof that certain Einstein orbifolds never appear as Gromov–Hausdorff limits of smooth Einstein manifolds in [29], bypassing certain technical restrictions, which were present in the earlier works on the question.

In the context of special holonomy, we note the works of Chan [9, 10], in which (real) 6-dimensional Calabi–Yau spaces with isolated conical singularities are resolved by asymptotically conical Calabi–Yau manifolds. Similarly to our case, there is a dichotomy depending on the rate of the asymptotically conical Calabi–Yau manifold; if the rate is smaller than -3, the problem is unobstructed. If the rate is exactly -3 an obstruction can be identified, which depends on a cohomology class on the link of the cone. Perhaps the article most similar to ours in subject matter is Karigiannis' [19], in which G_2 conifolds are desingularized by gluing in asymptotically conical G_2 manifolds. In contrast to our situation, this is possible if a topological condition is verified or else the rate of the asymptotically conical G_2 manifolds that are glued in is less than -4.

2 SU(3) and G_2 Geometry

The use of stable forms is a convenient approach to the description of SU(3) and G_2 structures with special properties, which originates in [18]. Let V be a real, n-dimensional vector space. A k-form $\omega \in \Lambda^k V^*$ is *stable*, if its $\mathrm{GL}(V)$ orbit in $\Lambda^k V^*$ is open. The stabilizer of ω is the subgroup $\mathrm{Stab}_\omega = \{A \in \mathrm{GL}(V) : A^*\omega = \omega\}$.

Three cases are of interest to us:

(1) If $n = 7$ and $k = 3$ or $k = 4$, then the stabilizer of a stable form is isomorphic to G_2.
(2) If $n = 6$ and $k = 2$ or $k = 4$, then the stabilizer of a stable form is isomorphic to $\mathrm{Sp}(6, \mathbb{R})$.
(3) If $n = 6$ and $k = 3$, then the stabilizer of a stable form is isomorphic to either $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$ or $\mathrm{SL}(3, \mathbb{C})$.

We denote by $\Lambda^k_+ V^*$ the set of all stable forms for $n = 7$, $k = 3, 4$ and for $n = 6$, $k = 2, 4$. In the last case, $n = 6$ and $k = 3$, $\Lambda^3_+ V^*$ is the set of all stable forms with stabilizer isomorphic to $\mathrm{SL}(3, \mathbb{C})$.

Each of the groups G_2, $\mathrm{Sp}(6, \mathbb{R})$ and $\mathrm{SL}(3, \mathbb{C})$ preserves a volume form in $\Lambda^n V^*$. Therefore, in these cases, we can define a $\mathrm{GL}(V)$-invariant, n/k-homogeneous map $\phi = \phi^n_k : \Lambda^k_+ V^* \to \Lambda^n V^*$. Given $\omega \in \Lambda^k_+ V^*$, the derivative $d\phi[\omega]$ can be expressed by an element $\hat{\omega} \in \Lambda^{n-k} V^*$ via the identity $d\phi[\omega]\nu = \hat{\omega} \wedge \nu$. This defines the *Hitchin duality map*

$$\Phi = \Phi^n_k : \Lambda^k V^* \to \Lambda^{n-k} V^*, \qquad \Phi(\omega) = \hat{\omega}.$$

In case $\dim V = 7$ we will denote $\Theta = \Phi_3^7$. For representation theoretic aspects of stable forms in the 7- and 6-dimensional settings, we refer to [7, 15], respectively. The interaction between the representation theory of the 6- and 7-dimensional cases, which is needed to understand G_2 cones, is well explained in [12, 32].

2.1 Torsion Free and Nearly G_2 Forms

Let V be a 7-dimensional real vector space and suppose $\varphi \in \Lambda^3_+ V^*$. The form φ induces a metric g_φ on V and a Hodge dual $*_{g_\varphi}$ on $\Lambda^* V^*$. This can be used to describe the Hitchin dual: $\Theta(\varphi) = *_{g_\varphi}\varphi$.

Let M be a 7-manifold. The fibre bundle of stable 3-forms on M is given by

$$\Lambda^3_+ T^*M = \bigcup_{x \in M} \Lambda^3_+ T^*_x M,$$

and the space of its sections will be denoted by $\Omega^3_+(M)$. The constructions described on a vector space in the previous paragraphs have their analogues on the manifold: an element φ of $\Omega^3_+(M)$ induces a Riemannian metric g_φ, a volume form vol_φ, the Hodge star $*_{g_\varphi}$. The map Θ induces a bundle map $\Lambda^3_+ T^*M \to \Lambda^4 T^*M$.

Definition 2.1 A section $\varphi \in \Omega^3_+(M)$ is called a G_2 *form* on M. A G_2 form φ is called *torsion free*, if

$$d\varphi = 0, \qquad d\Theta(\varphi) = 0. \tag{2.1}$$

A G_2 form φ is called *nearly parallel G_2 at scale* $\lambda > 0$, if

$$d\varphi = 4\lambda\Theta(\varphi). \tag{2.2}$$

It is called a *nearly parallel G_2 form* if 2.2 is satisfied with $\lambda = 1$.

A manifold M equipped with a torsion free G_2 form φ is called a G_2 *manifold* and is denoted by (M, g, φ), where $g = g_\varphi$.

A manifold M equipped with a nearly parallel G_2 form is called a *nearly G_2 manifold* and is denoted by (M, g, φ), where $g = g_\varphi$.

The significance of these conditions is that if φ is a torsion free G_2 form, then the reduced Riemannian holonomy group of g_φ is isomorphic to a subgroup of G_2 and if φ is a nearly parallel G_2 form, then the holonomy group of its Riemannian cone is $\mathrm{Spin}(7)$.

2.2 Calabi–Yau and Nearly Kähler 6-Manifolds

Observing that $\mathrm{SU}(3) = \mathrm{Sp}(6, \mathbb{R}) \cap \mathrm{SL}(3, \mathbb{C})$ allows us to parametrize an $\mathrm{SU}(3)$ structure on a six-dimensional vector space V by a stable forms $\omega \in \Lambda^2_+$ and $\mathrm{Re}\,\Omega \in \Lambda^3_+$. To ensure that the stabilizers of these forms intersect the same way as $\mathrm{Sp}(6, \mathbb{R})$ and $\mathrm{SL}(3, \mathbb{C})$, we impose the compatability constraints

$$\omega \wedge \mathrm{Re}\,\Omega = 0, \qquad \frac{1}{4}\,\mathrm{Re}\,\Omega \wedge \Phi(\mathrm{Re}\,\Omega) = \frac{1}{6}\omega^3. \tag{2.3}$$

The joint stabilizer of the pair $(\omega, \operatorname{Re}\Omega)$ is then isomorphic to SU(3) or SU(2, 1). In addition to the compatability constraints, we will assume that the joint stabilizer is isomorphic to SU(3). This is an open condition. Such a pair $(\omega, \operatorname{Re}\Omega)$ then induces a Riemannian metric and an almost complex structure on V. The complex 3-form $\operatorname{Re}\Omega + i\Phi(\operatorname{Re}\Omega)$ is a complex volume form and therefore we denote $\operatorname{Im}\Omega = \Phi(\operatorname{Re}\Omega)$.

We will describe an SU(3) structure on V by a pair (ω, Ω), where ω is a stable 2-form and Ω is a complex 3-form, such that $\operatorname{Re}\Omega$ is stable and $\operatorname{Im}\Omega = \Phi(\operatorname{Re}\Omega)$. We will denote the associated metric by $g_{\omega,\Omega}$ and the associated almost complex structure by $J_{\omega,\Omega}$.

The duality map on 2-forms can be described explicitly as $\Phi(\omega) = \frac{1}{2}\omega^2$.

On a 6-manifold M, one may define the bundles of stable 2-forms $\Lambda^2_+ T^*M = \bigcup_{x\in M} \Lambda^2_+ T^*_x M$ and analogously $\Lambda^3_+ T^*M$.

