



A Fundamental Condition for Harmonic Analysis in Anisotropic Generalized Orlicz Spaces

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Abstract

Anisotropic generalized Orlicz spaces have been investigated in many recent papers, but the basic assumptions are not as well understood as in the isotropic case. We study the greatest convex minorant of anisotropic Φ -functions and prove the equivalence of two widely used conditions in the theory of generalized Orlicz spaces, usually called (A1) and (M). This provides a more natural and easily verifiable condition for use in the theory of anisotropic generalized Orlicz spaces for results such as Jensen's inequality which we obtain as a corollary.

Keywords Generalized Orlicz space · Musielak–Orlicz spaces · Anisotropic · Nonstandard growth · Jensen's inequality

Mathematics Subject Classification 46E30 46A55 · 51M16

1 Introduction

This paper deals with generalized Orlicz spaces, also known as Musielak–Orlicz spaces. This is a very active field recently [2, 5, 6, 14, 16, 19, 29, 31, 32], boosted by work on the double phase problem by Baroni, Colombo and Mingione, e.g. [4, 13]. The generalized Orlicz case unifies the study of the double phase problem and the variable exponent growth, widely researched over the last 20 years [15]. Many studies deal with isotropic energies of the type

$$\int_{\Omega} \varphi(x, |\nabla u|) \, dx$$

but recently also the anisotropic case

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$$\int_{\Omega} \Phi(x, \nabla u) \, dx$$

has been considered e.g. in [1, 7–9, 24, 27]. As an example of an anisotropic energy with non-standard growth we could take a double-phase functional where the q -phase is directional:

$$\int_{\Omega} |\nabla u|^p + a(x) |\partial_{x_1} u|^q \, dx ;$$

here only variation in the x_1 -direction makes a contribution to the energy in the q -phase $\{a > 0\}$.

Counter examples (see, e.g., [3, 15]) show that more advanced results such as the boundedness of averaging operators or density of smooth functions require connecting $\Phi(x, \xi)$ for different values of x . To this end, we developed in the isotropic case the (A1) condition [18, 22] (see also [28]), which is essentially optimal for the boundedness of the maximal operator. In the anisotropic case Chlebicka, Gwiazda, Zatorska-Goldstein and co-authors [1, 7–12, 17] have developed a theory based on their (M) condition. To state and compare these conditions, let us define

$$\Phi_B^-(\xi) := \operatorname{ess\,inf}_{x \in B \cap \Omega} \Phi(x, \xi) \quad \text{and} \quad \Phi_B^+(\xi) := \operatorname{ess\,sup}_{x \in B \cap \Omega} \Phi(x, \xi). \quad (1.1)$$

In essence, the (A1) conditions says that Φ_B^+ can be bounded by Φ_B^- in small balls $B \subset \mathbb{R}^m$ in a quantitative way, whereas (M) say that it can be similarly bounded by the least convex minorant $(\Phi_B^-)^{\operatorname{conv}}$ of Φ_B^- (see Definition 3.2). Obviously, the latter is a stronger condition, and it is also more difficult to verify, since the relationship between $(\Phi_B^-)^{\operatorname{conv}}$ and Φ_B^- may be complicated in the anisotropic case.

In the isotropic case $\Phi_B^- \lesssim (\Phi_B^-)^{\operatorname{conv}}$ so (M) and (A1) are equivalent. In the anisotropic case this inequality does not hold (see Example 4.1), but we are nevertheless able to prove the equivalence of the conditions by a more careful analysis.

Theorem 1.2 *Let $\Phi : \Omega \times \mathbb{R}^m \rightarrow [0, \infty]$ be a strong Φ -function. Then (A1) and (M) are equivalent.*

This result and the techniques introduced in this paper will allow for the development of a theory of anisotropic generalized Orlicz spaces with more natural assumptions. As an example we prove the following Jensen-type inequality.

Corollary 1.3 (Jensen-type inequality) *Let Φ satisfy (A1) and $f \in L_{\mu}^{\Phi}(\Omega; \mathbb{R}^m)$. Then there exists $\beta > 0$ such that*

$$\Phi_B^+ \left(\beta \int_B f \, d\mu \right) \leq \int_B \Phi(x, f) \, d\mu + 1$$

when $\varrho_{\Phi}(f) \leq 1$ and $\mu(B) \leq 1$.

Although the extra assumption $\varrho_\Phi(f) \leq 1$ in the corollary may seem strange, it follows naturally for instance when dealing with local regularity and it is known that the anisotropic Jensen-inequality does not hold without restrictions.

Let us next define precisely the concepts we are using and characterize functions Φ for which the equivalence $\Phi^{\text{conv}} \simeq \Phi$ holds (Corollary 2.4). In Sect. 3, we define the conditions (A1) and (M) and give preliminary remarks regarding the definitions. Finally, in Sect. 4 we prove the main results mentioned above.

