Triangulating Surfaces with Bounded Energy

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Abstract

We show that if a closed Lipschitz surface in $\mathbb{R}^n$ has bounded Kolasinski–Menger energy, then it can be triangulated with triangles whose number is bounded by the energy and the area. Each of the triangles is an image of a subset of a plane under a diffeomorphism whose distortion is bounded by $\sqrt{2}$.

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1 Introduction

It is a general principle in the theory of energies of manifolds that small energy implies uncomplicated topology. Probably, the first instance of this principle is the Fáry–Milnor theorem [2, 8], stating that a knot in $\mathbb{R}^3$ whose total curvature is less than 4π is necessarily trivial.

For energies of curves in $\mathbb{R}^3$, there are bounds for the stick number and the average crossing number of a knot, see for example [9] and references therein.

For higher dimensional submanifolds some analogs exist, but are not abundant. The Fáry–Milnor theorem can be generalized to the case of surfaces [4]. In [7], delicate arguments involving compactness and stability (manifolds with bounded energy that are sufficiently close with respect to the Hausdorff distance are ambiently isotopic), are used to show that there are finitely many isotropy classes of submanifolds below some fixed energy level.

Motivated by [4], we give another bound on the complexity of a surface in $\mathbb{R}^n$ in terms of its energy. Namely, we bound from above the minimal number of triangles in

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a triangulation of the surface. In particular, we bound the genus of a surface in terms of its energy. Actually, our result goes further. For a surface with given energy we construct a triangulation in such a way that each triangle is a graph of a function with bounded derivative and distortion. In this sense, the triangles in the triangulation are "almost flat".

Noting that the energy $\mathcal{E}_p^\ell$ is introduced in Definition 2.1, we present now the main result of this paper,

Theorem 1.1 *Suppose $\Sigma \subset \mathbb{R}^n$ is a closed, Lipschitz surface. Let $\ell \in \{1, \ldots, 4\}$ and $p > 2\ell$. Suppose Σ has energy $\mathcal{E}_p^\ell(\Sigma) = E < \infty$ and area A. Then Σ can be triangulated with $C_2 A E^{2/(p-2\ell)}$ triangles, where C_2 is a universal constant depending only on p, ℓ and n.*

Each of the triangles in the triangulation is an image of an open subset of a plane under a function whose derivative has norm bounded by $\sqrt{2}$ and whose distortion is bounded by $\sqrt{2}$.

Combining Theorem 1.1 with [5, Theorem 1.1] stating that the minimal number $T(g)$ of triangles in a triangulation of a closed surface of genus g grows linearly with g, we obtain the following result.

Corollary 1.2 *There is a constant C_g (depending on ℓ, n, p) such that if $\Sigma \subset \mathbb{R}^n$ is a closed Lipschitz surface with $\mathcal{E}_p^\ell(\Sigma) = E < \infty$ and area A, where $p > 2\ell$, then $g(\Sigma) \leq C_g A E^{2/(p-2\ell)}$.*

The proof of Theorem 1.1 goes along the following lines. The key tool is the Regularity Theorem of [7], recalled here as Theorem 2.3, which states that an m-dimensional submanifold with bounded energy can be covered by so-called graph patches, that is, subsets that are graphs of functions from subsets of $\mathbb{R}^m$, and the derivatives of the functions are bounded. The size of graph patches is controlled by the energy, that is, the graph patches are not too small. An immediate corollary of Theorem 2.3 is an Ahlfors like inequality, Proposition 2.7, controlling from both sides the volume of the part of a submanifold cut out by a ball whose center is on the submanifold.

Next, assuming Σ is a surface of bounded energy and area, we find a cover of Σ by balls of some radius r (depending on the energy), such that each ball is a graph patch. We let the centers of the balls be $x_1, \ldots, x_N$. A simple topological argument in Sect. 3.1 allows us to control the number N of the centers in terms of the energy and area of the surface.

To construct the triangulation, we first connect all pairs x_i, x_j, such that $\operatorname{dist}(x_i, x_j) < 4r$, by an arc γ_{ij} on Σ. The requirement on γ_{ij}, spelled out in Definition 3.5, is that the length $\ell(\gamma_{ij})$ be bounded by a constant times $\operatorname{dist}(x_i, x_j)$. Unlike geodesics, two such curves can intersect at more than a single point. By a procedure called *bigon removal* we improve the collection of curves γ_{ij} to obtain a concrete bound on the number of intersection points between them; see Proposition 3.17.

We let Σ_0 be the complement of $\bigcup \gamma_{ij}$ in Σ. The triangulation is constructed by cutting connected components of Σ_0 into triangles. A second technicality, and chronologically the first we deal with in this article, appears. We need to ensure that

each connected component of Σ_0 is homeomorphic to a planar open set. We address this problem in Proposition 3.8. Given that result, we consider each component C of Σ_0. As it is planar, we cut C into triangles without adding new vertices. The number of triangles in the triangulation is estimated using the number of intersection points between curves γ_{ij}; see Corollary 4.6. The proof of Theorem 1.1 is carried out in Sect. 4.2.

2 Review of Surface Energies

In this section, we recall definitions of surface energy. References include [6, 7, 10].

2.1 Discrete Menger Energy for a Submanifold in $\mathbb{R}^n$

For points $(x_0, \ldots, x_m)$ in $\mathbb{R}^n$, we let $\Delta(x_0, \ldots, x_m)$ denote the m-dimensional simplex spanned by $x_0, \ldots, x_m$ (the convex hull of these points). We define

$$\mathcal{K}(x_0, \ldots, x_m) = \frac{1}{d^m} \operatorname{vol}_m \Delta(x_0, \ldots, x_m),$$

where d is the diameter of $\Delta(x_0, \ldots, x_m)$.

Suppose now $\Sigma \subset \mathbb{R}^n$ is a Lipschitz submanifold of dimension $m < n$. Let $\ell \in \{1, \ldots, m + 2\}$ and $p > 0$.

Definition 2.1 The *Kolasinski-Menger energy* of Σ is the integral

$$\mathcal{E}_p^\ell(\Sigma) = \int_{\Sigma^\ell} \sup_{x_\ell, \ldots, x_{m+2} \in \Sigma} \mathcal{K}(x_0, \ldots, x_{m+2})^p \, d\operatorname{vol}(\Sigma^\ell). \tag{2.1}$$

The integral is computed with respect to the variables $x_0, \ldots, x_{\ell-1}$.

2.2 Graph Patches

For $k \geq 0$ and $\alpha \in (0, 1]$, we let $\mathcal{C}_{m,n}^{k,\alpha}$ denote the set of all compact manifolds of dimension m, embedded $\mathcal{C}^{k,\alpha}$-smoothly in $\mathbb{R}^n$. The following definition is taken from [7, Definition 1]

Definition 2.2 (Graph patches) Suppose R, L, d are real positive and $\alpha \in (0, 1]$. The class $\mathcal{C}_{m,n}^{1,\alpha}(R, L, d)$ is the class of all m-dimensional $C^{1,\alpha}$-smooth submanifolds $\Sigma \subset \mathbb{R}^n$ such that:

(P-1) $\Sigma \subset B(0, d)$;
(P-2) for each $x \in \Sigma$, there exists a function $f_x \colon T_x\Sigma \to T_x\Sigma^\perp$ of class $C^{1,\alpha}$ with $f_x(0) = 0$, $Df_x(0) = 0$ and

$$\Sigma \cap B(x, R) = (x + \operatorname{graph}(f_x)) \cap B(x, R). \tag{2.2}$$

(P-3) the function f_x is Lipschitz with Lipschitz constant 1 and $||Df_x(\xi) - Df_x(\eta)|| \leq L||\xi - \eta||^\alpha$.

We quote now the following result

Theorem 2.3 ([7, Regularity Theorem]) *For $p > m\ell$, there exist constants $c_1(m, n, \ell, p)$ and $c_2(m, n, \ell, p)$ such that with $\alpha = 1 - m\ell/p$, any Lipschitz manifold $\Sigma \in C^{0,1}_{m,n}$ with energy $\mathcal{E}^\ell_p(\Sigma) = E < \infty$ satisfies*

$$\Sigma \in C^{1,\alpha}_{m,n}(c_1 E^{-1/(p-m\ell)}, c_2 E^{1/p}, d)$$

as long as $\Sigma \subset B(0, d)$.

We now introduce some notation regarding graph patches. Let $x \in \Sigma$ and $r < c_1 E^{-1/(p-m\ell)}$. We let $H_x \subset \mathbb{R}^n$ be the m-dimensional affine plane in $\mathbb{R}^n$, which is tangent to Σ at x. The map f_x induces a map $\phi_x \colon H_x \to \Sigma$ given by $y \mapsto (y, x + f_x(y))$. The inverse map $\pi_x \colon \Sigma \to H_x$ is the orthogonal projection. We will consider both maps ϕ_x and π_x as defined in open neighborhoods of x in H_x and Σ, respectively. This is made precise in Corollary 2.5.

Notation 2.4 *We use the notation B^H for a ball contained in H, the notation like $B(y, \rho)$ means an open ball in the ambient space $\mathbb{R}^n$.*

The distance of two points x, y in the ambient space $\mathbb{R}^n$ is denoted by $\text{dist}(x, y)$, *while the distance of two points in the tangent space H is denoted by $||x - y||_H$.*

We will use the following corollary of Theorem 2.3.