Definition 2.2 An SU(3) structure on M is given by a pair (ω, Ω), where $\omega \in \Omega^2_+(M)$ and $\Omega \in \Gamma(\Lambda^3_{\mathbb{C}} T^*M)$, such that $\operatorname{Re}\Omega \in \Omega^3_+(M)$, $\operatorname{Im}\Omega = \Phi(\operatorname{Re}\Omega)$,

$$\omega \wedge \operatorname{Re}\Omega = 0, \qquad \frac{1}{4}\operatorname{Re}\Omega \wedge \operatorname{Im}\Omega = \frac{1}{6}\omega^3,$$

and the symmetric, bilinear form defined by

$$g(v, w)\frac{1}{6}\omega^3 = (v \lrcorner \omega) \wedge (w \lrcorner \operatorname{Re}\Omega) \wedge \operatorname{Re}\Omega,$$

is positive definite.

Definition 2.3 An SU(3) structure (ω, Ω) is *torsion free* or *Calabi–Yau*, if

$$\mathrm{d}\omega = 0, \qquad \mathrm{d}\operatorname{Re}\Omega = 0, \qquad \mathrm{d}\operatorname{Im}\Omega = 0.$$

An SU(3) structure (ω, Ω) is *nearly Kähler* at scale $\lambda > 0$, if

$$\mathrm{d}\omega = -3\lambda \operatorname{Re}\Omega, \qquad \mathrm{d}\operatorname{Im}\Omega = 2\lambda\omega^2.$$

A 6-manifold M equipped with a torsion free SU(3) structure (ω, Ω) is called a *Calabi–Yau 6-manifold* and is denoted by (M, g, ω, Ω), where g denotes the associated Riemannian metric.

A 6-manifold M equipped with a nearly Kähler SU(3) structure (ω, Ω) of scale 1 is called a *nearly Kähler 6-manifold* and is denoted by (M, g, ω, Ω), where g denotes the associated Riemannian metric.

3 Cones, Conifolds and Asymptotically Conical Manifolds

This section gives definitions of the principal objects of this article: asymptotically conical and conically singular manifolds.

Definition 3.1 Let L be a closed manifold. The *cone* over L is denoted by $C(L)$ and is the product manifold $(0, \infty) \times L$. The manifold L is called the *link* of the cone.

Points in $C(L)$ are typically denoted by (r, x) and r can be considered as the canonical radial coordinate function. In this section, L will always denote a closed manifold.

Definition 3.2 A *dilation* by the factor $\lambda > 0$ on a cone $C(L)$ is the diffeomorphism

$$\mathrm{dil}_\lambda : C(L) \to C(L), \qquad \mathrm{dil}_\lambda(r, x) = (\lambda r, x).$$

The *dilation vector field* on a cone $C(L)$ is $r\partial_r$.

The dilation vector field generates the dilations.

Definition 3.3 Let L be a closed manifold. A form $\kappa \in \Omega^k(C(L))$ is *homogeneous of rate* λ if

$$\mathcal{L}_{r\partial_r}\kappa = (k + \lambda)\kappa.$$

Similarly, a symmetric 2-form $h \in \Gamma(\mathrm{Sym}^2 T^*C(L))$ is *homogeneous of rate* λ if $\mathcal{L}_{r\partial_r}h = (2 + \lambda)h$.

Definition 3.4 A *Riemannian cone* is a cone $C = C(L)$ together with a Riemannian metric g_C of the form $dr^2 + r^2 g_L$, where g_L is a Riemannian metric on L.

Note that the metric g_C is always homogeneous of rate 0, but a homogeneous Riemannian metric of rate 0 need not be of this form.

The normalization of the rates of forms and tensors is explained by the following observation. If $\kappa \in \Omega^k(C(L))$ is homogeneous of rate λ, then $|\kappa|_{g_C}$ is a homogeneous function of rate λ. In the same manner, if $h \in \Gamma(\mathrm{Sym}^2 T^*C(L))$ is homogeneous of rate λ, then $|h|_{g_C}$ is homogeneous of rate λ.

Definition 3.5 Let (L, g_L) be a closed Riemannian manifold. A complete Riemannian manifold (M, g) is *asymptotically conical* with *cone at infinity* (C, g_C) and *rate* $\nu < 0$, if there exists a compact set $K \subset M$ and a diffeomorphism $\Phi : (R, \infty) \times L \to M \backslash K$, such that for every k there exists $C_k > 0$, such that

$$r^k |\nabla^k(\Phi^* g - g_C)| \leq C_k r^\nu,$$

holds on $(R, \infty) \times L$. Here, ∇ is the Levi–Civita connection of g_C and $|\cdot|$ refers to the metric g_C.

A *radial function* on (M, g) is a smooth function $\rho : M \to \mathbb{R}$, such that $\rho \equiv 1$ on K, $\rho \geq 1$ on M and $\Phi^*\rho = r$ on $(R_0, \infty) \times L$ for some large enough R_0.

Definition 3.6 Let L be a closed manifold of dimension 6 with a nearly Kähler structure (ω, Ω). The G_2 *cone* over L is the manifold $C = (0, \infty) \times L$ with the G_2 form

$$\varphi_C = r^3 \operatorname{Re}\Omega - r^2 \mathrm{d}r \wedge \omega.$$

If (C, φ_C) is a G_2 cone, then the dual of φ_C is given by

$$\psi_C = \Theta(\varphi_C) = -r^3 \mathrm{d}r \wedge \operatorname{Im} \Omega - \frac{1}{2} r^4 \omega^2.$$

The associated metric g_φ is $dr^2 + r^2 g_{\omega,\Omega}$.

Definition 3.7 Let (C, φ_C) be a G_2 cone with link L. A G_2 structure φ on a manifold M is an *asymptotically conical (AC) G_2 structure* asymptotic to the cone (C, φ_C) at *rate* $\nu < 0$, if there exists a compact set $K \subset M$ and a diffeomorphism $\Psi : (R, \infty) \times L \to M \backslash K$, such that there exists for every $k \in \mathbb{N}_0$ a $C_k > 0$, for which the following inequality holds on $(R, \infty) \times L$:

$$r^k |\nabla_C^k (\Psi^* \varphi - \varphi_C)|_{g_C} \leq C_k r^\nu.$$

A pair (M, φ) is called an *asymptotically conical G_2 manifold*, if φ is a torsion free asymptotically conical G_2 structure on M.

For the sake of brevity, "asymptotically conical" will usually be abbreviated by AC. It is easy to generalize this definition to multiple ends, but connected AC G_2 manifolds automatically have only one end. An asymptotically conical G_2 manifold is automatically an asymptotically conical Riemannian manifold with the same rate.

In the case of conically singular manifolds, we do admit multiple conical singularities and each singularity may be modelled on a different G_2 cone.

Definition 3.8 Let $\overline{M}$ be a topological space, $\Sigma = \{x_1, \ldots, x_n\} \subset \overline{M}$ and suppose that $M = \overline{M} \backslash \Sigma$ is a smooth 7-dimensional manifold.

A *conically singular G_2 structure* on $\overline{M}$ with singular set Σ and cones (C_i, φ_{C_i}) of rate $\nu_i > 0$ at x_i, $1 \leq i \leq n$, is a G_2 form φ defined on M, such that:

(1) there exists a compact set $K \subset M$ and open sets $S_1, \ldots, S_n$, such that $M \backslash K = S_1 \cup \ldots \cup S_n$ and the closures of the S_i in $\overline{M}$ are pairwise disjoint,
(2) there exist diffeomorphisms $\Psi_i : L_i \times (0, \epsilon_i) \to S_i$, such that for every $k \in \mathbb{N}_0$ and $i \in \{1, \ldots, n\}$ there exists a $C_{k,i} > 0$ for which the following inequality holds on $(0, \epsilon_i) \times L_i$:

$$r^k |\nabla_{C_i}^k (\Psi_i^* \varphi - \varphi_{C_i})|_{g_{C_i}} \leq C_{k,i} r^\nu.$$

A space $\overline{M}$ equipped with a conically singular G_2 structure will also be called a *conifold with a G_2 structure*. If the G_2 structure is torsion free, we will call it a G_2 *conifold*. If the G_2 structure is nearly parallel, we will call $(\overline{M}, \varphi)$ a *nearly G_2 conifold*.