2 Almost Convexity and the Greatest Convex Minorant

I refer to the monographs [9, 18] for background on isotropic and anisotropic generalized Orlicz spaces, respectively. We consider functions $\Phi : \Omega \times \mathbb{R}^m \rightarrow [0, \infty]$; the capital letter Φ is used to highlight the distinction from the isotropic case L^φ in [18] where $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$. The idea is to define

$$\varrho_\Phi(v) := \int_\Omega \Phi(x, v) \, d\mu \quad \text{and} \quad \|v\|_\Phi := \inf\{\lambda > 0 \mid \varrho_\Phi\left(\frac{v}{\lambda}\right) \leq 1\}$$

for a vector field $v \in L^1_\mu(\Omega; \mathbb{R}^m)$. The space $L^\Phi_\mu(\Omega; \mathbb{R}^m)$ is defined by the requirement $\|v\|_\Phi < \infty$. We use the equivalence relation $\Phi \simeq \Psi$ which means that there exists $\beta > 0$ such that

$$\Phi(\beta\xi) \leq \Psi(\xi) \leq \Phi\left(\frac{\xi}{\beta}\right).$$

Here and in the rest of the paper β denotes a parameter which is given by by one or more conditions; if the conditions hold with different β_k , then we can use $\beta := \min_k \beta_k$ for all the conditions so that we may just as well use only the one common β . Since the parameter λ is inside Φ in the definition of $\|v\|_\Phi$, this is the natural way to compare functions Φ (cf. Example 2.2). To ensure that the integral in ϱ_Φ makes sense and $\|\cdot\|_\Phi$ is a norm we require some conditions.

Definition 2.1 Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $\Phi : \Omega \times \mathbb{R}^m \rightarrow [0, \infty]$ is a *strong Φ -function*, and write $\Phi \in \Phi_s(\Omega)$, if the following four conditions hold:

- (1) $x \mapsto \Phi(x, \xi)$ is measurable for every $\xi \in \mathbb{R}^m$.
- (2) $\Phi(x, 0) = \lim_{\xi \rightarrow 0} \Phi(x, \xi) = 0$ and $\lim_{\xi \rightarrow \infty} \Phi(x, \xi) = \infty$ for a.e. $x \in \Omega$.
- (3) $\xi \mapsto \Phi(x, \xi)$ is continuous in the topology of $[0, \infty]$ for a.e. $x \in \Omega$.
- (4) Φ is convex for a.e. $x \in \Omega$:

$$\Phi(x, \alpha\xi + \alpha'\xi') \leq \alpha\Phi(x, \xi) + \alpha'\Phi(x, \xi'), \quad \alpha, \alpha' \geq 0, \quad \alpha + \alpha' = 1.$$

With these conditions, $\|\cdot\|_\Phi$ is a norm. Note that continuity in ξ follows from convexity if Φ is real-valued and (3) is only needed to ensure that Φ does not jump to ∞ . Note also that this class of strong Φ -functions is broader than that studied [9] since we do not require that upper and lower bounds in terms of N -functions independent

of x . For instance, this definition allows for L^1 - and L^∞ -type growth. In [18] in the isotropic case we relaxed (3) and (4) further, and so used “strong” for this class, even though it is still less restrictive than N -functions.

For the study of Φ -functions depending on the space-variable x , we use local approximations with the functions Φ_B^+ and Φ_B^- from (1.1) [18, 22]. However, Φ_B^- need not be convex even if each $\Phi(x, \cdot)$ is (just think of $\min\{t, t^2\}$). In the isotropic case, φ_B^- nevertheless satisfies the following weaker variant of (4) above:

(W4) Φ is *almost convex* if there exists $\beta > 0$ such that

$$\Phi(x, \beta(\alpha\xi + \alpha'\xi')) \leq \alpha\Phi(x, \xi) + \alpha'\Phi(x, \xi'),$$

for a.e. $x \in \Omega$ and $\alpha, \alpha' \geq 0$ with $\alpha + \alpha' = 1$.

Unfortunately, even this does not hold for Φ_B^- in the anisotropic case (see Example 4.1).

The constant β in the almost convexity condition (W4) should be inside the function since we do not assume doubling, or even finite, functions, as the following example illustrates. A constant outside is possible, but too restrictive.

Example 2.2 Let $\varphi_\infty(t) := \infty\chi_{(1,\infty)}(t)$ be the function generating the space $L^\infty(\Omega)$. Define $\Phi(\xi) := \varphi_\infty(\|\xi\|_{1/2}) = \varphi_\infty(|\xi_1| + 2\sqrt{|\xi_1\xi_2|} + |\xi_2|)$ in \mathbb{R}^2 . Consider $\alpha = \frac{1}{2}$ and the basis vectors $\xi = e_1$ and $\xi' = e_2$ in (W4). Then $\|\frac{e_1+e_2}{2}\|_{1/2} = 2$ so $\Phi(\frac{e_1+e_2}{2}) = \infty$ and the inequality

$$\Phi\left(\frac{e_1+e_2}{2}\right) \leq \frac{L}{2} [\Phi(e_1) + \Phi(e_2)]$$

does not hold for any $L < \infty$. However, the almost convexity inequality (W4)

$$\Phi\left(\beta\frac{e_1+e_2}{2}\right) \leq \frac{1}{2} [\Phi(e_1) + \Phi(e_2)] = 0.$$

holds for $\beta \leq \frac{1}{2}$ since in this case $\Phi\left(\beta\frac{e_1+e_2}{2}\right) = 0$.