Corollary 2.5 *Let c_1, c_2 be the constants as in Theorem 2.3. Set*

$$R_0 = 2^{-1/2} \min(c_1, c_2^{-\alpha}) E^{-1/(p-m\ell)}. \tag{2.3}$$

If $r < R_0$, then:

(C-1) *ϕ_x is well-defined on $H_x \cap B^{H_x}(x, r)$;*
(C-2) *π_x is well-defined on $\Sigma \cap B(x, r\sqrt{2})$;*
(C-3) *for any $s < r\sqrt{2}$, the image $B^{H_x}(x, s) \subset \pi_x(\Sigma \cap B(x, s\sqrt{2})) \subset B^{H_x}(x, s\sqrt{2})$.*
(C-4) *$||D\phi_x(y)|| \leq \sqrt{2}$ for all $y \in H_x \cap B^{H_x}(x, r)$;*
(C-5) *ϕ_x is Lipschitz with Lipschitz constant $\sqrt{2}$ and π_x is Lipschitz with Lipschitz constant 1.*

Proof We begin with (C-4). By (P-2), we have $Df_x(0) = 0$. Condition (P-3) implies then that $||Df_x(y)|| \leq L||y||^\alpha$. By the definition, $||D\phi_x||^2 = 1 + ||Df_x||^2$. Therefore, item (C-4) holds, as long as $||y|| \leq L^{-\alpha} = (c_2 E^{1/p})^{-\alpha}$.

The map π_x is a projection, so it has Lipschitz constant 1. By (C-4), ϕ_x has Lipschitz constant $\sqrt{2}$. This proves (C-5).

Item (C-2) follows from (P-2). In fact, $\Sigma \cap B(x, r\sqrt{2})$ is in the image of ϕ_x by (2.2), thus π_x is defined on $\Sigma \cap B(x, r\sqrt{2})$ as the inverse of ϕ_x.

To prove (C-1), we note that the whole of $\overline{B}(x, r\sqrt{2}) \cap \Sigma$ is covered by a graph patch by (P-2). Here, we consider the closed ball, which is legitimate since $r < R_0$

(and not only $r \leq R_0$). Let $U \subset H_x$ be the preimage $\phi_x^{-1}(B(x, r\sqrt{2}) \cap \Sigma)$. The closure of U is the preimage $\phi_x^{-1}(\overline{B}(x, r\sqrt{2}) \cap \Sigma)$.

Take $u \in \partial U$. We know that $\phi_x(u)$ belongs to the boundary of $\overline{B}(x, r\sqrt{2}) \cap \Sigma$. As Σ is closed, we deduce that the boundary of $\overline{B}(x, r\sqrt{2}) \cap \Sigma$ belongs to the boundary of $\overline{B}(x, r\sqrt{2})$. That is to say, $\mathrm{dist}(\phi_x(u), x) = r\sqrt{2}$. As $\phi_x(x) = x$ and ϕ_x has Lipschitz constant $\sqrt{2}$, we infer that $\|u - x\| \geq r$.

Therefore, the boundary of U lies outside the (open) ball $B^{H_x}(x, r)$, and $x \in U$. Therefore, $B^{H_x}(x, r) \subset U$, proving (C-1).

Finally, item (C-3) is a direct consequence of the Lipschitz property (C-5). □

Corollary 2.6 *The distortion of ϕ_x is bounded by $\sqrt{2}$.*

Proof The distortion of ϕ_x at the point z is given by

$$D(z) = \limsup_{r \to 0} \frac{\max_{y\,:\,\|z-y\|_{H_x}=r} \mathrm{dist}(\phi_x(y), \phi_x(z))}{\min_{y\,:\,\|z-y\|_{H_x}=r} \mathrm{dist}(\phi_x(y), \phi_x(z))}.$$

The numerator in the formula is bounded from above by r times the Lipschitz constant of ϕ_x. The denominator is bounded from below by r times the Lipschitz constant of π_x. □

2.3 Local Volume Bound

Throughout Sect. 2.3, we let Σ be a submanifold of $\mathbb{R}^n$ in the class $\mathcal{C}_{m,n}^{1,\alpha}(R_0, L, d)$.

Proposition 2.7 (Local volume bound) *Suppose $r < R_0$. Then for any $x \in \Sigma$ we have*

$$2^{-n/2} V_m r^m < \mathrm{vol}(\Sigma \cap B(x, r)) < 2^{n/2} V_m r^m,$$

where V_m is the volume of the unit ball in dimension m.

Proof Let $U = \pi_x(\Sigma \cap B(x, r))$. By (C-3):

$$B^H(x, r/\sqrt{2}) \subset U \subset B^H(x, r). \tag{2.4}$$

As $\Sigma \cap B(x, r)$ is the image of ϕ_x, a classical result from multivariable calculus computes the volume of $\Sigma \cap B(x, r)$ in terms of the integral over U over the square root of the Gram determinant:

$$\mathrm{vol}(\Sigma \cap B(x, r)) = \int_U \sqrt{|\det G|},$$

where $G = D\phi_x D\phi_x^T$.

The derivative of ϕ_x has a block structure $D\phi_x = (I \; Df_x)$, so $G = I + Df_x Df_x^T$. On the one hand, since $Df_x Df_x^T$ is non-negative definite, $\det G \geq 1$. On the other hand, $\|Df_x\| \leq 1$, so $\|Df_x Df_x^T\| \leq 1$. Therefore, $\|G\| \leq 2$. This means that all

the eigenvalues of the symmetric matrix G have modulus less than or equal to 2, so $|\det G| \leq 2^n$. In particular,

$$\int_U 1 \leq \text{vol}(\Sigma \cap B(x, r)) \leq \int_U 2^{n/2}.$$

Combining this with (2.4) we quickly conclude the proof.   □

3 Geodesic-Like Systems of Curves

3.1 Nets of Points

We recall a classical notion of general topology. We will need the following technical definition, which makes sense under very mild assumptions on Σ.

Definition 3.1 Assume Σ is a metric space. Let $r > 0$. A finite set $\mathcal{X}$ of points in Σ is an *r-net* if

- the balls $B(x, r)$ for $x \in \mathcal{X}$ cover Σ;
- for any $x, x' \in \mathcal{X}$, $x \neq x'$, we have $\text{dist}(x, x') \geq r/2$.

The following result is classical in general topology.

Proposition 3.2 *If Σ is a compact metric space, then it admits an r-net for any $r > 0$.*

Proof For the reader's convenience we provide a quick proof. Cover first Σ by all balls $B(x, r/2)$ with $x \in \Sigma$. Choose a finite subcover, and let $\mathcal{Y} = \{y_1, \ldots, y_M\}$ be the set of centers of balls in this subcover.

We act inductively. Start with y_1 and define $x_1 = y_1$. Remove from $\mathcal{Y}$ all points $y_i \neq y_1$ such that $\text{dist}(y_1, y_j) < r/2$. After this procedure, the balls $B(x_1, r)$ and $B(y_i, r/2)$ for $i > 1$ still cover Σ.

For the inductive step assume that, for given n, the balls $B(x_1, r), \ldots, B(x_{n-1}, r)$ and $B(y_n, r/2), B(y_{n+1}, r/2), \ldots, B(y_M, r/2)$ cover Σ, and there are no indices i, j with $i < n$ and $j \geq n$ such that $\text{dist}(x_i, y_j) < r/2$. We remove from $\mathcal{Y}$ points y_j with $j > n$ such that $\text{dist}(x_n, y_j) < r/2$ and set $x_n = y_n$.

After a finite number of steps we are left with the set $\mathcal{X} \subset \mathcal{Y}$, such that $B(x_i, r)$, $x_i \in \mathcal{X}$ cover Σ and $\text{dist}(x_i, x_j) \geq r/2$ for all $x_i, x_j \in \mathcal{X}$.   □

We now return to our previous assumptions on Σ.

Proposition 3.3 (*Bounding $|\mathcal{X}|$*) *Suppose Σ is a closed Lipschitz m-dimensional submanifold of $\mathbb{R}^n$ with $\mathcal{E}_p^\ell(\Sigma) = E < \infty$. Assume that R_0 satisfies (2.3). Let $A = \text{vol}_m(\Sigma)$. If $r < R_0$, then any r-net $\mathcal{X}$ has $|\mathcal{X}| < 2^{n/2+2m} V_m^{-1} r^{-m} A$.*

Proof By the local volume bound (Proposition 2.7) the sets $B(x_i, r/4) \cap \Sigma$, $x_i \in \mathcal{X}$ have volume at least $2^{-n/2} V_m \left(\frac{r}{4}\right)^m$. As $\mathcal{X}$ is an r-net, these balls are pairwise disjoint. The total volume of $\bigcup_{x_i \in \mathcal{X}} B(x_i, r/4) \cap \Sigma$ is at least $2^{-n/2-2m} V_m r^m |\mathcal{X}|$. This quantity does not exceed the volume of Σ.   □

Essentially the same argument yields the following result.

Proposition 3.4 *Let Σ be as in Proposition 3.3. Suppose $\sigma > 0, r > 0$ are such that $(\sigma + 1/4)r < R_0$. Let $\mathcal{X}$ be an r-net. Each ball $B(x, \sigma r)$ for $x \in \Sigma$ contains at most $\mathcal{T}_m(\sigma)$ points from $\mathcal{X}$, where*

$$\mathcal{T}_m(\sigma) = 2^{n+2m}(\sigma + 1/4)^m.$$

Proof The ball $B(x, (\sigma + 1/4)r) \cap \Sigma$ has volume at most $2^{n/2} V_m(\sigma + 1/4)^m r^m$ by Proposition 2.7. All balls of radius $r/4$ with centers at $x_i \in \mathcal{X}$ such that $\mathrm{dist}(x, x_i) < \sigma r$ are pairwise disjoint, belong to $B(x, (\sigma + 1/4)r)$, and each of them has volume at least $2^{-n/2-2m} V_m r^m$. Hence, the number of points in $\mathcal{X}$ at distance at most σr to x is bounded by $2^{n+2m}(\sigma + 1/4)^m$. $\qquad\square$

From now on, we set

$$\mathcal{T}(\sigma) := \mathcal{T}_2(\sigma) = 2^{n+4}(\sigma + 1/4)^2.$$

3.2 Good Arcs

Throughout Sect. 3.2 we let Σ be a Lipschitz surface with area A and energy $\mathcal{E}_p^\ell(\Sigma) = E$. We note that by Regularity Theorem 2.3, Σ is C^1-smooth. We will be using smoothness of Σ for transversality arguments. Recall (see Notation 2.4), that $\mathrm{dist}(x, y)$ is the distance of two points in $\mathbb{R}^n$.