Note that our terminology is slightly different from the literature, where a conifold usually also allows for AC ends.

It is clear that similar definitions are possible for manifolds with SU(3) structures. To avoid needless repetitions, we leave out the precise definitions, but mention that to define a Calabi–Yau cone, one has to employ Sasaki–Einstein structures on 5-manifolds. Details on this may be found, for example, in Sect. 2 of [16].

4 Smooth Desingularizations and the Obstruction Equation

The following definition will formalize the idea of the desingularization of nearly G_2 conifolds by an AC G_2 manifold. It can be seen as an analogue of Cheeger–Gromov convergence for a smooth 1-parameter family instead of a sequence.

Suppose (C, φ_C) is a G_2 cone, (N, φ_{AC}) is an AC G_2 manifold with cone (C, φ_C) and $(\overline{X}, \varphi_{CS})$ is a nearly G_2 conifold, which has a singular point x_0 modelled on the cone (C, φ_C).

Definition 4.1 Suppose $\overline{M}$ is a topological space, $\Sigma \subset M$ a finite set and $\varphi(t)$, $t \in (0, \epsilon)$, is a smooth family of conically singular nearly parallel G_2 structures on $M = \overline{M} \setminus \Sigma$. This family is a *smooth desingularization* of $(\overline{X}, \varphi_{CS})$ at x_0 by (N, φ_{AC}), if the following conditions are met:

(1) For every $t \in (0, \epsilon)$, there exist open sets $U_t \subset X$, $\widetilde{U}_t \subset M$, such that

  (a) $U_s \subset U_t$ for every $s > t$,
  (b) $\bigcup_{t \in (0,\epsilon)} U_t = X$.

(2) For every $t \in (0, \epsilon)$, there exists a diffeomorphism $F_t : U_t \to \widetilde{U}_t$ with the following significance:

  (a) Let $\mathcal{U} = \bigcup_{t \in (0,\epsilon)} \{t\} \times U_t$. The forms $F_t^* \varphi(t)$ on U_t define a smooth section $\Phi \in \Gamma(\mathcal{U}, \pi^* \Lambda^3 T^* X)$, where $\pi : \mathcal{U} \to X$ is the canonical projection.
  (b) For any fixed t_0, the restriction of Φ to $U_{t_0} \times (0, t_0) \subset \mathcal{U}$ extends to a smooth section of $U_{t_0} \times [0, t_0)$ and $\Phi\big|_{\{0\} \times U_{t_0}} = \varphi_{CS}\big|_{U_{t_0}}$.

(3) For every $t \in (0, \epsilon)$, there exist open sets $V_t \subset N$ and $\widetilde{V}_t \subset M$, such that

  (a) $V_s \subset V_t$ for every $s > t$,
  (b) $\bigcup_{t \in (0,\epsilon)} V_t = N$.

(4) For every $t \in (0, \epsilon)$, there exists a diffeomorphism $G_t : V_t \to \widetilde{V}_t$ with the following significance:

  (a) Let $\mathcal{V} = \bigcup_{t \in (0,\epsilon)} \{t\} \times V_t$. The forms $t^{-3} G_t^* \varphi(t)$ on V_t define a smooth section $\hat{\Phi} \in \Gamma(\mathcal{V}, \pi^* \Lambda^3 T^* X)$, where $\pi : \mathcal{V} \to N$ is the canonical projection.
  (b) For any fixed t_0, the restriction of $\hat{\Phi}$ to $V_{t_0} \times (0, t_0)$ extends to a smooth section of $V_{t_0} \times [0, t_0) \subset \mathcal{W}$ and $\hat{\Phi}\big|_{\{0\} \times V_{t_0}} = \varphi_{AC}\big|_{V_{t_0}}$.

(5) For every t, the sets $\widetilde{U}_t$ and $\widetilde{V}_t$ cover M, i.e. $M = \widetilde{U}_t \cup \widetilde{V}_t$.

Since this definition is fairly technical, we explain it in informal terms. Some of the sets appearing in the definition are illustrated in Fig. 1 . Roughly speaking, we want the forms $\varphi(t)$ to converge in the smooth Cheeger–Gromov sense to the conifold $(\overline{X}, \varphi_{CS})$ as t goes to 0. Similarly, a rescaling of the forms $\varphi(t)$ should converge to the AC G_2 manifold (N, φ_{AC}) as t goes to 0. The form φ_{CS} is only defined on the non-compact manifold X and for this reason we pick an exhaustion of X in (1). This is an exhaustion that expands as t decreases. The second point then essentially says that $\varphi(t)$ can be pulled back to this exhaustion, so that $\varphi(t)$ defines a smooth family

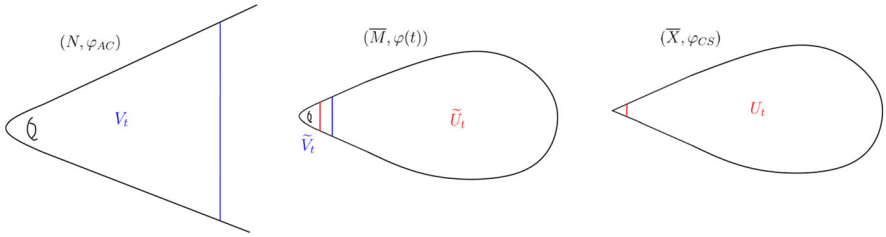

Fig. 1 Spaces and sets occuring in a desingularization

of forms on the "space time" of the exhaustion. The convergence as $t \to 0$ is then encoded as condition (2)(b), which says that the smooth family extends smoothly to the time 0 slice. It is crucial for our application that the family depends smoothly on the parameter t up to and including time 0.

Definition 4.1 can easily be adapted to the nearly Kähler setting. In the interest of space we refrain from giving the full definition again. Instead, the following definition only serves as a reference point and to pin down the terminology.

Definition 4.2 Suppose (C, ω_C, Ω_C) is a 6-dimensional Calabi–Yau cone, $(N, \omega_{AC}, \Omega_{AC})$ an asymptotically conical Calabi–Yau manifold asymptotic to (C, ω_C, Ω_C) and $(\overline{X}, \omega_{CS}, \Omega_{CS})$ a nearly Kähler conifold, which has a singular point x_0 modelled on the cone (C, ω_C, Ω_C). A *smooth desingularization* of $(\overline{X}, \omega_{CS}, \Omega_{CS})$ at x_0 by $(N, \omega_{AC}, \Omega_{AC})$ is a family of conically singular nearly Kähler structures $(\omega(t), \Omega(t))$ satisfying the conditions (1)–(5) in 4.1, changing the occurrences of $\varphi(t)$ by $\omega(t)$ and $\Omega(t)$ as necessary.

The existence of a smooth desingularization implies existence of a solution to an equation on the AC manifold, which we will call the obstruction equation. This will be used to prove the corollaries A and B.

We start with the G_2 case.

Suppose (C, φ_C) is a G_2 cone, (N, φ_{AC}) is an AC G_2 manifold with cone (C, φ_C) and $(\overline{X}, \varphi_{CS})$ is a conically singular nearly parallel G_2 manifold, which has a singular point x_0 modelled on the cone (C, φ_C).

Let $\overline{M}$ be a topological space, $\Sigma \subset \overline{M}$ a finite space and suppose $M = \overline{M} \backslash \Sigma$ is a smooth 7-manifold. Suppose $\varphi(t)$, $t \in (0, \epsilon)$, is a smooth desingularization of $(\overline{X}, \varphi_{CS})$ at x_0 by (N, φ_{AC}).

Then the family $\widetilde{\varphi}(t) = t^{-3}\varphi(t)$ satisfies

$$\mathrm{d}\widetilde{\varphi}(t) = 4t^{-3}\Theta(\varphi(t)) = 4t\Theta(\widetilde{\varphi}(t)),$$

where we use the homogeneity $\Theta(t^3\varphi) = t^4\Theta(\varphi)$.

