



Quantitative Sobolev Extensions and the Neumann Heat Kernel for Integral Ricci Curvature Conditions

Olaf Post¹ · Xavier Ramos Olivé² · Christian Rose³ 

Received: 11 April 2022 / Accepted: 5 October 2022 / Published online: 19 December 2022
© The Author(s) 2022

Abstract

We prove the existence of Sobolev extension operators for certain uniform classes of domains in a Riemannian manifold with an explicit uniform bound on the norm depending only on the geometry near their boundaries. We use this quantitative estimate to obtain uniform Neumann heat kernel upper bounds and gradient estimates for positive solutions of the Neumann heat equation assuming integral Ricci curvature conditions and geometric conditions on the possibly non-convex boundary. Those estimates also imply quantitative lower bounds on the first Neumann eigenvalue of the considered domains.

Keywords Sobolev extensions · Integral Ricci curvature bounds · Neumann heat equation · Gradient estimate

Mathematics Subject Classification 35P15 · 58J35 · 47B38 · 46E35

1 Introduction

The first aim of this article is to prove the existence of Sobolev extension operators for domains with smooth boundary in a Riemannian manifold whose norms depend

✉ Christian Rose
christian.rose@uni-potsdam.de

Olaf Post
olaf.post@uni-trier.de

Xavier Ramos Olivé
xramosolive@smith.edu

¹ FB 4 - Mathematik, Universität Trier, Universitätsring 15, 54296 Trier, Germany

² Department of Mathematics and Statistics, Smith College, 10 Elm Street, Northampton, MA 01063, USA

³ Institut für Mathematik, Universität Potsdam, Karl-Liebknecht-Straße 24-25, 14476 Potsdam, Germany

only on the geometry near the boundary. To the best of our knowledge, we give the first explicit construction, giving a quantitative bound on the norm, that depends only on sectional and principal curvature assumptions. Such an extension operator provides a tool for geometric applications, especially when working on *classes* of manifolds fulfilling certain geometric bounds. Our second aim is using these extension operators to derive quantitative upper bounds for the Neumann heat kernel, gradient estimates for positive solutions of the Neumann heat equation, and lower bounds on the first Neumann eigenvalue under L^p -Ricci curvature conditions for relatively compact domains with sufficiently regular possibly non-convex boundary.

Let $M = (M^n, g)$ be a complete Riemannian manifold of dimension $n \geq 2$ with possibly non-empty boundary ∂M and distance function d . Fix an open subset $\Omega \subset M$ such that $\bar{\Omega} \neq M$ is a smooth manifold with boundary. A linear and bounded operator $E_\Omega: H^1(\Omega) \rightarrow H^1(M)$ is called *extension operator* for Ω , if it is a bounded right inverse for the restriction operator of Ω , i.e., if E_Ω satisfies

$$E_\Omega u|_\Omega = u, \quad u \in H^1(\Omega), \quad (1.1)$$

and has bounded operator norm. Such extension operators have a long history, starting with the seminal work by Whitney [34] for extension operators of class C^k . It is well known, cf., e.g., Stein's monography [31, Thm. 5, Sec. VI.3.1], that for a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, such a Sobolev extension operator E_Ω exists. Its norm depends implicitly on the Lipschitz constants of the charts as well as other properties of the atlas of $\partial\Omega$. Therefore, this construction does not imply the existence of extension operators for classes of subsets of manifolds whose norms depend only on curvature restrictions. Sobolev extension operators especially for finite sets are constructed in [10], see also the references therein. The name "extension operator" is also used in a slightly different context, namely as a bounded right inverse of a Sobolev trace operator, i.e., of a restriction of functions to proper submanifolds, see, e.g., [12, 14] and the references therein.

However, to the best of our knowledge, apart from Stein's result for subsets in \mathbb{R}^n mentioned above, existing constructions of extension operators for $\Omega \subset M$ in the sense of (1.1) do not focus on quantitative bounds on $\|E_\Omega\|$. Our aim is to construct extension operators whose operator norms depend only on geometric quantities such as bounds on the second fundamental form and the sectional curvature in a suitable tubular neighborhood of $\partial\Omega$. Therefore, we introduce the following class of subsets of a manifold.

Definition 1.1 Let M be a Riemannian manifold of dimension $n \geq 2$, $r > 0$ and $H, K \geq 0$. An open subset $\Omega \subset M$ is called (r, H, K) -regular if

- (i) $\bar{\Omega} \neq M$ is a connected smooth manifold with (smooth) boundary $\partial\Omega$,
- (ii) the *exterior rolling r -ball condition*: for any $x \in \partial\Omega$, there is a point $p \in M \setminus \Omega$ such that $B(p, r) \subset M \setminus \Omega$ and $\overline{B(p, r)} \cap \partial\Omega = \{x\}$;
- (iii) the *interior rolling r -ball condition*: for any $x \in \partial\Omega$, there is a point $p \in \Omega$ such that $B(p, r) \subset \Omega$ and $\overline{B(p, r)} \cap \partial\Omega = \{x\}$;
- (iv) the second fundamental form Π w.r.t. the inward pointing normal of $\partial\Omega$ satisfies $-H \leq \Pi \leq H$ (see Remark 1.3 (iv)),

- (v) the sectional curvature satisfies $\text{Sec} \leq K$ on the tubular neighborhood $T(\partial\Omega, r)$ of $\partial\Omega$.

Our first main result is the following.

Theorem 1.2 Fix $K, H \geq 0$, and a complete Riemannian manifold M of dimension $n \geq 2$. There exists an explicitly computable $r_0 = r_0(K, H) > 0$ such that for any $r \in (0, r_0]$, there exists an explicitly computable constant $C(r, K, H) > 0$ and an extension operator $E_\Omega: H^1(\Omega) \rightarrow H^1(M)$ satisfying

$$\|E_\Omega\| \leq C(K, H, r)$$

for any open (r, H, K) -regular subset Ω .

Our construction of an extension operator aims on implementing the curvature properties of $\partial\Omega$ which are naturally encoded in the behavior of geodesics around $\partial\Omega$. We will present a purely differential geometric approach and use a parametrization of the tubular neighborhood by geodesics perpendicular to $\partial\Omega$, and control the behavior of the geodesics in terms of the geometric quantities given in $T(\partial\Omega, R)$. Then we use the reflection principle from [31] to construct extensions of functions and estimate the norm of the corresponding operator. Stein's approach for Lipschitz atlases, which yields a bound on the extension operator in terms of Lipschitz and covering constants of the atlas, uses a regularized distance depending on a Whitney cover. The regularized distance implements the Lipschitz properties of the chosen atlas of the boundary. Curvature quantities cannot be reflected by such a regularized distance such that this approach does not suffice for our purposes and has to be modified. We want to emphasize that we are not interested in the most general situation but extension operators with norms not depending on the chosen atlas but on curvature quantities. In fact, it is impossible to recover Lipschitz constant estimates for some atlas only by curvature restrictions, such that Stein's result does not yield the bounds we are aiming for. On the other hand, we are aware of other approaches to extension operators using Whitney's ideas which even work in metric measure spaces, see, e.g., [3]. It would be interesting to see if this yields an easier proof of our result yielding an extension operator with our desired curvature properties.

Remark 1.3 (i) Although we assume completeness of M in the above theorem, the proof also applies for incomplete M and $\Omega \subset M$ in any connected component M' satisfying the conditions of Theorem 1.2 such that $\overline{\Omega} \neq M'$ and such that $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$.

- (ii) We do not expect that our constant $C(K, H, r)$ is sharp. If $\Omega = B(0, R_0)$ is an open ball in \mathbb{R}^n , then we can choose $r_0 = r = R_0/2$ and $H = 1/R_0$. A simple scaling argument shows that $\|E_\Omega\|$ is bounded from above by the norm of the extension operator on the unscaled ball $B(0, 1)$ times $R_0^{-1} = r^{-1}$.
- (iii) The upper bound on the sectional curvature and the double sided bound on the second fundamental form ensure that the exponential map of $\partial\Omega$ is well defined on $T(\partial\Omega, r)$ for $r \leq r_0$. The distance r_0 is in fact bounded by the minimal focal distance of all points in $\partial\Omega$.

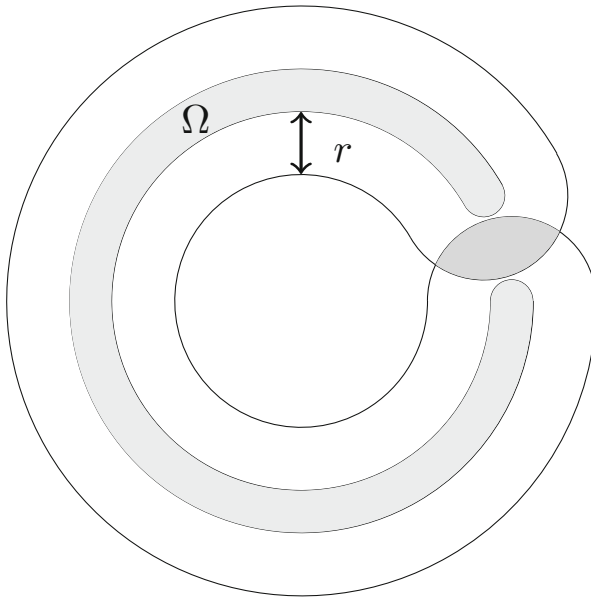


Fig. 1 A neighborhood not fulfilling the exterior rolling r -ball condition

- (iv) Note that a lower bound on the second fundamental form w.r.t. a fixed direction is equivalent to an upper bound of the second fundamental form w.r.t. the opposite direction, so the second fundamental form Π w.r.t. the outward pointing normal of $\partial\Omega$ also satisfies $-H \leq \Pi \leq H$.
- (v) The upper bound for the sectional curvature on $T(\partial\Omega, r)$ and the double sided bounds for the principal curvatures on $\partial\Omega$ can be replaced by two bounds on $T(\partial\Omega, r) \cap \Omega$ and on $T(\partial\Omega, r) \setminus \Omega$, separately: an upper bound for sectional curvature, and a lower bound for Π w.r.t. the corresponding inward normal of $\partial\Omega$ (which changes depending on which side of the tubular neighborhood you consider).
- (vi) The interior rolling r -ball condition ensures that Ω is "sufficiently thick": there always fits a ball of a controllable size into the interior, and geodesics emanating from different points stay unique up to r .
- (vii) The exterior rolling r -ball condition, cf. Fig. 2, is indeed necessary: the proof of Theorem 1.2 relies on a particular parametrization, so-called *Fermi-coordinates*, of the tubular neighborhood. The exterior rolling r -ball condition prevents the tubular neighborhood from self-overlapping, ensuring that the parametrization and the extension operator are well defined. For a problematic case, see Fig. 1.