Finally, let $r > 0$.

Definition 3.5 Let $\mathcal{X} = \{x_1, \ldots, x_N\}$ be an r-net. Let $\mathcal{I}$ be a subset of pairs (i, j) with $1 \leq i \neq j \leq N$. A collection $\mathcal{G}$ of arcs γ_{ij}, $(i, j) \in \mathcal{I}$ C^1–smoothly embedded in Σ and connecting x_i with x_j is called a collection of *good arcs associated to $\mathcal{X}$* if:

(G-1) $\mathcal{I} = \{(i, j) : i \neq j, \ \mathrm{dist}(x_i, x_j) < 4r\}$;
(G-2) for all $(i, j) \in \mathcal{I}$, γ_{ij} has length less than $2\,\mathrm{dist}(x_i, x_j)$;
(G-3) $\gamma_{ij} = \gamma_{ji}$;
(G-4) γ_{ij} is regularly embedded, in particular, has no self-intersections;
(G-5) if $\gamma_{ij} \neq \gamma_{k\ell}$, then γ_{ij} intersects $\gamma_{k\ell}$ transversally, in particular, there are no triple intersections;
(G-6) γ_{ij} has a well-defined tangent vector at x_i and x_j. At each point $x_i \in \mathcal{X}$, the tangent directions to all curves γ_{ij} are pairwise distinct.

A collection of good arcs is called *tame*, if it additionally satisfies the following two conditions.

(T-1) Every connected component of $\Sigma \setminus \Gamma$ with $\Gamma = \bigcup \gamma_{ij}$ is homeomorphic to an open set of $\mathbb{R}^2$;
(T-2) Unless $\gamma_{ij} = \gamma_{k\ell}$, the curves γ_{ij} and $\gamma_{k\ell}$ intersect transversally at most $\mathcal{T}(17)$ points.

One should think of a collection of good arcs as an analog of a collection of geodesics. Condition (T-2) is automatically satisfied if γ_{ij} are geodesics whose length is less than

the geodesic radius. However, we cannot use geodesics because this would require extra regularity assumptions on Σ.

Proposition 3.6 *Suppose* $4r < R_0$. *For each* r-*net* $\mathcal{X}$ *there exists an associated collection* $\mathcal{G}$ *of good arcs.*

Proof We construct all the curves in $\mathcal{G}$ one by one. Take two points $x_i, x_j \in \mathcal{X}$ with $\text{dist}(x_i, x_j) < 4r$ that are not yet connected by a curve. Recall that H_{x_i} is the affine plane tangent to Σ at x_i. Let $y_j = \pi_{x_i}(x_j)$, where π_{x_i} is the projection onto H_{x_i}; compare with Corollary 2.5. As $4r < R_0$, by (C-2), π_{x_i} is defined on x_j. Moreover, by (C-1), the straight segment ρ on H_{x_i} connecting x_i and y_j belongs to the domain of ϕ_{x_i}.

By (C-2), π_{x_i} has Lipschitz constant 1. Hence, $||x_i - y_j||_{H_{x_i}} \leq \text{dist}(x_i, x_j)$, so the length of ρ is less than or equal to $\text{dist}(x_i, x_j)$. If necessary, we perturb the segment ρ (keeping its endpoints fixed) to be transverse to the image under π_{x_i} of all curves that have already been constructed (here we use the fact that Σ is C^1 smooth). That is, by perturbation we achieve that $\gamma_{ij} := \phi_{x_i}(\rho)$ satisfies (G-4), (G-5) and (G-6)

A perturbation might increase the length of ρ, but we can choose a perturbation as small (in the C^1 sense) as we please, so the length of ρ is less than $\sqrt{2}\,\text{dist}(x_i, x_j)$. By (C-4) we infer that the length of $\gamma_{ij} := \phi_{x_i}(\rho)$ is at most $\sqrt{2}\,\text{dist}(x_i, x_j) < 2\,\text{dist}(x_i, x_j)$. □

Remark 3.7 It is clear from the proof that one can achieve a better bound on the length $\ell(\gamma_{ij}) < 1.5\,\text{dist}(x_i, x_j)$. This would eventually lead to a smaller constant C_2 in Theorem 1.1. However, choosing an integer factor 2 leads to more readable estimates.

Our goal is to show that there exists a tame collection of good arcs. In Sect. 3.3 we shall deal with Condition (T-1), while in Sect. 3.4 we deal with (T-2).

3.3 On the Property (T-1)

Proposition 3.8 *If* $22r < R_0$, *a collection of good arcs* $\mathcal{G}$ *satisfies* (T-1).

Proof We use the notation of Corollary 2.5. Let $x_i \in \mathcal{X}$. Denote by H_i the plane tangent to Σ at x_i. Let $\Sigma_{x_i} = \Sigma \cap B(x_i, 22r)$; see Fig. 1. As $22r < R_0$, properties (C-1)–(C-5) are satisfied with r replaced by $22r$. We let $U_i = \pi_{x_i}(\Sigma_{x_i})$. By (C-3), we have

$$B^{H_i}(x_i, 15r) \subset B^{H_i}(x_i, 22r/\sqrt{2}) \subset U_i \subset B^{H_i}(x_i, 22r).$$

Consider a regular 38-gon on H_i with center x_i and side length r. Denote by $u_1, \ldots, u_{38}$ its vertices, so that $||u_j - u_{j+1}||_{H_i} = r$. We will use the convention that the indices are cyclic, that is $u_{39} = u_1$ (and similarly for points y_j, z_j, w_j, which we will introduce in a while).

Assume H_i is oriented in such a way that the vertices are numbered in the counter-clockwise direction. We have $||u_j - x_i||_{H_i} = \frac{r}{2 \sin \frac{\pi}{38}} \sim 6.05r$. In particular, for any
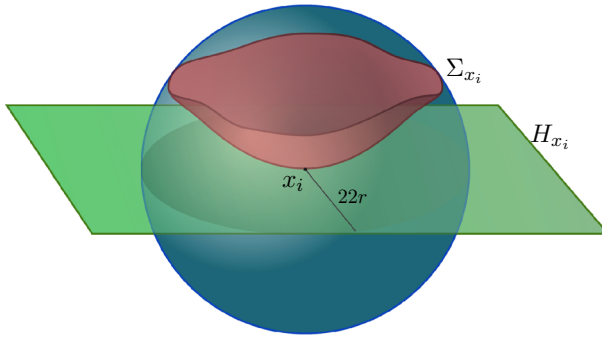
Fig. 1 The set H_{x_i} and Σ_{x_i} of the proof of Proposition 3.8

$j = 1, \ldots, 38$:

$$6r < ||u_j - x_i||_{H_i} < 7r. \tag{3.1}$$

Let $y_j = \phi_{x_i}(u_j) \in \Sigma_{x_i}$. By (C-5):

$$r \leq \text{dist}(y_j, y_{j+1}) \leq r\sqrt{2}. \tag{3.2}$$

By the definition of $\mathcal{X}$, for any y_j there exists an element $z_j \in \mathcal{X}$ such that $\text{dist}(z_j, y_j) < r$. In particular, by the triangle inequality and (3.2):

$$\text{dist}(z_j, z_{j+1}) < (2 + \sqrt{2})r, \quad \text{dist}(x_i, y_j) < 7r\sqrt{2} < 10r. \tag{3.3}$$

As $\mathcal{G}$ is a collection of good arcs, and $\text{dist}(z_j, z_{j+1}) < 4r$, there exists a curve $\lambda_j \in \mathcal{G}$ that connects z_j and z_{j+1}. The length of λ_j is at most $2 \, \text{dist}(z_j, z_{j+1}) < (4+2\sqrt{2})r < 7r$.

Denote $w_j = \pi_{x_i}(z_j)$, and let $\rho_j = \pi_{x_i}(\lambda_j)$. The next lemma controls the position of λ_j and ρ_j.

Lemma 3.9 *We have $z_j \in B(x_i, 11r)$. Moreover, $\lambda_j \subset B(x_i, 15r)$ and ρ_j belongs to $B^{H_i}(x_i, 15r)$.*

Proof From (3.3) we read off that $\text{dist}(x_i, y_j) < 10r$. As $\text{dist}(y_j, z_j) < r$, we conclude that $\text{dist}(x_i, z_j) < 11r$.

The curve λ_j has length at most $7r$. No point on λ_j can be further from x_i than $11r + \frac{7}{2}r < 15r$. Indeed, if x is outside $B(x_i, 15r)$, then the length of the part of λ_j from z_j to x and the length of the part of λ_j from x to z_{j+1} are both at least $4r$, contributing to the length of λ_j being at least $8r$. The contradiction shows that $\lambda_j \subset B(x_i, 15r)$. Now $\pi_{x_i}(B(x_i, 15r)) \subset B^{H_i}(x_i, 15r)$, so $\rho_j \subset B^{H_i}(x_i, 15r)$. □

As $\text{dist}(z_j, y_j) < r$, we have $||w_j - u_j||_{H_i} < r$ by (C-5). By (3.1) and the triangle inequality:

