



# Triangulating Surfaces with Bounded Energy

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Received: 2 August 2021 / Accepted: 28 June 2022 / Published online: 1 August 2022  
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## Abstract

We show that if a closed Lipschitz surface in  $\mathbb{R}^n$  has bounded Kolasinski–Menger energy, then it can be triangulated with triangles whose number is bounded by the energy and the area. Each of the triangles is an image of a subset of a plane under a diffeomorphism whose distortion is bounded by  $\sqrt{2}$ .

**Keywords** Menger curvature · Surface energy · Triangulation · Knot energy · Genus

**Mathematics Subject Classification** 53A05 · 28A75 · 49Q10 · 49Q20 · 53C21

## 1 Introduction

It is a general principle in the theory of energies of manifolds that small energy implies uncomplicated topology. Probably, the first instance of this principle is the Fáry–Milnor theorem [2, 8], stating that a knot in  $\mathbb{R}^3$  whose total curvature is less than  $4\pi$  is necessarily trivial.

For energies of curves in  $\mathbb{R}^3$ , there are bounds for the stick number and the average crossing number of a knot, see for example [9] and references therein.

For higher dimensional submanifolds some analogs exist, but are not abundant. The Fáry–Milnor theorem can be generalized to the case of surfaces [4]. In [7], delicate arguments involving compactness and stability (manifolds with bounded energy that are sufficiently close with respect to the Hausdorff distance are ambiently isotopic), are used to show that there are finitely many isotropy classes of submanifolds below some fixed energy level.

Motivated by [4], we give another bound on the complexity of a surface in  $\mathbb{R}^n$  in terms of its energy. Namely, we bound from above the minimal number of triangles in

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a triangulation of the surface. In particular, we bound the genus of a surface in terms of its energy. Actually, our result goes further. For a surface with given energy we construct a triangulation in such a way that each triangle is a graph of a function with bounded derivative and distortion. In this sense, the triangles in the triangulation are “almost flat”.

Noting that the energy  $\mathcal{E}_p^\ell$  is introduced in Definition 2.1, we present now the main result of this paper,

**Theorem 1.1** *Suppose  $\Sigma \subset \mathbb{R}^n$  is a closed, Lipschitz surface. Let  $\ell \in \{1, \dots, 4\}$  and  $p > 2\ell$ . Suppose  $\Sigma$  has energy  $\mathcal{E}_p^\ell(\Sigma) = E < \infty$  and area  $A$ . Then  $\Sigma$  can be triangulated with  $C_2 A E^{2/(p-2\ell)}$  triangles, where  $C_2$  is a universal constant depending only on  $p, \ell$  and  $n$ .*

*Each of the triangles in the triangulation is an image of an open subset of a plane under a function whose derivative has norm bounded by  $\sqrt{2}$  and whose distortion is bounded by  $\sqrt{2}$ .*

Combining Theorem 1.1 with [5, Theorem 1.1] stating that the minimal number  $T(g)$  of triangles in a triangulation of a closed surface of genus  $g$  grows linearly with  $g$ , we obtain the following result.

**Corollary 1.2** *There is a constant  $C_g$  (depending on  $\ell, n, p$ ) such that if  $\Sigma \subset \mathbb{R}^n$  is a closed Lipschitz surface with  $\mathcal{E}_p^\ell(\Sigma) = E < \infty$  and area  $A$ , where  $p > 2\ell$ , then  $g(\Sigma) \leq C_g A E^{2/(p-2\ell)}$ .*

The proof of Theorem 1.1 goes along the following lines. The key tool is the Regularity Theorem of [7], recalled here as Theorem 2.3, which states that an  $m$ -dimensional submanifold with bounded energy can be covered by so-called graph patches, that is, subsets that are graphs of functions from subsets of  $\mathbb{R}^m$ , and the derivatives of the functions are bounded. The size of graph patches is controlled by the energy, that is, the graph patches are not too small. An immediate corollary of Theorem 2.3 is an Ahlfors like inequality, Proposition 2.7, controlling from both sides the volume of the part of a submanifold cut out by a ball whose center is on the submanifold.

Next, assuming  $\Sigma$  is a surface of bounded energy and area, we find a cover of  $\Sigma$  by balls of some radius  $r$  (depending on the energy), such that each ball is a graph patch. We let the centers of the balls be  $x_1, \dots, x_N$ . A simple topological argument in Sect. 3.1 allows us to control the number  $N$  of the centers in terms of the energy and area of the surface.

To construct the triangulation, we first connect all pairs  $x_i, x_j$ , such that  $\text{dist}(x_i, x_j) < 4r$ , by an arc  $\gamma_{ij}$  on  $\Sigma$ . The requirement on  $\gamma_{ij}$ , spelled out in Definition 3.5, is that the length  $\ell(\gamma_{ij})$  be bounded by a constant times  $\text{dist}(x_i, x_j)$ . Unlike geodesics, two such curves can intersect at more than a single point. By a procedure called *bigon removal* we improve the collection of curves  $\gamma_{ij}$  to obtain a concrete bound on the number of intersection points between them; see Proposition 3.17.

We let  $\Sigma_0$  be the complement of  $\bigcup \gamma_{ij}$  in  $\Sigma$ . The triangulation is constructed by cutting connected components of  $\Sigma_0$  into triangles. A second technicality, and chronologically the first we deal with in this article, appears. We need to ensure that

each connected component of  $\Sigma_0$  is homeomorphic to a planar open set. We address this problem in Proposition 3.8. Given that result, we consider each component  $C$  of  $\Sigma_0$ . As it is planar, we cut  $C$  into triangles without adding new vertices. The number of triangles in the triangulation is estimated using the number of intersection points between curves  $\gamma_{ij}$ ; see Corollary 4.6. The proof of Theorem 1.1 is carried out in Sect. 4.2.

## 2 Review of Surface Energies

In this section, we recall definitions of surface energy. References include [6, 7, 10].

### 2.1 Discrete Menger Energy for a Submanifold in $\mathbb{R}^n$

For points  $(x_0, \dots, x_m)$  in  $\mathbb{R}^n$ , we let  $\Delta(x_0, \dots, x_m)$  denote the  $m$ -dimensional simplex spanned by  $x_0, \dots, x_m$  (the convex hull of these points). We define

$$\mathcal{K}(x_0, \dots, x_m) = \frac{1}{d^m} \text{vol}_m \Delta(x_0, \dots, x_m),$$

where  $d$  is the diameter of  $\Delta(x_0, \dots, x_m)$ .

Suppose now  $\Sigma \subset \mathbb{R}^n$  is a Lipschitz submanifold of dimension  $m < n$ . Let  $\ell \in \{1, \dots, m + 2\}$  and  $p > 0$ .

**Definition 2.1** The *Kolasinski-Menger energy* of  $\Sigma$  is the integral

$$\mathcal{E}_p^\ell(\Sigma) = \int_{\Sigma^\ell} \sup_{x_\ell, \dots, x_{m+2} \in \Sigma} \mathcal{K}(x_0, \dots, x_{m+2})^p d\text{vol}(\Sigma^\ell). \tag{2.1}$$

The integral is computed with respect to the variables  $x_0, \dots, x_{\ell-1}$ .

### 2.2 Graph Patches

For  $k \geq 0$  and  $\alpha \in (0, 1]$ , we let  $C_{m,n}^{k,\alpha}$  denote the set of all compact manifolds of dimension  $m$ , embedded  $C^{k,\alpha}$ -smoothly in  $\mathbb{R}^n$ . The following definition is taken from [7, Definition 1]

**Definition 2.2** (Graph patches) Suppose  $R, L, d$  are real positive and  $\alpha \in (0, 1]$ . The class  $C_{m,n}^{1,\alpha}(R, L, d)$  is the class of all  $m$ -dimensional  $C^{1,\alpha}$ -smooth submanifolds  $\Sigma \subset \mathbb{R}^n$  such that:

- (P-1)  $\Sigma \subset B(0, d)$  ;
- (P-2) for each  $x \in \Sigma$ , there exists a function  $f_x : T_x \Sigma \rightarrow T_x \Sigma^\perp$  of class  $C^{1,\alpha}$  with  $f_x(0) = 0, Df_x(0) = 0$  and

$$\Sigma \cap B(x, R) = (x + \text{graph}(f_x)) \cap B(x, R). \tag{2.2}$$

(P-3) the function  $f_x$  is Lipschitz with Lipschitz constant 1 and  $\|Df_x(\xi) - Df_x(\eta)\| \leq L\|\xi - \eta\|^\alpha$ .

We quote now the following result

**Theorem 2.3** ([7, Regularity Theorem]) *For  $p > m\ell$ , there exist constants  $c_1(m, n, \ell, p)$  and  $c_2(m, n, \ell, p)$  such that with  $\alpha = 1 - m\ell/p$ , any Lipschitz manifold  $\Sigma \in \mathcal{C}_{m,n}^{0,1}$  with energy  $\mathcal{E}_p^\ell(\Sigma) = E < \infty$  satisfies*

$$\Sigma \in \mathcal{C}_{m,n}^{1,\alpha}(c_1 E^{-1/(p-m\ell)}, c_2 E^{1/p}, d)$$

as long as  $\Sigma \subset B(0, d)$ .

We now introduce some notation regarding graph patches. Let  $x \in \Sigma$  and  $r < c_1 E^{-1/(p-m\ell)}$ . We let  $H_x \subset \mathbb{R}^n$  be the  $m$ -dimensional affine plane in  $\mathbb{R}^n$ , which is tangent to  $\Sigma$  at  $x$ . The map  $f_x$  induces a map  $\phi_x: H_x \rightarrow \Sigma$  given by  $y \mapsto (y, x + f_x(y))$ . The inverse map  $\pi_x: \Sigma \rightarrow H_x$  is the orthogonal projection. We will consider both maps  $\phi_x$  and  $\pi_x$  as defined in open neighborhoods of  $x$  in  $H_x$  and  $\Sigma$ , respectively. This is made precise in Corollary 2.5.

**Notation 2.4** *We use the notation  $B^H$  for a ball contained in  $H$ , the notation like  $B(y, \rho)$  means an open ball in the ambient space  $\mathbb{R}^n$ .*

*The distance of two points  $x, y$  in the ambient space  $\mathbb{R}^n$  is denoted by  $\text{dist}(x, y)$ , while the distance of two points in the tangent space  $H$  is denoted by  $\|x - y\|_H$ .*

We will use the following corollary of Theorem 2.3.

**Corollary 2.5** *Let  $c_1, c_2$  be the constants as in Theorem 2.3. Set*

$$R_0 = 2^{-1/2} \min(c_1, c_2^{-\alpha}) E^{-1/(p-m\ell)}. \tag{2.3}$$

If  $r < R_0$ , then:

- (C-1)  $\phi_x$  is well-defined on  $H_x \cap B^{H_x}(x, r)$ ;
- (C-2)  $\pi_x$  is well-defined on  $\Sigma \cap B(x, r\sqrt{2})$ ;
- (C-3) for any  $s < r\sqrt{2}$ , the image  $B^{H_x}(x, s) \subset \pi_x(\Sigma \cap B(x, s\sqrt{2})) \subset B^{H_x}(x, s\sqrt{2})$ .
- (C-4)  $\|D\phi_x(y)\| \leq \sqrt{2}$  for all  $y \in H_x \cap B^{H_x}(x, r)$ ;
- (C-5)  $\phi_x$  is Lipschitz with Lipschitz constant  $\sqrt{2}$  and  $\pi_x$  is Lipschitz with Lipschitz constant 1.