Our main motivation to study extension operators relies on our interest in the Neumann heat equation of compact (sub-)manifolds having possibly non-convex boundary. Denote by $\Delta \geq 0$ the Laplace–Beltrami operator on a Riemannian manifold M . Let

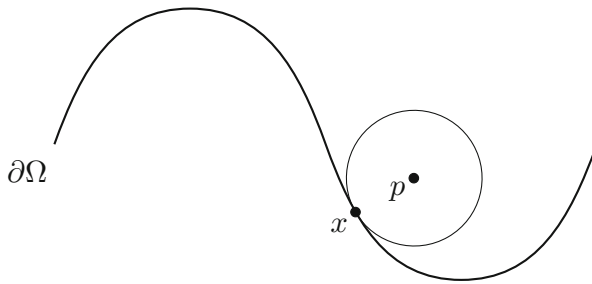


Fig. 2 The interior/exterior rolling r -ball

u be a positive solution of the heat equation

$$\partial_t u = -\Delta u, \tag{1.2}$$

where one assumes additionally $\partial_\nu u = 0$ in case $\partial M \neq \emptyset$, and ν the inward pointing normal. If $D > 0$, $K \geq 0$, then it was shown in [17] that there are $c_1, c_2, c_3 > 0$ such that for any compact M with *convex* smooth boundary, $\text{diam } M \leq D$ and Ricci curvature bounded from below by $-K$, the solution u satisfies

$$c_1 |\nabla \ln u|^2 - \partial_t \ln u \leq c_2 K + c_3 t^{-1}, \quad t > 0. \tag{1.3}$$

From such a gradient estimate, Li and Yau deduced a Harnack inequality as well as an upper bound for the Neumann heat kernel h of M having convex boundary of the form

$$h_t(x, y) \leq c'_1 \text{Vol}(B(x, \sqrt{t}))^{-1/2} \text{Vol}(B(y, \sqrt{t}))^{-1/2} \exp\left(c'_2 K t - \frac{d(x, y)^2}{c'_3 t}\right),$$

$$x, y \in M, t > 0. \tag{1.4}$$

Inequality (1.3), and in turn (1.4), has been generalized in [32] to compact manifolds M with smooth possibly non-convex boundary satisfying the interior rolling r -ball condition (cf. Definition 1.1 (iii) and also Remark 1.6), second fundamental form bounded from below, and Ricci curvature bounded from below by $-K$, $K \geq 0$. For an extensive treatment of Neumann heat kernel estimates on non-compact domains, see, e.g., [15].

During the last decades, there was an increasing interest in relaxing the uniform pointwise Ricci curvature lower bound to integral Ricci curvature bounds. Those provide estimates that are more stable under perturbations of the metric. In the following, we denote

$$\rho : M \rightarrow \mathbb{R}, \quad x \mapsto \min(\sigma(\text{Ric}_x)),$$

where the Ricci tensor Ric of M is viewed as a pointwise endomorphism on the cotangent bundle, and $\sigma(A)$ denotes the spectrum of an operator A . For $x \in \mathbb{R}$, the

negative part of $x \in \mathbb{R}$ will be denoted by $x_- = \max(0, -x) \geq 0$. For a subset $\Omega \subset M$, $p > n/2$, and $R > 0$, we let

$$\kappa_\Omega(p, R) := \sup_{x \in \Omega} \left(\frac{1}{\text{Vol}(B(x, R))} \int_{B(x, R)} \rho_-^p \, \text{dvol} \right)^{\frac{1}{p}}, \tag{1.5}$$

measuring the L^p -mean of the negative part of Ricci curvature uniformly in balls of radius R with center in Ω . It is convenient to work with the scale-invariant quantity $R^2 \kappa_\Omega(p, R)$. If M is complete with $\partial M = \emptyset$, assuming $\kappa_M(p, R)$ is small for $p > n/2$ led to several analytic and geometric generalizations of results that depend on pointwise lower Ricci curvature bounds, see, e.g., [1, 8, 11, 19–25, 28, 35, 36].

If M is allowed to have non-empty boundary $\partial M \neq \emptyset$ and small $\kappa_M(p, R)$, $p > n/2$, it is not known that a version of (1.3) can be derived by only adapting the techniques from [17, 32]. However, there is a way to obtain (1.3) for proper compact subsets of a manifold: recently, the second author [22] obtained a generalized version of (1.3) for positive solutions of (1.2) on compact submanifolds $\Omega \subset M$ with smooth boundary, where the volume measure of M is globally volume doubling and $\kappa_\Omega(p, \text{diam } \Omega)$ is small. Note that the latter does not imply global volume doubling unless M is compact. Additionally, the obtained gradient estimate relies a priori on Gaussian upper bounds of the Neumann heat kernel (1.4), that were obtained in [7]. Note that such gradient estimates depend implicitly on the norm of a Sobolev extension operator of Ω .

We will prove a uniform, quantitative, and localized version: if $\kappa_M(p, R)$ is small for some $p > n/2$, we obtain quantitative Gaussian Neumann heat kernel upper bounds (1.4) in the spirit of [7] depending only on geometric conditions as well as a generalization of (1.3) for all compact $\Omega \subset M$ with possibly non-convex boundary satisfying certain regularity conditions that does neither depend on global volume doubling nor on extension operators with unknown operator norm.

If M is complete and $\partial M = \emptyset$, we denote by $p \in C^\infty((0, \infty) \times M \times M)$ the heat kernel of M , that is, the minimal fundamental solution of (1.2). If $\Omega \subset M$ is a non-empty relatively compact domain with smooth boundary $\partial\Omega \neq \emptyset$, we let $h^\Omega \in C^\infty((0, \infty) \times M \times M)$ be the Neumann heat kernel, i.e., the minimal fundamental solution of (1.2) subject to Neumann boundary conditions on $\partial\Omega$.

Our second main theorem is the following (recall the definition of $\kappa_M(p, R)$ in (1.5)).

Theorem 1.4 *Let M be a complete Riemannian manifold of dimension $n \geq 2$, $p > n/2$, $R > 0$, and $K, H \geq 0$. There exists an explicitly computable $r_0 = r_0(H, K) > 0$ sufficiently small (cf. Remark 1.6) such that for any $r \in (0, r_0]$, there are explicitly computable constants $C = C(n, p, r, R, H, K) > 0$ and $\varepsilon = \varepsilon(n, p, r, H, K) > 0$ such that if*

$$R^2 \kappa_M(p, R) \leq \varepsilon,$$

then for any (r, H, K) -regular domain $\Omega \subset M$ with $\text{diam } \Omega \leq R/2$, the Neumann heat kernel h^Ω of Ω satisfies

$$h_t^\Omega(x, x) \leq \frac{C}{\text{Vol}_\Omega(x, \sqrt{t})}, \quad x \in \Omega, \quad t > 0, \tag{1.6}$$

where $\text{Vol}_\Omega(B(x, s)) := \text{Vol}(\Omega \cap B(x, s))$.

We prove this theorem via a careful analysis of the results obtained in [2, 7] and Theorem 1.2. We want to emphasize again that Theorem 1.4 applies to domains with non-convex boundary.

Theorem 1.4 provides a tool to prove quantitative gradient estimates of type (1.3) for integral Ricci curvature assumptions. Moreover, such estimates give the opportunity to provide quantitative lower bounds on the first (non-zero) Neumann eigenvalue of (r, H, K) -regular subsets. Recently, the third author and G. Wei obtained in [30] gradient and Neumann eigenvalue estimates assuming only the interior rolling r -ball condition, a lower bound on the second fundamental form, and a Kato-type condition on the negative part of the Ricci curvature defined by the Neumann heat semigroup. It is known that the latter condition is more general than assuming $\kappa_M(p, R)$ is small. We refer to [4, 5, 26, 27, 29, 30] for more information about Kato-type curvature assumptions. While those results are very general, it is hard to check that the assumptions are indeed satisfied for compact manifolds with boundary and small $\kappa_M(p, R)$. However, Theorem 1.4 gives such an opportunity for relatively compact subdomains, as the following corollary shows.

Corollary 1.5 *Let M be a complete Riemannian manifold of dimension $n \geq 2$, $p > n/2$, $R > 0$, and $K, H \geq 0$. There exists an explicitly computable $r_0 = r_0(K, H) > 0$ such that for any $r \in (0, r_0]$ there are explicitly computable constants $C_1 = C_1(n, p, r, R, H, K) > 0$, $C_2 = C_2(n, p, r, R, H, K) > 0$, $\varepsilon = \varepsilon(n, p, r, H, K) > 0$, and a function $J = J_{n,p,r,R,H,K}: (0, \infty) \rightarrow (0, \infty)$ such that if*

$$R^2 \kappa_M(p, R) \leq \varepsilon,$$

then for any (r, H, K) -regular domain $\Omega \subset M$ with $\text{diam } \Omega \leq R/2$, any positive solution of (1.2) with Neumann boundary conditions satisfies

$$J(t)|\nabla \ln u|^2 - \partial_t \ln u \leq C_1 + \frac{C_2}{tJ(t)}, \quad t > 0.$$

Moreover, there is an explicitly computable constant $C_3 = C_3(n, p, r, R, H, K) > 0$ such that if η_1^Ω denotes the first (non-zero) Neumann eigenvalue of Ω , we have

$$\eta_1^\Omega \geq C_3 \text{diam}(\Omega)^{-2}.$$

The above corollary generalizes many recent results on Neumann eigenvalues, e.g., [13].

Remark 1.6 As pointed out in [6, 32], the assumption that the interior rolling r -ball condition holds for $r > 0$ small enough in Corollary 1.5 depends implicitly on upper bounds on the sectional curvature in $T(\partial\Omega, r) \cap \Omega$ and the lower bound on the second fundamental form $\text{II} \geq -H$ w.r.t. the inward pointing normal. More precisely, $r \in (0, 1)$ has to be chosen such that

$$\sqrt{K} \tan\left(r\sqrt{K}\right) \leq \frac{1}{2}(1 + H) \quad \text{and} \quad \frac{H}{\sqrt{K}} \tan\left(r\sqrt{K}\right) \leq \frac{1}{2}.$$

Thus, the restriction on the sectional curvature in the tubular neighborhood is a natural assumption in our context.

The structure of this paper is as follows. In Sect. 2, we provide geometric criteria for the tubular neighborhood that imply conditions necessary to prove Theorem 1.2. To prove the main theorem, we establish a new estimate for norms of vector fields along geodesics perpendicular to hypersurfaces. Some proofs of auxiliary comparison estimates for hypersurfaces that we used in the proof can be found in the appendix. In Sect. 3, we construct an extension operator using Jacobi field techniques; this construction depends on geometric assumptions that may or may not be satisfied for a given manifold. In Sect. 4, we carefully adapt the main results of [2, 7] to our setting and prove Theorem 1.4. At the end, we provide proofs of Corollary 1.5 by showing that the Kato condition is indeed satisfied in our situation.