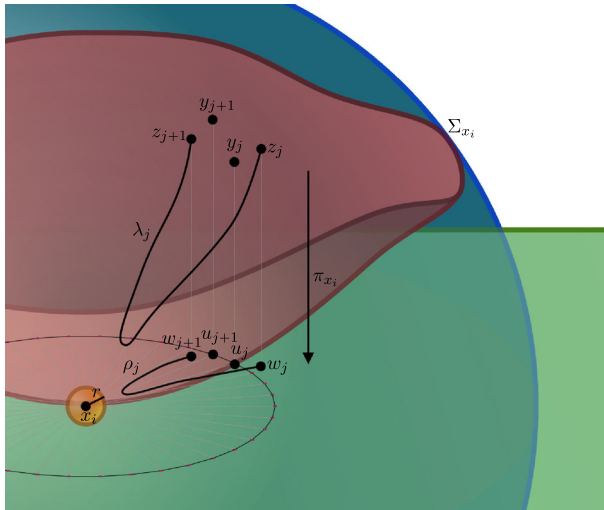$$5r < ||w_j - x_i||_{H_i} < 8r \text{ for all } j = 1, \ldots, 38. \tag{3.4}$$
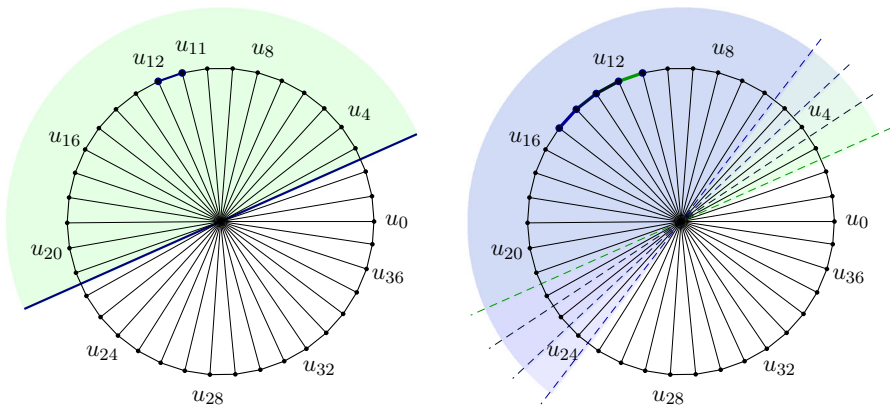
Fig. 2 Notation of Sect. 3.3

Fig. 3 The set H_{ij}^+ for $j = 11$ (left) and V_{ijs}^+ for $j = 11$ and $s = 4$ (right) of the proof of Lemmas 3.10 and 3.11

The bounds on lengths of λ_j and ρ_j will be used to show that these curves do not come close to the point x_i. More precisely, we have the following result.

Lemma 3.10 *Let H_{ij}^+ be the half-plane cut off from H_i by the line parallel to the segment joining u_j and u_{j+1} and passing through x_i such that $u_j, u_{j+1} \in H_{ij}^+$; see Fig. 3. The curve ρ_j misses both $B^{H_i}(x_i, r)$ and $H_{ij}^- := H_i \setminus H_{ij}^+$. In particular, λ_j is disjoint from $B(x_i, r)$.*

Proof of Lemma 3.10 Note that π_{x_i} having Lipschitz constant 1 implies that the length of ρ_j is at most $7r$; see (C-5). Suppose towards contradiction that ρ_j passes through a point $z \in B(x_i, r)$. We have $||w_j - z||_{H_i} > 4r$ and $||w_{j+1} - z||_{H_i} > 4r$ by (3.4) and

the triangle inequality, hence the length of ρ_j is at least $8r$. The contradiction shows that ρ_j misses the ball $B^{H_i}(x_i, r)$.

Suppose now λ_j hits the ball $B(x_i, r)$, that is, there exists a point $x \in \lambda_j \cap B(x_i, r)$. Then, $\pi_{x_i}(x) \in B^{H_i}(x_i, r)$ by (C-5) and so ρ_j hits $B^{H_i}(x_i, r)$, contradicting what we have already proved. This shows that λ_j is disjoint from $B(x_i, r)$.

To prove that ρ_j misses H_{ij}^+ is analogous. The distance of u_j to the boundary of H_{ij}^+ (computed on H_i) is equal to $r\sqrt{1/(4\sin^2(\pi/38)) - 1/4}$, and it is greater than $6r$. Hence, the distance of w_j to ∂H_{ij}^+ is at least $5r$. That is, if ρ_j leaves H_{ij}^+, then its length must be at least $10r$. Contradiction. □

Continuing the proof of Proposition 3.8 Recall that the orientation was chosen in such a way that the vertices $u_1, \ldots, u_{38}$ are in counterclockwise direction, put differently, the oriented angle at x_i between segments $\overline{x_i u_j}$ and $\overline{x_i u_{j+1}}$ is positive.

We will use the notion of the increment of the argument along a curve. This notion is usually used in complex analysis, see for example [1, Sect. 4.2.1]. A choice of orientation of H_i and the Euclidean metric on H_i specify the complex structure on H_i: the increment of the argument along ρ_j with respect to the point x_i is equal to the imaginary part of $\int_{\rho_j} \frac{1}{z - x_i} dz$.

Denote by δ_j the increment of the argument along ρ_j with respect to x_i. Intuitively, δ_j should be positive, especially that ρ_j cannot 'go around' the center point x_i. The precise result is slightly more complicated.

Lemma 3.11 *For any j and $s = 3, 4, \ldots, 18$, $\delta_j + \delta_{j+1} + \cdots + \delta_{j+s-1}$ is positive.*

The oriented angle at x_i between the segments $\overline{x_i u_j}$ and $\overline{x_i u_{j+s}}$ is equal to $\frac{s\pi}{19} \geq \frac{3\pi}{19} > 0.49$. By (3.1) and (3.4) $6r < ||x_i - u_j||$, $5r < ||x_i - w_j||$. Moreover, using (C-5), we showed that $||u_j - w_j|| < r$. By the law of cosines, the angle α_j between $\overline{x_i w_j}$ and $\overline{x_i u_j}$ satisfies

$$\cos \alpha_j = \frac{||x_i - u_j||^2 + ||x_i - w_j||^2 - ||u_j - w_j||^2}{2||x_i - u_j|| ||x_i - w_j||} == \frac{||x_i - u_j||^2 + ||x_i - w_j||^2}{2||x_i - u_j|| ||x_u - w_j||}$$

$$- \frac{||u_j - w_j||^2}{2||x_i - u_j|| ||x_i - w_j||} \geq$$

$$\geq 1 - \frac{r^2}{2||x_i - u_j|| ||x_i - w_j||} \geq \frac{59}{60},$$

where we have used a standard inequality $\frac{x^2 + y^2}{2xy} = \frac{x}{2y} + \frac{y}{2x} \geq 1$ valid for all real positive numbers x, y. In particular, $|\alpha_j| \leq \arccos \frac{59}{60} < 0.19$. Therefore, the oriented angle between the lines $\overline{x_i w_j}$ and $\overline{x_i w_{j+s-1}}$ is at least $0.49 - 2 \cdot 0.19 = 0.11 > 0$; compare Fig. 4. Therefore, it is enough to prove that $\overline{\rho}$ does not make a full negative turn while going from w_j to w_{j+s}.

To this end, let $V_{ijs}^+ = H_{ij}^+ \cup H_{i,j+1}^+ \cup \cdots \cup H_{i,j+s-1}^+$ and $V_{ijs}^- = H_i \setminus V_{ijs}^+$; see Fig. 3. If $s < 19$, V_{ijs}^- is non-empty. Moreover, $\overline{\rho}$ misses V_{ijs}^- by Lemma 3.10. That is, the curve $\overline{\rho}$ does not go around the point x_j. The increment of the argument along $\overline{\rho}$ is the same as the oriented angle between $\overline{x_i u_j}$ and $\overline{x_i u_{j+s}}$, so it is positive. □

Fig. 4 Proof of Lemma 3.11.
Case $s = 3$ and $j = 2$. The
points w_2 and w_5 belong to the
disks with centers at u_2,
respectively u_5. No matter
where the points z_2 and z_5 are,
the oriented angle η is positive
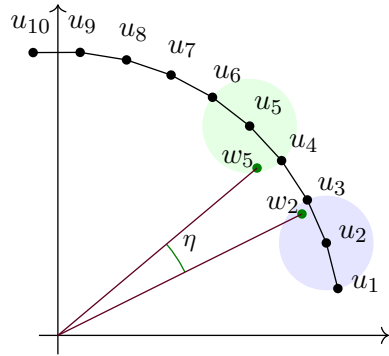
Corollary 3.12 *The sum $\delta_1 + \cdots + \delta_{38}$ is a positive multiple of 2π.*

Proof By Lemma 3.11, $\delta_1 + \cdots + \delta_{10} > 0$, $\delta_{11} + \cdots + \delta_{20} > 0$ and $\delta_{21} + \cdots + \delta_{30} > 0$ and $\delta_{31} + \cdots + \delta_{38} > 0$. That is, $\delta_1 + \cdots + \delta_{38}$ is positive.
Define

$$\Delta = \rho_1 \cup \cdots \cup \rho_{38}. \tag{3.5}$$

This is a closed curve on H_i. The increment of argument along Δ is $\delta_1 + \cdots + \delta_{38} > 0$ on the one hand, and it is a multiple of 2π on the other. $\qquad\square$

The following result, following from Corollary 3.12, is the final step in the proof of Proposition 3.8. Let S_i be a connected component of $H_i \setminus \Delta$ containing x_i.

Lemma 3.13 $B^{H_i}(x_i, r) \subset S_i$. *Moreover, $S_i \subset B^{H_i}(x_i, 15r)$.*

Proof Consider the function $\iota \colon H_i \setminus \Delta \to \mathbb{Z}$, assigning to a point $x \notin \Delta$, the winding number of Δ around x. As Δ is compact, ι is zero away from a compact subset of H_i. On the other hand, $\iota(x_i) = \frac{1}{2\pi}(\delta_1 + \cdots + \delta_{38})$ is positive by Corollary 3.12. The winding number is locally constant, so $\iota > 0$ on S_i. This means that S_i is a bounded subset of H_i.