If we choose $\xi' = 0$ in the almost convexity condition (W4), then we obtain (aInc)₁:

$$\Phi(x, \beta\alpha\xi) \leq \alpha\Phi(x, \xi) \quad \text{for any } \alpha \in [0, 1]. \tag{aInc}_1$$

In the special case $\beta = 1$, i.e. for a convex function, we have (Inc)₁:

$$\Phi(x, \alpha\xi) \leq \alpha\Phi(x, \xi) \quad \text{for any } \alpha \in [0, 1]. \tag{Inc}_1$$

These inequalities mean that the function $t \mapsto \frac{\Phi(x,t\xi)}{t}$ is almost increasing or increasing, hence the notation (aInc)₁ and (Inc)₁. In [18] we showed that these inequalities are useful substitutes for convexity in the isotropic case. In particular, it is easy to see that Φ_B^+ and Φ_B^- satisfy (aInc)₁ or (Inc)₁ if Φ does. For the anisotropic case the almost convexity is more appropriate since it also carries information about non-radial behavior.

Let us denote by Φ^{conv} the greatest convex minorant of Φ . This function is often denoted by Φ^{**} , since it can be obtained by applying the conjugation operation $*$ twice [9, Corollary 2.1.42], but we will not use this fact here. We next show a connection between the greatest convex minorant and the almost convexity condition. The following is a version of Carathéodory’s Theorem from convex analysis. Probably it is known, but a proof is included for completeness, since I could not find a reference.

Lemma 2.3 *Let $\Phi : \mathbb{R}^m \rightarrow [0, \infty]$. Then*

$$\Phi^{\text{conv}}(\xi) = \min \left\{ \sum_{k=1}^{m+1} \alpha_k \Phi(\xi_k) \mid \sum_{k=1}^{m+1} \alpha_k \xi_k = \xi, \sum_{k=1}^{m+1} \alpha_k = 1, \alpha_k \geq 0 \right\}.$$

Proof Consider the epigraph of Φ ,

$$E := \{(\xi, t) \in \mathbb{R}^m \times \mathbb{R} \mid \Phi(\xi) \leq t\} \subset \mathbb{R}^{m+1}.$$

By Carathéodory’s Theorem (see, e.g., [30, Theorem 2.1.3]), every point in the convex hull of E can be represented as a convex combination of at most $m + 2$ points ξ_k from E . Furthermore, we observe that if any of the points ξ_k are from the interior of E , then the convex combination is also in the interior of the convex hull. Thus the points of the boundary, i.e. the graph of Φ^{conv} , are given as a convex combination of points in the boundary of E , i.e. on the graph of Φ . Hence

$$\Phi^{\text{conv}}(\xi) = \sum_{k=1}^{m+2} \alpha_k \Phi(\xi_k) \quad \text{for some} \quad \sum_{k=1}^{m+2} \alpha_k \xi_k = \xi, \quad \sum_{k=1}^{m+2} \alpha_k = 1, \quad \alpha_k \geq 0.$$

This is the claim, except with one extra point ξ_{m+2} .

However, ξ lies in the convex hull of $\xi_1, \dots, \xi_{m+2} \in \mathbb{R}^m$. Thus by Carathéodory’s Theorem in \mathbb{R}^m , ξ can be expressed as the convex combination of at most $m + 1$ of the points ξ_1, \dots, ξ_{m+2} . By re-labeling if necessary, we obtain

$$\sum_{k=1}^{m+1} \alpha'_k \xi_k = \xi, \quad \sum_{k=1}^{m+1} \alpha'_k = 1, \quad \alpha'_k \geq 0.$$

Since $(\xi_k, \Phi(\xi_k))_{k=1}^{m+2}$, all lie on the same hyper-plane for a boundary point, we also have

$$\Phi^{\text{conv}}(\xi) = \sum_{k=1}^{m+1} \alpha'_k \Phi(\xi_k).$$

□

We can now show that $\Phi \simeq \Phi^{\text{conv}}$ for almost convex functions.

Corollary 2.4 *The function $\Phi : \mathbb{R}^m \rightarrow [0, \infty]$ is almost convex if and only if $\Phi \simeq \Phi^{\text{conv}}$.*

Proof Clearly, $\Phi^{\text{conv}} \leq \Phi$ since Φ^{conv} is defined as a minorant of Φ . Let $2^i \geq m + 1$ and set $\alpha_k := 0$ and $\xi_k := 0$ for $k > m + 1$. By the almost convexity condition,

$$\Phi\left(\beta^i \sum_{k=1}^{2^i} \alpha_k \xi_k\right) \leq \alpha_{i,1} \Phi\left(\beta^{i-1} \sum_{k=1}^{2^{i-1}} \frac{\alpha_k}{\alpha_{i,1}} \xi_k\right) + \alpha_{i,2} \Phi\left(\beta^{i-1} \sum_{k=2^{i-1}}^{2^i} \frac{\alpha_k}{\alpha_{i,2}} \xi_k\right)$$

where

$$\alpha_{i,1} := \sum_{k=1}^{2^{i-1}} \alpha_k \quad \text{and} \quad \alpha_{i,2} := \sum_{k=2^{i-1}+1}^{2^i} \alpha_k.$$

Iterating this i times, we obtain that

$$\Phi\left(\beta^i \sum_{k=1}^{2^i} \alpha_k \xi_k\right) \leq \sum_{k=1}^{2^i} \alpha_k \Phi(\xi_k)$$

By Lemma 2.3 and this inequality,

$$\Phi^{\text{conv}}(\xi) \geq \min \left\{ \sum_{k=1}^{2^i} \alpha_k \Phi(\xi_k) \mid \sum_{k=1}^{2^i} \alpha_k \xi_k = \xi, \sum_{k=1}^{2^i} \alpha_k = 1, \alpha_k \geq 0 \right\} \geq \Phi(\beta^i \xi).$$

Thus the almost convexity implies that $\Phi \simeq \Phi^{\text{conv}}$.