2 Geometry of Tubular Neighborhoods of Hypersurfaces

In order to prove Theorem 1.2, we need geometric assumptions in order to bound the norm of the extension operator, that will depend on the geometry of the tubular neighborhood of the boundary of the considered domain. Let M be a Riemannian manifold of dimension $n \geq 2$, and $N \subset M$ a hypersurface in M . In order to provide a quantitative estimate depending on curvature restrictions and the size of the tubular neighborhood, it is necessary to justify the regularity of the coordinate maps we use in our calculations. Our aim is to prove that under certain curvature restrictions in the tubular neighborhood of N , there exists $r > 0$ such that for any $p \in N$,

$$\exp_p tv, \quad t \in (-r, r)$$

is non-singular. On one hand, \exp is non-singular only up to the cut-locus. On the other hand, geodesics emanating from different points starting in N must not intersect. If such two geodesics intersect in a point, this point is called a focal point. It is known that focal points appear not later than cut points [33]. Thus, we can restrict our investigation to the absence of focal points along any geodesic emanating from N . Below we provide estimates for the focal distance along a geodesic. Note that those are local considerations which do not focus on geodesics with starting points lying far away from each other inside the boundary.

Let $\gamma : [0, l] \rightarrow M$ be a distance minimizing geodesic such that $\gamma(0) \in N$ and $\gamma'(0) \in (T_{\gamma(0)}N)^\perp$. Denote by Sec the sectional curvature of M and II the second

fundamental form of N w.r.t. the unit normal ν with the same orientation as $\gamma'(0)$ at $\gamma(0)$. As explained above, a point $q \in \gamma$ is called a *focal point* if the exponential map at $\gamma(0)$ is singular in q . The focal distance is bounded below in geometric terms by the following theorem.

Lemma 2.1 [33, Corollary 4.2] *Let M, N, γ as above and $H, K \in \mathbb{R}$. Suppose $\Pi \geq H$ in $\gamma(0)$ and $\text{Sec} \leq K$. Then, there are no focal points along γ on $[0, \min(r_0, l))$, where $r_0 > 0$ is the smallest positive number r such that one of the three conditions below is satisfied:*

$$\begin{cases} \cot(\sqrt{Kr}) = \frac{H}{\sqrt{K}}, & \text{if } K > 0, \\ r = \frac{1}{H}, & \text{if } K = 0, \\ \coth(\sqrt{Kr}) = \frac{H}{\sqrt{K}} & \text{if } K < 0. \end{cases} \tag{2.1}$$

Note that the equations (2.1) come from a comparison result with the first zero of the N -Jacobi field equation in a space of constant curvature for some hypersurface with constant second fundamental form. If no positive solution exists, there are no focal points along any geodesic. Warner’s result ensures that a Fermi parametrization of the r_0 -tubular neighborhood of N is well defined if we assume uniform upper bounded sectional curvature in the r_0 -tubular neighborhood and uniform lower bounds for the principal curvatures of N . In our formulation above, we are also taking into account that the curve γ might only be defined up to length l , which may or may not be larger than r_0 as defined by (2.1). This becomes useful in our application of the result below, where we restrict our attention to geodesics of length r , where r is the radius of the rolling r -balls.

An additional ingredient for our proof is the following theorem stating that the norm of a vector along a geodesic can be estimated in terms of a geometric constant and Fourier coefficients w.r.t. a frame consisting of the so-called *N -Jacobi fields*. The difficulty is the non-orthonormality of the frame along the geodesic. Recall that according to [33], an N -Jacobi field J along γ is a Jacobi field satisfying $S(J(0), \nu) - J'(0) \perp T_{\gamma(0)}N$, where S denotes the shape operator of N .

Theorem 2.2 *Let γ be a distance minimizing geodesic perpendicular to a hypersurface $N \subset M$, $X_1 := \gamma', X_1(0), e_2, \dots, e_n$ an orthonormal basis of $T_{\gamma(0)}M$, and $X_i, i = 2, \dots, n$, N -Jacobi fields with $X_i(0) = e_i$ and t_F be the focal distance of N in $\gamma(0)$. Assume $\text{Sec} \leq K$ along γ and $\Pi \geq -H$ in $\gamma(0)$. There exist explicitly computable constants $\beta_{K,H,n}$ and $t_\beta \in (0, t_F/2]$ such that for any vector field v along γ we have*

$$|v|_{\gamma(s)} \leq \beta_{K,H,n} \sum_{i=1}^n |\langle v, X_i \rangle_{\gamma(s)}|, \quad s \in [0, t_\beta].$$

Proof For any $s \in [0, t_F)$, $X_i(\gamma(s)), i = 1, \dots, n$, spans $T_{\gamma(s)}M$. Let $E_i(\gamma(s))$ be the parallel orthonormal frame along $\gamma(s)$ with $E_i(0) = X_i(0)$. Denote by $X(s)$ the matrix of change of basis from $\{E_i^*(\gamma(s))\}_{i=1}^n$ to $\{X_i^*(\gamma(s))\}_{i=1}^n$, i.e., the matrix with

the i th row given by the coordinates of $X_i(\gamma(s))$ in the basis $\{E_i(\gamma(s))\}_{i=1}^n$. In other words, $X_{ij} = E_j^*(X_i)$. In particular, X^{-1} will have the coordinates of X_i^* in the basis $\{E_i^*\}_{i=1}^n$ in the i th column, $(X^{-1})_{i,j} = X_j^*(E_i)$.

By definition of the dual basis, we have that

$$v = \sum_{i=1}^n X_i^*(v)X_i,$$

thus we have

$$\begin{aligned} |v|_{\gamma(s)}^2 &= \left\langle v, \sum_{i=1}^n X_i^*(v)X_i \right\rangle_{\gamma(s)} \\ &= \sum_{i=1}^n X_i^*(v)\langle v, X_i \rangle_{\gamma(s)} \\ &\leq \sum_{i=1}^n |X_i^*|_{op} |v|_{\gamma(s)} |\langle v, X_i \rangle_{\gamma(s)}|, \end{aligned}$$

where $|\cdot|_{op}$ denotes the operator norm. Denoting

$$|A|_F^2 = \sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2$$

the square of the Frobenius norm of a matrix $A_{n \times m}$ and using that

$|A|_{op} \leq |A|_F \leq \sqrt{n}|A|_{op}$, we get

$$|v|_{\gamma(s)} \leq \sum_{i=1}^n |X_i^*|_{\gamma(s),F} |\langle v, X_i \rangle_{\gamma(s)}| \leq |X^{-1}|_{\gamma(s),F} \sum_{i=1}^n |\langle v, X_i \rangle_{\gamma(s)}|,$$

where we used that $|X_i^*|_{\gamma(s),F}^2 = \sum_{j=1}^n [X_i^*(E_j)]^2 \leq \sum_{i,j=1}^n [X_i^*(E_j)]^2 = |X^{-1}|_{\gamma(s),F}^2$.

To estimate $|X^{-1}|_{\gamma(s),F}^2$, we employ the Neumann criterion as follows. Let $I = E_i \otimes E_i^*$ along γ . Then

$$|I - X|_{\gamma(s)}^2 \leq |I - X|_{\gamma(s),F}^2 = \sum_{i=1}^n |E_i - X_i|_{\gamma(s)}^2.$$

Moreover, we have $E_1 - X_1 = 0$, and for $i = 2, \dots, n$

$$\begin{aligned} \frac{d}{ds}|E_i - X_i|_{\gamma(s)}^2 &= -2\frac{d}{ds}\langle E_i, X_i \rangle_{\gamma(s)} + \frac{d}{ds}|X_i|_{\gamma(s)}^2 \\ &= -2\langle E_i, \dot{X}_i \rangle_{\gamma(s)} + 2\langle \dot{X}_i, X_i \rangle_{\gamma(s)} \end{aligned}$$

$$\leq 2|\dot{X}_i|_{\gamma(s)}(1 + |X_i|_{\gamma(s)}).$$

Integrating w.r.t. s and using $X_i(0) = E_i(0)$, we arrive at

$$|E_i - X_i|_{\gamma(s)}^2 \leq \int_0^s 2|\dot{X}_i|_{\gamma(s)}(1 + |X_i|_{\gamma(s)})ds.$$

Recalling that S denotes the shape operator of N , we have $\dot{X}_i(s) = S(X_i(s), \gamma'(s))$, $s \in [0, t_F]$. Since $\text{Sec} \leq K$ and $\text{II} \geq -H$ there exist functions $f = f_{K,H,n}$ and $g = g_{K,H}$ such that

$$|X_i|_{\gamma(s)} \leq |f(s)|, \quad |\dot{X}_i|_{\gamma(s)} \leq |g(s)|,$$

cf. [16, p. 211 pp.], [33]. Hence

$$|E_i - X_i|_{\gamma(s)}^2 \leq 2s \max_{[0,t_F]} g(1 + f).$$

Set

$$t_\beta := \min \left(t_F/2, \left(4n \max_{[0,t_F]} g(1 + f) \right)^{-1} \right)$$

such that

$$\alpha_{K,H,n} := 2t_\beta \max_{[0,t_F]} g(1 + f) \leq \frac{1}{2} < 1,$$

and hence von Neumann’s criterion yields $|X^{-1}|_\gamma \leq \sqrt{n}(1 - \alpha_{K,H,n})^{-1}$. □

We also need comparison estimates for the metric tensor along distance hypersurfaces. Suppose that $\gamma: [0, r_0) \rightarrow M$, $\gamma(0) = p \in N$ is as above. Although the proof is straightforward and adapted from the proof in [18] for distance spheres, we are not aware of any result in the literature that gives our comparison estimates below, so we decided to include a full proof in the appendix using the Riccati comparison technique. The tubular neighborhood $T(N, r)$ can be parametrized by the distance function $s := \text{dist}(\cdot, N)$ to N : Set

$$\psi: (-r, r) \times N \rightarrow M, (s, x) \mapsto \psi(s, x) = \exp_x(vs). \tag{2.2}$$

If N is the smooth boundary $\partial\Omega$ of $\Omega \subset M$, then ψ satisfies for all $x \in \partial\Omega$

$$d(\psi(s, x), x) = \begin{cases} s, & \text{if } \psi(s, x) \in M \setminus \Omega, \\ -s, & \text{if } \psi(s, x) \in \Omega. \end{cases}$$

The parametrization ψ is defined only for r small enough; more precisely, the size of r depends on the focal set of $\partial\Omega$, that is, the set of points where the exponential map

is non-regular. We can decompose

$$g = ds^2 + g_s, \quad (2.3)$$

where g_s is the metric on N evolving with respect to s as long as there are no focal points along the corresponding distance minimizing geodesic γ perpendicular to N up to r_0 .

Proposition 2.3 *Let $k, K \in \mathbb{R}$, and γ, r_0 as above. If $\text{Sec} \geq k$ along γ on the interval $[0, r_0)$, then for almost all $s \in [0, r_0)$,*

$$g_s \leq \mu_{k, H_+}^2(s) g_0, \quad (2.4)$$

and if $\text{Sec} \leq K$, then for almost all $s \in [0, r_0)$,

$$\mu_{K, H_-}^2(s) g_0 \leq g_s, \quad (2.5)$$

where H_+ and H_- are the maximum resp. minimum principal curvatures of N in $\gamma(0)$, and $\mu_{k, H}(s)$ are functions that arise from the solution of a Riccati-type ODE (cf. Appendix A, Eq. (A.2)).