Now let $U_t, \mathcal{U}$ and G_t be as in Definition 4.1 and define $\hat{\varphi}(t) = t^{-3}G_t^*\varphi(t)$. Then $\hat{\varphi}(t)$ satisfies

$$\mathrm{d}\hat{\varphi}(t) = 4t\Theta(\hat{\varphi}(t)),$$

on every $U_{t_0} \times [0, t_0)$. A priori the second equation is only satisfied on $U_{t_0} \times (0, t_0)$, but since $\hat{\varphi}$ is smooth on $U_{t_0} \times [0, t_0)$, it also holds on $U_{t_0} \times \{0\}$.

Taking the derivative of the first equation in t-direction yields $d\partial_t\hat{\varphi}(t) = 4\Theta(\hat{\varphi}(t)) + 4td\Theta[\hat{\varphi}(t)]\partial_t\hat{\varphi}(t)$. Evaluating at time $t = 0$ we obtain $d\partial_t\hat{\varphi}(0) = 4\Theta(\hat{\varphi}(0))$. Note that this equations hold on any U_{t_0}. Since U_{t_0} exhausts N as $t_0 \to 0$, it follows that the equation holds on all of N. According to Definition 4.1 at $t = 0$ the form $\hat{\varphi}(t)$ becomes the form φ_{AC}. Now let $\eta = \partial_t\hat{\varphi}(0)$. Then we have derived the *obstruction equation*

$$\mathrm{d}\eta = 4\Theta(\varphi_{AC}) = 4 *_{g_{AC}} \varphi_{AC}.$$

Therefore $*_{g_{AC}}\varphi_{AC}$ is exact. This proves Theorem A.

The obstruction equation admits a different interpretation coming from the associated gluing problem. If one tries to glue the asymptotically conical G_2 manifold into the nearly G_2 conifold one faces two problems. The first is that the equations satisfied on each piece are obviously different. Given a solution of the obstruction equation η, we find that

$$\mathrm{d}(\varphi_{AC} + \epsilon\eta) = 4\epsilon\Theta(\varphi_{AC}),$$

i.e. $\varphi_{AC} + \epsilon\eta$ solves the nearly G_2 equation at scale ϵ to first order. The second issue is that φ_{AC} at infinity matches φ_{CS} at the singularity only to the zeroth order. We could now derive a condition on the expansion of η at infinity, which ensures that $\varphi_{AC} + \eta$ matches φ_{CS} at least to the first order. As we will see later, even without this extra condition, it is difficult to solve the equation $d\eta = 4\Theta(\varphi_{AC})$ and therefore we will not pursue this here.

Note that in principle, one can derive obstruction equations to any order by taking higher derivatives. However, as we will find that already the first equation is not soluble in many cases, this will also not be pursued.

This interpretation of the obstruction equations is closely related to the works of Biquard [3, 4], Morteza–Viaclovsky [24], Ozuch [26, 27, 29]. As explained in the introduction, the work of Ozuch suggests that much stronger results may be true. The interpretation of solutions of the obstruction equations as a starting point for a refined gluing construction is crucial for Biquard's and Ozuch's method of proof.

For nearly Kähler manifolds, the derivation of the obstruction equations is very similar. To this end, we now assume that $(\overline{X}, \omega_{CS}, \Omega_{CS})$ is a nearly Kähler conifold with a singularity at x_0 modelled on the Calabi–Yau cone $(C = C(L), \omega_C, \Omega_C)$. Suppose $(N, \omega_{AC}, \Omega_{AC})$ is an asymptotically conical Calabi–Yau manifold asymptotic to (C, ω_C, Ω_C) and suppose that $(\overline{M}, \omega(t), \Omega(t))$ is a desingularization of $(\overline{X}, \omega_{CS}, \Omega_{CS})$ at x_0 by $(N, \omega_{AC}, \Omega_{AC})$.

To derive the obstruction equations in this case, observe that by the construction above, we obtain a family of forms $(\hat{\omega}(t), \hat{\Omega}(t))$, such that

$$\mathrm{d}\hat{\omega}(t) = -3t\,\mathrm{Re}\,\hat{\Omega}(t), \qquad \mathrm{d}\,\mathrm{Im}\,\hat{\Omega}(t) = 2t\hat{\omega}(t)^2.$$

Taking the time derivative at $t = 0$, we obtain a 2-form ν and a 3-form η satisfying

$$\mathrm{d}\nu = -3\,\mathrm{Re}\,\Omega_{AC}, \qquad \mathrm{d}\eta = 2\omega_{AC}^2.$$

This proves Theorem B. As in the G_2 case, we may also interpret this obstruction equation coming from the associated gluing problem.

5 Analysis on Asymptotically Conical Manifolds

The proof of Theorems C and D will require an understanding of certain analytical and topological properties of AC manifolds. For the convenience of the reader, we summarize the required results here.

Let (M, g) be an asymptotically conical Riemannian manifold and let $\rho : M \to \mathbb{R}$ be a radial function.

For any tensor T, any integer $l \in \mathbb{N}_0$, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$ the $L^p_{l,\lambda}$-Sobolev norm is defined by

$$\|u\|_{L^p_{l,\lambda}} = \left(\sum_{j \le l} \int_M \left| \rho^{j-\lambda} \nabla^j T \right|_g^p \rho^{-n} \,\mathrm{vol}_M \right)^{1/p}.$$

By $L^p_{l,\lambda}$, we denote the completion of the space of smooth tensors with compact support. If necessary, we will be more precise about the space of tensors considered by putting it in brackets. For example, the Sobolev space of differential k-forms will be denoted by $L^p_{l,\lambda}(\Lambda^k T^* M)$. The parameter λ is called the *weight* of the Sobolev space. Note that for $\lambda = -n/2$ and $p = 2$, we recover the unweighted L^2 space, $L^2_{0,-n/2} = L^2$.

There is a well-developed theory of elliptic operators acting on these spaces. We will only need results for the Laplacian acting on functions, $\Delta_g = -\,\mathrm{tr}_g\,\mathrm{Hess}\, g$. It is easy to see that Δ_g is a bounded operator

$$\Delta_g : L^p_{l+2,\lambda} \to L^p_{l,\lambda-2},$$

for any choice of p, l, λ as above. The set of *critical rates* on the cone (C, g_C) is defined to be

$$\mathcal{D} = \left\{ \lambda \in \mathbb{R} : \exists f \in C^\infty(C) : \Delta_g \kappa = 0,\ \mathcal{L}_{r\partial r} f = \lambda \kappa \right\}.$$

This means that $\lambda \in \mathbb{R}$ is a critical rate of the Laplacian, if there exists a homogeneous harmonic function of rate λ on the cone. This set is crucial for describing the mapping properties of Δ_g, as the following theorem explains.

Theorem 5.1 *The Laplacian Δ_g has the following properties.*

(1) If $\lambda \notin \mathcal{D}$, the operator $\Delta_g : L^p_{l+2,\lambda} \to L^p_{l,\lambda-2}$ is Fredholm.

(2) *The kernel* $\ker \Delta_g : L^p_{l+2,\lambda} \to L^p_{l,\lambda-2}$ *does not depend on* p *or* l *and is denoted by* $\ker_\lambda \Delta_g$.
(3) *If* $\lambda_1 < \lambda_2 \in \mathbb{R}$ *are such that* $[\lambda_1, \lambda_2] \cap \mathcal{D}_k = \varnothing$, *then* $\ker_{\lambda_1} \Delta = \ker_{\lambda_2} \Delta$.

A close reference for this result is Sect. 4.2 of [20]. (In this case, the theory is discussed in dimension 7, but it holds in any dimension.) This particular approach to the Fredholm theory of elliptic operators on non-compact manifolds goes back to Lockhart and McOwen, see [23].

On a closed Riemannian manifold, one consequence of Hodge theory is that any coclosed, exact differential form must vanish. On non-compact manifolds, this statement fails in general. The following theorem, due to Lockhart, gives a partial replacement for this statement on asymptotically conical Riemannian manifolds.