The properties of S_i follow from properties of Δ, because $\partial S_i \subset \Delta$. First, each of the ρ_j misses the ball $B^{H_i}(x_i, r)$; see Lemma 3.10. As $x_i \in S_i$, the whole ball $B^{H_i}(x_i, r) \subset S_i$. Next, $\rho_j \subset B^{H_i}(x_i, 15r)$ by Lemma 3.9. Hence, $\Delta \subset B^{H_i}(x_i, 15r)$. As S_i is bounded, $S_i \subset B^{H_i}(x_i, 15r)$. $\qquad\square$

Finishing the proof of Proposition 3.8. Let $x \in \Sigma \setminus \Gamma$ (we recall that $\Gamma = \bigcup \gamma_{ij}$). By the definition of $\mathcal{X}$, there exists a point $x_i \in \mathcal{X}$ such that $x \in B^{H_i}(x_i, r)$. Set $y = \pi_{x_i}(x) \in H_i$. As π_{x_i} has Lipschitz constant 1, $y \in B^{H_i}(x_i, r)$. In particular, $y \in S_i$. Take $\Gamma_i = \pi_{x_i}(\Gamma \cap B(x_i, 22r))$, the projection of the set Γ to H_i. Note that the curve Δ constructed in (3.5) is contained in Γ_i. Hence, the connected component P_i of $H_i \setminus \Gamma_i$ containing y satisfies $P_i \subset S_i$. The map ϕ_{x_i} takes P_i to the connected component of $\Sigma \setminus \Gamma$ containing x. Moreover, P_i is an open planar set and ϕ_{x_i} is homeomorphism between P_i and its image. This is precisely the property (T-1). $\qquad\square$

For future use we note the following corollary of the proof of Proposition 3.8.

Corollary 3.14 *Any connected component of $\Sigma \setminus \Gamma$ belongs to $B(x_i, 22r)$ for some $x_i \in \mathcal{X}$.*

Proof We use the notation of the proof of Proposition 3.8. We have that $S_i \subset B^{H_i}(x_i, 15r)$ by Lemma 3.13, hence $P_i \subset B^{H_i}(x_i, 15r)$ as well. By (C-5), ϕ_i takes $B^{H_i}(x_i, 15r)$ to $B(x_i, 15r\sqrt{2}) \cap \Sigma \subset B(x_i, 22r) \cap \Sigma$. □

3.4 The Property (T-2)

Before we deal with property (T-2), we will need two technical results

Lemma 3.15 *Suppose $(i, j) \in \mathcal{I}$. The curve $\gamma_{ij} \in \mathcal{G}$ satisfies $\gamma_{ij} \subset B(x_i, 6r) \cap B(x_j, 6r)$.*

Proof As $(i, j) \in \mathcal{I}$, we have $\text{dist}(x_i, x_j) < 4r$ (see (G-1)). Then, (G-2) gives the bound on the length $\ell(\gamma_{ij}) < 8r$. Now, if $x \in \gamma_{ij}$ is outside $B(x_i, 6r)$, then, by the triangle inequality $x \notin B(x_j, 2r)$. Therefore, $\ell(\gamma_{ij}) \geq \text{dist}(x_i, x) + \text{dist}(x, x_j) > 8r$, contradiction. □

A similar argument works for two intersecting curves.

Lemma 3.16 *Suppose $(i, j), (k, l) \in \mathcal{I}$, and γ_{ij} intersects γ_{kl}. Then either $\gamma_{ij} \cup \gamma_{kl} \subset B(x_i, 12r)$, or $\gamma_{ij} \cup \gamma_{kl} \subset B(x_j, 12r)$. In the first case, at least one of the x_k, x_l belongs to $B(x_i, 8r)$; in the second case, at least one of the x_k, x_l belongs to $B(x_j, 8r)$.*

Proof Let $x \in \gamma_{ij} \cap \gamma_{kl}$. If $x \notin B(x_i, 4r)$ and $x \notin B(x_j, 4r)$, we conclude that the length of γ_{ij} is at least $8r$, contradiction. Without losing generality, we may assume that $x \in B(x_i, 4r)$. The length of γ_{kl} is less than $8r$, hence $\gamma_{kl} \subset B(x, 8r) \subset B(x_i, 12r)$. We also know that $\gamma_{ij} \subset B(x_i, 12r)$, so $\gamma_{ij} \cup \gamma_{kl} \subset B(x_i, 12r)$.

To prove the last part, note that if $x \in \gamma_{ij} \cap \gamma_{kl}$, and—as above—$x \in B(x_i, 4r)$, then at least one of the points x_k, x_l needs to belong to $B(x, 4r) \subset B(x_i, 8r)$. Indeed, the length of γ_{kl} is bounded from below by $\text{dist}(x_k, x) + \text{dist}(x, x_l)$ and $\ell(\gamma_{kl}) < 8r$. The last part follows promptly. □

We will now show that under slightly stronger conditions as in Proposition 3.8 we can improve the collection $\mathcal{G}$ in such a way that (T-2) is satisfied.

Proposition 3.17 *Suppose $\mathcal{G}$ is a collection of good arcs. If $29r < R_0$, then there exists a collection of good arcs $\widetilde{\gamma}_{ij}$ satisfying (T-2).*

Proof Choose $\varepsilon > 0$ such that for any $i, j \in \mathcal{I}$ we have $\ell(\gamma_{ij}) + \varepsilon < 2\,\text{dist}(x_i, x_j)$. We introduce the following notion.

Definition 3.18 A *boundary bigon* is a pair of two arcs α and β with common end points and disjoint interiors such that α and β are parts of some curves γ_{ij} and γ_{kl}.

A *bigon* (D, α, β) is a triple (D, α, β), where the pair (α, β) forms a boundary bigon and $D \subset \Sigma$ is a properly, C^1-smoothly embedded disk with the property that $\partial D = \alpha \cup \beta$ and D belongs to $B(x_i, 17r)$ for some x_i, which is an end point of a curve in $\mathcal{G}$ whose part is either α or β.
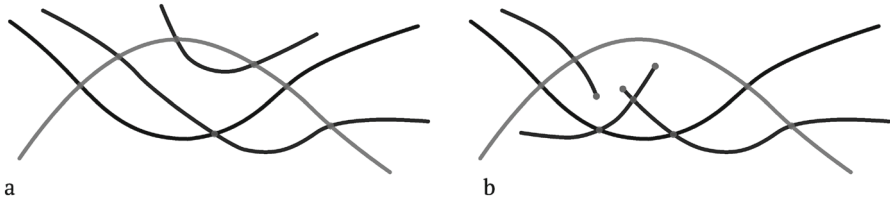
Fig. 5 A desolate and an inhabited bigon

Suppose γ_{ij} and γ_{kl} are distinct. Let $s \geq 1$ be the number of their intersection points. Then, by transversality, γ_{ij} and γ_{kl} form $s - 1$ boundary bigons.

The proof of Proposition 3.17 relies on successively removing boundary bigons.

Lemma 3.19 *Every boundary bigon can be uniquely extended to a bigon.*

Proof Let (α, β) be a boundary bigon. Let γ_{ij}, respectively γ_{kl} be the curves in $\mathcal{G}$ such that $\alpha \subset \gamma_{ij}$ and $\beta \subset \gamma_{kl}$.

As $\gamma_{kl} \cap \gamma_{ij}$ is not empty, from Lemma 3.16, we conclude that $\gamma_{kl} \cup \gamma_{ij} \subset B(x_i, 12r)$ or $\gamma_{kl} \cup \gamma_{ij} \subset B(x_j, 12r)$. We will assume the first possibility. As $12r < R_0$, π_{x_i} is defined on $\Sigma \cap B(x_i, 12r)$. Let $(\widehat{\alpha}, \widehat{\beta}) = (\pi_{x_i}(\alpha), \pi_{x_i}(\beta))$. Then $\widehat{\alpha} \cup \widehat{\beta}$ is a simple closed curve on H_{x_i} contained in $B^{H_{x_i}}(x_i, 12r)$. By the Jordan curve theorem (see e.g. [3, Sect. 2.5]), there exists a disk $\widehat{D}$ in $B^{H_{x_i}}(x_i, 12r)$ such that $\partial \widehat{D} = \widehat{\alpha} \cup \widehat{\beta}$. The desired disk D is constructed as $\phi_{x_i}(\widehat{D})$. By (C-5), $D \subset B(x_i, 12\sqrt{2}r) \subset B(x_i, 17r)$.

It remains to prove uniqueness. Suppose that D_1 and D_2 are two different disks such that (D_1, α, β) and (D_2, α, β) form bigons. The interiors of D_1 and D_2 are disjoint, for otherwise Σ has self-intersections. Assume that $D_1 \subset B(x_i, 17r)$ and $D_2 \subset B(x_{i'}, 17r)$, where i' is any of the i, j, k, ℓ. Note that $\mathrm{dist}(x_i, x_j) < 4r$ and $\mathrm{dist}(x_i, x_k), \mathrm{dist}(x_i, x_\ell) < 12r$, again by Lemma 3.16. Therefore, in the worst case scenario, when $D_2 \subset B(x_k, 17r)$ or $D_2 \subset B(x_\ell, 17r)$, we still have that $D_2 \subset B(x_i, 29r)$. Then $D_1 \cup D_2$ glue to a two-dimensional sphere in $B(x_i, 29r)$. But $29r < R_0$ and so the union $D_1 \cup D_2$ belongs to a graph patch. This is impossible. □

Continuation of the proof of Proposition 3.17. We introduce more terminology. Let $D = (D, \alpha, \beta)$ be a bigon. We say that

- D is *minimal* if D does not contain any smaller bigon;
- D is *desolate* if D does not contain any point x_i;
- D is *inhabited* if D contains at least one of x_i; see Fig. 5.