The volume form dvol of M decomposes accordingly into

$$\text{dvol} = ds \wedge \text{dvol}_s, \quad s \in (-r, r),$$

where dvol_s denotes the volume element of the distance hypersurface $\psi^{-1}(s, \cdot)$.

Proposition 2.4 *Under the assumptions as in Proposition 2.3, we have*

$$\begin{aligned} \text{dvol}_s &\leq \text{dvol}_s^{k, H_+} = D(s) \text{dvol}_0, \quad s \in [0, r_0), \\ \text{and } \text{dvol}_{-s} &\geq \text{dvol}_{-s}^{K, H_-} = d(s) \text{dvol}_0, \quad s \in [0, r_0). \end{aligned}$$

with

$$D(s) = \mu_{k, H_+}^{n-1}(s), \quad \text{and } d(s) = \mu_{K, H_-}^{n-1}(s).$$

For a self-contained proof, see the Appendix.

3 Quantitative Sobolev Extensions

The existence of an extension operator $E_\Omega: H^1(\Omega) \rightarrow H^1(M)$ with bounded operator norm will follow by constructing an extension operator along geodesics perpendicular to $\partial\Omega$ with bounded operator norm depending on the behavior of the geodesics in a tubular neighborhood. E_Ω can then be defined on M via Fermi-coordinates on $\partial\Omega$. The operator norm will then be controlled by the behavior of the geodesics in the tubular neighborhood and of the volume element as explained in Sect. 2.

For $U \subset M$ be open, we denote by $\|\cdot\|_{H^1(U)}$ the H^1 -norm in U , i.e., for $u \in C^1(U)$,

$$\|u\|_{H^1(U)}^2 := \|u\|_{L^2(U)}^2 + \|\nabla u\|_{L^2(U)}^2,$$

and by $C^1(\overline{U})$ the set of all $f \in C^1(U)$ with continuous zeroth and first derivatives up to the boundary of U . Let $\Omega \subset M$ be (r, H, K) -regular.

Theorem 2.2 can be applied to all geodesics perpendicular to $\partial\Omega$. Let $R_\beta := \min(t_\beta, r)$ be the resulting minimal width for the tubular neighborhood. We abbreviate for $x \in T(\partial\Omega, R_\beta)$ and $s \in (-R_\beta, R)$

$$|\cdot|_x := |\cdot|_{g(x)} \quad \text{and} \quad |\cdot|_{s,x} := |\cdot|_{g_s(x)},$$

For $x \in \partial\Omega$, let

$$\gamma = \gamma_x : (-R_\beta, R_\beta) \rightarrow M, \quad s \mapsto \exp_x(sv)$$

be the unique geodesic perpendicular to $\partial\Omega$ at x . Let $\varphi_x \in C_c^\infty(M)$ be a cut-off function such that $0 \leq \varphi \leq 1$, $\text{supp } \varphi \subset B(x, R_\beta)$, $\varphi = 1$ on $B(x, R_\beta/2)$, and

$$|\nabla\varphi_x| \leq G/R_\beta$$

for some dimension constant $G > 0$ which always exists by completeness of M . Moreover, let $u \in C^1(\overline{\Omega})$. We define the one-dimensional extension of u along γ by

$$(E_x u)(s) := \begin{cases} u(\gamma(s)), & : s \in (-R_\beta, 0], \\ (-3u(\gamma(-s)) + 4u(\gamma(-s/2))) \varphi_x(\gamma(s)), & : s \in (0, R_\beta). \end{cases}$$

Proposition 3.1 *$E_x u$ is continuously differentiable along γ . Furthermore, there exists an explicitly computable $\theta_{K,H,n} \geq 1$ such that*

$$\|E_x u\|_{H^1(\gamma \cap (M \setminus \Omega))}^2 \leq \theta_{K,H,n} \left(164 \|\nabla u\|_{L^2(\gamma \cap \Omega)}^2 + (82 + 164G^2 R_\beta^{-2}) \|u\|_{L^2(\gamma \cap \Omega)}^2 \right). \tag{3.1}$$

Proof The continuity of $E_x u$ is obvious. Moreover, we can restrict to the case $s \geq 0$ since $E_x u$ coincides with u for $s < 0$. We compute the gradient of $E_x u$ by calculating its directional derivatives. The regularity of our parametrization allows to define a variation of $E_x u$ along γ such that we can compute the partial derivatives in directions perpendicular to γ . The difficulty here is that we cannot use just a parallel frame along γ to compute the gradient because we do not know that curves with initial tangent vectors given by the frame in small neighborhoods of the point under consideration do not intersect. Thus, we need to define an appropriate frame consisting of Jacobi fields whose curves lie in the distance hypersurfaces and that span the tangent spaces along γ . The directional derivatives in the directions given by the Jacobi fields will be obviously continuous and the gradient therefore exists.

For $\gamma = \gamma_x$ as above, we get for free

$$\partial_s E_x u(s) = (3(\partial_s u)(\gamma(-s)) - 2(\partial_s u)(\gamma(-s/2))) \varphi_x(\gamma(s)) \tag{3.2}$$

$$+ (-3u(\gamma(-s)) + 4u(\gamma(-s/2))) (\partial_s \varphi_x)(\gamma(s)). \tag{3.3}$$

To compute the partial derivatives perpendicular to γ , we introduce the following variation of γ : Let $e_i \in T_x M, i = 2, \dots, n$, be a completion of ν to a basis. For $i \in \{2, \dots, n\}$, let $x_i(t)$ be a curve in $\partial\Omega$ such that $x_i(0) = x, x'(0) = e_i$. We define the variation

$$\gamma_i(t, s) = \exp_{x_i(t)}(s\nu), \quad t \in (-\varepsilon, \varepsilon), \quad s \in (-R_\beta, R_\beta).$$

Varying t gives a variation through geodesics emanating perpendicularly from $\partial\Omega$. In particular, for fixed s , the curves $\gamma_i(\cdot, s)$ lie in the distance hypersurface with distance s . The vector field

$$X_i(s) := \frac{\partial}{\partial t} \gamma_i(t, s)|_{t=0}$$

is a Jacobi field along γ and satisfies $X_i(0) = e_i, X'_i(0) = S(e_i, \nu)$, where S denotes the shape operator of $\partial\Omega$. In particular, we have

$$\langle X_i, \gamma' \rangle_{\gamma(s)} = 0, \quad X_i(0) \in T_x \Omega, \quad S(X_i(0), \nu) - X'(0) = 0 \perp T_x \partial\Omega,$$

so X_i is a $\partial\Omega$ -Jacobi field along γ for any $i \in \{2, \dots, n\}$. According to [33], $\{v(s)\} \cup \{X_i(s)\}_{i=2}^n$ spans $T_{\gamma(s)}M$. For $i \in \{2, \dots, n\}$, define the following variation of Eu :

$$E_x^i u(t, s) := \begin{cases} u(\gamma_i(t, s)), & : t \in (-\varepsilon, \varepsilon), s \in (-R_\beta, 0], \\ (-3u(\gamma_i(t, -s)) + 4u(\gamma_i(t, -s/2))) \varphi_x(\gamma_i(t, s)), & : t \in (-\varepsilon, \varepsilon), s \in (0, R_\beta). \end{cases} \tag{3.4}$$

The directional derivative of $Eu(s)$ in direction $X_i(s)$ is given by

$$\begin{aligned} \langle X_i, \nabla E_x u \rangle_{\gamma(s)} &= \frac{d}{dt} E_x^i u(t, s)|_{t=0} \\ &= \frac{d}{dt} (-3u(\gamma_i(t, -s)) + 4u(\gamma_i(t, -s/2))) \varphi_x(\gamma_i(t, s)) \\ &\quad + (-3u(\gamma_i(t, -s)) + 4u(\gamma_i(t, -s/2))) \frac{d}{dt} \varphi_x(\gamma_i(t, s))|_{t=0} \end{aligned}$$

and thus

$$\begin{aligned}
 \langle X_i, \nabla E_x u \rangle_{\gamma(s)} &= \left(-3 \langle \nabla u, \frac{\partial \gamma_i}{\partial t} \rangle_{\gamma_i(t, -s)} + 4 \langle \nabla u, \frac{\partial \gamma_i}{\partial t} \rangle_{\gamma_i(t, -s/2)} \right) \varphi_x(\gamma_i(t, s)) \\
 &\quad + (-3u(\gamma_i(t, -s)) + 4u(\gamma_i(t, -s/2))) \langle \nabla \varphi_x, \frac{\partial \gamma_i}{\partial t} \rangle_{\gamma_i(t, s)}|_{t=0} \\
 &= \left(-3 \langle \nabla u, \frac{\partial \gamma_i}{\partial t} |_{t=0} \rangle_{\gamma(-s)} + 4 \langle \nabla u, \frac{\partial \gamma_i}{\partial t} |_{t=0} \rangle_{\gamma(-s/2)} \right) \varphi_x(\gamma(s)) \\
 &\quad + (-3u(\gamma(-s)) + 4u(\gamma(-s/2))) \langle \nabla \varphi_x, \frac{\partial \gamma_i}{\partial t} |_{t=0} \rangle_{\gamma(s)} \\
 &= (-3 \langle \nabla u, X_i \rangle_{\gamma(-s)} + 4 \langle \nabla u, X_i \rangle_{\gamma(-s/2)}) \varphi_x(\gamma(s)) \\
 &\quad + (-3u(\gamma(-s)) + 4u(\gamma(-s/2))) \langle \nabla \varphi_x, X_i \rangle_{\gamma(s)}. \tag{3.5}
 \end{aligned}$$

In particular, we have for the right limits

$$\begin{aligned}
 \partial_s E_x u(0+) &= 3 \partial_s u(\gamma(0)) - 2 \partial_s u(\gamma(0)) \\
 &= \partial_s u(\gamma(0)) = \partial_s u(x), \\
 \langle X_i, \nabla E_x u \rangle_{\gamma(0+)} &= -3 \partial_i u(\gamma(0)) + 4 \partial_i u(\gamma(0)) = \partial_i u(x).
 \end{aligned}$$

It is easily seen that for the pointwise norm $E_x u$ along γ , for $s \geq 0$ we have

$$|E_x u(s)|^2 \leq 18|u(\gamma(-s))|^2 + 32|u(\gamma(-s/2))|^2. \tag{3.6}$$

The pointwise norm of $\nabla E_x u$ along γ is more involved. Since $E_x u$ coincides with u in Ω , we restrict the computations for the norm to $M \setminus \Omega$.