Theorem 5.2 *Suppose* (M, g) *is an* n*-dimensional asymptotically conical manifold. If* $\kappa \in L^2(\Lambda^k T^* M)$ *with* $k \geq n/2$ *is exact and coclosed, then* $\kappa \equiv 0$.

See [22], Theorem 7.4 and example 0.15.

Investigating the critical rates of the operator $d + d^*$, one can sometimes improve Theorem 5.2 beyond the L^2 range. The following situation is relevant for our application.

Corollary 5.3 *Suppose* (M, φ) *is an asymptotically conical* G_2 *manifold and* $\lambda < -3$. *If* $\kappa \in L^2_\lambda(\Lambda^4 T^* M)$ *is exact and coclosed, then* $\kappa \equiv 0$.

This is a consequence of corollary 4.29 in [20], which says that $\mathcal{H}^k_\lambda = \{\kappa \in L^2_{2,\lambda}(\Lambda^k T^* M) : d\kappa = 0,\ d^*\kappa = 0\}$ is independent of λ for $\lambda \in (-4, -3)$. Since theorem 5.2 says that $\mathcal{H}^4_{-7/2}$ does not contain any exact forms besides the zero form, this remains valid for $\mathcal{H}^4_\lambda$ for any $\lambda \in (-4, -3)$.

6 Approximate Potentials and Their Asymptotics

Let (L, g_L) be a closed Riemannian manifold. The cone metric $g_C = dr^2 + r^2 g_C$ on $C = (0, \infty) \times L$ has a *potential*: the function $\frac{1}{2}r^2$ satisfies

$$\mathrm{Hess}_{g_C} \frac{1}{2} r^2 = g_C,$$

or equivalently $\mathcal{L}_{\mathrm{grad}_{g_C} \frac{1}{2} r^2} g_C = 2 g_C$. Conversely, the existence of a function whose Hessian is the metric implies that the metric is locally conical. The only complete manifold admitting such a function is Euclidean space. This is the content of the following theorem, due to Tashiro [30].

Theorem 6.1 *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold. If there exists* $u \in C^2(M)$ *with*

$$\mathrm{Hess}_g\, u = g,$$

then (M, g) *is isometric to* $(\mathbb{R}^n, g_{\mathrm{eucl}})$.

On an asymptotically conical manifold (M, g) we can study *approximate potentials*, that is functions $u \in C^\infty(M)$, which are asymptotic to $\frac{1}{2}r^2$ and satisfy $\Delta_g u = -n$. These functions and understanding their precise asymptotics will be fundamental for Theorems C and D. The following theorem ensures existence of such a function and describes its asymptotics.

Theorem 6.2 *Let (M, g) be an asymptotically conical Riemannian manifold with cone at infinity (C, g_C) and rate $\nu < -2$. Suppose that $\Psi : (R, \infty) \times K \to M \backslash K$ are coordinates at infinity and let $\rho : M \to \mathbb{R}$ be a radial function.*

There exists a unique function $u \in C^\infty(M)$, such that

$$\Delta_g u = -n, \qquad u - \frac{1}{2}\rho^2 \in L^2_{2,-\epsilon},$$

for some $\epsilon > 0$. This function satisfies

$$u - \frac{1}{2}\rho^2 \in L^2_{2,\lambda},$$

with $\lambda = 2 + \nu + \epsilon$ for every $\epsilon > 0$.

Lemma 6.3 *Let (M, g) be an n-dimensional AC manifold.*

For $\lambda < 0$ the operator

$$\Delta_g : L^2_{2,\lambda} \to L^2_{0,\lambda-2},$$

is an isomorphism is an isomorphism if λ is not a critical rate. There are no critical rates in the interval $(2 - n, 0)$.

Proof First observe that by partial integration for $\lambda < \frac{1}{2}(2-n)$ any harmonic function on M is trivial and therefore $\ker_\lambda \Delta_g = 0$ for this range of weights.

Let (C, g_C) denote the cone at infinity of (M, g). Let (L, g_L) be its link. To see that $\lambda \in (2-n, 0)$ is not a critical rate, observe that for any function $r^\lambda f$ with $f \in C^\infty(L)$ we have

$$\Delta_{g_C} r^\lambda f = r^{\lambda-2}(\Delta_{g_L} f - \lambda(\lambda + n - 2) f).$$

Therefore $r^\lambda f$ can only be harmonic if $\Delta_{g_L} f = \lambda(\lambda+n-2) f$. But for $\lambda \in (2-n, 0)$ the coefficient $\lambda(\lambda + n - 2)$ is negative. On the other hand, Δ_{g_L} is a non-negative operator and therefore $f = 0$. Thus there are no non-trivial harmonic, homogeneous functions of rate $\lambda \in (2 - n, 0)$.

This also shows that $\ker_\lambda \Delta_g = 0$ for every $\lambda < 0$ by Theorem 5.1 (3), since $2 - n < \frac{1}{2}(2 - n) < 0$.

Whenever λ is not a critical rate $\Delta_g : L^2_{2,\lambda} \to L^2_{0,\lambda-2}$ is a Fredholm operator and therefore when $\lambda < 0$ it is an isomorphism.

Lemma 6.4 *Let (M, g) be an asymptotically Riemannian manifold with cone at infinity (C, g_C) and rate $\nu < -2$. Suppose that $\Psi : (R, \infty) \times K \to M \backslash K$ are coordinates at infinity and let $\rho : M \to \mathbb{R}$ be a radial function. Then for every $\epsilon > 0$*

$$\Delta_g \rho^2 + 2n \in L^2_{0,\nu+\epsilon}.$$

Proof Denote the pullback of g to the cone via Ψ by $\widetilde{g} = \Psi^* g$. Then

$$\Psi^* \Delta_g \rho^2 = \Delta_{\widetilde{g}} r^2 + 2n,$$

because $\Psi^* \rho^2 = r^2$. Since $\Delta_{g_C} r^2 = -2n$, we may rewrite this as

$$\Psi^* \Delta_g \rho^2 = \Delta_{\widetilde{g}} r^2 - \Delta_{g_C} r^2.$$

To estimate the difference $\Delta_{\widetilde{g}} - \Delta_{g_C}$, recall the general formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Delta_{g+th} u = -2g(\mathrm{Hess}_g\, u, h) - g(B_g h, \mathrm{d}u),$$

where g is any Riemannian metric and h a symmetric 2-tensor and u is any function.

For $g = g_C$ and $h = \widetilde{g} - g_C$, we therefore get

$$\Delta_{\widetilde{g}} u = \Delta_{g_C} u - 2g_C(\mathrm{Hess}_{g_C}\, u, h) - g(B_g h, \mathrm{d}u) + Q(h, \mathrm{d}u, \nabla^{g_C} \mathrm{d}u),$$

for some Q satisfying

$$|Q(h, \mathrm{d}u, \nabla^{g_C} \mathrm{d}u)| \leq C|h|^2(|\mathrm{d}u| + |\nabla^{g_C} \mathrm{d}u|).$$

The absence of a term depending on the zero-order term u is explained by the definition $\Delta_g = -\operatorname{tr}_g \nabla^g du$ and the fact that the operator $u \mapsto du$ does not depend on the metric.

If we now let $u = \Psi^* \rho^2 = r^2$, we have $\mathrm{Hess}_{g_C}\, u = 2g_C$ and since $g_C(g_C, h) = \operatorname{tr}_{g_C} h$, this yields

$$\Delta_{\widetilde{g}} r^2 = \Delta_{g_C} u - 4 \operatorname{tr}_{g_C} h - g_C(B_{g_C} h, 2r\mathrm{d}r) + Q(h, \mathrm{d}u, \nabla^{g_C} \mathrm{d}u).$$

By definition of AC manifolds, we then have that there exists a constant C_0, such that

$$|h|_{g_C} = |\widetilde{g} - g_C|_{g_C} \leq C_0 r^\nu.$$

This implies $|\operatorname{tr}_{g_C} h| \leq nC_0 r^\nu$. Similarly, we have $|\nabla^{g_C} h|_{g_C} \leq C_1 r^{\nu-1}$ and therefore $|g_C(B_{g_C} h, 2rdr)| \leq |B_{g_C} h|_{g_C} |2rdr|_{g_C} \leq \widetilde{C} r^\nu$ for some $\widetilde{C} > 0$. Therefore, $|\Delta_{\widetilde{g}} r^2 - 2n| \leq \hat{C}(r^\nu + r^{2\nu+1})$ for some constant $\hat{C} > 0$. This yields the result, since $2\nu + 1 < \nu$ and $r^\nu \in L^2_{0,\nu+\epsilon}$ for every $\epsilon > 0$.