We will now describe a procedure called bigon removal, sketched in Fig. 6. Suppose (D, α, β) is a minimal, desolate bigon. We can swap the roles of α and β, if needed, to ensure that α is not longer than β. The curve β is replaced by a curve β' parallel to the curve α and not much longer than α. By this we mean that:

(B-1) the endpoints of β' are on γ_α, the curve in $\mathcal{G}$ containing α;
(B-2) the length of $\ell(\beta')$ is less than $\ell(\alpha) + \varepsilon/2$;
(B-3) suppose $\gamma \in \mathcal{G}$ does not contain β or α; there is a bijection between points $\gamma \cap \alpha$ and $\gamma \cap \beta'$.
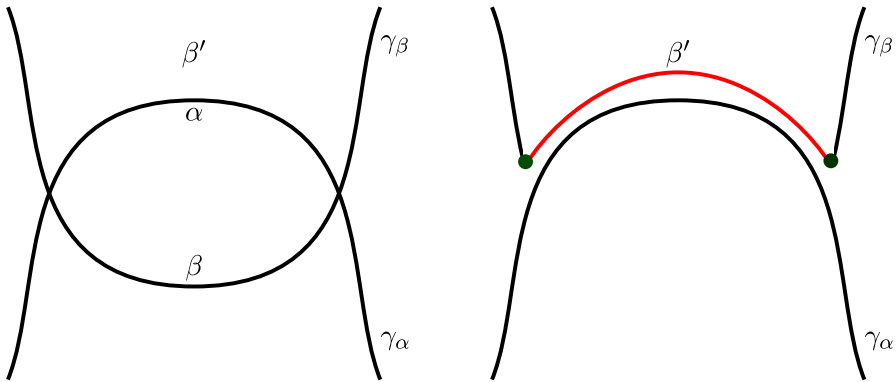
Fig. 6 Bigon removal: **a** output minimal desolate bigon, **b** fragment of one curve replaced with the approximation of the second

Lemma 3.20 *If* (D, α, β) *is a minimal, desolate bigon, then there exists a curve* β' *satisfying* (B-1), (B-2), *and* (B-3).

Proof There are two cases of the proof, depending on whether $\alpha \cap \beta$ contains an element of $\mathcal{X}$. We give a rather detailed account of the first case; then we point out necessary modifications for the second case.

Case 1 Assume that none of the two points of $\alpha \cap \beta$ is an element of the net $\mathcal{X}$. We denote by γ_α and γ_β, the curves in $\mathcal{G}$ such that $\alpha \subset \gamma_\alpha$, $\beta \subset \gamma_\beta$. Set also $\mathcal{G}' = \mathcal{G} \setminus \{\gamma_\alpha, \gamma_\beta\}$. For $\theta > 0$ sufficiently small, let U_θ be the set of points on Σ at distance to α less than or equal to θ. For the sake of the argument, we use the extrinsic distance, that is, on $\mathbb{R}^n$, but this is not compulsory, because our discussion is purely local.

We may and will assume that U_θ is disjoint from $\mathcal{X}$.

The following is an elementary result from calculus. We state it without proof.

Lemma 3.21 *There exists* $\theta_0 > 0$ *such that if* $\theta < \theta_0$, *then* ∂U_θ *is a smooth manifold (of dimension 1).*

Continuation of the proof of Lemma 3.20. The curve ∂U_θ intersects γ_β at four points, which we denote by a_1, a_2, a_3, a_4. The numbering convention is clockwise. We require that the part of ∂U_θ between a_3 and a_4 belongs to the bigon. On shrinking θ, we may and will assume, that the only curve in $\mathcal{G}$ hitting ∂U_θ along the arcs connecting a_4 and a_1, as well as along a_2 and a_3, is α. In Fig. 7, the arcs between a_4 and a_1 and between a_2 and a_3 are 'vertical' parts of ∂U_θ.

We let β' be the arc on ∂U_θ connecting a_1 and a_2; see Fig. 7. We need to make sure that with this choice, if θ is sufficiently small, (B-1)—(B-3) are satisfied.

Suppose $\gamma \in \mathcal{G}'$. Parametrize γ as the image $\kappa : [0, 1] \to \Sigma$ by a C^1-smooth map with non-vanishing derivative. Let $h(t)$ be the square of the distance of $\kappa(t)$ to the curve α. Note that $h(t)$ has an isolated minimum at each t_0 such that $\kappa(t_0) \in \alpha$. On shrinking U_θ if needed (by decreasing the value of θ) we may achieve that whenever $[a, b]$ is an interval such that $\kappa((a, b)) \subset U_\theta$ and κ maps a, b to the boundary of U_θ, we have:

Fig. 7 Proof of Lemma 3.20. Construction of curve β'
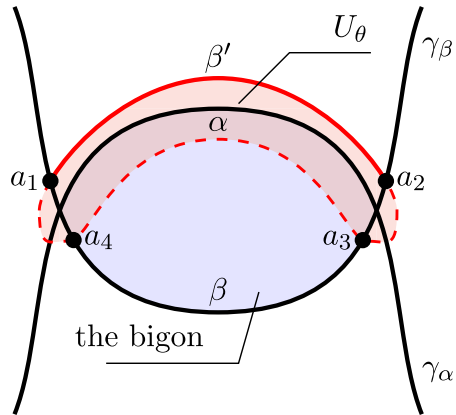
- $h(t)$ has no local maxima on (a, b) and the derivative of h does not vanish at a and b;
- $h(t)$ has precisely one local minimum t_0 on $[a, b]$ and $\kappa(t_0) \in \alpha$.

Finding such θ is straightforward. Namely, we look at all zeros of $h(t)$ (there are finitely many of them), choose a sufficiently small neighborhood U_h of the zeros in such a way that h has no critical points in U_h except for its zero locus. Any number θ smaller than $\inf_{t \notin U_h} \sqrt{h(t)}$ will be enough. Note also, that the two conditions in the itemized list imply that h has non-vanishing derivative at each point where it hits ∂U_θ. This is equivalent to saying that $\kappa(t)$ is transverse to U_θ. Moreover, note that any intersection of $\kappa([a, b])$ with α is a local minimum of $h(t)$. As h has a single local minimum on $[a, b]$, to each intersection point of γ with α (that is to each local minimum of h on $[a, b]$, there correspond precisely two points of intersections of γ with ∂U_θ: these are $\kappa(a)$ and $\kappa(b)$. We claim that out of these two intersections points $\kappa(a), \kappa(b)$, precisely one belongs to β' and one belongs to the segment of ∂U_θ connecting a_3 with a_4.

To justify the claim, note that none of the $\kappa(a), \kappa(b)$ can belong to the segment connecting a_4 with a_1 or a_2 with a_3. Second, α cuts $\overline{U_\theta}$ into two parts, one of which contains β', the other one contains the part of ∂U_θ between a_3 and a_4. If $\kappa([a, b])$ is transverse to α, $\kappa(a)$ and $\kappa(b)$ must belong to different connected components of $\overline{U_\theta} \setminus \alpha$. This shows, that for the particular choice of $\gamma \in \mathcal{G}'$, item (B-3) is satisfied. Item (B-1) follows by construction. Verification that (B-2) holds for $\theta \ll 1$, is left to the reader.

We have chosen θ for a concrete curve $\gamma \in \mathcal{G}'$, but now, we can choose the minimum of all such θ's over a finite set $\mathcal{G}'$.

Case 2 One of the points of the intersection $\alpha \cap \beta$ is in $\mathcal{X}$. Call this point x. As in Case 1, we let U_θ for $\theta \ll 1$ be the set of points at distance to α less than or equal to θ. The main difference with the previous case is that ∂U_θ intersects γ_β at two points and not at four points as above, compare Fig. 8. Call these points a_2 and a_3. We choose the numbering so that a_3 belongs to the boundary of the bigon.

Our aim is to construct the missing two points a_1 and a_4 and use the argumentation from the previous case. To this end, let $v_1, \ldots, v_m$ be the unit tangent vectors at x to

those curves in $\mathcal{G}'$ that emanate from x. Let v_α and v_β be the unit tangent vectors to α and β at x. Choose a cyclic order on the unit vectors in $T_x\Sigma$ in such a way that all unit vectors $v \in T_x\Sigma$ between v_β and v_α point into the bigon (the vectors between v_α and v_β do not point into the bigon then). With this choice, there cannot be a vector v_i between v_β and v_α, because otherwise the curve $\gamma_i \in \mathcal{G}$, which is tangent to v_i, would enter the bigon. This would contradict minimality of the bigon: γ_i cannot be contained in the bigon, so it has to intersect either α or β, creating a smaller bigon.

Without loosing generality, we may and will assume that $v_\beta, v_\alpha, v_1, \ldots, v_m$ are cyclically ordered. In this case, we choose an arc $\widetilde{\alpha}$, passing through x satisfying the following properties.

- $\widetilde{\alpha}$ is a union of two smooth, embedded subarcs $\widetilde{\alpha}_+$ and $\widetilde{\alpha}_-$; each of the $\widetilde{\alpha}_\pm$ has one end point at x and another end point outside U_θ;
- The tangent vector to $\widetilde{\alpha}_+$ lies between v_α and v_1;
- The tangent vector to $\widetilde{\alpha}_-$ lies between v_m and v_β.

A sketch of $\widetilde{\alpha}$ is presented in Fig. 9. There exist points $a_1 \in \widetilde{\alpha}_+ \cap \partial U_\theta$ and $a_4 \in \widetilde{\alpha}_- \cap \partial U_\theta$ such that the whole part of $\widetilde{\alpha}_+$ between x and a_1 belongs to U_θ and the whole part of $\widetilde{\alpha}_-$ between x and a_4 belongs to U_θ. Moreover, if $\theta \ll 1$ is sufficiently small, then no curve $\gamma \in \mathcal{G}$ emanating from the point x intersects $\widetilde{\alpha}_+$ between x and a_1 or $\widetilde{\alpha}_-$ between x and a_4 (except possibly at the point x). Define α' to be the union of the part of $\widetilde{\alpha}_+$ between x and a_1 and the part of $\widetilde{\alpha}_-$ between x and a_4. Then, $\alpha' \subset U_\theta$ and α' separates U_θ into two connected components U_θ^0, U_θ^1 (that is $U_\theta^0 \cup U_\theta^1 = U_\theta \setminus \alpha'$). We number U_θ^0, U_θ^1 in such a way that U_θ^0 is disjoint from the bigon. In Fig. 8, the part U_θ^0 is to the left of $\widetilde{\alpha}$. Any curve $\gamma \in \mathcal{G}$ hitting U_θ^0 eventually terminates at x. Define now β' to be the union of:

- the portion of ∂U_θ between a_1 and a_2;
- the part of $\widetilde{\alpha}_+$ from x to a_1.