Denote $X_1 = v$. Theorem 2.2 yields the existence of $\beta_{K,H,n} > 0$ such that for any $s \in [0, R_\beta]$, we have

$$\begin{aligned}
 |\nabla E_x u|_{\gamma(s)}^2 &\leq \beta_{K,H,n} \left(\sum_{i=1}^n |\langle \nabla E_x u, X_i \rangle_{\gamma(s)}| \right)^2 \leq \beta_{K,H,n} n \sum_{i=1}^n |\langle \nabla E_x u, X_i \rangle_{\gamma(s)}|^2 \\
 &= \beta_{K,H,n} n \left(|\partial_s E_x u(s)|^2 + \sum_{i=2}^n \langle \nabla E_x u, X_i \rangle_{\gamma(s)}^2 \right).
 \end{aligned}$$

We have

$$\begin{aligned}
 |\partial_s E_x u(s)|^2 &+ \sum_{i=2}^n \langle \nabla E_x u, X_i \rangle_{\gamma(s)}^2 \\
 &\leq [3|\partial_s u(\gamma(-s))| + 2|\partial_s u(\gamma(-s/2))| \\
 &\quad + (3u(\gamma(-s)) + 4u(\gamma(-s/2))) |\partial_s \varphi_x(\gamma(s))|]^2 \\
 &+ \sum_{i=2}^n [(-3 \langle \nabla u, X_i \rangle_{\gamma(-s)} + 4 \langle \nabla u, X_i \rangle_{\gamma(-s/2)}) \varphi_x(\gamma(s))]^2
 \end{aligned}$$

$$\begin{aligned}
 &+ (-3u(\gamma(-s)) + 4u(\gamma(-s/2))) \langle \nabla \varphi_x, X_i \rangle_{\gamma(s)}]^2 \\
 \leq &36|\partial_s u(\gamma(-s))|^2 + 16|\partial_s u(\gamma(-s/2))|^2 \\
 &+ \left(36u(\gamma(-s))^2 + 64u(\gamma(-s/2))^2\right) |\partial_s \varphi_x(\gamma(s))|^2 \\
 &+ \sum_{i=2}^n 36\langle \nabla u, X_i \rangle_{\gamma(-s)}^2 + 64\langle \nabla u, X_i \rangle_{\gamma(-s/2)}^2 \\
 &+ \sum_{i=2}^n (36u(\gamma(-s))^2 + 64u(\gamma(-s/2))^2) \langle \nabla \varphi_x, X_i \rangle_{\gamma(s)}^2.
 \end{aligned}$$

By assumption on the sectional and principal curvatures, there exists a constant $\vartheta = \vartheta_{K,H} \geq 1$ that bounds the norms of the Jacobi fields from above, cf. the discussion in the proof of Theorem 2.2. This together with Cauchy–Schwarz yields a $\theta = \theta_{K,H,n} > 0$ such that

$$\begin{aligned}
 |\nabla E_x u|_{\gamma(s)}^2 &\leq 36\theta |\nabla u|_{\gamma(-s)}^2 + 64\theta |\nabla u|_{\gamma(-s/2)}^2 \\
 &+ \theta \left(36u(\gamma(-s))^2 + 64u(\gamma(-s/2))^2\right) |\nabla \varphi|_{\gamma(s)}^2.
 \end{aligned}$$

Denote by χ_I the characteristic function of $I \subset \mathbb{R}$. The assumption on the cut-off function implies

$$\begin{aligned}
 |\nabla E_x u|_{\gamma(s)}^2 &\leq 36\theta |\nabla u|_{\gamma(-s)}^2 + 64\theta |\nabla u|_{\gamma(-s/2)}^2 \\
 &+ \theta \left(36u(\gamma(-s))^2 + 64u(\gamma(-s/2))^2\right) \chi_{[R_\beta/2, R_\beta]} G^2 R_\beta^{-2}. \tag{3.7}
 \end{aligned}$$

Now we can compute the H^1 -norm of $E_x u$. We already mentioned above that we only care about the norm in the complement of Ω . By (3.6), for the L^2 -norm of $E_x u$, we have

$$\begin{aligned}
 \|E_x u\|_{L^2(\gamma \cap (M \setminus \Omega))}^2 &= \int_0^{R_\beta} |E_x u(s)|^2 ds \\
 &\leq \int_0^{R_\beta} 18|u(\gamma(-s))|^2 + 32|u(\gamma(-s/2))|^2 ds \\
 &\leq \int_{-R_\beta}^0 18|u(\gamma(s))|^2 + 32|u(\gamma(s/2))|^2 ds \\
 &= \int_{-R_\beta}^0 18|u(\gamma(s))|^2 ds + \int_{-R_\beta/2}^0 64|u(\gamma(s))|^2 ds \\
 &\leq 82 \int_{-R_\beta}^0 |u(\gamma(s))|^2 ds \\
 &= 82 \|u\|_{L^2(\gamma \cap \Omega)}^2 \tag{3.8}
 \end{aligned}$$

By (3.7), for the L^2 -norm of $\nabla E_x u$, we have

$$\begin{aligned} & \|\nabla E_x u\|_{L^2(\gamma \cap (M \setminus \Omega))}^2 \\ &= \int_0^{R_\beta} |\nabla E_x u(s)|_{\gamma(s)}^2 ds \\ &\leq \int_0^{R_\beta} 36\theta |\nabla u|_{\gamma(-s)}^2 + 64\theta |\nabla u|_{\gamma(-s/2)}^2 \\ &\quad + \theta \left(36u(\gamma(-s))^2 + 64u(\gamma(-s/2))^2 \right) \chi_{[R_\beta/2, R_\beta]} G^2 R_\beta^{-2} ds \\ &\leq \int_0^{R_\beta} 36\theta |\nabla u|_{\gamma(-s)}^2 ds + \int_0^{R_\beta/2} 128\theta |\nabla u|_{\gamma(-s)}^2 ds \\ &\quad + \theta G^2 R_\beta^{-2} \left(\int_{R_\beta/2}^{R_\beta} 36u(\gamma(-s))^2 ds + \int_{R_\beta/4}^{R_\beta/2} 128u(\gamma(-s))^2 ds \right) \\ &\leq 164\theta \|\nabla u\|_{L^2(\gamma \cap \Omega)}^2 + 164\theta G^2 R_\beta^{-2} \|u\|_{L^2(\gamma \cap \Omega)}^2. \end{aligned}$$

Hence, the claim follows. □

Proof of Theorem 1.2 Denote by x' the distance minimizing point of $x \in T(\partial\Omega, R_\beta) \setminus \overline{\Omega}$ to $\partial\Omega$. This point always exists and is unique due to our uniquely defined parametrization. For $u \in C^1(\overline{\Omega})$, the extension $E_\Omega u$ is given by

$$E_\Omega u(x) := \begin{cases} u(x), & \text{if } x \in \overline{\Omega}, \\ E_{x'} u(d(x, x')), & \text{if } x \in T(\partial\Omega, r) \setminus \overline{\Omega}, \\ 0, & \text{otherwise.} \end{cases}$$

Claim 1 E_Ω defined above is linear and continuous from $H^1(\Omega)$ to $H^1(M)$ with operator norm

$$\|E_\Omega\|^2 \leq 1 + 164\theta_{K, H, n} \max_{s, t \in [0, R_\beta]} \frac{D(t)}{d(s)} (1 + G^2 R_\beta^{-2}). \tag{3.9}$$

We can restrict our considerations to $u \in C^1(\overline{\Omega})$ since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$. To compute the operator norm of E_Ω , note that

$$\|E_\Omega u\|_{H^1(M)}^2 = \|u\|_{H^1(\Omega)}^2 + \|E_\Omega u\|_{H^1(T(\partial\Omega, R_\beta) \setminus \Omega)}^2. \tag{3.10}$$

Proposition 2.4 on the volume element of the hypersurfaces on the interval $[0, R_\beta)$ implies

$$\begin{aligned} \|E_\Omega u\|_{H^1(T(\partial\Omega, R_\beta) \setminus \Omega)}^2 &= \|E_\Omega u\|_{L^2(T(\partial\Omega, R_\beta) \setminus \Omega)}^2 + \|\nabla E_\Omega u\|_{L^2(T(\partial\Omega, R_\beta) \setminus \Omega)}^2 \\ &= \int_0^{R_\beta} \int_{\partial\Omega} \left(|E_\theta u(s)|^2 + |\nabla E_\theta u(s)|_{\gamma(s)}^2 \right) d\text{vol}_s \theta ds \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^{R_\beta} \int_{\partial\Omega} \left(|E_\theta u(s)|^2 + |\nabla E_\theta u(s)|_{\gamma(s)}^2 \right) D(s) \, d\text{vol}_0 \theta \, ds \\
 &= \int_{\partial\Omega} \int_0^{R_\beta} \left(|E_\theta u(s)|^2 + |\nabla E_\theta u(s)|_{\gamma(s)}^2 \right) D(s) \, ds \, d\text{vol}_0 \theta \\
 &\leq \max_{s \in [0, R_\beta]} D(s) \int_{\partial\Omega} \|E_\theta u\|_{H^1(\gamma_\theta \cap M \setminus \Omega)}^2 \, d\text{vol}_0 \theta
 \end{aligned}$$

Using Proposition 3.1, the last integral can be interpreted as an integral over $T(\partial\Omega, R_\beta) \cap \Omega$, and we get with Proposition 2.4

$$\begin{aligned}
 &\|E_\Omega u\|_{H^1(T(\partial\Omega, R_\beta) \setminus \Omega)}^2 \\
 &\leq 164\theta_{K,H,n} \max_{s \in [0, R_\beta]} D(s) \int_{\partial\Omega} \|\nabla u\|_{L^2(\gamma_\theta \cap \Omega)}^2 \, d\text{vol}_0 \theta \\
 &\quad + \theta_{K,H,n} (82 + 164G^2 R_\beta^{-2}) \max_{s \in [0, R_\beta]} D(s) \int_{\partial\Omega} \|u\|_{L^2(\gamma_\theta \cap \Omega)}^2 \, d\text{vol}_0 \theta \\
 &\leq 164\theta_{K,H,n} \max_{s \in [0, R_\beta]} D(s) \int_{\partial\Omega} \|\nabla u\|_{L^2(\gamma_\theta \cap \Omega)}^2 \frac{1}{d(s)} \, d\text{vol}_{-s} \theta \\
 &\quad + \theta_{K,H,n} (82 + 164G^2 R_\beta^{-2}) \max_{s \in [0, R_\beta]} D(s) \int_{\partial\Omega} \|u\|_{L^2(\gamma_\theta \cap \Omega)}^2 \frac{1}{d(s)} \, d\text{vol}_{-s} \theta \\
 &\leq 164\theta_{K,H,n} \max_{s,t \in [0,r]} \frac{D(t)}{d(s)} \int_{\partial\Omega} \|\nabla u\|_{L^2(\gamma_\theta \cap \Omega)}^2 \, d\text{vol}_{-s} \theta \\
 &\quad + \theta_{K,H,n} (82 + 164G^2 R_\beta^{-2}) \max_{s,t \in [0, R_\beta]} \frac{D(t)}{d(s)} \int_{\partial\Omega} \|u\|_{L^2(\gamma_\theta \cap \Omega)}^2 \, d\text{vol}_{-s} \theta \\
 &= 164\theta_{K,H,n} \max_{s,t \in [0, R_\beta]} \frac{D(t)}{d(s)} \|\nabla u\|_{L^2(\Omega \cap T(\partial\Omega, R_\beta))}^2 \\
 &\quad + \theta_{K,H,n} (82 + 164G^2 R_\beta^{-2}) \max_{s,t \in [0, R_\beta]} \frac{D(t)}{d(s)} \|u\|_{L^2(\Omega \cap T(\partial\Omega, R_\beta))}^2.
 \end{aligned}$$