**Proof** We begin with (C-4). By (P-2), we have  $Df_x(0) = 0$ . Condition (P-3) implies then that  $\|Df_x(y)\| \leq L\|y\|^\alpha$ . By the definition,  $\|D\phi_x\|^2 = 1 + \|Df_x\|^2$ . Therefore, item (C-4) holds, as long as  $\|y\| \leq L^{-\alpha} = (c_2 E^{1/p})^{-\alpha}$ .

The map  $\pi_x$  is a projection, so it has Lipschitz constant 1. By (C-4),  $\phi_x$  has Lipschitz constant  $\sqrt{2}$ . This proves (C-5).

Item (C-2) follows from (P-2). In fact,  $\Sigma \cap B(x, r\sqrt{2})$  is in the image of  $\phi_x$  by (2.2), thus  $\pi_x$  is defined on  $\Sigma \cap B(x, r\sqrt{2})$  as the inverse of  $\phi_x$ .

To prove (C-1), we note that the whole of  $\overline{B}(x, r\sqrt{2}) \cap \Sigma$  is covered by a graph patch by (P-2). Here, we consider the closed ball, which is legitimate since  $r < R_0$

(and not only  $r \leq R_0$ ). Let  $U \subset H_x$  be the preimage  $\phi_x^{-1}(B(x, r\sqrt{2}) \cap \Sigma)$ . The closure of  $U$  is the preimage  $\phi_x^{-1}(\overline{B}(x, r\sqrt{2}) \cap \Sigma)$ .

Take  $u \in \partial U$ . We know that  $\phi_x(u)$  belongs to the boundary of  $\overline{B}(x, r\sqrt{2}) \cap \Sigma$ . As  $\Sigma$  is closed, we deduce that the boundary of  $\overline{B}(x, r\sqrt{2}) \cap \Sigma$  belongs to the boundary of  $\overline{B}(x, r\sqrt{2})$ . That is to say,  $\text{dist}(\phi_x(u), x) = r\sqrt{2}$ . As  $\phi_x(x) = x$  and  $\phi_x$  has Lipschitz constant  $\sqrt{2}$ , we infer that  $\|u - x\| \geq r$ .

Therefore, the boundary of  $U$  lies outside the (open) ball  $B^{H_x}(x, r)$ , and  $x \in U$ . Therefore,  $B^{H_x}(x, r) \subset U$ , proving (C-1).

Finally, item (C-3) is a direct consequence of the Lipschitz property (C-5). □

**Corollary 2.6** *The distortion of  $\phi_x$  is bounded by  $\sqrt{2}$ .*

**Proof** The distortion of  $\phi_x$  at the point  $z$  is given by

$$D(z) = \limsup_{r \rightarrow 0} \frac{\max_{y: \|z-y\|_{H_x}=r} \text{dist}(\phi_x(y), \phi_x(z))}{\min_{y: \|z-y\|_{H_x}=r} \text{dist}(\phi_x(y), \phi_x(z))}.$$

The numerator in the formula is bounded from above by  $r$  times the Lipschitz constant of  $\phi_x$ . The denominator is bounded from below by  $r$  times the Lipschitz constant of  $\pi_x$ . □

### 2.3 Local Volume Bound

Throughout Sect. 2.3, we let  $\Sigma$  be a submanifold of  $\mathbb{R}^n$  in the class  $\mathcal{C}_{m,n}^{1,\alpha}(R_0, L, d)$ .

**Proposition 2.7** (Local volume bound) *Suppose  $r < R_0$ . Then for any  $x \in \Sigma$  we have*

$$2^{-n/2} V_m r^m < \text{vol}(\Sigma \cap B(x, r)) < 2^{n/2} V_m r^m,$$

where  $V_m$  is the volume of the unit ball in dimension  $m$ .

**Proof** Let  $U = \pi_x(\Sigma \cap B(x, r))$ . By (C-3):

$$B^H(x, r/\sqrt{2}) \subset U \subset B^H(x, r). \tag{2.4}$$

As  $\Sigma \cap B(x, r)$  is the image of  $\phi_x$ , a classical result from multivariable calculus computes the volume of  $\Sigma \cap B(x, r)$  in terms of the integral over  $U$  over the square root of the Gram determinant:

$$\text{vol}(\Sigma \cap B(x, r)) = \int_U \sqrt{|\det G|},$$

where  $G = D\phi_x D\phi_x^T$ .

The derivative of  $\phi_x$  has a block structure  $D\phi_x = (I \ Df_x)$ , so  $G = I + Df_x Df_x^T$ . On the one hand, since  $Df_x Df_x^T$  is non-negative definite,  $\det G \geq 1$ . On the other hand,  $\|Df_x\| \leq 1$ , so  $\|Df_x Df_x^T\| \leq 1$ . Therefore,  $\|G\| \leq 2$ . This means that all

the eigenvalues of the symmetric matrix  $G$  have modulus less than or equal to 2, so  $|\det G| \leq 2^n$ . In particular,

$$\int_U 1 \leq \text{vol}(\Sigma \cap B(x, r)) \leq \int_U 2^{n/2}.$$

Combining this with (2.4) we quickly conclude the proof. □

### 3 Geodesic-Like Systems of Curves

#### 3.1 Nets of Points

We recall a classical notion of general topology. We will need the following technical definition, which makes sense under very mild assumptions on  $\Sigma$ .

**Definition 3.1** Assume  $\Sigma$  is a metric space. Let  $r > 0$ . A finite set  $\mathcal{X}$  of points in  $\Sigma$  is an  $r$ -net if

- the balls  $B(x, r)$  for  $x \in \mathcal{X}$  cover  $\Sigma$ ;
- for any  $x, x' \in \mathcal{X}, x \neq x'$ , we have  $\text{dist}(x, x') \geq r/2$ .

The following result is classical in general topology.

**Proposition 3.2** *If  $\Sigma$  is a compact metric space, then it admits an  $r$ -net for any  $r > 0$ .*

**Proof** For the reader’s convenience we provide a quick proof. Cover first  $\Sigma$  by all balls  $B(x, r/2)$  with  $x \in \Sigma$ . Choose a finite subcover, and let  $\mathcal{Y} = \{y_1, \dots, y_M\}$  be the set of centers of balls in this subcover.

We act inductively. Start with  $y_1$  and define  $x_1 = y_1$ . Remove from  $\mathcal{Y}$  all points  $y_i \neq y_1$  such that  $\text{dist}(y_1, y_j) < r/2$ . After this procedure, the balls  $B(x_1, r)$  and  $B(y_i, r/2)$  for  $i > 1$  still cover  $\Sigma$ .

For the inductive step assume that, for given  $n$ , the balls  $B(x_1, r), \dots, B(x_{n-1}, r)$  and  $B(y_n, r/2), B(y_{n+1}, r/2), \dots, B(y_M, r/2)$  cover  $\Sigma$ , and there are no indices  $i, j$  with  $i < n$  and  $j \geq n$  such that  $\text{dist}(x_i, y_j) < r/2$ . We remove from  $\mathcal{Y}$  points  $y_j$  with  $j > n$  such that  $\text{dist}(x_n, y_j) < r/2$  and set  $x_n = y_n$ .

After a finite number of steps we are left with the set  $\mathcal{X} \subset \mathcal{Y}$ , such that  $B(x_i, r), x_i \in \mathcal{X}$  cover  $\Sigma$  and  $\text{dist}(x_i, x_j) \geq r/2$  for all  $x_i, x_j \in \mathcal{X}$ . □

We now return to our previous assumptions on  $\Sigma$ .

**Proposition 3.3** (*Bounding  $|\mathcal{X}|$* ) *Suppose  $\Sigma$  is a closed Lipschitz  $m$ -dimensional submanifold of  $\mathbb{R}^n$  with  $\mathcal{E}_p^\ell(\Sigma) = E < \infty$ . Assume that  $R_0$  satisfies (2.3). Let  $A = \text{vol}_m(\Sigma)$ . If  $r < R_0$ , then any  $r$ -net  $\mathcal{X}$  has  $|\mathcal{X}| < 2^{n/2+2m} V_m^{-1} r^{-m} A$ .*

**Proof** By the local volume bound (Proposition 2.7) the sets  $B(x_i, r/4) \cap \Sigma, x_i \in \mathcal{X}$  have volume at least  $2^{-n/2} V_m \left(\frac{r}{4}\right)^m$ . As  $\mathcal{X}$  is an  $r$ -net, these balls are pairwise disjoint. The total volume of  $\bigcup_{x_i \in \mathcal{X}} B(x_i, r/4) \cap \Sigma$  is at least  $2^{-n/2-2m} V_m r^m |\mathcal{X}|$ . This quantity does not exceed the volume of  $\Sigma$ . □

Essentially the same argument yields the following result.

**Proposition 3.4** *Let  $\Sigma$  be as in Proposition 3.3. Suppose  $\sigma > 0, r > 0$  are such that  $(\sigma + 1/4)r < R_0$ . Let  $\mathcal{X}$  be an  $r$ -net. Each ball  $B(x, \sigma r)$  for  $x \in \Sigma$  contains at most  $\mathcal{T}_m(\sigma)$  points from  $\mathcal{X}$ , where*

$$\mathcal{T}_m(\sigma) = 2^{n+2m}(\sigma + 1/4)^m.$$

**Proof** The ball  $B(x, (\sigma + 1/4)r) \cap \Sigma$  has volume at most  $2^{n/2}V_m(\sigma + 1/4)^m r^m$  by Proposition 2.7. All balls of radius  $r/4$  with centers at  $x_i \in \mathcal{X}$  such that  $\text{dist}(x, x_i) < \sigma r$  are pairwise disjoint, belong to  $B(x, (\sigma + 1/4)r)$ , and each of them has volume at least  $2^{-n/2-2m}V_m r^m$ . Hence, the number of points in  $\mathcal{X}$  at distance at most  $\sigma r$  to  $x$  is bounded by  $2^{n+2m}(\sigma + 1/4)^m$ .  $\square$

From now on, we set

$$\mathcal{T}(\sigma) := \mathcal{T}_2(\sigma) = 2^{n+4}(\sigma + 1/4)^2.$$

### 3.2 Good Arcs

Throughout Sect. 3.2 we let  $\Sigma$  be a Lipschitz surface with area  $A$  and energy  $\mathcal{E}_p^l(\Sigma) = E$ . We note that by Regularity Theorem 2.3,  $\Sigma$  is  $C^1$ -smooth. We will be using smoothness of  $\Sigma$  for transversality arguments. Recall (see Notation 2.4), that  $\text{dist}(x, y)$  is the distance of two points in  $\mathbb{R}^n$ .

Finally, let  $r > 0$ .

