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Properties of Bounded Holomorphic Functions: A Survey

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Abstract
We discuss interrelations between $\mathcal{H}^\infty$-convex domains and $\mathcal{H}^\infty$-domains of holomorphy for various classes of domains in $\mathbb{C}^n$.

Keywords Hartogs domains · Reinhardt domains · Balanced domains · Extension of $\mathcal{H}^\infty$-functions · $\mathcal{H}^\infty$-convexity

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1 Introduction

For a domain $D \subset \mathbb{C}^n$ we denote by $\mathcal{O}(D)$, respectively $\mathcal{H}^\infty(D)$, the space of all holomorphic, respectively all bounded holomorphic functions on D. It is well known that a domain $D \subset \mathbb{C}^n$ is holomorphically convex iff it is a domain of holomorphy (Cartan–Thullen theorem).

(1) There are similar notions of convexity with respect to $\mathcal{H}^\infty(D)$, namely D is $\mathcal{H}^\infty$-convex or D is an $\mathcal{H}^\infty$-domain of holomorphy. In dimension $n = 1$ both notions are the same (see [1]); but in higher dimensions these are, in general, two different properties as we will see later.

(2) To have an idea how large $\mathcal{H}^\infty(D)$ is the Carathéodory pseudodistance $\mathfrak{c}_D$ may be used as a tool (for details see [9]). Recall that

In Memory of Nessim Sibony (1947–2021).

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$$\mathfrak{c}_D(z_1, z_2) = \sup\left\{\tanh^{-1}\left|\frac{f(z_1) - f(z_2)}{1 - f(z_1)\overline{f(z_2)}}\right| : f \in \mathcal{O}(D), |f| < 1\right\}, \quad z_1, z_2 \in D.$$

Obviously, $\mathfrak{c}_D$ is a metric iff $\mathcal{H}^\infty(D)$ separates the points of D (e.g. if D is bounded). Then D is said to be $\mathfrak{c}$-hyperbolic.

(a) Note that for a $\mathfrak{c}$-hyperbolic domain D in dimension $n = 1$ the standard topology of D coincides with the topology induced by the metric $\mathfrak{c}_D$; a fact which does not remain true in higher dimensions.

(b) Moreover, there are two notions of completeness: D is $\mathfrak{c}_D$-Cauchy complete or D is $\mathfrak{c}_D$-finitely compact. D $\mathfrak{c}_D$-finitely compact means that it is $\mathfrak{c}$-hyperbolic and all $\mathfrak{c}$-balls with center in D and finite radius are relatively compact subsets in D. Obviously, $\mathfrak{c}_D$-finitely compact always implies $\mathfrak{c}_D$-Cauchy complete. In dimension $n = 1$ both notions even coincide ([14–16]). In higher dimensions equality is still an open problem. Moreover, switching to complex spaces, there is an example of a complex space X which is $\mathfrak{c}_X$-Cauchy complete but not $\mathfrak{c}_X$ finitely compact.

The main aim of this note is to discuss the topics (1) and (2) from above in higher dimensions.

2 $\mathcal{H}^\infty$-Convexity vs $\mathcal{H}^\infty$-Domain of Holomorphy

Let $\mathcal{F} \subset \mathcal{O}(D)$. Then D is said to be *$\mathcal{F}$-convex* if for any compact set $K \subset D$ the so called $\mathcal{F}$-convex hull $\widehat{K}(\mathcal{F}) := \{z \in D : |f(z)| \le \|f\|_K \text{ for all } f \in \mathcal{F}\}$ is relatively compact in D. Moreover, D is said to be an *$\mathcal{F}$-domain of holomorphy*, if there does not exist a pair of open sets U, V with $\varnothing \ne U \subset D \cap V$, $V \not\subset D$ connected, such that for every $f \in \mathcal{F}$ there exists a function $\widehat{f} \in \mathcal{O}(V)$ with $\widehat{f}|_U = f|_U$. Therefore, any $\mathcal{H}^\infty$-domain of holomorphy is automatically a domain of holomorphy. Observe that more is true, namely: D is an $\mathcal{H}^\infty$-domain of holomorphy iff there exists a function $f \in \mathcal{H}^\infty(D)$ such that for any pair (U, V) as above $f|_U$ is never the restriction of a holomorphic function on V

Note that the property to be an $\mathcal{H}^\infty$-domain of holomorphy is not invariant under biholomorphic mappings; e.g. the Hartogs triangle is an $\mathcal{H}^\infty$-domain of holomorphy but its biholomorphic image $\mathbb{D}_* \times \mathbb{D}$ is not ($\mathbb{D}$ stands for the unit disc, $\mathbb{D}_* := \mathbb{D} \setminus \{0\}$).