Hence, (3.10) becomes

$$\begin{aligned}
 \|E_\Omega u\|_{H^1(M)}^2 &\leq \left(1 + 164\theta_{K,H,n} \max_{s,t \in [0, R_\beta]} \frac{D(t)}{d(s)} \right) \|\nabla u\|_{L^2(\Omega)}^2 \\
 &\quad + \left(1 + \theta_{K,H,n} (82 + 164G^2 R_\beta^{-2}) \max_{s,t \in [0, R_\beta]} \frac{D(t)}{d(s)} \right) \|u\|_{L^2(\Omega)}^2.
 \end{aligned}$$

□

Remark 3.2 The extension operator defined above can be used in [22] to obtain a Li-Yau-type estimate on the Neumann heat kernel for (r, H, K) -regular domains under integral curvature conditions with appropriately chosen $r > 0$, where the estimate would only depend on geometric parameters. However, one would still need to require the ambient space to be globally doubling, since in general only local volume doubling

holds under integral Ricci curvature assumptions. To remove the necessity of this condition, we develop in the next section local estimates for the Neumann heat kernel that will only require local volume doubling.

4 Localizing Estimates for the Neumann Heat Kernel

In [7], the authors observed that the main result of [2] can be used to prove full Gaussian upper bounds for the Neumann heat kernel of relatively compact domains with Lipschitz boundary satisfying the volume doubling property provided the ambient space is globally volume doubling and its heat kernel has full Gaussian upper bounds. In the proof they use the existence of a bounded extension operator. Although they show as an example that their result holds for relatively compact domains with Lipschitz boundary in \mathbb{R}^n and hyperbolic spaces, they do not discuss conditions for the extension operator to be uniformly bounded in geometric parameters. In Sect. 2, we have shown the existence of extension operators with this property, such that those extension operators can be used to give a quantitative version of the results in [7].

Furthermore, the paper does not discuss any localized results, i.e., if local volume doubling and small-time upper bound for the heat kernel on M imply a small-time Gaussian upper bound for the Neumann heat kernel on a domain $\Omega \subset M$ with sufficiently regular boundary. This is of independent interest for other quantitative applications. In fact, their main theorem is based on [2, Theorem 1.1], which does not apply to small-time Gaussian upper bounds. While it is indicated on p. 308 of the latter article that their results hold in the localized situation as well, [2, Theorem 1.1], as it is formulated, does not hold assuming only local instead of global volume doubling.

We will show here Neumann heat kernel upper bounds for (r, H, K) -regular bounded subsets of M , while $R^2\kappa_M(p, R)$ is small for $p > n/2$, i.e., Theorem 1.4. As explained above, this does neither follow directly from [7] nor from [2].

The results presented below are adaptations from [2, 7]. Although the proofs are almost the same, we give a complete outline, because the differences are quite subtle. The main improvements are that we only use local volume doubling and localized heat kernel estimates and our extension operators from Theorem 1.2. First, we will show that a local volume doubling and upper heat kernel estimate imply a family of localized Gagliardo–Nirenberg inequalities. Although we work directly on Riemannian manifolds, the proof is the same for metric measure spaces. Second, we show that Theorem 1.2 implies local Gagliardo–Nirenberg inequalities on (r, H, K) -regular bounded domains with appropriate r with a proof slightly different from [7]. The desired upper Neumann heat kernel bound then follows directly from [2, Theorem 1.1], since the Neumann Laplace operator satisfies the finite speed propagation property.

In the following, we use notation adapted from [2]. Let M be a Riemannian manifold of dimension $n \geq 2$. For $x \in M$, $r \geq 0$, denote $v_r(x) := \text{Vol}(B(x, r))$. Let $x_0 \in M$, $r_0 > 0$ and $B_0 := B(x_0, r_0)$. We say that B_0 satisfies the *local volume doubling condition*

if there are $C_D > 0$ and $R > 0$ such that

$$v_r(x) \leq C_D \left(\frac{r}{s}\right)^n v_s(x), \quad x \in B_0, \quad 0 < s \leq r \leq R. \tag{4.1}$$

We say that the Dirichlet heat kernel p^{B_0} of B_0 satisfies *local upper bounds* if there are $C, R > 0$ such that

$$p_t^{B_0}(x, y) \leq \frac{C}{v_{\sqrt{t}}(x)^{\frac{1}{2}} v_{\sqrt{t}}(y)^{\frac{1}{2}}}, \quad x, y \in B_0, \quad t \in (0, R^2/4]. \tag{4.2}$$

Furthermore, we define the following family of Gagliardo–Nirenberg inequalities on B_0 : there exists $q \in (2, \infty]$, $\frac{q-2}{q}n < 2$, such that

$$\left\| v_r^{\frac{1}{2} - \frac{1}{q}} f \right\|_q \leq C \left(\|f\|_2 + r \|\Delta^{1/2} f\|_2 \right), \quad f \in C_c^\infty(B(x_0, r)) \quad r \leq r_0. \tag{4.3}$$

One could also choose another function v instead of the volume measure satisfying similar properties as in (4.1) and (4.2), while the underlying volume measure fulfills (4.1) separately, but we decided not to do so for the sake of presentation.

We will often write f to denote the multiplication operator by the function f . Given $1 \leq p, q \leq +\infty$ and $\gamma, \delta \geq 0$ such that $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$, we will say that the condition $(vEv_{p,q,\gamma})$ holds if there exist $t_0 > 0$ such that

$$\sup_{0 < t \leq t_0} \|v_{\sqrt{t}}^\gamma P_t^{B_0} v_{\sqrt{t}}^\delta\|_{p,q} < \infty, \tag{vEv_{p,q,\gamma}}$$

where $\|\cdot\|_{p,q}$ denotes the operator norm from $L^p(B_0)$ to $L^q(B_0)$, $P_t^{B_0} = e^{-t\Delta^{B_0}}$, $t \geq 0$, and Δ^{B_0} is the Dirichlet Laplacian on B_0 . As with the assumption (4.2), the difference between our assumption $(vEv_{p,q,\gamma})$ and the one in [2] is that our assumption is local, for short time, as opposed to a global assumption for all values of $t > 0$. Note that if p' and q' are the conjugate exponents of p and q , respectively, then by duality $(vEv_{p,q,\gamma})$ is equivalent to $(vEv_{p',q',\delta})$.

Proposition 4.1 (cf. [2, Proposition 2.1.1 and Corollary 2.1.2]) *Assume that (4.1) is satisfied. Then, the following conditions are equivalent.*

- (4.2) holds up to $t_0 = R^2/4$,
- $(vEv_{\infty,\infty,\frac{1}{2}})$ is satisfied up to time t_0 ,
- $(vEv_{1,\infty,\frac{1}{2}})$ is satisfied up to time t_0 ,
- $(vEv_{1,2,0})$ is satisfied up to time $t_0/2$,
- $(vEv_{2,\infty,\frac{1}{2}})$ is satisfied up to time $t_0/2$.

Proof The proof is analogous to the one of Proposition 2.1.1 and Corollary 2.1.2 in [2]. The only difference is that the last two conditions will hold for different time ranges

than the first two. The equivalence between (4.2) and $(vEv_{1,\infty,\frac{1}{2}})$ follows directly from the Dunford–Pettis theorem: for all $t \in (0, t_0]$, we have

$$\|v_{\sqrt{t}}^{1/2} P_t^{B_0} v_{\sqrt{t}}^{1/2}\|_{1,\infty} = \sup_{x,y \in B_0} v_{\sqrt{t}}^{1/2}(x) p_t^{B_0}(x,y) v_{\sqrt{t}}^{1/2}(y) \leq C < \infty.$$

Since the proof in [2] is for a fixed value of t , the same proof gives us the equivalence in our case. The equivalence between $(vEv_{1,2,0})$ and $(vEv_{2,\infty,\frac{1}{2}})$ follows by duality. It suffices to show that $(vEv_{1,\infty,\frac{1}{2}})$ holds up to time t_0 if and only if $(vEv_{1,2,0})$ holds up to time $t_0/2$. Note that if $T : L^1 \rightarrow L^2$, then

$$\|T^*T\|_{1,\infty} = \|T^*\|_{2,\infty}^2 = \|T\|_{1,2}^2,$$

so by taking $T = P_{t/2}^{B_0} v_{\sqrt{t}}^{1/2}$, we get that

$$\|v_{\sqrt{t}}^{1/2} P_{t/2}^{B_0} v_{\sqrt{t}}^{1/2}\|_{1,\infty} = \|v_{\sqrt{t}}^{1/2} P_{t/2}^{B_0}\|_{2,\infty}^2 = \|P_{t/2}^{B_0} v_{\sqrt{t}}^{1/2}\|_{1,2}^2.$$

Hence, $(vEv_{1,\infty,\frac{1}{2}})$ is equivalent to

$$\sup_{0 < t \leq t_0} \|v_{\sqrt{t}}^{1/2} P_{t/2}^{B_0}\|_{2,\infty}^2 < \infty \quad \text{and} \quad \sup_{0 < t \leq t_0} \|P_{t/2}^{B_0} v_{\sqrt{t}}^{1/2}\|_{1,2}^2 < \infty.$$

If $(vEv_{1,\infty,\frac{1}{2}})$ holds up to time t_0 , then defining $\tilde{t} = 2t$ for any $0 < t \leq t_0/2$, we have that

$$\sup_{0 < t \leq t_0/2} \|P_t^{B_0} v_{\sqrt{t}}^{1/2}\|_{1,2} = \sup_{0 < \tilde{t} \leq t_0} \|P_{\tilde{t}/2}^{B_0} v_{\sqrt{\tilde{t}/2}}^{1/2}\|_{1,2} \leq \sup_{0 < \tilde{t} \leq t_0} \|P_{\tilde{t}/2}^{B_0} v_{\sqrt{\tilde{t}}}^{1/2}\|_{1,2} < +\infty$$

where we used that $v_r(x)$ is non-decreasing in r . Thus, $(vEv_{1,2,0})$ holds up to time $t_0/2$. Conversely, if $(vEv_{1,2,0})$ holds up to time $t_0/2$, then we have

$$\sup_{0 < \tilde{t} \leq t_0} \|P_{\tilde{t}/2}^{B_0} v_{\sqrt{\tilde{t}}}^{1/2}\|_{1,2} = \sup_{0 < t \leq t_0/2} \|P_t^{B_0} v_{\sqrt{2t}}^{1/2}\|_{1,2} \leq \sup_{0 < t \leq t_0/2} C \|P_t^{B_0} v_{\sqrt{t}}^{1/2}\|_{1,2} < +\infty,$$

where we used that $v_{\sqrt{2t}}(x) \leq v_{2\sqrt{t}}(x) \leq C v_{\sqrt{t}}(x)$ by the non-decreasing and the v -doubling (4.1) properties of $v_r(x)$. Hence, $(vEv_{1,\infty,\frac{1}{2}})$ holds up to time t_0 , and this completes the proof. □