**Definition 3.5** Let  $\mathcal{X} = \{x_1, \dots, x_N\}$  be an  $r$ -net. Let  $\mathcal{I}$  be a subset of pairs  $(i, j)$  with  $1 \leq i \neq j \leq N$ . A collection  $\mathcal{G}$  of arcs  $\gamma_{ij}, (i, j) \in \mathcal{I}$   $C^1$ -smoothly embedded in  $\Sigma$  and connecting  $x_i$  with  $x_j$  is called a collection of *good arcs associated to  $\mathcal{X}$*  if:

- (G-1)  $\mathcal{I} = \{(i, j) : i \neq j, \text{dist}(x_i, x_j) < 4r\}$ ;
- (G-2) for all  $(i, j) \in \mathcal{I}$ ,  $\gamma_{ij}$  has length less than  $2 \text{dist}(x_i, x_j)$ ;
- (G-3)  $\gamma_{ij} = \gamma_{ji}$ ;
- (G-4)  $\gamma_{ij}$  is regularly embedded, in particular, has no self-intersections;
- (G-5) if  $\gamma_{ij} \neq \gamma_{k\ell}$ , then  $\gamma_{ij}$  intersects  $\gamma_{k\ell}$  transversally, in particular, there are no triple intersections;
- (G-6)  $\gamma_{ij}$  has a well-defined tangent vector at  $x_i$  and  $x_j$ . At each point  $x_i \in \mathcal{X}$ , the tangent directions to all curves  $\gamma_{ij}$  are pairwise distinct.

A collection of good arcs is called *tame*, if it additionally satisfies the following two conditions.

- (T-1) Every connected component of  $\Sigma \setminus \Gamma$  with  $\Gamma = \bigcup \gamma_{ij}$  is homeomorphic to an open set of  $\mathbb{R}^2$ ;
- (T-2) Unless  $\gamma_{ij} = \gamma_{k\ell}$ , the curves  $\gamma_{ij}$  and  $\gamma_{k\ell}$  intersect transversally at most  $\mathcal{T}(17)$  points.

One should think of a collection of good arcs as an analog of a collection of geodesics. Condition (T-2) is automatically satisfied if  $\gamma_{ij}$  are geodesics whose length is less than

the geodesic radius. However, we cannot use geodesics because this would require extra regularity assumptions on  $\Sigma$ .

**Proposition 3.6** *Suppose  $4r < R_0$ . For each  $r$ -net  $\mathcal{X}$  there exists an associated collection  $\mathcal{G}$  of good arcs.*

**Proof** We construct all the curves in  $\mathcal{G}$  one by one. Take two points  $x_i, x_j \in \mathcal{X}$  with  $\text{dist}(x_i, x_j) < 4r$  that are not yet connected by a curve. Recall that  $H_{x_i}$  is the affine plane tangent to  $\Sigma$  at  $x_i$ . Let  $y_j = \pi_{x_i}(x_j)$ , where  $\pi_{x_i}$  is the projection onto  $H_{x_i}$ ; compare with Corollary 2.5. As  $4r < R_0$ , by (C-2),  $\pi_{x_i}$  is defined on  $x_j$ . Moreover, by (C-1), the straight segment  $\rho$  on  $H_{x_i}$  connecting  $x_i$  and  $y_j$  belongs to the domain of  $\phi_{x_i}$ .

By (C-2),  $\pi_{x_i}$  has Lipschitz constant 1. Hence,  $\|x_i - y_j\|_{H_{x_i}} \leq \text{dist}(x_i, x_j)$ , so the length of  $\rho$  is less than or equal to  $\text{dist}(x_i, x_j)$ . If necessary, we perturb the segment  $\rho$  (keeping its endpoints fixed) to be transverse to the image under  $\pi_{x_i}$  of all curves that have already been constructed (here we use the fact that  $\Sigma$  is  $C^1$  smooth). That is, by perturbation we achieve that  $\gamma_{ij} := \phi_{x_i}(\rho)$  satisfies (G-4), (G-5) and (G-6)

A perturbation might increase the length of  $\rho$ , but we can choose a perturbation as small (in the  $C^1$  sense) as we please, so the length of  $\rho$  is less than  $\sqrt{2} \text{dist}(x_i, x_j)$ . By (C-4) we infer that the length of  $\gamma_{ij} := \phi_{x_i}(\rho)$  is at most  $\sqrt{2} \text{dist}(x_i, x_j) < 2 \text{dist}(x_i, x_j)$ . □

**Remark 3.7** It is clear from the proof that one can achieve a better bound on the length  $\ell(\gamma_{ij}) < 1.5 \text{dist}(x_i, x_j)$ . This would eventually lead to a smaller constant  $C_2$  in Theorem 1.1. However, choosing an integer factor 2 leads to more readable estimates.

Our goal is to show that there exists a tame collection of good arcs. In Sect. 3.3 we shall deal with Condition (T-1), while in Sect. 3.4 we deal with (T-2).

### 3.3 On the Property (T-1)

**Proposition 3.8** *If  $22r < R_0$ , a collection of good arcs  $\mathcal{G}$  satisfies (T-1).*

**Proof** We use the notation of Corollary 2.5. Let  $x_i \in \mathcal{X}$ . Denote by  $H_i$  the plane tangent to  $\Sigma$  at  $x_i$ . Let  $\Sigma_{x_i} = \Sigma \cap B(x_i, 22r)$ ; see Fig. 1. As  $22r < R_0$ , properties (C-1)–(C-5) are satisfied with  $r$  replaced by  $22r$ . We let  $U_i = \pi_{x_i}(\Sigma_{x_i})$ . By (C-3), we have

$$B^{H_i}(x_i, 15r) \subset B^{H_i}(x_i, 22r/\sqrt{2}) \subset U_i \subset B^{H_i}(x_i, 22r).$$

Consider a regular 38-gon on  $H_i$  with center  $x_i$  and side length  $r$ . Denote by  $u_1, \dots, u_{38}$  its vertices, so that  $\|u_j - u_{j+1}\|_{H_i} = r$ . We will use the convention that the indices are cyclic, that is  $u_{39} = u_1$  (and similarly for points  $y_j, z_j, w_j$ , which we will introduce in a while).

Assume  $H_i$  is oriented in such a way that the vertices are numbered in the counter-clockwise direction. We have  $\|u_j - x_i\|_{H_i} = \frac{r}{2 \sin \frac{\pi}{38}} \sim 6.05r$ . In particular, for any



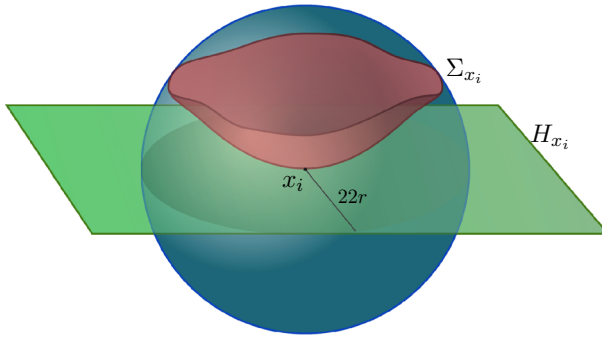


Fig. 1 The set  $H_{x_i}$  and  $\Sigma_{x_i}$  of the proof of Proposition 3.8

$j = 1, \dots, 38$ :

$$6r < \|u_j - x_i\|_{H_i} < 7r. \tag{3.1}$$

Let  $y_j = \phi_{x_i}(u_j) \in \Sigma_{x_i}$ . By (C-5):

$$r \leq \text{dist}(y_j, y_{j+1}) \leq r\sqrt{2}. \tag{3.2}$$

By the definition of  $\mathcal{X}$ , for any  $y_j$  there exists an element  $z_j \in \mathcal{X}$  such that  $\text{dist}(z_j, y_j) < r$ . In particular, by the triangle inequality and (3.2):

$$\text{dist}(z_j, z_{j+1}) < (2 + \sqrt{2})r, \quad \text{dist}(x_i, y_j) < 7r\sqrt{2} < 10r. \tag{3.3}$$

As  $\mathcal{G}$  is a collection of good arcs, and  $\text{dist}(z_j, z_{j+1}) < 4r$ , there exists a curve  $\lambda_j \in \mathcal{G}$  that connects  $z_j$  and  $z_{j+1}$ . The length of  $\lambda_j$  is at most  $2 \text{dist}(z_j, z_{j+1}) < (4 + 2\sqrt{2})r < 7r$ .

Denote  $w_j = \pi_{x_i}(z_j)$ , and let  $\rho_j = \pi_{x_i}(\lambda_j)$ . The next lemma controls the position of  $\lambda_j$  and  $\rho_j$ .

**Lemma 3.9** *We have  $z_j \in B(x_i, 11r)$ . Moreover,  $\lambda_j \subset B(x_i, 15r)$  and  $\rho_j$  belongs to  $B^{H_i}(x_i, 15r)$ .*

**Proof** From (3.3) we read off that  $\text{dist}(x_i, y_j) < 10r$ . As  $\text{dist}(y_j, z_j) < r$ , we conclude that  $\text{dist}(x_i, z_j) < 11r$ .

The curve  $\lambda_j$  has length at most  $7r$ . No point on  $\lambda_j$  can be further from  $x_i$  than  $11r + \frac{7}{2}r < 15r$ . Indeed, if  $x$  is outside  $B(x_i, 15r)$ , then the length of the part of  $\lambda_j$  from  $z_j$  to  $x$  and the length of the part of  $\lambda_j$  from  $x$  to  $z_{j+1}$  are both at least  $4r$ , contributing to the length of  $\lambda_j$  being at least  $8r$ . The contradiction shows that  $\lambda_j \subset B(x_i, 15r)$ . Now  $\pi_{x_i}(B(x_i, 15r)) \subset B^{H_i}(x_i, 15r)$ , so  $\rho_j \subset B^{H_i}(x_i, 15r)$ .  $\square$

As  $\text{dist}(z_j, y_j) < r$ , we have  $\|w_j - u_j\|_{H_i} < r$  by (C-5). By (3.1) and the triangle inequality:

$$5r < \|w_j - x_i\|_{H_i} < 8r \text{ for all } j = 1, \dots, 38. \tag{3.4}$$

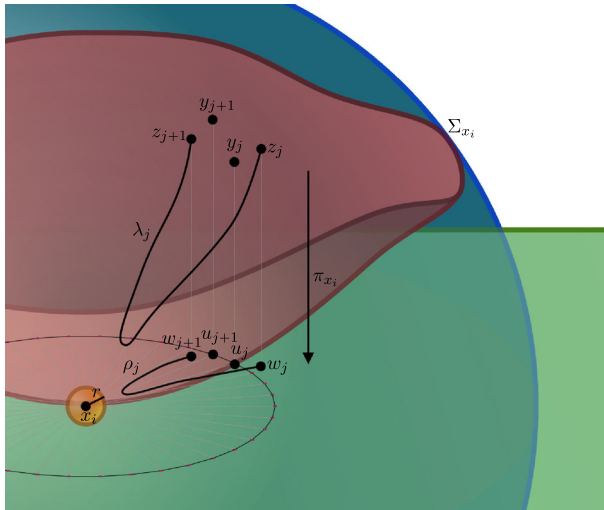


Fig. 2 Notation of Sect. 3.3

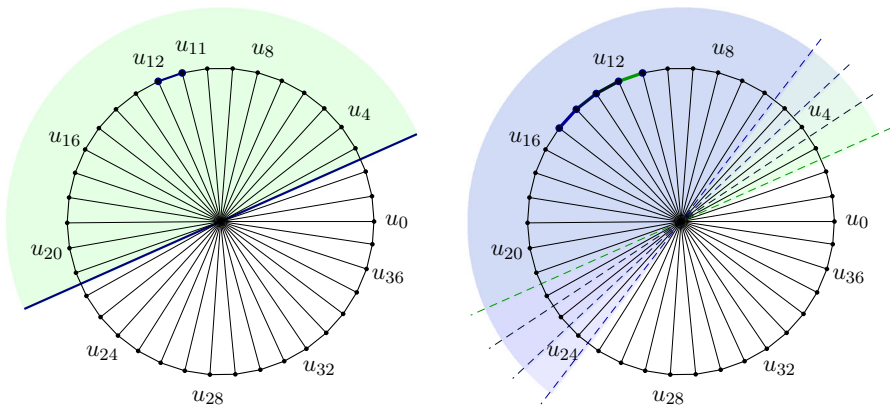


Fig. 3 The set  $H_{ij}^+$  for  $j = 11$  (left) and  $V_{ijs}^+$  for  $j = 11$  and  $s = 4$  (right) of the proof of Lemmas 3.10 and 3.11

The bounds on lengths of  $\lambda_j$  and  $\rho_j$  will be used to show that these curves do not come close to the point  $x_i$ . More precisely, we have the following result.