The properties (B-1) and (B-2) hold trivially as long as θ is sufficiently small. We show (B-3). Take $\gamma \in \mathcal{G}'$. We need to show the bijection of intersection points of α with γ and β' with γ.

Note that x is the common intersection point of α with γ and β' with γ. Therefore, in the remaining part of the proof, we focus on intersection points different than x. The reasoning is very similar to the one in Case 1, therefore we present only a sketch.

Suppose $\gamma \in \mathcal{G}$ intersects β' at a point $x_1 \neq x$. As γ is disjoint from $\widetilde{\alpha}_+ \setminus \{x\}$, we infer that x_1 belongs to the part of ∂U_θ between a_1 and a_2. In particular, $x_1 \in U_\theta^1$. Hence, the whole connected component γ' of $\gamma \cap U_\theta$ containing x_1, belongs to U_θ^1. The same argument as in Case 1 implies that γ' intersects α at precisely one point x_2 and $x_2 \neq x$.

On the other hand, if $\gamma \in \mathcal{G}$ intersects α at a point $x_2 \neq x$, we again consider γ', the connected component of $\gamma \cap U_\theta$ containing x_2. Clearly, $\gamma' \subset U_\theta^1$. As in Case 1, precisely one of the end points of γ' belongs to the part of ∂U_θ between a_1 and a_2, that is, to β'.

This shows the bijection between intersection points of $\gamma \cap \alpha$ and $\gamma \cap \beta'$, which is required by (B-3). $\qquad\square$

Fig. 8 Proof of Lemma 3.20. Second case. Dashed curves are other curves from $\mathcal{G}$ emanating from the point $x \in \mathcal{X}$
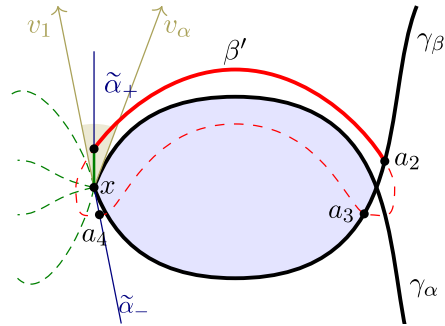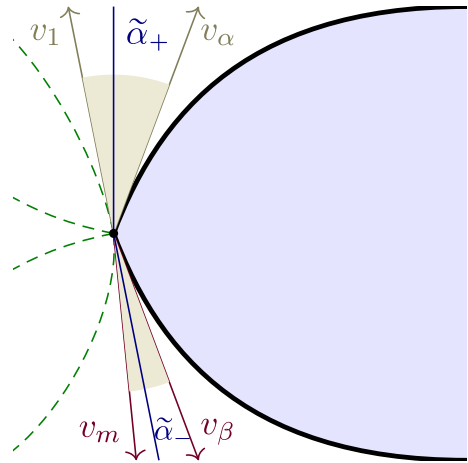

Fig. 9 Proof of Lemma 3.20. Case 2. Position of $\widetilde{\alpha}$. The tangent vectors at x to $\widetilde{\alpha}_+$ and $\widetilde{\alpha}_-$ are required to be in the shaded cones

Lemma 3.22 *If (D, α, β) is a desolate minimal bigon, bigon removal procedure applied to (D, α, β) decreases the number of desolate bigons by 1 and creates no other bigons.*

Proof The number of bigons between two different curves $\gamma, \gamma' \in \mathcal{G}$ is equal to $|\gamma \cap \gamma'| - 1$. Therefore, we will strive to show that the total number of intersection points between all curves in $\mathcal{G}$ decreases after bigon removal.

Let $\gamma_\alpha, \gamma_\beta \in \mathcal{G}$ be such that $\alpha \subset \gamma_\alpha$ and $\beta \subset \gamma_\beta$. Let γ'_β be the curve γ_β with β replaced by β'. We have

$$|\gamma_\alpha \cap \gamma'_\beta| = |\gamma_\alpha \cap \gamma_\beta| - 2.$$

Suppose γ_i is another curve in $\mathcal{G}$. If it does not hit the bigon D, it is also disjoint from the closure of U_θ, for θ sufficiently small, so $\gamma_\beta \cap \gamma_i = \gamma'_\beta \cap \gamma_i$, that is, the number of intersection points is preserved. If γ_i hits the bigon D, we look at connected components of $\gamma_i \cap D$. Each such connected component δ is an arc, and if $|\delta \cap \alpha| = 2$ or $|\delta \cap \beta| = 2$, the arc δ and the relevant part of α or β form a bigon contained in D, contradicting minimality of D. If $|\delta \cap (\alpha \cup \beta)| = 1$, one of the end points of δ is
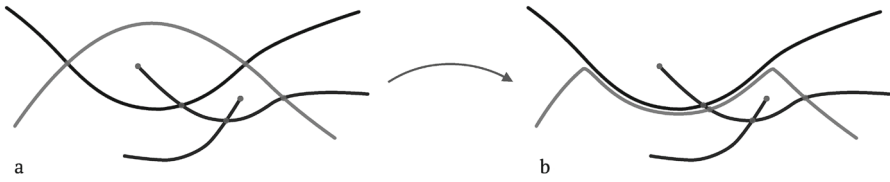
Fig. 10 An attempt to remove an inhabited bigon results in creating another inhabited bigon. There is a little control on the number of bigons that can be produced in this way

inside D, but such an end point must be an end point of γ_β, that is, it must be a point from $\mathcal{X}$. This contradicts the condition that D be desolate.

The only remaining possibility is that $|\delta \cap \alpha| = |\delta \cap \beta| = 1$. This, in turn, shows that $|\gamma_i \cap \alpha| = |\gamma_i \cap \beta|$. Now, by (B-3), $|\gamma_i \cap \beta'| = |\gamma_i \cap \alpha|$. Eventually $|\gamma_i \cap \beta| = |\gamma_i \cap \beta'|$, that is, $|\gamma_i \cap \gamma'_\beta| = |\gamma_i \cap \gamma_\beta|$. In other words, no new bigons are created. □

Remark 3.23 The statement of Lemma 3.22 need not hold if D is inhabited. A bigon removal procedure can create new bigons, both inhabited and desolate, whose number is rather hard to control; see Fig. 10.

After a single bigon removal procedure, the length of one of the curves γ_{ij} can increase, but in a controlled manner, because of (B-2). We decrease ε so that $\ell(\gamma_{ij}) + \varepsilon < 2 \operatorname{dist}(x_i, x_j)$ for all $(i, j) \in \mathcal{I}$.

We now apply inductively the bigon removal procedure to all minimal desolate bigons, until there are no minimal desolate bigons. This requires a finite number of steps. We make the following trivial observation.

Lemma 3.24 *If there are no minimal desolate bigons, there are no desolate bigons at all.*

From now on we will assume that the set $\mathcal{G}$ of curves is such that there are no desolate bigons. The following lemma concludes the proof of Proposition 3.17.

Lemma 3.25 *Suppose curves γ_{ij} and γ_{kl} do not form desolate bigons. Then, the number of intersection points between γ_{ij} and γ_{kl} is bounded by $\mathcal{T}(17)$.*

Proof Suppose γ_{ij} and γ_{kl} are not disjoint. Each bigon formed by γ_{ij} and γ_{kl} belongs to $B(x_i, 17r)$. All such bigons have pairwise disjoint interiors. Moreover, each of the bigons is inhabited, so to each of them we can associate an element of $\mathcal{X} \setminus \{x_i, x_j, x_k, x_l\}$ that belongs to $B(x_i, 17r)$; see the proof of Lemma 3.19. The number of the bigons is bounded from above by the total number of points of $\mathcal{X}$ in $B(x_i, 17r)$ distinct from $\{x_i, x_j, x_k, x_l\}$. According to Proposition 3.4, the number of points of $\mathcal{X}$ in $B(x_i, 17r)$ is bounded by $\mathcal{T}(17)$, so the number of the bigons is smaller than $\mathcal{T}(17) - 3$ (it is -3, not -4, because x_i, x_j, x_k, x_l need not be all distinct).

The number of intersection points is equal to one plus the number of the bigons. The lemma follows. □

The proof of Proposition 3.17 is complete. □

Corollary 3.26 *If $29r < R_0$, there exists a collection of good arcs satisfying both* (T-1) *and* (T-2)*.*

Proof The collection of arcs constructed in Proposition 3.17 satisfies (T-2). As $22r < R_0$, by Proposition 3.8, this collection satisfies also (T-1).   □

4 Triangulation

4.1 Bounding the Number of Triangles

Proposition 4.1 *Let $\mathcal{X}$ be an r-net with $29r < R_0$. Suppose $\mathcal{G}$ is a good tame collection of arcs. Then Σ can be triangulated with at most $S(\Sigma)$ triangles with*

$$S(\Sigma) = \frac{1}{12}\mathcal{T}(4)^2\mathcal{T}(8)^2\mathcal{T}(12)^2\mathcal{T}(17)^2\mathcal{T}(26)^2|\mathcal{X}|. \tag{4.1}$$

Proof The proof of Proposition 4.1 takes the rest of Sect. 4.1. The triangulation is constructed based on $\Gamma = \bigcup_{(i,j)\in\mathcal{I}} \gamma_{ij}$. The vertices are going to be the points in $\mathcal{X}$ as well as the intersection points $\gamma_{ij} \cap \gamma_{kl}$. To begin with, we bound the total number of intersection points of γ_{ij}.