Proposition 4.2 *Assume that B_0 satisfies (4.1) and (4.2) for $R = 2r_0$ and let $q \in (2, \infty]$, $\frac{q-2}{q}n < 2$. Then, (4.3) holds on B_0 .*

Proof The proof is adapted from the beginning of Section 2 and Proposition 2.3.2 of [2]. According to Proposition 4.1, (4.1) and (4.2) for $R > 0$, together with an interpolation

argument for bounded operators between L^p -spaces (see, e.g., [2, Corollary 2.1.6]), imply the existence of a $C > 0$ such that

$$H := \sup_{t \in (0, R/2]} \|v^{\frac{1}{2} - \frac{1}{q}} P_t^{B_0}\|_{2,q} \leq C. \tag{4.4}$$

The fundamental theorem gives for any $f \in C_c^\infty(B_0)$

$$f = P_t^{B_0} f + \int_0^t \Delta^{B_0} P_s^{B_0} f ds.$$

Thus, putting $\alpha = \frac{1}{2} - \frac{1}{q}$,

$$\|v^\alpha_{\sqrt{t}} f\|_q \leq \|v^\alpha_{\sqrt{t}} P_t^{B_0}\|_{2,q} \|f\|_2 + \int_0^t \|v^\alpha_{\sqrt{t}} P_{s/2}^{B_0}\|_{2,q} \|\Delta^{B_0} P_{s/2}^{B_0} f\|_2 ds.$$

Using (4.1), we get

$$\begin{aligned} \|v^\alpha_{\sqrt{t}} f\|_q &\leq \|v^\alpha_{\sqrt{t}} P_t^{B_0}\|_{2,q} \|f\|_2 + \int_0^t \left\| \frac{v_{\sqrt{t}}}{v_{\sqrt{s/2}}} \right\|_\infty^\alpha \|v^\alpha_{\sqrt{s/2}} P_{s/2}^{B_0}\|_{2,q} \|\Delta^{B_0} P_{s/2}^{B_0} f\|_2 ds \\ &\leq H \|f\|_2 + C_D 2^{n/2} \int_0^t \left(\frac{t}{s}\right)^{n\alpha/2} \|\Delta^{1/2} P_{s/2}^{B_0} \Delta^{1/2} f\|_2 ds \\ &\leq H \|f\|_2 + G H t^{n\alpha/2} \int_0^t s^{-n\alpha/2 - 1/2} \|\Delta^{1/2} f\|_2 ds, \end{aligned}$$

and the last integral is finite by assumption on q . Hence, for all $\sqrt{t} \in (0, R/2]$,

$$\|v^\alpha_{\sqrt{t}} f\|_q \leq C(\|f\|_2 + \sqrt{t} \|\Delta^{1/2} f\|_2)$$

for some C depending only on q, C_D, n , and the upper bound on H . Putting $r = \sqrt{t}$ yields the result. □

Corollary 4.3 *Suppose $R > 0$, and $2p > n \geq 2$. There is an $\varepsilon > 0$ such that if a manifold M of dimension n satisfies*

$$\kappa_M(p, R) \leq \varepsilon,$$

then (4.3) holds on B_0 with $r_0 = R/2$.

Proof According to [19, 21], (4.1) holds up to radius R for some $\varepsilon > 0$. Since Dirichlet heat kernels are always less than the heat kernel of the manifold, (4.2) follows from [8, 25] by choosing ε possibly smaller. The claim follows from Proposition 4.2. □

Proof of Theorem 1.4 The proof is adapted from [7]. If $\Omega \subset M$ is (r, H, K) -regular, then it satisfies in particular the rolling r -ball condition for some $r \leq r_0$, and r_0 depends on K and H only. According to [22], there exists a $C > 0$ depending on R, p, n such that

$$\text{Vol}_\Omega(B(x, t)) \leq C \left(\frac{t}{s}\right)^n \text{Vol}_\Omega(B(x, s)), \quad x \in \Omega, 0 < s \leq t \leq \text{diam } \Omega \leq R. \tag{4.5}$$

Moreover, according to Theorem 1.2, (r, H, K) -regularity implies that there is an extension operator E_Ω with norm bounded in terms of H, K, r . Note that by construction, $\|E_\Omega\|_{L^2(\Omega), L^2(M)}$ is bounded as well and does not depend on any curvature restrictions but on the rolling r -ball condition. Moreover, we have

$$\text{Vol}_\Omega(B(x, t)) \leq \text{Vol}(B(x, t)), \quad t > 0.$$

Abbreviate $A = \|E_\Omega\|_{L^2(\Omega), L^2(M)}$ and $B = \|E_\Omega\|_{H^1(\Omega), H^1(M)}$. By the local Gagliardo–Nirenberg inequality, Corollary 4.3 and the existence of an extension operator E_Ω from Theorem 1.2, for any $s \in (0, R/2]$, $q \in [2, \infty]$, $\frac{q-2}{q}n < 2$, $f \in C^1(\overline{\Omega})$, we have

$$\begin{aligned} & \| \text{Vol}_\Omega(B(x, s))^{1/2-1/q} f \|_{L^q(\Omega)} \\ & \leq \| v_s^{1/2-1/q} E_\Omega f \|_{L^q(\Omega)} \\ & \leq C(\|E_\Omega f\|_{L^2(M)} + s \|\nabla E_\Omega f\|_{L^2(M)}) \\ & \leq C(A\|f\|_{L^2(\Omega)} + Bs(\|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)})) \\ & \leq C \max(A, B)((1+s)\|f\|_{L^2(\Omega)} + s\|\nabla f\|_{L^2(\Omega)}) \\ & \leq C \max(A, B)(1 + \text{diam}(\Omega))(\|f\|_{L^2(\Omega)} + s\|\nabla f\|_{L^2(\Omega)}) \\ & \leq C(n, p, r, K, H, R)(\|f\|_{L^2(\Omega)} + s\|\nabla f\|_{L^2(\Omega)}), \end{aligned}$$

i.e., the Gagliardo–Nirenberg inequality on Ω for all $s \in (0, R/2]$. In the second inequality, we use that since $\text{diam } \Omega \leq R/2$ we can choose the extension operator in such a way that the extended function has support in $B(x, R/2)$. For the case $s > R/2$, note that $\text{Vol}_\Omega(B(x, s)) = \text{Vol}(\Omega)$ for $s \geq \text{diam } \Omega$, hence the case $s > \text{diam}(\Omega)$ reduces to the case $s = \text{diam}(\Omega)$. Thus, the Gagliardo–Nirenberg inequality holds for all $s > 0$ on Ω . To derive the desired Neumann heat kernel upper bound, we want to apply [2, Theorem 1.1] directly. More precisely, the latter theorem shows that global volume doubling and global Gagliardo–Nirenberg inequalities on Ω yield an all-time upper bound for the Neumann heat kernel. The only thing that is still needed to check is whether inside Ω , volumes of different balls of the same radius are comparable, i.e., condition (D'_v) in the notation of the latter paper for $v = \text{Vol}_\Omega$. If $s > 0$ and $x, y \in \Omega$, $d(x, y) \leq s$, then $B(y, s) \subset B(x, 2s)$, such that (4.5) implies

$$\text{Vol}_\Omega(B(y, s)) \leq \text{Vol}_\Omega(B(x, 2s)) \leq 2^n C \text{Vol}_\Omega(B(x, s)).$$

Hence, the theorem follows, and the constants appearing depend only on the dimension, the doubling constant and radius, and the heat kernel upper bound. \square

Proof of Corollary 1.5 According to [30], there exists an explicit constant $\varepsilon = \varepsilon(n, r, H, K) > 0$ such that if

$$\kappa_T(\rho_-) \leq \int_0^T \|H_t^\Omega \rho_-\|_\infty dt \leq \varepsilon, \quad (4.6)$$

then all the conclusions of the corollary hold. Here $(H_t^\Omega)_{t \geq 0}$ is the Neumann heat semigroup of Ω . First, note that by the Dunford–Pettis theorem, Theorem 1.4, and volume doubling on Ω , we have for all $t \leq R^2$

$$\begin{aligned} \|H_t^\Omega\|_{1,\infty} &= \sup_{x,y \in \Omega} h_t^\Omega(x,y) \\ &\leq \sup_{x,y \in \Omega} \frac{\bar{C}}{\text{Vol}_\Omega(B(x, \sqrt{t}))^{1/2} \text{Vol}_\Omega(B(y, \sqrt{t}))^{1/2}} \leq \frac{\tilde{C}}{\text{Vol}(\Omega)} \left(\frac{R}{\sqrt{t}}\right)^n. \end{aligned}$$

Thus, since $\|H_t^\Omega\|_{\infty,\infty} \leq 1$ and by duality, the Riesz–Thorin interpolation theorem implies

$$\|H_t^\Omega\|_{p,\infty} \leq C_R \text{Vol}(\Omega)^{-1/p} t^{-n/2p}.$$

Hence,

$$\begin{aligned} \kappa_T(\rho_-) &\leq \int_0^T \|H_t^\Omega \rho_-\|_\infty dt \\ &\leq \int_0^T \|H_t^\Omega\|_{p,\infty} \|\rho_-\|_{p,\Omega} dt \\ &\leq C_R \text{Vol}(\Omega)^{-1/p} \|\rho_-\|_{p,\Omega} \int_0^T t^{-n/2p} dt, \end{aligned}$$

and the latter function is integrable provided $n/2p < 1$, i.e., $p > n/2$. Thus, the result follows by taking $T = R^2$ and forcing the right-hand side to be smaller than ε . \square

Acknowledgements We want to thank Leonhard Frerick, Rostislav Matveev, and Jürgen Jost for useful remarks. C.R. wants to thank O.P. and the University of Trier for their hospitality and Sebastian Boldt for pointing out a gap in an earlier version of the article.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors did not receive support from any organization for the submitted work and have no relevant financial or non-financial interests to disclose.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Appendix: Proofs of Propositions 2.3 and 2.4

We derive a differential inequality for the metric tensor along a geodesic depending on upper and lower bounds of the sectional and mean curvature that we compare with the solution λ of the differential equation

$$\lambda' + \lambda^2 = -k, \quad \lambda(0) = h, \quad (\text{A.1})$$

where $k, h \in \mathbb{R}$ are a lower (resp. upper) bound for the sectional curvature along the geodesic and upper (resp. lower) bound on the principal curvatures of N . By substituting $\lambda = \frac{\mu'}{\mu+C}$ for $C \in \mathbb{R}$, this equation transforms into the solvable differential equation

$$\mu'' + k\mu = -kC, \quad \mu'(0) = h(\mu(0) + C). \quad (\text{A.2})$$

Choosing $\mu(0) = 1$ and C appropriately guarantees the existence of a $t_0 > 0$ and a unique solution $\mu_{k,H}$ that is positive on an interval $(0, t_0]$.