**Lemma 3.10** *Let  $H_{ij}^+$  be the half-plane cut off from  $H_i$  by the line parallel to the segment joining  $u_j$  and  $u_{j+1}$  and passing through  $x_i$  such that  $u_j, u_{j+1} \in H_{ij}^+$ ; see Fig. 3. The curve  $\rho_j$  misses both  $B^{H_i}(x_i, r)$  and  $H_{ij}^- := H_i \setminus H_{ij}^+$ . In particular,  $\lambda_j$  is disjoint from  $B(x_i, r)$ .*

**Proof of Lemma 3.10** Note that  $\pi_{x_i}$  having Lipschitz constant 1 implies that the length of  $\rho_j$  is at most  $7r$ ; see (C-5). Suppose towards contradiction that  $\rho_j$  passes through a point  $z \in B(x_i, r)$ . We have  $\|w_j - z\|_{H_i} > 4r$  and  $\|w_{j+1} - z\|_{H_i} > 4r$  by (3.4) and

the triangle inequality, hence the length of  $\rho_j$  is at least  $8r$ . The contradiction shows that  $\rho_j$  misses the ball  $B^{H_i}(x_i, r)$ .

Suppose now  $\lambda_j$  hits the ball  $B(x_i, r)$ , that is, there exists a point  $x \in \lambda_j \cap B(x_i, r)$ . Then,  $\pi_{x_i}(x) \in B^{H_i}(x_i, r)$  by (C-5) and so  $\rho_j$  hits  $B^{H_i}(x_i, r)$ , contradicting what we have already proved. This shows that  $\lambda_j$  is disjoint from  $B(x_i, r)$ .

To prove that  $\rho_j$  misses  $H_{ij}^+$  is analogous. The distance of  $u_j$  to the boundary of  $H_{ij}^+$  (computed on  $H_i$ ) is equal to  $r\sqrt{1/(4 \sin^2(\pi/38))} - 1/4$ , and it is greater than  $6r$ . Hence, the distance of  $w_j$  to  $\partial H_{ij}^+$  is at least  $5r$ . That is, if  $\rho_j$  leaves  $H_{ij}^+$ , then its length must be at least  $10r$ . Contradiction.  $\square$

*Continuing the proof of Proposition 3.8* Recall that the orientation was chosen in such a way that the vertices  $u_1, \dots, u_{38}$  are in counterclockwise direction, put differently, the oriented angle at  $x_i$  between segments  $\overline{x_i u_j}$  and  $\overline{x_i u_{j+1}}$  is positive.

We will use the notion of the increment of the argument along a curve. This notion is usually used in complex analysis, see for example [1, Sect. 4.2.1]. A choice of orientation of  $H_i$  and the Euclidean metric on  $H_i$  specify the complex structure on  $H_i$ : the increment of the argument along  $\rho_j$  with respect to the point  $x_i$  is equal to the imaginary part of  $\int_{\rho_j} \frac{1}{z-x_i} dz$ .

Denote by  $\delta_j$  the increment of the argument along  $\rho_j$  with respect to  $x_i$ . Intuitively,  $\delta_j$  should be positive, especially that  $\rho_j$  cannot ‘go around’ the center point  $x_i$ . The precise result is slightly more complicated.

**Lemma 3.11** *For any  $j$  and  $s = 3, 4, \dots, 18$ ,  $\delta_j + \delta_{j+1} + \dots + \delta_{j+s-1}$  is positive.*

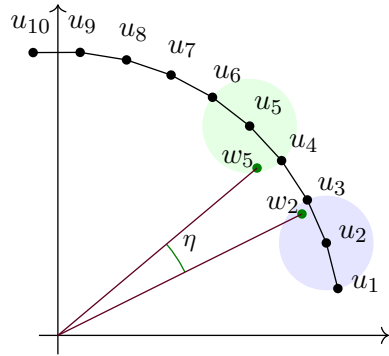
The oriented angle at  $x_i$  between the segments  $\overline{x_i u_j}$  and  $\overline{x_i u_{j+s}}$  is equal to  $\frac{s\pi}{19} \geq \frac{3\pi}{19} > 0.49$ . By (3.1) and (3.4)  $6r < \|x_i - u_j\|$ ,  $5r < \|x_i - w_j\|$ . Moreover, using (C-5), we showed that  $\|u_j - w_j\| < r$ . By the law of cosines, the angle  $\alpha_j$  between  $\overline{x_i w_j}$  and  $\overline{x_i u_j}$  satisfies

$$\begin{aligned} \cos \alpha_j &= \frac{\|x_i - u_j\|^2 + \|x_i - w_j\|^2 - \|u_j - w_j\|^2}{2\|x_i - u_j\|\|x_i - w_j\|} = \frac{\|x_i - u_j\|^2 + \|x_i - w_j\|^2}{2\|x_i - u_j\|\|x_i - w_j\|} \\ &\quad - \frac{\|u_j - w_j\|^2}{2\|x_i - u_j\|\|x_i - w_j\|} \geq \\ &\geq 1 - \frac{r^2}{2\|x_i - u_j\|\|x_i - w_j\|} \geq \frac{59}{60}, \end{aligned}$$

where we have used a standard inequality  $\frac{x^2+y^2}{2xy} = \frac{x}{2y} + \frac{y}{2x} \geq 1$  valid for all real positive numbers  $x, y$ . In particular,  $|\alpha_j| \leq \arccos \frac{59}{60} < 0.19$ . Therefore, the oriented angle between the lines  $\overline{x_i w_j}$  and  $\overline{x_i w_{j+s-1}}$  is at least  $0.49 - 2 \cdot 0.19 = 0.11 > 0$ ; compare Fig. 4. Therefore, it is enough to prove that  $\bar{\rho}$  does not make a full negative turn while going from  $w_j$  to  $w_{j+s}$ .

To this end, let  $V_{ijs}^+ = H_{ij}^+ \cup H_{i,j+1}^+ \cup \dots \cup H_{i,j+s-1}^+$  and  $V_{ijs}^- = H_i \setminus V_{ijs}^+$ ; see Fig. 3. If  $s < 19$ ,  $V_{ijs}^-$  is non-empty. Moreover,  $\bar{\rho}$  misses  $V_{ijs}^-$  by Lemma 3.10. That is, the curve  $\bar{\rho}$  does not go around the point  $x_j$ . The increment of the argument along  $\bar{\rho}$  is the same as the oriented angle between  $\overline{x_i u_j}$  and  $\overline{x_i u_{j+s}}$ , so it is positive.  $\square$

**Fig. 4** Proof of Lemma 3.11. Case  $s = 3$  and  $j = 2$ . The points  $w_2$  and  $w_5$  belong to the disks with centers at  $u_2$ , respectively  $u_5$ . No matter where the points  $z_2$  and  $z_5$  are, the oriented angle  $\eta$  is positive



**Corollary 3.12** *The sum  $\delta_1 + \dots + \delta_{38}$  is a positive multiple of  $2\pi$ .*

**Proof** By Lemma 3.11,  $\delta_1 + \dots + \delta_{10} > 0$ ,  $\delta_{11} + \dots + \delta_{20} > 0$  and  $\delta_{21} + \dots + \delta_{30} > 0$  and  $\delta_{31} + \dots + \delta_{38} > 0$ . That is,  $\delta_1 + \dots + \delta_{38}$  is positive.

Define

$$\Delta = \rho_1 \cup \dots \cup \rho_{38}. \tag{3.5}$$

This is a closed curve on  $H_i$ . The increment of argument along  $\Delta$  is  $\delta_1 + \dots + \delta_{38} > 0$  on the one hand, and it is a multiple of  $2\pi$  on the other.  $\square$

The following result, following from Corollary 3.12, is the final step in the proof of Proposition 3.8. Let  $S_i$  be a connected component of  $H_i \setminus \Delta$  containing  $x_i$ .

**Lemma 3.13**  $B^{H_i}(x_i, r) \subset S_i$ . Moreover,  $S_i \subset B^{H_i}(x_i, 15r)$ .

**Proof** Consider the function  $\iota: H_i \setminus \Delta \rightarrow \mathbb{Z}$ , assigning to a point  $x \notin \Delta$ , the winding number of  $\Delta$  around  $x$ . As  $\Delta$  is compact,  $\iota$  is zero away from a compact subset of  $H_i$ . On the other hand,  $\iota(x_i) = \frac{1}{2\pi}(\delta_1 + \dots + \delta_{38})$  is positive by Corollary 3.12. The winding number is locally constant, so  $\iota > 0$  on  $S_i$ . This means that  $S_i$  is a bounded subset of  $H_i$ .

The properties of  $S_i$  follow from properties of  $\Delta$ , because  $\partial S_i \subset \Delta$ . First, each of the  $\rho_j$  misses the ball  $B^{H_i}(x_i, r)$ ; see Lemma 3.10. As  $x_i \in S_i$ , the whole ball  $B^{H_i}(x_i, r) \subset S_i$ . Next,  $\rho_j \subset B^{H_i}(x_i, 15r)$  by Lemma 3.9. Hence,  $\Delta \subset B^{H_i}(x_i, 15r)$ . As  $S_i$  is bounded,  $S_i \subset B^{H_i}(x_i, 15r)$ .  $\square$

*Finishing the proof of Proposition 3.8.* Let  $x \in \Sigma \setminus \Gamma$  (we recall that  $\Gamma = \bigcup \gamma_{ij}$ ). By the definition of  $\mathcal{X}$ , there exists a point  $x_i \in \mathcal{X}$  such that  $x \in B^{H_i}(x_i, r)$ . Set  $y = \pi_{x_i}(x) \in H_i$ . As  $\pi_{x_i}$  has Lipschitz constant 1,  $y \in B^{H_i}(x_i, r)$ . In particular,  $y \in S_i$ . Take  $\Gamma_i = \pi_{x_i}(\Gamma \cap B(x_i, 22r))$ , the projection of the set  $\Gamma$  to  $H_i$ . Note that the curve  $\Delta$  constructed in (3.5) is contained in  $\Gamma_i$ . Hence, the connected component  $P_i$  of  $H_i \setminus \Gamma_i$  containing  $y$  satisfies  $P_i \subset S_i$ . The map  $\phi_{x_i}$  takes  $P_i$  to the connected component of  $\Sigma \setminus \Gamma$  containing  $x$ . Moreover,  $P_i$  is an open planar set and  $\phi_{x_i}$  is homeomorphism between  $P_i$  and its image. This is precisely the property (T-1).  $\square$

For future use we note the following corollary of the proof of Proposition 3.8.