If, on the other hand, $\Phi \simeq \Phi^{\text{conv}}$ with constant β , then we directly obtain

$$\Phi(\beta(\alpha\xi + \alpha'\xi')) \leq \Phi^{\text{conv}}(\alpha\xi + \alpha'\xi') \leq \alpha\Phi^{\text{conv}}(\xi) + \alpha'\Phi^{\text{conv}}(\xi') \leq \alpha\Phi(\xi) + \alpha'\Phi(\xi').$$

□

For almost convex functions we easily obtain a Jensen inequality with an extra constant.

Corollary 2.5 (Jensen’s inequality) *Let $E \subset \mathbb{R}^m$ have positive, finite measure $\mu(E)$. If $\Phi \in C(E; [0, \infty])$ is almost convex, then there exists β such that*

$$\Phi\left(\beta \int_E f \, d\mu\right) \leq \int_E \Phi(f) \, d\mu$$

for every $f \in L^\Phi_\mu(E; \mathbb{R}^m)$.

Proof By Corollary 2.4 and Jensen’s inequality for the convex function Φ^{conv} ,

$$\Phi\left(\beta \int_E f \, d\mu\right) \leq \Phi^{\text{conv}}\left(\int_E f \, d\mu\right) \leq \int_E \Phi^{\text{conv}}(f) \, d\mu \leq \int_E \Phi(f) \, d\mu.$$

□

3 Definition of and Remarks on Conditions

The (aInc)₁ and almost convexity (W4) conditions connect $\Phi(x, \xi)$ for different values of ξ with x fixed. However, more advanced results such as the density of smooth functions in Sobolev spaces require connecting $\Phi(x, \xi)$ for different values of x , cf. [6]. This is the purpose of the conditions (A1-Ψ) and (M-Ψ), which generalize (A1) and (M).

However, let us first start with the more elementary condition (A0): there exists $\beta > 0$ such that

$$\Phi(x, \beta\xi) \leq 1 \leq \Phi\left(x, \frac{1}{\beta}\xi\right)$$

for all $\xi \in \mathbb{R}^m$ with $|\xi| = 1$ and all $x \in \Omega$. Note that (A0) is implicit in the assumption $m_1(|\xi|) \leq \Phi(x, \xi) \leq m_2(|\xi|)$ for N -functions m_1 and m_2 used in [1, 7–11, 17]. This property is inherited by other versions of Φ :

Lemma 3.1 *If $\Phi \in \Phi_s(\Omega)$ satisfies (A0), then so do Φ_B^+ , Φ_B^- and $(\Phi_B^-)^{\text{conv}}$.*

Proof Taking the supremum or infimum over $x \in \Omega$ in (A0) of Φ gives (A0) for Φ_B^+ and Φ_B^- . Since $(\Phi_B^-)^{\text{conv}} \leq \Phi_B^-$, the left inequality of (A0) follows for $(\Phi_B^-)^{\text{conv}}$. If $|\xi| \geq \frac{1}{\beta}$, then by (Inc)₁ and (A0) we conclude that

$$\Phi(x, \xi) \geq \beta |\xi| \Phi\left(x, \frac{1}{\beta} \frac{\xi}{|\xi|}\right) \geq \beta |\xi|.$$

Hence, for all $\xi \in \mathbb{R}^m$, $\Phi(x, \xi) \geq (\beta |\xi| - 1)_+$ (since $(\beta |\xi| - 1)_+ = 0$ when $|\xi| \leq \frac{1}{\beta}$) and so $\Phi_B^-(\xi) \geq (\beta |\xi| - 1)_+$. But the right-hand side is a convex function, so it follows that $(\Phi_B^-)^{\text{conv}}(\xi) \geq (\beta |\xi| - 1)_+$ since $(\Phi_B^-)^{\text{conv}}$ is defined as the greatest convex minorant. Consequently,

$$(\Phi_B^-)^{\text{conv}}\left(\frac{2}{\beta}\xi\right) \geq \left(\beta \frac{2}{\beta} - 1\right)_+ = 1,$$

when $\xi \in \mathbb{R}^m$ with $|\xi| = 1$, so $(\Phi_B^-)^{\text{conv}}$ satisfies (A0) with constant $\frac{\beta}{2}$. □

The condition (A1) was introduced in [22] (see also [18, 28]) and is essentially optimal for the boundedness of the maximal operator in isotropic generalized Orlicz spaces. It also implies the Hölder continuity of solutions and (quasi)minimizers [5, 20, 21]. For higher regularity, we introduced in [23] a vanishing-(A1) condition along

the same lines. These previous studies apply to the isotropic case, i.e. $m = 1$. In [24, 25] we generalized the (A1)-conditions to the anisotropic case, although only the quasi-isotropic case was considered in the main results.

Chlebicka, Gwiazda, Zatorska-Goldstein and co-authors [1, 7–12, 17] considered the assumption (M) in the anisotropic case; in the next definition their condition is reformulated to make it easier to compare with the (A1) condition (see also Lemma 3.4); also note that some of the earlier works included additional restrictions in the condition.