Proof of Proposition 2.3 We only show Equation (2.4) by following the proof of [18, Theorem 27]. The upper bound for the metric can be proven similarly. Observe that the initial conditions for the Hessian of s are given by

$$\text{Hess } s(0) = \Pi_{c(0)}.$$

Fix $\theta \in N$ and define

$$\lambda(s) := \lambda(s, \theta) := \max_{v \perp \partial_s} \frac{\text{Hess } s(v, v)}{g(v, v)},$$

that is Lipschitz and hence absolutely continuous. Let v be a vector such that

$$\text{Hess } s(v, v) = \lambda(s_0)g_s(v, v),$$

where s_0 is a point where λ is differentiable, and extend v to a parallel field V . The function $\varphi(s) := \text{Hess } s(V, V)$ satisfies $\varphi(s) \leq \lambda(s)$ and $\varphi(s_0) = \lambda(s_0)$. This yields, since V is parallel,

$$\lambda'(s_0) + \lambda^2(s_0) = \partial_s \text{Hess } s(v, v) + \text{Hess } s^2(v, v) = \nabla_{\partial_s} \text{Hess } s(v, v) + \text{Hess } s^2(v, v)$$

$$= -g(R(v, \partial_s)\partial_s, v) \leq -k.$$

Moreover, $\lambda(0) = H_+$. Thus, by Riccati comparison, the Hessian satisfies

$$\text{Hess } s \leq \frac{\mu'_{k,H_+}(s)}{\mu_{k,H_+}(s)} g_s.$$

In general, we have

$$\partial_s g_s = 2 \text{Hess } s,$$

yielding

$$\partial_s g_s \leq 2 \frac{\mu'_{k,H_+}(s)}{\mu_{k,H_+}(s)} g_s. \tag{A.3}$$

To get the desired estimate for the metric, we compare this differential inequality with the conformal variation

$$h_s = \mu_{k,H_+}^2(s) g_0, \quad \mu_{k,H_+}(0) = 1, \quad s \in (0, r_0).$$

This variation satisfies

$$\partial_s h_s = 2 \frac{\mu'_{k,H_+}(s)}{\mu_{k,H_+}(s)} h_s, \quad h_0 = g_0.$$

Comparing this equality with Equation (A.3) yields the claim. □

To get Proposition 2.4, we adopt the comparison argument from [18] to our situation. In general for the decomposition (2.3) of our metric, we have

$$\partial_s \text{dvol} = \Delta s \text{dvol}$$

where $\Delta s = \text{Tr } \Pi$ denotes the mean curvature of the distance hypersurface. If we decompose the volume element into

$$\text{dvol} = \lambda(s, \theta) ds d\text{vol}_s,$$

we see that the equation above for fixed $\theta \in N$ reduces to

$$\lambda'(s) = \Delta s \lambda(s),$$

where $\Delta s(0) = H := \text{Tr } \Pi(0)$. We want to compare this metric with the conformal variation

$$h_s = \mu_{k,H}^2(s) g_0.$$

on N . Note that we have

$$\mathrm{dvol}_s^{k,H} = \mu_{k,H}^{n-1}(s) \mathrm{dvol}_0,$$

and

$$\partial_s(\mathrm{dvol}_s^{k,H}) = \partial_s \mu_{k,H}^{n-1} \mathrm{dvol}_0 = (n-1) \frac{\mu'_{k,H}(s)}{\mu_{k,H}(s)} \mathrm{dvol}_s^{k,H}, \quad \frac{\mu'_{k,H}(0)}{\mu_{k,H}(0)} = HC.$$

The mean curvature can be controlled by the following result by Eschenburg.

Lemma A.1 [9, Theorem 4.1] *For M, N, c, r_0, H as above, we have*

$$\Delta s \leq \mu_{k,H}(s), \quad s \in [0, r_0), \quad m_{k,H}(0) = \mathrm{Tr} \, \Pi(0). \quad (\text{A.4})$$

For $s < 0$, the opposite inequality holds.

References

1. Aubry, E.: Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature. *Ann. Sci. École Norm. Sup. (4)* **40**(4), 675–695 (2007)
2. Boutayeb, S., Coulhon, T., Sikora, A.: A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces. *Adv. Math.* **270**, 302–374 (2015)
3. Brudnyi, A.: *Methods of Geometric Analysis in Extension and Trace Problems: Monographs in Mathematics*, vol. 2. Springer, Basel (2011)
4. Carron, G.: Geometric inequalities for manifolds with Ricci curvature in the Kato class. *Ann. Inst. Fourier (Grenoble)* **69**(7), 3095–3167 (2019)
5. Carron, G., Rose, C.: Geometric and spectral estimates based on spectral Ricci curvature assumptions. *J. Reine Angew. Math.* (2020). <https://doi.org/10.1515/crelle-2020-0026>
6. Chen, R.: Neumann eigenvalue estimate on a compact Riemannian manifold. *Proc. Am. Math. Soc.* **108**(4), 961–970 (1990). (4)
7. Choulli, M., Kayser, L., Ouhabaz, E.M.: Observations on Gaussian upper bounds for Neumann heat kernels. *Bull. Aust. Math. Soc.* **92**(3), 429–439 (2015)
8. Dai, X., Wei, G., Zhang, Z.: Local Sobolev constant estimate for integral Ricci curvature bounds. *Adv. Math.* **325**, 1–33 (2018)
9. Eschenburg, J.H.: Comparison theorems and hypersurfaces. *Manuscr. Math.* **59**(3), 295–323 (1987)
10. Fefferman, C.L., Israel, A., Luli, G.K.: Sobolev extension by linear operators. *J. Am. Math. Soc.* **27**, 69–145 (2014)
11. Gallot, S.: Isoperimetric inequalities based on integral norms of Ricci curvature. *Astérisque* **157–158**, 191–216 (1988). (*Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987)*)
12. Gluck, M., Zhu, M.: An extension operator on bounded domains and applications. *Calc. Var. Partial Differ. Equ.* **58**(2), 79 (2019)
13. Gol'dshtein, V., Pchelintsev, V., Ukhlov, A.: Sobolev extension operators and Neumann Eigenvalues. *J. Spectr. Theory* **10**, 337–353 (2020)
14. Große, N., Schneider, C.: Sobolev spaces on Riemannian manifolds with bounded geometry: General coordinates and traces. *Math. Nachr.* **286**(16), 1586–1613 (2013)
15. Gyrya, P., Saloff-Coste, L.: *Neumann and Dirichlet Heat Kernels in Inner Uniform Domains. Asterisque Series*. American Mathematical Society, Providence, RI (2011)
16. Jost, J.: *Riemannian geometry and geometric analysis*, 5th edn. Universitext, Springer-Verlag, Berlin Heidelberg (2008)
17. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156**(3–4), 153–201 (1986)
18. Petersen, P.: *Riemannian Geometry. Graduate Texts in Mathematics*. Springer, New York (2006)

19. Petersen, P., Wei, G.: Relative volume comparison with integral curvature bounds. *Geom. Funct. Anal.* **7**(6), 1031–1045 (1997)
20. Petersen, P., Sprouse, C.: Integral curvature bounds, distance estimates and applications. *J. Differ. Geom.* **50**(2), 269–298 (1998)
21. Petersen, P., Wei, G.: Analysis and geometry on manifolds with integral Ricci curvature bounds. II. *Trans. Am. Math. Soc.* **353**(2), 457–478 (2001)
22. Ramos Olivé, X.: Neumann Li-Yau gradient estimate under integral Ricci curvature bounds. *Proc. Am. Math. Soc.* **147**(1), 411–426 (2019)
23. Ramos Olivé, X., Seto, S., Wei, G., Zhang, Q.S.: Zhong-Yang type eigenvalue estimate with integral curvature condition. *Math. Z.* **296**, 595–613 (2019)
24. Rose, C.: Heat kernel estimates based on Ricci curvature integral bounds. PhD thesis, Technische Universität Chemnitz (2017)
25. Rose, C.: Heat kernel upper bound on Riemannian manifolds with locally uniform Ricci curvature integral bounds. *J. Geom. Anal.* **27**, 1737–1750 (2017)
26. Rose, C.: Almost positive Ricci curvature in Kato sense - an extension of Myers' theorem. *Math. Res. Lett.* **28**(6), 1841–1849 (2021)
27. Rose, C.: Li-Yau gradient estimate for compact manifolds with negative part of Ricci curvature in the Kato class. *Ann. Glob. Anal. Geom.* **55**(3), 443–449 (2019)
28. Rose, C., Stollmann, P.: The Kato class on compact manifolds with integral bounds of Ricci curvature. *Proc. Am. Math. Soc.* **145**, 2199–2210 (2017)
29. Rose, C., Stollmann, P.: Manifolds with Ricci curvature in the Kato class: heat kernel bounds and applications. In: Keller, M., Lenz, D., Wojciechowski, R.K. (eds.) *Analysis and Geometry on Graphs and Manifolds*, London Mathematical Society Lecture Note Series, vol. 461. Cambridge University Press, Cambridge (2020)
30. Rose, C., Wei, G.: Eigenvalue estimates under Kato-type Ricci curvature conditions. [arXiv:2003.07075v2](https://arxiv.org/abs/2003.07075v2) [math.DG] (2020)
31. Stein, E.M.: *Singular Integrals and Differentiability Properties of Functions*. Monographs in Harmonic Analysis. Princeton University Press, Princeton (1970)
32. Wang, J.: Global heat kernel estimates. *Pac. J. Math.* **178**(2), 377–398 (1997)
33. Warner, F.W.: Extension of the Rauch comparison theorem to submanifolds. *Trans. Am. Math. Soc.* **122**(2), 341–356 (1966)
34. Whitney, H.: Functions differentiable on the boundaries of regions. *Ann. Math.* **35**(3), 482–485 (1934)
35. Zhang, Q.S., Zhu, M.: Li-Yau gradient bound for collapsing manifolds under integral curvature condition. *Proc. Am. Math. Soc.* **145**(7), 3117–3126 (2017)
36. Zhang, Q.S., Zhu, M.: Li-Yau gradient bounds on compact manifolds under nearly optimal curvature conditions. *J. Funct. Anal.* **275**(2), 478–515 (2018)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.