**Corollary 3.14** Any connected component of  $\Sigma \setminus \Gamma$  belongs to  $B(x_i, 22r)$  for some  $x_i \in \mathcal{X}$ .

**Proof** We use the notation of the proof of Proposition 3.8. We have that  $S_i \subset B^{H_i}(x_i, 15r)$  by Lemma 3.13, hence  $P_i \subset B^{H_i}(x_i, 15r)$  as well. By (C-5),  $\phi_i$  takes  $B^{H_i}(x_i, 15r)$  to  $B(x_i, 15r\sqrt{2}) \cap \Sigma \subset B(x_i, 22r) \cap \Sigma$ .  $\square$

### 3.4 The Property (T-2)

Before we deal with property (T-2), we will need two technical results

**Lemma 3.15** Suppose  $(i, j) \in \mathcal{I}$ . The curve  $\gamma_{ij} \in \mathcal{G}$  satisfies  $\gamma_{ij} \subset B(x_i, 6r) \cap B(x_j, 6r)$ .

**Proof** As  $(i, j) \in \mathcal{I}$ , we have  $\text{dist}(x_i, x_j) < 4r$  (see (G-1)). Then, (G-2) gives the bound on the length  $\ell(\gamma_{ij}) < 8r$ . Now, if  $x \in \gamma_{ij}$  is outside  $B(x_i, 6r)$ , then, by the triangle inequality  $x \notin B(x_j, 2r)$ . Therefore,  $\ell(\gamma_{ij}) \geq \text{dist}(x_i, x) + \text{dist}(x, x_j) > 8r$ , contradiction.  $\square$

A similar argument works for two intersecting curves.

**Lemma 3.16** Suppose  $(i, j), (k, l) \in \mathcal{I}$ , and  $\gamma_{ij}$  intersects  $\gamma_{kl}$ . Then either  $\gamma_{ij} \cup \gamma_{kl} \subset B(x_i, 12r)$ , or  $\gamma_{ij} \cup \gamma_{kl} \subset B(x_j, 12r)$ . In the first case, at least one of the  $x_k, x_l$  belongs to  $B(x_i, 8r)$ ; in the second case, at least one of the  $x_k, x_l$  belongs to  $B(x_j, 8r)$ .

**Proof** Let  $x \in \gamma_{ij} \cap \gamma_{kl}$ . If  $x \notin B(x_i, 4r)$  and  $x \notin B(x_j, 4r)$ , we conclude that the length of  $\gamma_{ij}$  is at least  $8r$ , contradiction. Without losing generality, we may assume that  $x \in B(x_i, 4r)$ . The length of  $\gamma_{kl}$  is less than  $8r$ , hence  $\gamma_{kl} \subset B(x, 8r) \subset B(x_i, 12r)$ . We also know that  $\gamma_{ij} \subset B(x_i, 12r)$ , so  $\gamma_{ij} \cup \gamma_{kl} \subset B(x_i, 12r)$ .

To prove the last part, note that if  $x \in \gamma_{ij} \cap \gamma_{kl}$ , and—as above— $x \in B(x_i, 4r)$ , then at least one of the points  $x_k, x_l$  needs to belong to  $B(x, 4r) \subset B(x_i, 8r)$ . Indeed, the length of  $\gamma_{kl}$  is bounded from below by  $\text{dist}(x_k, x) + \text{dist}(x, x_l)$  and  $\ell(\gamma_{kl}) < 8r$ . The last part follows promptly.  $\square$

We will now show that under slightly stronger conditions as in Proposition 3.8 we can improve the collection  $\mathcal{G}$  in such a way that (T-2) is satisfied.

**Proposition 3.17** Suppose  $\mathcal{G}$  is a collection of good arcs. If  $29r < R_0$ , then there exists a collection of good arcs  $\tilde{\gamma}_{ij}$  satisfying (T-2).

**Proof** Choose  $\varepsilon > 0$  such that for any  $i, j \in \mathcal{I}$  we have  $\ell(\gamma_{ij}) + \varepsilon < 2 \text{dist}(x_i, x_j)$ . We introduce the following notion.

**Definition 3.18** A boundary bigon is a pair of two arcs  $\alpha$  and  $\beta$  with common end points and disjoint interiors such that  $\alpha$  and  $\beta$  are parts of some curves  $\gamma_{ij}$  and  $\gamma_{kl}$ .

A bigon  $(D, \alpha, \beta)$  is a triple  $(D, \alpha, \beta)$ , where the pair  $(\alpha, \beta)$  forms a boundary bigon and  $D \subset \Sigma$  is a properly,  $C^1$ -smoothly embedded disk with the property that  $\partial D = \alpha \cup \beta$  and  $D$  belongs to  $B(x_i, 17r)$  for some  $x_i$ , which is an end point of a curve in  $\mathcal{G}$  whose part is either  $\alpha$  or  $\beta$ .

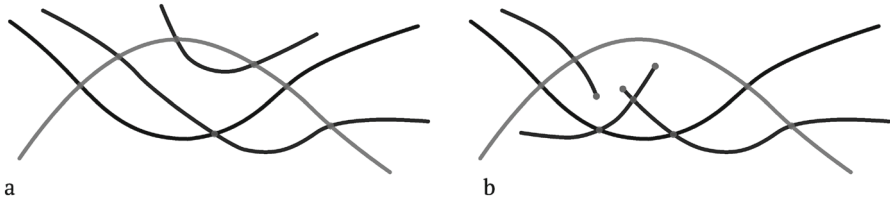


Fig. 5 A desolate and an inhabited bigon

Suppose  $\gamma_{ij}$  and  $\gamma_{kl}$  are distinct. Let  $s \geq 1$  be the number of their intersection points. Then, by transversality,  $\gamma_{ij}$  and  $\gamma_{kl}$  form  $s - 1$  boundary bigons.

The proof of Proposition 3.17 relies on successively removing boundary bigons.

**Lemma 3.19** *Every boundary bigon can be uniquely extended to a bigon.*

**Proof** Let  $(\alpha, \beta)$  be a boundary bigon. Let  $\gamma_{ij}$ , respectively  $\gamma_{kl}$  be the curves in  $\mathcal{G}$  such that  $\alpha \subset \gamma_{ij}$  and  $\beta \subset \gamma_{kl}$ .

As  $\gamma_{kl} \cap \gamma_{ij}$  is not empty, from Lemma 3.16, we conclude that  $\gamma_{kl} \cup \gamma_{ij} \subset B(x_i, 12r)$  or  $\gamma_{kl} \cup \gamma_{ij} \subset B(x_j, 12r)$ . We will assume the first possibility. As  $12r < R_0$ ,  $\pi_{x_i}$  is defined on  $\Sigma \cap B(x_i, 12r)$ . Let  $(\hat{\alpha}, \hat{\beta}) = (\pi_{x_i}(\alpha), \pi_{x_i}(\beta))$ . Then  $\hat{\alpha} \cup \hat{\beta}$  is a simple closed curve on  $H_{x_i}$  contained in  $B^{H_{x_i}}(x_i, 12r)$ . By the Jordan curve theorem (see e.g. [3, Sect. 2.5]), there exists a disk  $\hat{D}$  in  $B^{H_{x_i}}(x_i, 12r)$  such that  $\partial \hat{D} = \hat{\alpha} \cup \hat{\beta}$ . The desired disk  $D$  is constructed as  $\phi_{x_i}(\hat{D})$ . By (C-5),  $D \subset B(x_i, 12\sqrt{2}r) \subset B(x_i, 17r)$ .

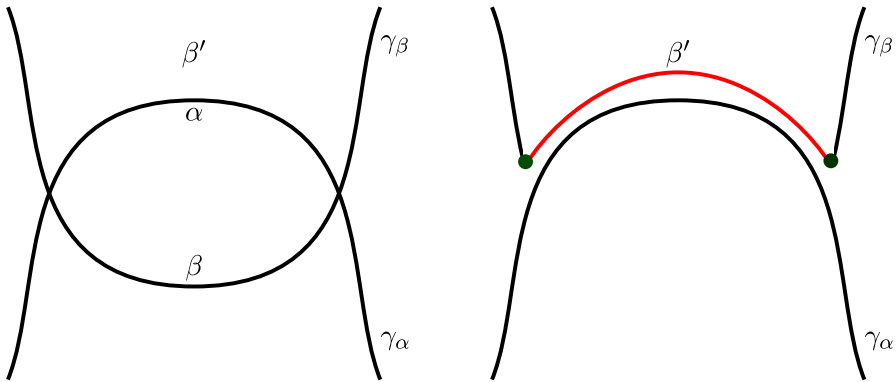
It remains to prove uniqueness. Suppose that  $D_1$  and  $D_2$  are two different disks such that  $(D_1, \alpha, \beta)$  and  $(D_2, \alpha, \beta)$  form bigons. The interiors of  $D_1$  and  $D_2$  are disjoint, for otherwise  $\Sigma$  has self-intersections. Assume that  $D_1 \subset B(x_i, 17r)$  and  $D_2 \subset B(x_{i'}, 17r)$ , where  $i'$  is any of the  $i, j, k, \ell$ . Note that  $\text{dist}(x_i, x_j) < 4r$  and  $\text{dist}(x_i, x_k), \text{dist}(x_i, x_\ell) < 12r$ , again by Lemma 3.16. Therefore, in the worst case scenario, when  $D_2 \subset B(x_k, 17r)$  or  $D_2 \subset B(x_\ell, 17r)$ , we still have that  $D_2 \subset B(x_i, 29r)$ . Then  $D_1 \cup D_2$  glue to a two-dimensional sphere in  $B(x_i, 29r)$ . But  $29r < R_0$  and so the union  $D_1 \cup D_2$  belongs to a graph patch. This is impossible.  $\square$

*Continuation of the proof of Proposition 3.17.* We introduce more terminology. Let  $D = (D, \alpha, \beta)$  be a bigon. We say that

- $D$  is *minimal* if  $D$  does not contain any smaller bigon;
- $D$  is *desolate* if  $D$  does not contain any point  $x_i$ ;
- $D$  is *inhabited* if  $D$  contains at least one of  $x_i$ ; see Fig. 5.