Definition 3.2 Let $\Phi, \Psi \in \Phi_s(\Omega)$. We say that Φ satisfies (A1- Ψ) or (M- Ψ) if for any $K > 0$ there exists $\beta > 0$ such that

$$\Phi_B^+(\beta\xi) \leq \Phi_B^-(\xi) + 1 \quad \text{when } \Psi_B^-(\xi) \leq \frac{K}{\mu(B)}$$

or

$$\Phi_B^+(\beta\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi) + 1 \quad \text{when } (\Psi_B^-)^{\text{conv}}(\xi) \leq \frac{K}{\mu(B)}$$

for all balls $B \subset \mathbb{R}^n$ with $\mu(B) \leq 1$ and $\xi \in \mathbb{R}^n$.

When $\Psi(t) := t^s$ and $\Psi := \Phi$ we use the abbreviations (A1- s), (A1), (M- s) and (M).

The role of Ψ is to calibrate the almost continuity requirement with the information on the function we are interested in and was developed from the initial condition (A1) over the course of several studies [5, 20, 21]. For instance, we showed in [5, Theorem 3.9] that the weak Harnack inequality holds for non-negative supersolutions of $\text{div} \left(\varphi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = 0$ if the isotropic Φ -function φ satisfies (A1- Ψ) and the supersolution satisfies $u \in W^{1,\psi}(\Omega)$, where $\psi \in \Phi_w(\Omega)$ is a potentially different function. Note that this involves a trade-off, since larger ψ means more restriction on u and less restriction on φ .

As far as I know, Chlebicka, Gwiazda, Zatorska-Goldstein and co-authors considered (M) only in the case $\Psi(t) := t$ and $\Psi(t) := t^p$ (i.e. (M-1) and (M- p) in the notation above). However, the next example illustrates why this does not lead to optimal results.

Example 3.3 (Variable exponent double phase) Let $\varphi(x, t) := t^{p(x)} + a(x)t^{q(x)}$ where $a \in C^{0,\alpha}(\Omega)$, $a \geq 0$ and $1 < p \leq q$. Now the (A1) or (M) conditions reduce to

$$\frac{q(x)}{p(x)} \leq 1 + \frac{\alpha}{n} \iff \left(\frac{q}{p}\right)^+ \leq 1 + \frac{\alpha}{n}$$

Let $p^- := \inf_{x \in \Omega} p(x)$ and $p^+ := \sup_{x \in \Omega} p(x)$. If we only use fixed exponent gauges such as (A1- p^-) or (M- p^-), then we instead end up with the condition

$$\frac{q(x)}{p^-} \leq 1 + \frac{\alpha}{n} \iff \frac{q^+}{p^-} \leq 1 + \frac{\alpha}{n}$$

which is worse, and quite unnatural as the largest value of q is bounded by the smallest value of p .

As a final remark about the formulation, we note that earlier papers used a form without the “+1” and instead restricted the range of Ψ_B^- . However, if (A0) holds, then these formulations are equivalent. We prove it for (M), the same applies to (A1).

Lemma 3.4 *Let $\Phi \in \Phi_s(\Omega)$ satisfy (A0). Then (M) holds if and only if*

$$\Phi_B^+(\beta\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi) \quad \text{when } (\Phi_B^-)^{\text{conv}}(\xi) \in \left[1, \frac{K}{\mu(B)}\right].$$

Proof If the condition of the lemma holds, then (M) needs only to be checked when $(\Phi_B^-)^{\text{conv}}(\xi) \leq 1$. This inequality and (A0) imply that $|\xi| \leq \frac{1}{\beta}$. Thus $\Phi_B^+(\beta^2\xi) \leq 1$ by (A0), so (M) holds with constant β^2 .

Assume conversely that (M) holds and $(\Phi_B^-)^{\text{conv}}(\xi) \in \left[1, \frac{K}{\mu(B)}\right]$. Then it follows that

$$\Phi_B^+(\beta\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi) + 1 \leq 2(\Phi_B^-)^{\text{conv}}(\xi).$$

Then (Inc)₁ implies that $\Phi_B^+\left(\frac{\beta}{2}\xi\right) \leq \frac{1}{2}\Phi_B^+(\beta\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi)$, so the condition of the lemma holds with constant $\frac{\beta}{2}$. □

4 Equivalence of Conditions

In the previous section we introduced and motivated the conditions (A1) and (M) and their variants. We now move on to the main result, and consider their relation to one another.

Since $(\Phi_B^-)^{\text{conv}} \leq \Phi_B^-$, (M-Ψ) implies (A1-Ψ). If φ is isotropic and satisfies (aInc)₁, then I showed in [22] that $\varphi_B^-(\beta t) \leq (\varphi_B^-)^{\text{conv}}(t)$. Hence the two conditions are equivalent in this case. However, as pointed out in [9, Remarks 2.3.14 and 3.7.6], this approach is not possible in the anisotropic case. Since I did not understand the examples implicit in these remarks without consulting the authors, I include here an explicit example based on ideas of Piotr Nayar communicated to me by Iwona Chlebicka.

Example 4.1 Let $m = 2$ and $\Phi_k(\xi_1 e_1 + \xi_2 e_2) := \xi_k^2$. Then both Φ_1 and Φ_2 are convex and $\Phi_1(e_2) = \Phi_2(e_1) = 0$. Denote $\Phi := \min\{\Phi_1, \Phi_2\}$. It follows that

$$\Phi^{\text{conv}}(\alpha_1 e_1 + \alpha_2 e_2) \leq \alpha_1 \Phi^{\text{conv}}(e_1) + \alpha_2 \Phi^{\text{conv}}(e_2) \leq \alpha_1 \Phi(e_1) + \alpha_2 \Phi(e_2) = 0,$$

where $\alpha_1 + \alpha_2 = 1$ and $\alpha_1, \alpha_2 \geq 0$. Thus we see that $\Phi^{\text{conv}} \equiv 0$. Since $\Phi(\beta(e_1 + e_2)) = \Phi_1(\beta(e_1 + e_2)) = \beta^2$ but $\Phi^{\text{conv}} \equiv 0$, the relation $\Phi \simeq \Phi^{\text{conv}}$ does not hold.