Proof Define $p = \Delta_g \frac{1}{2}\rho^2 + n$. By lemma 6.3, the operator

$$\Delta_{g_{AC}} : L^2_{2,\lambda} \to L^2_{0,\lambda-2},$$

is an isomorphism for every $\lambda < 0$ that is not a critical rate.

Lemma 6.4 says that $p \in L^2_{0,\nu+\epsilon}$ for every $\epsilon > 0$. Since the set of critical rates is discrete, it follows that $\Delta_g : L^2_{2,\nu+\epsilon} \to L^2_{0,\nu+\epsilon}$ is an isomorphism for every $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$, even if ν is a critical rate. Therefore $v = \Delta_g^{-1} p \in L^2_{2,\nu+2+\epsilon}$ is well defined for every $\epsilon \in (0, \epsilon_0)$.

Now consider $u = \frac{1}{2}\rho^2 - v$. Then we have

$$\Delta_g u = \frac{1}{2}\Delta_g \rho^2 - \Delta_g v = p - p - n = -n.$$

This shows existence and the decay property.

For uniqueness, assume that there exists another function $\hat{u}$ with $\hat{u} - \frac{1}{2}\rho^2 \in L^2_{2,-\widetilde{\epsilon}}$ for some $\widetilde{\epsilon} > 0$. Let $\lambda = \max\{\nu + 2 + \epsilon, -\widetilde{\epsilon}\}$. Then $\hat{u} - u \in L^2_{2,\lambda}$ and $\Delta_g(\hat{u} - u) = 0$. By lemma 6.3 it follows that $\hat{u} - u = 0$.

Remark. As pointed out by the anonymous reviewer, if the cone at infinity (C, g_C) is assumed to be Ricci flat, the condition on the rate $\nu < -2$ may be improved to $\nu < -1 + \delta$ for some $\delta > 0$. Indeed, as a consequence of Obata's theorem the only critical rates of the Laplacian acting on functions on a Ricci flat cone in the interval $[1 - n, 1]$ are $2 - n$ and 0. The critical rate $2 - n$ is not relevant for our discussion, since a $L^2_{2-n} \subset L^2$. When crossing the rate zero, the kernel $\ker_\lambda \Delta_g$ is enlarged by the constant functions. Except for the uniqueness, this change does not affect the proof of Theorem 6.2. In particular, we obtain an approximate potential up to a constant function. This ambiguity is not relevant to the applications below. On the other hand, in all applications below the rate will be assumed to be smaller than -2 for other reasons.

7 The Action of Sym^2 on Λ^*

Let (M, g) be a Riemannian manifold. If $h \in \Gamma(\mathrm{Sym}^2 T^*M)$ and $\kappa \in \Omega^k(M)$, then we define

$$h_*\kappa = -\sum_{j=1}^{n} h(e_j, \cdot) \wedge \iota_{e_j}\kappa,$$

where $e_1, \ldots, e_n$ is any orthonormal frame. Using the standard basis of $\Lambda^* T^*M$ induced by the dual frame $e^1, \ldots, e^n$ it is easy to see that

$$g_*\kappa = -k\kappa.$$

Recall that for any differential form $\eta \in \Omega^k(M)$, we have

$$d^*\eta = -\sum_{j=1}^{n} \iota_{e_i} \nabla_{e_i} \eta.$$

Proposition 7.1 *If $\kappa \in \Omega^k(M)$ is parallel and $u \in C^\infty(M)$, then*

$$d^*(du \wedge \kappa) = (d^*du)\kappa - (\operatorname{Hess} u)_*\kappa.$$

Proof This follows from repeated use of the product rule and the definition $\operatorname{Hess} u = \nabla du$:

$$\begin{aligned} d^*(du \wedge \kappa) &= -\sum_{j=1}^{n} \iota_{e_i} \nabla_{e_i}(du \wedge \kappa) = -\sum_{j=1}^{n} \iota_{e_i}[(\nabla_{e_i} du) \wedge \kappa] \\ &= -\sum_{j=1}^{n} \left[(\iota_{e_i} \nabla_{e_i} du) \wedge \kappa - (\nabla_{e_i} du) \wedge \iota_{e_i}\kappa\right] \\ &= (d^*du)\kappa - (\nabla du)_*\kappa. \end{aligned}$$

In general, not much can be said about $h_*\kappa$. For the forms associated to G_2 and SU(3) structures, the action of Sym^2 can be described more precisely.

Suppose that M is a 7-manifold and $\varphi \in \Omega^3_+(M)$ is a G_2 structure. Let g be the associated Riemannian metric. Then we can decompose

$$\operatorname{Sym}^2 T^*M = \mathbb{R}g \oplus \operatorname{Sym}^2_0 T^*M,$$

where $\operatorname{Sym}^2_0 T^*M = \{h \in \operatorname{Sym}^2 T^*M : \operatorname{tr}_g h = 0\}$. It turns out that the map

$$\operatorname{Sym}^2_0 T^*M \to \Lambda^3 T^*M, \qquad h \mapsto h_*\varphi,$$

is injective. This can be understood much better from the representation theoretic point of view. For this we refer to [7].

If, on the other hand, M is a 6-manifold and (ω, Ω) is an SU(3) structure, then let g denote the associated metric and J the associated almost complex structure. In this case, the space $\operatorname{Sym}^2 T^*M$ can be decomposed as

$$\operatorname{Sym}^2 T^*M = \mathbb{R}g \oplus \operatorname{Sym}^2_8 \oplus \operatorname{Sym}^2_{12},$$

where

$$\begin{aligned} \operatorname{Sym}^2_8 &= \{h \in \operatorname{Sym}^2 T^*M : \operatorname{tr}_g h = 0,\ J^*h = h\}, \\ \operatorname{Sym}^2_{12} &= \{h \in \operatorname{Sym}^2 T^*M : J^*h = -h\}. \end{aligned}$$

The subscripts denote the dimension of each of these spaces. Given $h \in \mathrm{Sym}^2_0 T^*M$, we may decompose

$$h = \underbrace{[h]_8}_{\in \mathrm{Sym}^2_8} + \underbrace{[h]_{12}}_{\in \mathrm{Sym}^2_{12}} .$$

With this notation, one finds that

$$h_*\omega = ([h]_8)_*\omega,$$
$$h_* \operatorname{Re}\Omega = ([h]_{12})_* \operatorname{Re}\Omega,$$

and the maps

$$\mathrm{Sym}^2_8 \to \Lambda^2 T^*M, \qquad h \mapsto h_*\omega,$$
$$\mathrm{Sym}^2_{12} \to \Lambda^3 T^*M, \qquad h \mapsto h_* \operatorname{Re}\Omega,$$

are injective. This decomposition is also best understood from the representation theoretic point of view and we refer to [15] for more details.

8 Proofs of Theorems C and D

This section contains the proofs of Theorems C and D. Before we enter the proofs, let us give two (closely related) conceptual motivations for the proof.

First, observe that on a G_2 cone (C, φ_C) the form $*\varphi_C$ is exact. Indeed, since φ_C is parallel Proposition 7.1 applies and with $u = r^/2$, we find that

$$d^*(dr^2/2 \wedge \varphi_C) = -\varphi_C - g_*\varphi_C = -4\varphi_C.$$

Therefore, the form $\frac{1}{4} * (dr^2/2 \wedge \varphi_C)$ is a primitive of $*\varphi_C$. On an asymptotically conical G_2 manifold (M, φ_{AC}), the form $\frac{1}{4} *(du \wedge \varphi_{AC})$ for u an approximate potential is then an *approximate* primitive. Provided the rate is small enough, it will then turn out that by Hodge theory, this approximate primitive is a primitive if and only if u is an actual potential function.