We will now describe a procedure called bigon removal, sketched in Fig. 6. Suppose  $(D, \alpha, \beta)$  is a minimal, desolate bigon. We can swap the roles of  $\alpha$  and  $\beta$ , if needed, to ensure that  $\alpha$  is not longer than  $\beta$ . The curve  $\beta$  is replaced by a curve  $\beta'$  parallel to the curve  $\alpha$  and not much longer than  $\alpha$ . By this we mean that:

- (B-1) the endpoints of  $\beta'$  are on  $\gamma_\alpha$ , the curve in  $\mathcal{G}$  containing  $\alpha$ ;
- (B-2) the length of  $\ell(\beta')$  is less than  $\ell(\alpha) + \varepsilon/2$ ;
- (B-3) suppose  $\gamma \in \mathcal{G}$  does not contain  $\beta$  or  $\alpha$ ; there is a bijection between points  $\gamma \cap \alpha$  and  $\gamma \cap \beta'$ .



**Fig. 6** Bigon removal: **a** output minimal desolate bigon, **b** fragment of one curve is replaced with the approximation of the second

**Lemma 3.20** *If  $(D, \alpha, \beta)$  is a minimal, desolate bigon, then there exists a curve  $\beta'$  satisfying (B-1), (B-2), and (B-3).*

**Proof** There are two cases of the proof, depending on whether  $\alpha \cap \beta$  contains an element of  $\mathcal{X}$ . We give a rather detailed account of the first case; then we point out necessary modifications for the second case.

**Case 1** Assume that none of the two points of  $\alpha \cap \beta$  is an element of the net  $\mathcal{X}$ . We denote by  $\gamma_\alpha$  and  $\gamma_\beta$ , the curves in  $\mathcal{G}$  such that  $\alpha \subset \gamma_\alpha, \beta \subset \gamma_\beta$ . Set also  $\mathcal{G}' = \mathcal{G} \setminus \{\gamma_\alpha, \gamma_\beta\}$ . For  $\theta > 0$  sufficiently small, let  $U_\theta$  be the set of points on  $\Sigma$  at distance to  $\alpha$  less than or equal to  $\theta$ . For the sake of the argument, we use the extrinsic distance, that is, on  $\mathbb{R}^n$ , but this is not compulsory, because our discussion is purely local.

We may and will assume that  $U_\theta$  is disjoint from  $\mathcal{X}$ .

The following is an elementary result from calculus. We state it without proof.

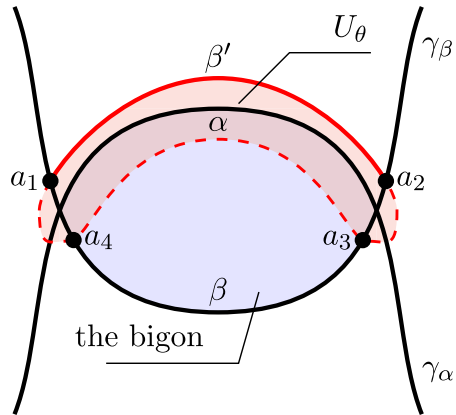
**Lemma 3.21** *There exists  $\theta_0 > 0$  such that if  $\theta < \theta_0$ , then  $\partial U_\theta$  is a smooth manifold (of dimension 1).*

*Continuation of the proof of Lemma 3.20.* The curve  $\partial U_\theta$  intersects  $\gamma_\beta$  at four points, which we denote by  $a_1, a_2, a_3, a_4$ . The numbering convention is clockwise. We require that the part of  $\partial U_\theta$  between  $a_3$  and  $a_4$  belongs to the bigon. On shrinking  $\theta$ , we may and will assume, that the only curve in  $\mathcal{G}$  hitting  $\partial U_\theta$  along the arcs connecting  $a_4$  and  $a_1$ , as well as along  $a_2$  and  $a_3$ , is  $\alpha$ . In Fig. 7, the arcs between  $a_4$  and  $a_1$  and between  $a_2$  and  $a_3$  are ‘vertical’ parts of  $\partial U_\theta$ .

We let  $\beta'$  be the arc on  $\partial U_\theta$  connecting  $a_1$  and  $a_2$ ; see Fig. 7. We need to make sure that with this choice, if  $\theta$  is sufficiently small, (B-1)—(B-3) are satisfied.

Suppose  $\gamma \in \mathcal{G}'$ . Parametrize  $\gamma$  as the image  $\kappa : [0, 1] \rightarrow \Sigma$  by a  $C^1$ -smooth map with non-vanishing derivative. Let  $h(t)$  be the square of the distance of  $\kappa(t)$  to the curve  $\alpha$ . Note that  $h(t)$  has an isolated minimum at each  $t_0$  such that  $\kappa(t_0) \in \alpha$ . On shrinking  $U_\theta$  if needed (by decreasing the value of  $\theta$ ) we may achieve that whenever  $[a, b]$  is an interval such that  $\kappa((a, b)) \subset U_\theta$  and  $\kappa$  maps  $a, b$  to the boundary of  $U_\theta$ , we have:

**Fig. 7** Proof of Lemma 3.20.  
Construction of curve  $\beta'$



- $h(t)$  has no local maxima on  $(a, b)$  and the derivative of  $h$  does not vanish at  $a$  and  $b$ ;
- $h(t)$  has precisely one local minimum  $t_0$  on  $[a, b]$  and  $\kappa(t_0) \in \alpha$ .

Finding such  $\theta$  is straightforward. Namely, we look at all zeros of  $h(t)$  (there are finitely many of them), choose a sufficiently small neighborhood  $U_h$  of the zeros in such a way that  $h$  has no critical points in  $U_h$  except for its zero locus. Any number  $\theta$  smaller than  $\inf_{t \notin U_h} \sqrt{h(t)}$  will be enough. Note also, that the two conditions in the itemized list imply that  $h$  has non-vanishing derivative at each point where it hits  $\partial U_\theta$ . This is equivalent to saying that  $\kappa(t)$  is transverse to  $U_\theta$ . Moreover, note that any intersection of  $\kappa([a, b])$  with  $\alpha$  is a local minimum of  $h(t)$ . As  $h$  has a single local minimum on  $[a, b]$ , to each intersection point of  $\gamma$  with  $\alpha$  (that is to each local minimum of  $h$  on  $[a, b]$ ), there correspond precisely two points of intersections of  $\gamma$  with  $\partial U_\theta$ : these are  $\kappa(a)$  and  $\kappa(b)$ . We claim that out of these two intersections points  $\kappa(a), \kappa(b)$ , precisely one belongs to  $\beta'$  and one belongs to the segment of  $\partial U_\theta$  connecting  $a_3$  with  $a_4$ .

To justify the claim, note that none of the  $\kappa(a), \kappa(b)$  can belong to the segment connecting  $a_4$  with  $a_1$  or  $a_2$  with  $a_3$ . Second,  $\alpha$  cuts  $\overline{U_\theta}$  into two parts, one of which contains  $\beta'$ , the other one contains the part of  $\partial U_\theta$  between  $a_3$  and  $a_4$ . If  $\kappa([a, b])$  is transverse to  $\alpha$ ,  $\kappa(a)$  and  $\kappa(b)$  must belong to different connected components of  $\overline{U_\theta} \setminus \alpha$ . This shows, that for the particular choice of  $\gamma \in \mathcal{G}'$ , item (B-3) is satisfied. Item (B-1) follows by construction. Verification that (B-2) holds for  $\theta \ll 1$ , is left to the reader.

We have chosen  $\theta$  for a concrete curve  $\gamma \in \mathcal{G}'$ , but now, we can choose the minimum of all such  $\theta$ 's over a finite set  $\mathcal{G}'$ .

**Case 2** One of the points of the intersection  $\alpha \cap \beta$  is in  $\mathcal{X}$ . Call this point  $x$ . As in Case 1, we let  $U_\theta$  for  $\theta \ll 1$  be the set of points at distance to  $\alpha$  less than or equal to  $\theta$ . The main difference with the previous case is that  $\partial U_\theta$  intersects  $\gamma_\beta$  at two points and not at four points as above, compare Fig. 8. Call these points  $a_2$  and  $a_3$ . We choose the numbering so that  $a_3$  belongs to the boundary of the bigon.

Our aim is to construct the missing two points  $a_1$  and  $a_4$  and use the argumentation from the previous case. To this end, let  $v_1, \dots, v_m$  be the unit tangent vectors at  $x$  to



those curves in  $\mathcal{G}'$  that emanate from  $x$ . Let  $v_\alpha$  and  $v_\beta$  be the unit tangent vectors to  $\alpha$  and  $\beta$  at  $x$ . Choose a cyclic order on the unit vectors in  $T_x \Sigma$  in such a way that all unit vectors  $v \in T_x \Sigma$  between  $v_\beta$  and  $v_\alpha$  point into the bigon (the vectors between  $v_\alpha$  and  $v_\beta$  do not point into the bigon then). With this choice, there cannot be a vector  $v_i$  between  $v_\beta$  and  $v_\alpha$ , because otherwise the curve  $\gamma_i \in \mathcal{G}$ , which is tangent to  $v_i$ , would enter the bigon. This would contradict minimality of the bigon:  $\gamma_i$  cannot be contained in the bigon, so it has to intersect either  $\alpha$  or  $\beta$ , creating a smaller bigon.

Without loosing generality, we may and will assume that  $v_\beta, v_\alpha, v_1, \dots, v_m$  are cyclically ordered. In this case, we choose an arc  $\tilde{\alpha}$ , passing through  $x$  satisfying the following properties.

- $\tilde{\alpha}$  is a union of two smooth, embedded subarcs  $\tilde{\alpha}_+$  and  $\tilde{\alpha}_-$ ; each of the  $\tilde{\alpha}_\pm$  has one end point at  $x$  and another end point outside  $U_\theta$ ;
- The tangent vector to  $\tilde{\alpha}_+$  lies between  $v_\alpha$  and  $v_1$ ;
- The tangent vector to  $\tilde{\alpha}_-$  lies between  $v_m$  and  $v_\beta$ .

A sketch of  $\tilde{\alpha}$  is presented in Fig. 9. There exist points  $a_1 \in \tilde{\alpha}_+ \cap \partial U_\theta$  and  $a_4 \in \tilde{\alpha}_- \cap \partial U_\theta$  such that the whole part of  $\tilde{\alpha}_+$  between  $x$  and  $a_1$  belongs to  $U_\theta$  and the whole part of  $\tilde{\alpha}_-$  between  $x$  and  $a_4$  belongs to  $U_\theta$ . Moreover, if  $\theta \ll 1$  is sufficiently small, then no curve  $\gamma \in \mathcal{G}$  emanating from the point  $x$  intersects  $\tilde{\alpha}_+$  between  $x$  and  $a_1$  or  $\tilde{\alpha}_-$  between  $x$  and  $a_4$  (except possibly at the point  $x$ ). Define  $\alpha'$  to be the union of the part of  $\tilde{\alpha}_+$  between  $x$  and  $a_1$  and the part of  $\tilde{\alpha}_-$  between  $x$  and  $a_4$ . Then,  $\alpha' \subset U_\theta$  and  $\alpha'$  separates  $U_\theta$  into two connected components  $U_\theta^0, U_\theta^1$  (that is  $U_\theta^0 \cup U_\theta^1 = U_\theta \setminus \alpha'$ ). We number  $U_\theta^0, U_\theta^1$  in such a way that  $U_\theta^0$  is disjoint from the bigon. In Fig. 8, the part  $U_\theta^0$  is to the left of  $\tilde{\alpha}$ . Any curve  $\gamma \in \mathcal{G}$  hitting  $U_\theta^0$  eventually terminates at  $x$ . Define now  $\beta'$  to be the union of:

- the portion of  $\partial U_\theta$  between  $a_1$  and  $a_2$ ;
- the part of  $\tilde{\alpha}_+$  from  $x$  to  $a_1$ .