Even though $\Phi_B^-(\beta\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi)$ does not hold in general, we next construct an almost convex minorant which is comparable to Φ_B^- when (A1) holds which can

be used in (M). We prove that (A1) implies (M) in the main case $\Psi := \Phi$, which corresponds to the natural energy space L^Φ or $W^{1,\Phi}$. The implication for (A1- Ψ) and (M- Ψ) when $\Psi \neq \Phi$ remains an open problem.

Let $\Phi : \mathbb{R}^m \rightarrow [0, \infty]$ be a strong Φ -function independent of x . Denote $K_s := \{\Phi \leq s\}$ and observe that it is a convex compact set which includes 0 in its interior. Define

$$\|\xi\|_{K_s} := \inf\{\lambda > 0 \mid \frac{\xi}{\lambda} \in K_s\} \quad \text{and} \quad N_s(\xi) := s \max\{1, \|\xi\|_{K_s}\}.$$

Here $\|\cdot\|_{K_s}$ is the Minkowski functional of the set K_s , first studied by Kolmogorov [26]. The Luxemburg norm $\|\cdot\|_\Phi$ defined previously is another example of a Minkowski functional. Note that N_s is a convex function with $\{N_s \leq s\} = K_s$. Since Φ is convex, $\Phi(\lambda\xi) \leq \lambda\Phi(\xi)$ for $\lambda \leq 1$. Thus $N_s \leq \Phi$ outside K_s , $N_s \geq \Phi$ in K_s and $N_s = \Phi$ on the boundary ∂K_s . In other words, we take the s -level set of Φ and replace Φ outside of it by the function N_s which grows linearly.

not be even almost convex. However, in the next proposition we show that $\min\{\Phi, N_s\}$ is almost convex, since the two functions are somehow compatible. This will be used to construct a convex minorant of Φ_B^- . The proposition also demonstrates the utility of the almost convexity condition, as it seems much more difficult to choose N_s to make the minimum convex while still being a minorant of Φ_B^- .

Proposition 4.2 *Let $\Phi : \mathbb{R}^m \rightarrow [0, \infty]$ be a strong Φ -function. Then $M_s := \min\{\Phi, N_s\}$ is almost convex.*

Proof Note that $M_s = \Phi\chi_{K_s} + N_s\chi_{\mathbb{R}^m \setminus K_s}$ and let $\alpha, \alpha' > 0$ with $\alpha + \alpha' = 1$. If $\xi, \xi' \notin K_s$, then the convexity of N_s implies that

$$M_s(\beta(\alpha\xi + \alpha'\xi')) \leq N_s(\beta(\alpha\xi + \alpha'\xi')) \leq \alpha N_s(\xi) + \alpha' N_s(\xi') = \alpha M_s(\xi) + \alpha' M_s(\xi').$$

If $\xi, \xi' \in K_s$, then the inequality follows from the convexity of Φ , which holds by assumption. Therefore it suffices to show that

$$M_s(\beta(\alpha\xi + \alpha'\xi')) \leq \alpha\Phi(\xi) + \alpha'N_s(\xi')$$

when $\xi \in K_s$ and $\xi' \notin K_s$. Define $\tilde{\xi} := \alpha\xi + \alpha'\xi'$ and $\tilde{\zeta} := \frac{1}{2}\tilde{\xi}$. We will show that

$$M_s(\tilde{\zeta}) \leq C(\alpha\Phi(\xi) + \alpha'N_s(\xi')). \tag{4.3}$$

Observe that M_s satisfies **(Inc)₁**, since N_s and Φ do. By **(Inc)₁**, (4.3) implies the previous inequality with constant $\beta := \frac{1}{2C}$ and concludes the proof. We consider two cases to prove (4.3).

Case 1 $\tilde{\zeta} \in K_s$. Then $M_s(\tilde{\zeta}) \leq s \leq N_s(\xi')$ and so (4.3) holds with $C = 2$ when $\alpha' > \frac{1}{2}$. Thus we may assume that $\alpha \geq \frac{1}{2}$. Now if $M_s(\tilde{\zeta}) \leq 2\Phi(\xi)$, then (4.3) holds with $C = 4$. Hence we further assume that $M_s(\tilde{\zeta}) > 2\Phi(\xi)$.