Lemma 4.2 *Suppose $x_i \in \mathcal{X}$. The total number of triples j, k, l such that $\gamma_{ij} \cap \gamma_{kl} \neq \emptyset$ is less than or equal to $2\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)$.*

Proof Assume that $\gamma_{kl} \cap \gamma_{ij} \neq \emptyset$. By Lemma 3.16, either $\gamma_{kl} \cup \gamma_{ij} \subset B(x_i, 12r)$, or $\gamma_{kl} \cup \gamma_{ij} \subset B(x_j, 12r)$.

In the first case, $x_j \in B(x_i, 4r)$, one of the x_k, x_l is in $B(x_i, 8r)$, and the other one is in $B(x_i, 12r)$. The total number of possibilities for x_j, x_k, x_l is therefore bounded by $\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)$.

The case $\gamma_{kl} \cup \gamma_{ij} \subset B(x_i, 12r)$ leads to the same number of possibilities.   □

Let now

$$\mathcal{Z} = \bigcup_{(l,k)\neq(i,j)\neq(k,l)} \{\gamma_{ij} \cap \gamma_{kl}\}.$$

Note that $\mathcal{X} \subset \mathcal{Z}$. Indeed, it is not hard to see that for any i there are at least two points j, l such that $(i, j), (i, l) \in \mathcal{I}$ and then $x_i \in \gamma_{ij} \cap \gamma_{il}$.

Lemma 4.3 *Let $\sigma > 0$ be such that $(\sigma + 4 + 1/4)r < R_0$. Let $\mathcal{Z}_{i_0}^\sigma$ be the number of points $z \in \mathcal{Z}$ such that $z \in B(x_{i_0}, \sigma r)$. Then, $|\mathcal{Z}_{i_0}^\sigma| < \mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)\mathcal{T}(17)\mathcal{T}(\sigma+4)$.*

Proof Take a point $z \in \mathcal{Z} \cap B(x_{i_0}, \sigma r)$. It is in the intersection of two curves $\gamma_{ij} \cap \gamma_{kl}$. We know that for at least one of the indices i, j and for at least one of the indices k, l, the distance of z to the respective endpoint on γ_{ij} and on γ_{kl} is at most $4r$. Hence, at least two of the points x_i, x_j, x_k, x_l belong to $B(x_{i_0}, (\sigma + 4)r)$.

For a fixed i such that $x_i \in \mathcal{X} \cap B(x_{i_0}, (\sigma + 4)r)$, the total number of intersection points $\gamma_{ij} \cap \gamma_{kl}$ for arbitrary j, k, l is $2\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)\mathcal{T}(17)$ by Lemma 4.2 and property (T-2). We sum over all points $x_i \in \mathcal{X} \cap B(x_{i_0}, (\sigma + 4)r)$. We obtain the bound of $2\mathcal{T}(4)\mathcal{T}(8)\mathcal{T}(12)\mathcal{T}(17)\mathcal{T}(\sigma + 4)$. Note, however, that each intersection
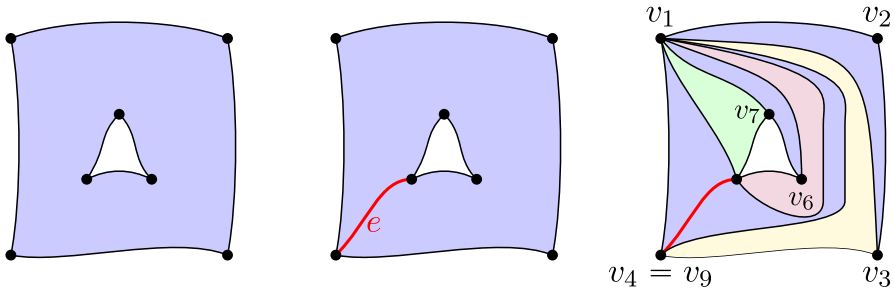
Fig. 11 An example of fan triangulation of C. Left: the starting situation. Middle: step (i) adds an edge. Right: step (ii) triangulates a polygon with connected boundary. The non-labelled vertex of the triangulation in the figure on the right is v_5 (and also v_8)

point was counted at least twice, because at least two points x_i, x_j, x_k, x_l belong to $B(x_{i_0}, (\sigma + 4)r)$. □

Now, we pass to the construction of the triangulation. By the property (T-1), each connected component of $\Sigma \setminus \Gamma$ is homeomorphic to an open a subset of $\mathbb{R}^2$. Take such a connected component C. Its boundary is a union of (parts of) curves γ_{ij} intersecting at points of $\mathcal{Z}$. We think of C as a polygon with vertices in $\mathcal{Z}$ whose edges are parts of curves from $\mathcal{G}$, though we do not necessarily assume that C has connected boundary. An elementary geometric argument allows us to triangulate C by adding pairwise non-intersecting arcs connecting vertices of C. Below we sketch one of possible ways of doing this, called a *fan triangulation*; see Fig. 11.

(i) take two vertices $v, w \in \mathcal{Z}$ of C belonging to different connected components of ∂C. Create an edge e (an arc in C) connecting these vertices. Replace C by $C \setminus e$. Then C is still connected, but its boundary has one less connected component than before;

(ii) once ∂C is connected, suppose $v_1, \ldots, v_M \in \mathcal{Z}$ are its vertices enumerated successively (some vertices might be repeated). Choose pairwise non-intersecting arcs connecting v_1 with $v_3, \ldots, v_{M-1}$. Then, C is triangulated; the triangles have vertices (v_1, v_2, v_3), (v_1, v_3, v_4), ..., (v_1, v_{M-1}, v_M).

This provides us with a triangulation. In particular, the triangulation has the following property.

Property 4.4 *An edge of the triangulation connects two elements in $\mathcal{Z}$, which belong to the closure of the same connected component of $\Sigma \setminus \Gamma$.*

To estimate the number of triangles we use the following lemma.

Lemma 4.5 *Suppose $z_1, z_2 \in \mathcal{Z}$ and z_1, z_2 belong to the closure of the same connected component S_z of $\Sigma \setminus \Gamma$. Then, there exists $x_i \in \mathcal{X}$ such that $z_1, z_2 \in \Sigma \cap B(x_i, 22r)$.*

Proof Let x_i be a point on Σ such that $\text{dist}(x_i, z_1) < r$. Set $y_1 = \pi_{x_i}(z_1)$, $y_2 = \pi_{x_i}(z_2)$. Consider the cycle Δ on H_i as in Corollary 3.12. The image $\pi_{x_i}(S_z)$ belongs to a connected component of $H_i \setminus \Delta$. Now $||y_1 - x_i|| < r$, so y_1 belongs to S_i, the connected

component of $H_i \setminus \Delta$ containing x_i as in the proof of Proposition 3.8. Therefore, $\pi_{x_i}(S_z) \subset S_i$. In particular, by Lemma 3.13, $\pi_{x_i}(S_z) \subset B^{H_i}(x_i, 15r)$. That is to say $\pi_{x_i}(z_2) \subset \overline{B}^{H_i}(x_i, 15r)$. Via (C-5), this implies that $\mathrm{dist}(z_2, x_i) \leq 15r\sqrt{2} < 22r$. □

Corollary 4.6 *The total number of edges in the triangulation is bounded from above by*

$$\frac{1}{8} \mathcal{T}(4)^2 \mathcal{T}(8)^2 \mathcal{T}(12)^2 \mathcal{T}(15)^2 \mathcal{T}(17)^2 \mathcal{T}(26)^2 |\mathcal{X}|.$$

Proof By Lemma 4.5 and Property 4.4, any two edges connecting points z_1 and z_2 belong to the same $B(x_i, 22r) \cap \Sigma$ for some $x_i \in \mathcal{Z}$. We bound the number of points $\mathcal{Z} \cap B(x_i, 22r)$ by Lemma 4.3. Half the square of this bound, that is, $\frac{1}{2}|\mathcal{Z}_i^{22}|^2 \leq \frac{1}{8} \mathcal{T}(4)^2 \mathcal{T}(8)^2 \mathcal{T}(12)^2 \mathcal{T}(17)^2 \mathcal{T}(26)^2$, estimates the number of unordered pairs z_1, z_2 of points $\mathcal{Z}$ in $B(x_i, 22r)$. Summing up over all $x_i \in \mathcal{X}$ we get the result. □

The rest of the proof of Proposition 4.1 is straightforward. Each edge belongs to precisely two triangles and each triangle has three edges. □

4.2 Proof of Theorem 1.1

Suppose $\mathcal{E}_p^\ell(\Sigma) = E$ and let $C_0 = 2^{-1/2} \min(c_1, c_2^{-\alpha})$; compare (2.3) in Corollary 2.5. Here, $\alpha = 1 - 2\ell/p$. Set $r = \frac{1}{29} C_0 E^{-1/p\alpha}$. Choose a net of points $\mathcal{X}$ with this given r. By Proposition 3.3, we have

$$|\mathcal{X}| < C_1 A E^{2/p\alpha}, \tag{4.2}$$

where $C_1 = \frac{1}{\pi} 2^{n/2+4} 29^2 C_0^{-2}$.

Let $\mathcal{G}$ be a collection of good arcs associated with $\mathcal{X}$; such collection exists by Proposition 3.6, because $4r < 29r < R_0 = C_0 E^{-1/p\alpha}$. The collection $\mathcal{G}$ can be improved to a tame collection of good arcs by Corollary 3.26, which works because $29r < R_0$. A tame collection of arcs provides a triangulation with the number of triangles bounded above by $S(\Sigma)$ triangles; see Proposition 4.1. In total, the number of triangles is bounded above by $C_2 A E^{2/p\alpha}$, where

$$C_2 = \frac{1}{12} \mathcal{T}(4)^2 \mathcal{T}(8)^2 \mathcal{T}(12)^2 \mathcal{T}(17)^2 \mathcal{T}(26)^2 C_1.$$

By construction, each triangle is an image of a subset of the tangent plane H_{x_i} to Σ at x_i (for some i) under the map ϕ_{x_i}, which has bounded derivative and bounded distortion by Corollaries 2.5 and 2.6.

Declarations

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