Second, as pointed out by the anonymous referee, deformation theory gives another conceptual explanation of the proof. The conditions that a manifold has holonomy G_2 is scaling invariant. Therefore, a given G_2 manifold (M, φ) is part of the 1-parameter family $(M, \lambda^3\varphi)$. The infinitesimal deformation at φ is then given by 3φ. On a G_2 cone (C, φ_C) the G_2 form is homogeneous of rate 0, i.e.

$$\mathcal{L}_{r\partial_r}\varphi_C = 3\varphi_C.$$

This means that on a G_2 cone the scaling deformation is trivial, in the sense that it is generated by diffeomorphisms. On an asymptotically conical G_2 manifold (M, φ)

asymptotic to (C, φ_C) at rate ν, consider the form $\mathcal{L}_{\operatorname{grad} u}\varphi - 3\varphi$, where u is an approximate potential. This is an infinitesimal deformation, which preserves the asymptotic cone (C, φ_C). By the deformation theory developed in [20], we may identify such a deformation with a closed and coclosed 3-form of rate ν. These properties can also be computed explicitly, as done in the proof below. Provided the rate is small enough, Hodge theory will here yield that the Hodge dual of the form $\mathcal{L}_{\operatorname{grad} u}\varphi - 3\varphi$ vanishes, if and only if φ is exact.

Note also that the formula $d^*(dr^2/2 \wedge \varphi_C) = -4\varphi_C$ is closely related to the formula $\mathcal{L}_{r\partial_r} * \varphi_C = 4 * \varphi_C$. Indeed, we have

$$\begin{aligned}\mathcal{L}_{r\partial_r} * \varphi_C &= \mathcal{L}_{\operatorname{grad}_{g_C} r^2/2} * \varphi_C = d(\operatorname{grad}_{g_C} r^2/2 \lrcorner * \varphi_C)\\ &= d * (dr^2/2 \wedge *^2\varphi_C) = *^2 d * (dr^2/2 \wedge \varphi_C)\\ &= - * d^*(dr^2/2 \wedge \varphi_C)\end{aligned}$$

and more generally, we have $\mathcal{L}_{\operatorname{grad} u} * \varphi = - * d^*(du \wedge \varphi)$ on any G_2 manifold (M, φ).

In the Calabi–Yau setting, we can argue similarly. In particular, on a Calabi–Yau cone, we find

$$\begin{aligned}d^*(dr^2/2 \wedge \omega_C) &= -4\omega_C = -\mathcal{L}_{r\partial_r}\omega_C^2,\\ d^*(dr^2/2 \wedge \operatorname{Im}\Omega_C) &= -3\operatorname{Re}\Omega_C = -\mathcal{L}_{r\partial_r}\operatorname{Re}\Omega_C.\end{aligned}$$

Proof of Theorem C Suppose that (M, g, φ) is an asymptotically conical G_2 manifold of rate $\nu < -3$.

If (M, g) is isometric to $(\mathbb{R}^7, g_{\text{eucl}})$, then the closed form $*\varphi$ is exact by the Poincaré lemma.

For the converse, assume that $*\varphi$ is exact. We will show that (M, g) is isometric to $(\mathbb{R}^7, g_{\text{eucl}})$.

By Theorem 6.2, there exists a function $u : N \to \mathbb{R}$ with

$$\Delta_{g_\varphi} u = -7, \qquad \Psi^* u - \frac{1}{2}\rho^2 \in L^2_{2,\lambda},$$

where $\lambda = 2 + \nu + \epsilon$ for any $\epsilon > 0$. The tracefree part of the Hessian of u, $\mathring{\operatorname{Hess}} u$, is then in $L^2_{0,\lambda-2}$. The form

$$\kappa = *(\mathring{\operatorname{Hess}} u)_* \varphi,$$

is then also in $L^2_{0,\lambda-2}$. Note that for sufficiently small $\epsilon > 0$, the weight $\lambda - 2$ will be smaller than -3, so that corollary 5.3 applies to κ. We will now show that κ is closed and coclosed. The assumption that $*\varphi$ is exact will imply that κ is exact.

Applying Proposition 7.1 yields

$$\mathrm{d}^*(\mathrm{d}u \wedge \varphi) = -7\varphi - (\operatorname{Hess} u)_* \varphi.$$

The Hessian of u can be split into a tracefree part $\mathring{\mathrm{Hess}}u$ and its trace part,

$$\mathrm{Hess}\, u = \mathring{\mathrm{Hess}}u + \frac{1}{7}(\mathrm{tr}_g \,\mathrm{Hess}\, u)g = \mathring{\mathrm{Hess}}u + g,$$

where we use $\mathrm{tr}_g \,\mathrm{Hess}\, u = -\Delta_g u = -7$. Since φ is a 3-form, we have $g_*\varphi = -3\varphi$ and therefore $(\mathrm{Hess}\, u)_*\varphi = (\mathring{\mathrm{Hess}}u)_*\varphi - 3\varphi$. Therefore,

$$(\mathring{\mathrm{Hess}}u)_*\varphi = 4\varphi - \mathrm{d}^*(\mathrm{d}u \wedge \varphi).$$

Since $d^*\varphi = 0$ and $(d^*)^2 = 0$, we obtain $d^*(\mathring{\mathrm{Hess}}u)_*\varphi = 0$. This is equivalent to $d\kappa = 0$. Applying the Hodge star operator to this equation, we see that

$$\kappa = 4 * \varphi - \mathrm{d} * (\mathrm{d}u \wedge \varphi)$$

using that $*d^* = d*$ on $\Omega^4(M)$. (Note that in view of the discussion before the proof $\kappa = 4*\varphi - \mathcal{L}_{\mathrm{grad}\, u} *\varphi$.) Therefore, if $*\varphi$ is exact, then so is κ. To see that κ is coclosed, we compute

$$\mathrm{d}^*\kappa = -\mathrm{d}^* * \mathrm{d}^*(\mathrm{d}u \wedge \varphi) = - * \mathrm{d}\mathrm{d}^*(\mathrm{d}u \wedge \varphi) = - * \Delta(\mathrm{d}u \wedge \varphi),$$

where we used that $d^*\varphi = 0$ and that $du \wedge \varphi$ is closed. Formula (10) in [20] says that $\Delta(du \wedge \varphi) = (\Delta du) \wedge \varphi$. Moreover, $\Delta du = d\Delta u = 0$, since Δu is constant and therefore $d^*\kappa = 0$ as claimed.

According to Theorem 5.3 coclosed, exact 4-forms in L^2_λ with $\lambda \in (-4, -3)$ are trivial. Since we checked all of these conditions for κ, we conclude that $\kappa = 0$. Thus $\mathring{\mathrm{Hess}}u = 0$ and since $\Delta_g u = -n$, this means

$$\mathrm{Hess}\, u = g.$$

Therefore Tashiro's Theorem 6.1 implies that (M, g) is isometric to $(\mathbb{R}^7, g_{\mathrm{eucl}})$.

Proof of Theorem D Let (M, g, ω, Ω) be an asymptotically conical 6-dimensional Calabi–Yau manifold of rate $\nu < -3$.

If (M, g) is isometric to $(\mathbb{R}^6, g_{\mathrm{eucl}})$, then obviously ω and $\mathrm{Im}\,\Omega$ are exact.

For the other direction, we assume that ω^2 and $\mathrm{Im}\,\Omega$ are exact. We will show that then (M, g) is isometric to $(\mathbb{R}^6, g_{\mathrm{eucl}})$.