We may assume that ξ, ξ' and 0 are not collinear since in the collinear case we can choose $\xi'_k \rightarrow \xi'$ such that ξ, ξ'_k and 0 are not collinear and use the

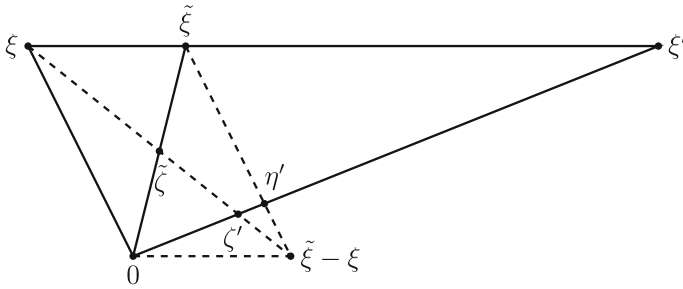


Fig. 1 Construction of auxiliary points

continuity of M_s and N_s . Let ζ' be the intersection of the segment $[0, \xi']$ and the line through ξ and $\tilde{\zeta}$ (see Fig. 1). If $\zeta' \in K_s$, then $\tilde{\zeta} = \theta\xi + (1 - \theta)\zeta'$ for some $\theta \in (0, 1)$. By the convexity of Φ and $M_s(\tilde{\zeta}) > 2\Phi(\xi)$ we have

$$M_s(\tilde{\zeta}) = \Phi(\tilde{\zeta}) \leq \theta\Phi(\xi) + (1 - \theta)\Phi(\zeta') \leq \frac{1}{2}M_s(\tilde{\zeta}) + M_s(\zeta').$$

Thus $M_s(\zeta') = \Phi(\zeta') \geq \frac{1}{2}M_s(\tilde{\zeta})$. If, on the other hand, $\zeta' \notin K_s$, then $M_s(\zeta') \geq s \geq M_s(\tilde{\zeta})$. In either case, we have $M_s(\zeta') \geq \frac{1}{2}M_s(\tilde{\zeta})$.

Consider the parallelogram $(0, \xi, \tilde{\xi}, \tilde{\xi} - \xi)$. Let η' be the intersection of the segments $[\tilde{\xi}, \tilde{\xi} - \xi]$ and $[0, \xi']$ (see Fig. 1). From $\tilde{\xi} = (1 - \alpha')\xi + \alpha'\xi'$ we observe that

$$\alpha' = \frac{|\tilde{\xi} - \xi|}{|\xi - \xi'|} = \frac{|\eta'|}{|\xi'|} \geq \frac{|\zeta'|}{|\xi'|} =: \nu \in (0, 1);$$

the second equality follows since the triangles $(\xi', \tilde{\xi}, \eta')$ and $(\xi', \xi, 0)$ are similar. Thus, by $M_s(\zeta') \geq \frac{1}{2}M_s(\tilde{\zeta})$ from the previous paragraph and (Inc)₁ of M_s ,

$$N_s(\xi') = M_s(\xi') \geq \frac{1}{\nu}M_s(\nu\xi') = \frac{1}{\nu}M_s(\zeta') \geq \frac{1}{2\nu}M_s(\tilde{\zeta}) \geq \frac{1}{2\alpha'}M_s(\tilde{\zeta}),$$

where we used the conclusion of the previous paragraph in the penultimate step. The inequality

$$M_s(\tilde{\zeta}) \leq 2\alpha'N_s(\xi')$$

follows, so (4.3) holds with $C = 2$.

Case 2 $\tilde{\zeta} \notin K_s$. Let $\nu := \|\tilde{\zeta}\|_{K_s}^{-1} < 1$. Since K_s is closed, it follows from the definition of $\|\cdot\|_{K_s}$ that $\Phi(\nu\tilde{\zeta}) = s$ and $\nu\tilde{\zeta} \in \partial K_s$. Furthermore, $N_s(\nu\tilde{\zeta}) = s = M_s(\nu\tilde{\zeta})$ and so

$$M_s(\tilde{\zeta}) = s\|\tilde{\zeta}\|_{K_s} = \frac{1}{\nu}s = \frac{1}{\nu}M_s(\nu\tilde{\zeta}) \leq \frac{4}{\nu}(\alpha\Phi(\nu\xi) + \alpha'N_s(\nu\xi')) \leq 4(\alpha\Phi(\xi) + \alpha'N_s(\xi')),$$

where we used the previous case for $v\tilde{\xi} \in K_s$ in the first inequality and **(Inc)₁** for the last step. □

We are ready to prove the main theorem, i.e. the equivalence of (A1) and (M).

Proof of Theorem 1.2 Since $(\Phi_B^-)^{\text{conv}} \leq \Phi_B^-$, it follows from (M) that

$$\Phi_B^+(\beta\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi) + 1 \leq \Phi_B^-(\xi) + 1,$$

when $\xi \in \mathbb{R}^m$ with $\Phi_B^-(\xi) \leq \frac{K}{\mu(B)}$, where $B \subset \mathbb{R}^n$ is a ball, which gives (A1).

Assume now conversely that (A1) holds and let $s := \frac{K}{\mu(B)} + 1$ for a ball $B \subset \mathbb{R}^n$ with $\mu(B) \leq 1$. Define N_s as before based on $K_s := \{\xi \in \mathbb{R}^m \mid \Phi_B^+(\beta\xi) \leq s\}$ and set $M_s(\xi) := \min\{\Phi_B^+(\beta\xi), N_s(\xi)\}$. By Proposition 4.2, M_s is almost convex so $M_s(\beta'\xi) \leq (M_s)^{\text{conv}}(\xi)$ by Corollary 2.4.