The properties (B-1) and (B-2) hold trivially as long as  $\theta$  is sufficiently small. We show (B-3). Take  $\gamma \in \mathcal{G}'$ . We need to show the bijection of intersection points of  $\alpha$  with  $\gamma$  and  $\beta'$  with  $\gamma$ .

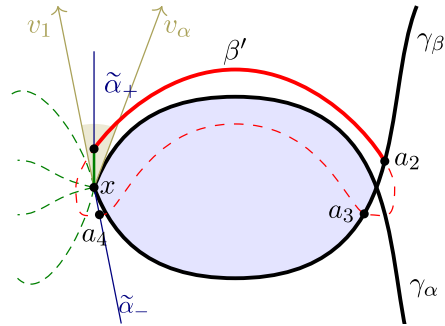
Note that  $x$  is the common intersection point of  $\alpha$  with  $\gamma$  and  $\beta'$  with  $\gamma$ . Therefore, in the remaining part of the proof, we focus on intersection points different than  $x$ . The reasoning is very similar to the one in Case 1, therefore we present only a sketch.

Suppose  $\gamma \in \mathcal{G}$  intersects  $\beta'$  at a point  $x_1 \neq x$ . As  $\gamma$  is disjoint from  $\tilde{\alpha}_+ \setminus \{x\}$ , we infer that  $x_1$  belongs to the part of  $\partial U_\theta$  between  $a_1$  and  $a_2$ . In particular,  $x_1 \in U_\theta^1$ . Hence, the whole connected component  $\gamma'$  of  $\gamma \cap U_\theta$  containing  $x_1$ , belongs to  $U_\theta^1$ . The same argument as in Case 1 implies that  $\gamma'$  intersects  $\alpha$  at precisely one point  $x_2$  and  $x_2 \neq x$ .

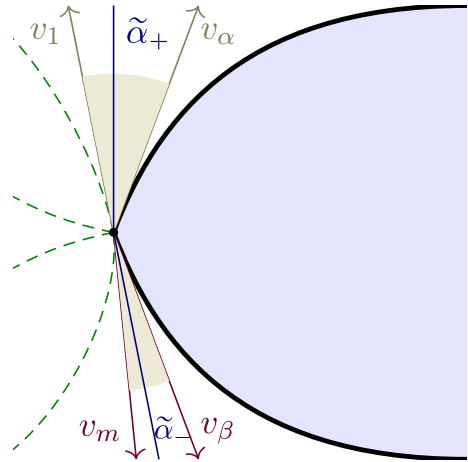
On the other hand, if  $\gamma \in \mathcal{G}$  intersects  $\alpha$  at a point  $x_2 \neq x$ , we again consider  $\gamma'$ , the connected component of  $\gamma \cap U_\theta$  containing  $x_2$ . Clearly,  $\gamma' \subset U_\theta^1$ . As in Case 1, precisely one of the end points of  $\gamma'$  belongs to the part of  $\partial U_\theta$  between  $a_1$  and  $a_2$ , that is, to  $\beta'$ .

This shows the bijection between intersection points of  $\gamma \cap \alpha$  and  $\gamma \cap \beta'$ , which is required by (B-3). □

**Fig. 8** Proof of Lemma 3.20. Second case. Dashed curves are other curves from  $\mathcal{G}$  emanating from the point  $x \in \mathcal{X}$



**Fig. 9** Proof of Lemma 3.20. Case 2. Position of  $\tilde{\alpha}$ . The tangent vectors at  $x$  to  $\tilde{\alpha}_+$  and  $\tilde{\alpha}_-$  are required to be in the shaded cones



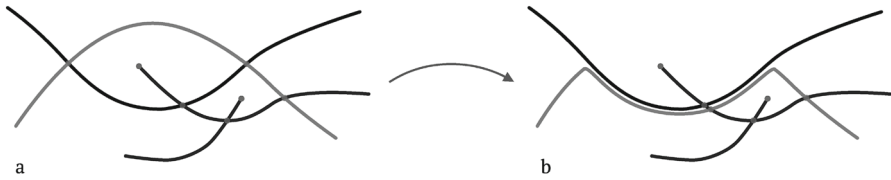
**Lemma 3.22** *If  $(D, \alpha, \beta)$  is a desolate minimal bigon, bigon removal procedure applied to  $(D, \alpha, \beta)$  decreases the number of desolate bigons by 1 and creates no other bigons.*

**Proof** The number of bigons between two different curves  $\gamma, \gamma' \in \mathcal{G}$  is equal to  $|\gamma \cap \gamma'| - 1$ . Therefore, we will strive to show that the total number of intersection points between all curves in  $\mathcal{G}$  decreases after bigon removal.

Let  $\gamma_\alpha, \gamma_\beta \in \mathcal{G}$  be such that  $\alpha \subset \gamma_\alpha$  and  $\beta \subset \gamma_\beta$ . Let  $\gamma'_\beta$  be the curve  $\gamma_\beta$  with  $\beta$  replaced by  $\beta'$ . We have

$$|\gamma_\alpha \cap \gamma'_\beta| = |\gamma_\alpha \cap \gamma_\beta| - 2.$$

Suppose  $\gamma_i$  is another curve in  $\mathcal{G}$ . If it does not hit the bigon  $D$ , it is also disjoint from the closure of  $U_\theta$ , for  $\theta$  sufficiently small, so  $\gamma_\beta \cap \gamma_i = \gamma'_\beta \cap \gamma_i$ , that is, the number of intersection points is preserved. If  $\gamma_i$  hits the bigon  $D$ , we look at connected components of  $\gamma_i \cap D$ . Each such connected component  $\delta$  is an arc, and if  $|\delta \cap \alpha| = 2$  or  $|\delta \cap \beta| = 2$ , the arc  $\delta$  and the relevant part of  $\alpha$  or  $\beta$  form a bigon contained in  $D$ , contradicting minimality of  $D$ . If  $|\delta \cap (\alpha \cup \beta)| = 1$ , one of the end points of  $\delta$  is



**Fig. 10** An attempt to remove an inhabited bigon results in creating another inhabited bigon. There is a little control on the number of bigons that can be produced in this way

inside  $D$ , but such an end point must be an end point of  $\gamma_\beta$ , that is, it must be a point from  $\mathcal{X}$ . This contradicts the condition that  $D$  be desolate.

The only remaining possibility is that  $|\delta \cap \alpha| = |\delta \cap \beta| = 1$ . This, in turn, shows that  $|\gamma_i \cap \alpha| = |\gamma_i \cap \beta|$ . Now, by (B-3),  $|\gamma_i \cap \beta'| = |\gamma_i \cap \alpha|$ . Eventually  $|\gamma_i \cap \beta| = |\gamma_i \cap \beta'|$ , that is,  $|\gamma_i \cap \gamma'_\beta| = |\gamma_i \cap \gamma_\beta|$ . In other words, no new bigons are created.  $\square$

**Remark 3.23** The statement of Lemma 3.22 need not hold if  $D$  is inhabited. A bigon removal procedure can create new bigons, both inhabited and desolate, whose number is rather hard to control; see Fig. 10.

After a single bigon removal procedure, the length of one of the curves  $\gamma_{ij}$  can increase, but in a controlled manner, because of (B-2). We decrease  $\varepsilon$  so that  $\ell(\gamma_{ij}) + \varepsilon < 2 \text{dist}(x_i, x_j)$  for all  $(i, j) \in \mathcal{I}$ .

We now apply inductively the bigon removal procedure to all minimal desolate bigons, until there are no minimal desolate bigons. This requires a finite number of steps. We make the following trivial observation.

**Lemma 3.24** *If there are no minimal desolate bigons, there are no desolate bigons at all.*

From now on we will assume that the set  $\mathcal{G}$  of curves is such that there are no desolate bigons. The following lemma concludes the proof of Proposition 3.17.

**Lemma 3.25** *Suppose curves  $\gamma_{ij}$  and  $\gamma_{kl}$  do not form desolate bigons. Then, the number of intersection points between  $\gamma_{ij}$  and  $\gamma_{kl}$  is bounded by  $\mathcal{T}(17)$ .*

**Proof** Suppose  $\gamma_{ij}$  and  $\gamma_{kl}$  are not disjoint. Each bigon formed by  $\gamma_{ij}$  and  $\gamma_{kl}$  belongs to  $B(x_i, 17r)$ . All such bigons have pairwise disjoint interiors. Moreover, each of the bigons is inhabited, so to each of them we can associate an element of  $\mathcal{X} \setminus \{x_i, x_j, x_k, x_l\}$  that belongs to  $B(x_i, 17r)$ ; see the proof of Lemma 3.19. The number of the bigons is bounded from above by the total number of points of  $\mathcal{X}$  in  $B(x_i, 17r)$  distinct from  $\{x_i, x_j, x_k, x_l\}$ . According to Proposition 3.4, the number of points of  $\mathcal{X}$  in  $B(x_i, 17r)$  is bounded by  $\mathcal{T}(17)$ , so the number of the bigons is smaller than  $\mathcal{T}(17) - 3$  (it is  $-3$ , not  $-4$ , because  $x_i, x_j, x_k, x_l$  need not be all distinct).

The number of intersection points is equal to one plus the number of the bigons. The lemma follows.  $\square$

The proof of Proposition 3.17 is complete.  $\square$

**Corollary 3.26** *If  $29r < R_0$ , there exists a collection of good arcs satisfying both (T-1) and (T-2).*

**Proof** The collection of arcs constructed in Proposition 3.17 satisfies (T-2). As  $22r < R_0$ , by Proposition 3.8, this collection satisfies also (T-1).  $\square$

## 4 Triangulation

### 4.1 Bounding the Number of Triangles

**Proposition 4.1** *Let  $\mathcal{X}$  be an  $r$ -net with  $29r < R_0$ . Suppose  $\mathcal{G}$  is a good tame collection of arcs. Then  $\Sigma$  can be triangulated with at most  $S(\Sigma)$  triangles with*

$$S(\Sigma) = \frac{1}{12} \mathcal{T}(4)^2 \mathcal{T}(8)^2 \mathcal{T}(12)^2 \mathcal{T}(17)^2 \mathcal{T}(26)^2 |\mathcal{X}|. \tag{4.1}$$

**Proof** The proof of Proposition 4.1 takes the rest of Sect. 4.1. The triangulation is constructed based on  $\Gamma = \bigcup_{(i,j) \in \mathcal{I}} \gamma_{ij}$ . The vertices are going to be the points in  $\mathcal{X}$  as well as the intersection points  $\gamma_{ij} \cap \gamma_{kl}$ . To begin with, we bound the total number of intersection points of  $\gamma_{ij}$ .