By Theorem 6.2, there exists a function $u : M \to \mathbb{R}$ with

$$\Delta_g u = -6, \qquad \Psi^* u - \frac{1}{2}\rho^2 \in L^2_{2,\lambda},$$

where $\lambda = 2 + \nu + \epsilon$ for any $\epsilon > 0$. The tracefree part of the Hessian of u, $\mathring{\mathrm{Hess}}u$, is then in $L^2_{0,\lambda-2}$. The forms

$$\eta = *(\mathring{\mathrm{Hess}}u)_* \,\mathrm{Re}\,\Omega, \qquad \kappa = *(\mathring{\mathrm{Hess}}u)_*\omega,$$

are then also in $L^2_{0,\lambda-2}$. If ϵ is sufficiently small, they will also be in the unweighted L^2 space, since $\nu < -3$. We will now show that η and κ are closed and coclosed. The assumption that $\operatorname{Im}\Omega$ is exact will imply that η is exact. Likewise, the assumption that $*\omega = \frac{1}{2}\omega^2$ is exact will imply that κ is exact.

Using Proposition 7.1, we obtain the formulas

$$d^*(du \wedge \omega) = -6\omega - (\operatorname{Hess} u)_*\omega, \qquad d^*(du \wedge \operatorname{Re}\Omega) = -6\operatorname{Re}\Omega - (\operatorname{Hess} u)_* \operatorname{Re}\Omega.$$

Using these formula, we may argue exactly as in the previous proof to show that η and κ are exact, coclosed forms.

By Theorem 5.2 coclosed, exact L^2 forms of form degree ≥ 3 are trivial. Since we checked all these conditions for η and κ, we conclude $\eta = 0$ and $\kappa = 0$.

If we decompose

$$\mathring{\operatorname{Hess}} u = \underbrace{[\mathring{\operatorname{Hess}} u]_8}_{\in \operatorname{Sym}^2_8} + \underbrace{[\mathring{\operatorname{Hess}} u]_{12}}_{\in \operatorname{Sym}^2_{12}},$$

then we have

$$\begin{aligned}(\mathring{\operatorname{Hess}} u)_*\omega &= ([\mathring{\operatorname{Hess}} u]_8)_*\omega,\\ (\mathring{\operatorname{Hess}} u)_* \operatorname{Re}\Omega &= ([\mathring{\operatorname{Hess}} u]_{12})_* \operatorname{Re}\Omega.\end{aligned}$$

Moreover, since these forms are 0, it follows that

$$[\mathring{\operatorname{Hess}} u]_8 = 0 \qquad \text{and} \qquad [\mathring{\operatorname{Hess}} u]_{12} = 0.$$

This shows that $\mathring{\operatorname{Hess}} u \equiv 0$ and by Tashiro's Theorem 6.1, this means (M, g) is isometric to $(\mathbb{R}^6, g_{\text{eucl}})$.

Acknowledgements The author wishes to thank M. Freibert and H. Weiß for many inspiring discussions, which led the author to consider the problems addressed in this article. The author also wishes to thank the anonymous referee, whose suggestions improved the exposition and sharpened several results.

Author Contributions The author was employed by Leibniz university of Hannover while preparing the manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL. No funding was received for conducting this study

Declarations

Conflict of interest The author has no competing interests to declare that are relevant to the content of this article.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included

in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Bär, C.: Real killing spinors and holonomy. Commun. Math. Phys. **154**, 509–521 (1993)
2. Bilal, A., Metzger, S.: Compact weak G_2-manifolds with conical singularities. Nucl. Phys. B **663**(1–2), 343–364 (2003)
3. Biquard, O.: Désingularisation de métriques d'Einstein. I. Invent. Math. **192**(1), 197–252 (2013)
4. Biquard, O.: Désingularisation de métriques d'Einstein. II. Invent. Math. **204**(2), 473–504 (2016)
5. Biquard, O., Hein, H.-J.: The renormalized volume of a 4-dimensional Ricci-flat ALE space, J. Differ. Geom. to appear. arXiv:1901.03647
6. Böhm, C.: Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. Invent. Math. **134**(1), 145–176 (1998)
7. Bryant, R.: Some remarks on G_2-structures. In: Proceedings of Gökova Geometry-Topology Conference 2005, pp 75–109, Gökova Geometry/Topology Conference (GGT), Gökova, (2006)
8. Bryant, R.L., Salamon, S.: On the construction of some complete metrics with exceptional holonomy. Duke Math. J. **58**(3), 829–850 (1989)
9. Chan, Y.M.: Desingularizations of Calabi-Yau 3-folds with a conical singularity. Q. J. Math. **57**(2), 151–181 (2006)
10. Chan, Y.M.: Desingularizations of Calabi-Yau 3-folds with conical singularities. II. The obstructed case. Q. J. Math. **60**(1), 1–44 (2009)
11. Cheeger, J., Tian, G.: On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay. Invent. Math. **118**(3), 493–571 (1994)
12. Chiossi, S., Salamon, S.: The intrinsic torsion of SU(3) and G2 structures. Differential geometry, Valencia,: pp 115–133, p. 2002. World Sci. Publ, River Edge, NJ (2001)
13. Conlon, R.J., Hein, H.-J.: Asymptotically conical Calabi–Yau manifolds, I. Duke Math. J. **162**(15), 2855–2902 (2013)
14. Dwivedi, S., Singhal, R.: Deformation theory of nearly G_2 manifolds, to appear in Comm. Anal. Geom
15. Foscolo, L.: Deformation theory of nearly Kähler manifolds. J. Lond. Math. Soc. **95**(2), 586–612 (2017)
16. Foscolo, L., Haskins, M.: New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$,. Ann. Math. **185**(2), 59–130 (2017)
17. Foscolo, L., Haskins, M., Nordström, J.: Infinitely many new families of complete cohomogeneity one G_2-manifolds: G_2 analogues of the Taub-NUT and Eguchi–Hanson spaces. J. Eur. Math. Soc. (JEMS) **23**(7), 2153–2220 (2021)
18. Hitchin, N.: Stable forms and special metrics, Contemp. Math. 288, Journal of the American Mathematical Society, Providence, RI (2001)
19. Karigiannis, S.: Desingularization of G_2 manifolds with isolated conical singularities. Geom. Topol. **13**(3), 1583–1655 (2009)
20. Karigiannis, S., Lotay, J.D.: Deformation of G_2 conifolds. Commun. Anal. Geom. **28**(5), 1057–1210 (2020)
21. Kröncke, K., Szabó, Á.: Optimal coordinates for Ricci-flat conifolds, arXiv:2203.01711
22. Lockhart, R.: Fredholm, Hodge and Liouville theorems on noncompact manifolds. Trans. Am. Math. Soc. **301**(1), 1–35 (1987)
23. Lockhart, R., McOwen, R.: Elliptic differential operators on noncompact manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4**(12), 409–447 (1985)
24. Morteza, P., Viaclovsky, J.A.: The Calabi metric and desingularization of Einstein orbifolds. J. Eur. Math. Soc. (JEMS) **22**(4), 1201–1245 (2020)
25. Nagy, P.A., Semmelmann, U.: Deformations of nearly G_2 structures. J. Lond. Math. Soc. **104**(2), 1795–1811 (2021)
26. Ozuch, T.: Noncollapsed degeneration of Einstein 4-manifolds I, Geom. Topol., to appear. arXiv:1909.12957

27. Ozuch, T.: Noncollapsed degeneration of Einstein 4-manifolds II, Geom. Topol., to appear. arXiv:1909.12960
28. Ozuch, T.: Higher order obstructions to the desingularization of Einstein metrics, Camb. J. Math. to appear. arXiv:2012.13316
29. Ozuch, T.: Integrability of Einstein deformations and desingularizations, Commun. Pure Appl. Math., to appear. arXiv:2105.13193
30. Tashiro, Y.: Complete Riemannian manifolds and some vector fields. Trans. Am. Math. Soc. **117**, 251–275 (1965)
31. van Coevering, C.: Ricci-flat Kähler metrics on crepant resolutions of Kähler cones. Math. Ann. **347**(3), 581–611 (2010)
32. Weiss, H., Witt, F.: A heat flow for special metrics. Adv. Math. **231**, 3288–3322 (2012)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.