If $\xi \in K_s$, then $M_s(\xi) = \Phi_B^+(\beta\xi) \leq s$. Now either $\Phi_B^-(\xi) \leq \frac{K}{\mu(B)}$ in which case (A1) implies that $\Phi_B^+(\beta\xi) \leq \Phi_B^-(\xi) + 1$, or $\Phi_B^-(\xi) > \frac{K}{\mu(B)}$ in which case $\Phi_B^+(\beta\xi) \leq s \leq \Phi_B^-(\xi) + 1$. Combining the two cases, we find that

$$\Phi_B^+(\beta\xi) \leq \Phi_B^-(\xi) + 1 \quad \text{for all } \xi \in K_s.$$

If $\xi \notin K_s$, then $v := \|\xi\|_{K_s}^{-1} < 1$. As in Case 2 of the previous proof $v\xi \in \partial K_s$ and $\Phi_B^+(\beta v\xi) = s$. If $\Phi_B^-(v\xi) < \frac{K}{\mu(B)}$, then (A1) implies that $\Phi_B^+(\beta v\xi) \leq \Phi_B^-(v\xi) + 1 < s$, which is a contradiction. Therefore $\Phi_B^-(v\xi) \geq \frac{K}{\mu(B)} = s - 1$ and so

$$M_s(\xi) = N_s(\xi) = \frac{1}{v}s \leq \frac{1}{v} \frac{s}{s-1} \Phi_B^-(v\xi) \leq \frac{s}{s-1} \Phi_B^-(\xi),$$

where we used **(Inc)₁** of Φ_B^- in the last step. Note that $\frac{s}{s-1} = 1 + \frac{\mu(B)}{K} \leq 1 + \frac{1}{K}$ since we assumed that $\mu(B) \leq 1$.

In the previous paragraph we have shown that $M_s \leq (1 + \frac{1}{K}) \Phi_B^- + 1$. Therefore, the convex minorant of M_s is also a convex minorant of $(1 + \frac{1}{K}) \Phi_B^- + 1$, and we conclude that $(M_s)^{\text{conv}} \leq (1 + \frac{1}{K}) (\Phi_B^-)^{\text{conv}} + 1$ since $(\Phi_B^-)^{\text{conv}}$ is the greatest convex minorant of Φ_B^- . We noted above that $M_s(\beta'\xi) \leq (M_s)^{\text{conv}}(\xi)$. Therefore,

$$M_s(\beta'\xi) \leq (1 + \frac{1}{K}) (\Phi_B^-)^{\text{conv}}(\xi) + 1 \quad \text{for all } \xi \in \mathbb{R}^m.$$

Let us show that (M) holds. Assume that $(\Phi_B^-)^{\text{conv}}(\xi) \leq \frac{K}{\mu(B)}$. By **(Inc)₁** and the conclusion of the previous paragraph,

$$M_s\left(\frac{K}{K+1}\beta'\xi\right) \leq \frac{K}{K+1}M_s(\beta'\xi) \leq (\Phi_B^-)^{\text{conv}}(\xi) + 1 \leq s.$$

Therefore $\frac{K}{K+1}\beta'\xi \in K_s$ and $M_s\left(\frac{K}{K+1}\beta'\xi\right) = \Phi_B^+\left(\frac{K}{K+1}\beta'\xi\right)$. Thus

$$\Phi_B^+\left(\frac{K}{K+1}\beta'\xi\right) \leq (\Phi_B^-)^{\text{conv}}(\xi) + 1$$

and we have established (M) with constant $\frac{K}{K+1} \beta \beta'$. □

Remark 4.4 From the proof of Proposition 4.2, we see that the almost convexity constant equals 8. From the proof of Corollary 2.4, we see that $\beta' = 8^{-i}$, where i is the smallest integer with $2^i \geq m + 1$. Thus $2^{-i} > \frac{1}{2^{(m+1)}}$ so that $\beta' > (2(m + 1))^{-3}$. Then we see from the proof or the previous theorem that the constant in (M) can be chosen as $(2(m + 1))^{-3} \frac{K}{K+1} \beta$ when β is the constant from (A1). In other words, the constants from the two conditions are comparable up to a constant depending on the dimension when $K \geq 1$.

The assumption $\Phi_B^-(\xi) \leq \frac{K}{\mu(B)}$ from (A1) is somewhat difficult to verify. In the isotropic case, if we assume that $\varrho_\varphi(f) \leq 1$, then we can conclude from Jensen’s inequality that

$$\varphi_B^-\left(\beta \int_B f \, d\mu\right) \leq \int_B \varphi(x, |f|) \, d\mu \leq \frac{1}{\mu(B)}. \tag{4.5}$$

Thus we may apply (A1) to conclude that

$$\varphi_B^+\left(\beta^2 \int_B f \, d\mu\right) \leq \varphi_B^-\left(\beta \int_B f \, d\mu\right) + 1.$$

This argument is not possible in the anisotropic case, since $(\Phi_B^-)^{\text{conv}}$ is not comparable to Φ_B^- . Fortunately, the condition of (M) is easier to use.

Proof of Corollary 1.3 By Theorem 1.2, Φ satisfies (M). Since $(\Phi_B^-)^{\text{conv}}$ is convex, it follows by Jensen’s inequality that

$$(\Phi_B^-)^{\text{conv}}\left(\int_B f \, d\mu\right) \leq \int_B (\Phi_B^-)^{\text{conv}}(f) \, d\mu \leq \int_B \Phi(x, f) \, d\mu \leq \frac{1}{\mu(B)}.$$

Therefore we can use (M) with $\xi = \int_B f \, d\mu$ and the previous inequality to conclude that

$$\Phi_B^+\left(\beta \int_B f \, d\mu\right) \leq (\Phi_B^-)^{\text{conv}}\left(\int_B f \, d\mu\right) + 1 \leq \int_B \Phi(x, f) \, d\mu + 1.$$

□

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