**Lemma 4.2** *Suppose  $x_i \in \mathcal{X}$ . The total number of triples  $j, k, l$  such that  $\gamma_{ij} \cap \gamma_{kl} \neq \emptyset$  is less than or equal to  $2\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)$ .*

**Proof** Assume that  $\gamma_{kl} \cap \gamma_{ij} \neq \emptyset$ . By Lemma 3.16, either  $\gamma_{kl} \cup \gamma_{ij} \subset B(x_i, 12r)$ , or  $\gamma_{kl} \cup \gamma_{ij} \subset B(x_j, 12r)$ .

In the first case,  $x_j \in B(x_i, 4r)$ , one of the  $x_k, x_l$  is in  $B(x_i, 8r)$ , and the other one is in  $B(x_i, 12r)$ . The total number of possibilities for  $x_j, x_k, x_l$  is therefore bounded by  $\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)$ .

The case  $\gamma_{kl} \cup \gamma_{ij} \subset B(x_i, 12r)$  leads to the same number of possibilities.  $\square$

Let now

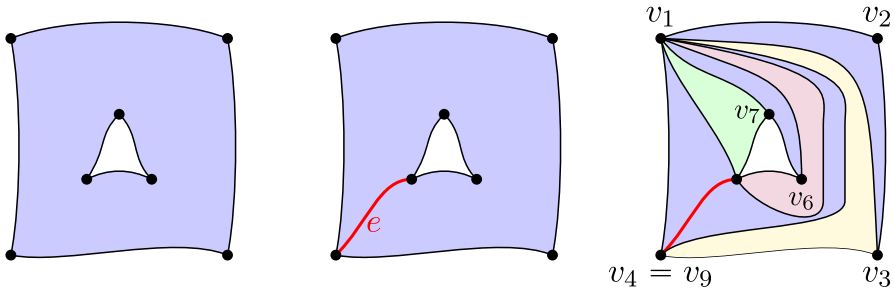
$$\mathcal{Z} = \bigcup_{(l,k) \neq (i,j) \neq (k,l)} \{\gamma_{ij} \cap \gamma_{kl}\}.$$

Note that  $\mathcal{X} \subset \mathcal{Z}$ . Indeed, it is not hard to see that for any  $i$  there are at least two points  $j, l$  such that  $(i, j), (i, l) \in \mathcal{I}$  and then  $x_i \in \gamma_{ij} \cap \gamma_{il}$ .

**Lemma 4.3** *Let  $\sigma > 0$  be such that  $(\sigma + 4 + 1/4)r < R_0$ . Let  $\mathcal{Z}_{i_0}^\sigma$  be the number of points  $z \in \mathcal{Z}$  such that  $z \in B(x_{i_0}, \sigma r)$ . Then,  $|\mathcal{Z}_{i_0}^\sigma| < \mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)\mathcal{T}(17)\mathcal{T}(\sigma + 4)$ .*

**Proof** Take a point  $z \in \mathcal{Z} \cap B(x_{i_0}, \sigma r)$ . It is in the intersection of two curves  $\gamma_{ij} \cap \gamma_{kl}$ . We know that for at least one of the indices  $i, j$  and for at least one of the indices  $k, l$ , the distance of  $z$  to the respective endpoint on  $\gamma_{ij}$  and on  $\gamma_{kl}$  is at most  $4r$ . Hence, at least two of the points  $x_i, x_j, x_k, x_l$  belong to  $B(x_{i_0}, (\sigma + 4)r)$ .

For a fixed  $i$  such that  $x_i \in \mathcal{X} \cap B(x_{i_0}, (\sigma + 4)r)$ , the total number of intersection points  $\gamma_{ij} \cap \gamma_{kl}$  for arbitrary  $j, k, l$  is  $2\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)\mathcal{T}(17)$  by Lemma 4.2 and property (T-2). We sum over all points  $x_i \in \mathcal{X} \cap B(x_{i_0}, (\sigma + 4)r)$ . We obtain the bound of  $2\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)\mathcal{T}(17)\mathcal{T}(\sigma + 4)$ . Note, however, that each intersection



**Fig. 11** An example of fan triangulation of  $C$ . Left: the starting situation. Middle: step (i) adds an edge. Right: step (ii) triangulates a polygon with connected boundary. The non-labelled vertex of the triangulation in the figure on the right is  $v_5$  (and also  $v_8$ )

point was counted at least twice, because at least two points  $x_i, x_j, x_k, x_l$  belong to  $B(x_{i_0}, (\sigma + 4)r)$ . □

Now, we pass to the construction of the triangulation. By the property (T-1), each connected component of  $\Sigma \setminus \Gamma$  is homeomorphic to an open subset of  $\mathbb{R}^2$ . Take such a connected component  $C$ . Its boundary is a union of (parts of) curves  $\gamma_{ij}$  intersecting at points of  $\mathcal{Z}$ . We think of  $C$  as a polygon with vertices in  $\mathcal{Z}$  whose edges are parts of curves from  $\mathcal{G}$ , though we do not necessarily assume that  $C$  has connected boundary. An elementary geometric argument allows us to triangulate  $C$  by adding pairwise non-intersecting arcs connecting vertices of  $C$ . Below we sketch one of possible ways of doing this, called a *fan triangulation*; see Fig. 11.

- (i) take two vertices  $v, w \in \mathcal{Z}$  of  $C$  belonging to different connected components of  $\partial C$ . Create an edge  $e$  (an arc in  $C$ ) connecting these vertices. Replace  $C$  by  $C \setminus e$ . Then  $C$  is still connected, but its boundary has one less connected component than before;
- (ii) once  $\partial C$  is connected, suppose  $v_1, \dots, v_M \in \mathcal{Z}$  are its vertices enumerated successively (some vertices might be repeated). Choose pairwise non-intersecting arcs connecting  $v_1$  with  $v_3, \dots, v_{M-1}$ . Then,  $C$  is triangulated; the triangles have vertices  $(v_1, v_2, v_3), (v_1, v_3, v_4), \dots, (v_1, v_{M-1}, v_M)$ .

This provides us with a triangulation. In particular, the triangulation has the following property.

**Property 4.4** *An edge of the triangulation connects two elements in  $\mathcal{Z}$ , which belong to the closure of the same connected component of  $\Sigma \setminus \Gamma$ .*

To estimate the number of triangles we use the following lemma.

**Lemma 4.5** *Suppose  $z_1, z_2 \in \mathcal{Z}$  and  $z_1, z_2$  belong to the closure of the same connected component  $S_z$  of  $\Sigma \setminus \Gamma$ . Then, there exists  $x_i \in \mathcal{X}$  such that  $z_1, z_2 \in \Sigma \cap B(x_i, 22r)$ .*

**Proof** Let  $x_i$  be a point on  $\Sigma$  such that  $\text{dist}(x_i, z_1) < r$ . Set  $y_1 = \pi_{x_i}(z_1), y_2 = \pi_{x_i}(z_2)$ . Consider the cycle  $\Delta$  on  $H_i$  as in Corollary 3.12. The image  $\pi_{x_i}(S_z)$  belongs to a connected component of  $H_i \setminus \Delta$ . Now  $\|y_1 - x_i\| < r$ , so  $y_1$  belongs to  $S_i$ , the connected

component of  $H_i \setminus \Delta$  containing  $x_i$  as in the proof of Proposition 3.8. Therefore,  $\pi_{x_i}(S_z) \subset S_i$ . In particular, by Lemma 3.13,  $\pi_{x_i}(S_z) \subset B^{H_i}(x_i, 15r)$ . That is to say  $\pi_{x_i}(z_2) \subset \overline{B}^{H_i}(x_i, 15r)$ . Via (C-5), this implies that  $\text{dist}(z_2, x_i) \leq 15r\sqrt{2} < 22r$ .  $\square$

**Corollary 4.6** *The total number of edges in the triangulation is bounded from above by*

$$\frac{1}{8}T(4)^2T(8)^2T(12)^2T(15)^2T(17)^2T(26)^2|\mathcal{X}|.$$

**Proof** By Lemma 4.5 and Property 4.4, any two edges connecting points  $z_1$  and  $z_2$  belong to the same  $B(x_i, 22r) \cap \Sigma$  for some  $x_i \in \mathcal{Z}$ . We bound the number of points  $\mathcal{Z} \cap B(x_i, 22r)$  by Lemma 4.3. Half the square of this bound, that is,  $\frac{1}{2}|\mathcal{Z}_i^{22}|^2 \leq \frac{1}{8}T(4)^2T(8)^2T(12)^2T(17)^2T(26)^2$ , estimates the number of unordered pairs  $z_1, z_2$  of points  $\mathcal{Z}$  in  $B(x_i, 22r)$ . Summing up over all  $x_i \in \mathcal{X}$  we get the result.  $\square$

The rest of the proof of Proposition 4.1 is straightforward. Each edge belongs to precisely two triangles and each triangle has three edges.  $\square$

#### 4.2 Proof of Theorem 1.1

Suppose  $\mathcal{E}_p^\ell(\Sigma) = E$  and let  $C_0 = 2^{-1/2} \min(c_1, c_2^{-\alpha})$ ; compare (2.3) in Corollary 2.5. Here,  $\alpha = 1 - 2\ell/p$ . Set  $r = \frac{1}{29}C_0E^{-1/p\alpha}$ . Choose a net of points  $\mathcal{X}$  with this given  $r$ . By Proposition 3.3, we have

$$|\mathcal{X}| < C_1AE^{2/p\alpha}, \tag{4.2}$$

where  $C_1 = \frac{1}{\pi}2^{n/2+4}29^2C_0^{-2}$ .

Let  $\mathcal{G}$  be a collection of good arcs associated with  $\mathcal{X}$ ; such collection exists by Proposition 3.6, because  $4r < 29r < R_0 = C_0E^{-1/p\alpha}$ . The collection  $\mathcal{G}$  can be improved to a tame collection of good arcs by Corollary 3.26, which works because  $29r < R_0$ . A tame collection of arcs provides a triangulation with the number of triangles bounded above by  $S(\Sigma)$  triangles; see Proposition 4.1. In total, the number of triangles is bounded above by  $C_2AE^{2/p\alpha}$ , where

$$C_2 = \frac{1}{12}T(4)^2T(8)^2T(12)^2T(17)^2T(26)^2C_1.$$

By construction, each triangle is an image of a subset of the tangent plane  $H_{x_i}$  to  $\Sigma$  at  $x_i$  (for some  $i$ ) under the map  $\phi_{x_i}$ , which has bounded derivative and bounded distortion by Corollaries 2.5 and 2.6.

**Acknowledgements** The paper is based on the Master’s thesis of the second named author under supervision of the first author. The authors are grateful to Sławomir Kolański for fruitful discussions. They are also grateful to the referees for their remarks.

**Funding** Maciej Borodzik was supported by the Polish National Science Grant 2019/B/35/ST1/01120. Monika Szczepanowska was supported by the National Science Center grant 2016/22/E/ST1/00040.

## Declarations

**Conflict of interest** The authors have no competing interests to declare that are relevant to the content of this article.

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