Observe that the punctured disc is neither $\mathcal{H}^\infty$-convex nor it is an $\mathcal{H}^\infty$-domain of holomorphy (use the Riemann theorem on removable singularities). On the other hand any bounded fat plane domain D (i.e. $\operatorname{int}\overline{D} = D$) is an $\mathcal{H}^\infty$-domain of holomorphy and therefore also $\mathcal{H}^\infty$-convex. But in higher dimensions, as we will see later, Sibony [16] constructed a fat domain of holomorphy $D \subset \mathbb{D}^2$, $D \ne \mathbb{D}^2$, such that every $f \in \mathcal{H}^\infty(D)$ is the restriction of an $\widehat{f} \in \mathcal{O}(\mathbb{D}^2)$, i.e. D is not an $\mathcal{H}^\infty$-domain of holomorphy.

2.1 Hartogs Domains

In 1972, Sibony (see [15], Séminaire Pierre Lelong 1972–1973, and also [16]) started to investigate which kind of Hartogs' domains are $\mathcal{H}^\infty$-domains of holomorphy. His starting point was a fat pseudoconvex Hartogs domain which is $\mathcal{H}^\infty$-convex, but not an $\mathcal{H}^\infty$-domain of holomorphy.

Example 2.1 [16] This domain is given as follows:

$$D := \{(z_1, z_2) \in \mathbb{D} \times \mathbb{C} : |z_2| < e^{-u(z_1)}\},$$

where $u := \exp(\widetilde{u})$, $\widetilde{u}(\zeta) := \sum_{j=1}^\infty \alpha_j \log|\frac{\zeta - a_j}{2}|$ $(\zeta \in \mathbb{C})$, $(a_j)_j \subset \mathbb{D}$ such that each $b \in \mathbb{T}$ $(\mathbb{T} := \partial\mathbb{D})$ is the non-tangential limit of a certain subsequence of $(a_j)_j$, and $(\alpha_j)_j \subset (0, \infty)$ is such that $\widetilde{u} \not\equiv -\infty$.

Note that $\widetilde{u}$ is a negative subharmonic function on $\mathbb{D}$ with $\widetilde{u}(a_j) = -\infty$ and u is continuous and subharmonic; in particular, D is fat and a domain of holomorphy. Moreover, every bounded holomorphic function on D extends to an holomorphic function on $\mathbb{D}^2$, thus D is not an $\mathcal{H}^\infty$ domain of holomorphy.

Indeed, let $f \in \mathcal{H}^\infty(D)$. We may assume that $|f| \leq 1$. Then $f(z, w) = \sum_{j=0}^\infty g_j(z) w^j$ with $g_j \in \mathcal{O}(\mathbb{D})$ (the Hartogs series of f) and $|g_j| \leq \exp(ju)$ (use Cauchy inequalities). Thus, g_j is bounded and $|g_j(a_k)| \leq 1$ for all $k \in \mathbb{N}$. Using the Fatou theorem it follows $|g_k| \leq 1$. Therefore the series representing f is locally uniformly convergent on $\mathbb{D}^2$, i.e. it gives a holomorphic extension of f to $\mathbb{D}^2$.

To see that D is $\mathcal{H}^\infty$-convex it suffices to mention that any $f \in \mathcal{O}(D)$ can be locally uniformly approximated by polynomials (use the representation of f by its Hartogs series as above).

In [13] it is shown that using different sequences $(a_j)_j$ and $(\alpha_j)_j$ as above there is an infinite family of such Sibony domains pairwise not biholomorphically equivalent.

To conclude the discussion of the above example we mention, using Fatou theorem, that

$$\widehat{u} := \sup\{\tfrac{1}{j} \log|f| : j \in \mathbb{N},\ f \in \mathcal{O}(\mathbb{D}) \text{ with } \log|f| \leq ju\} = 0,$$

while $\exp(-\widehat{u}) \gneqq \exp(-u)$. Soon it will be clear why this expression plays an important role to describe $\mathcal{H}^\infty$-domains of holomorphy.

Thus there is a need to characterize such Hartogs domains that are $\mathcal{H}^\infty$-domains of holomorphy. Before discussing such properties repeat the following definition of a special class of Hartogs domains. Let $\Omega \subset \mathbb{C}^n$ be a domain and $H : \Omega \times \mathbb{C}^m \longrightarrow \mathbb{R}_+$ upper semicontinuous satisfying $H(z, \lambda w) = |\lambda| H(z, w)$, $z \in \Omega$, $w \in \mathbb{C}^m$, and $\lambda \in \mathbb{C}$. Then the following domain

$$D := D(\Omega, H) := \{(z, w) \in \Omega \times \mathbb{C}^m : H(z, w) < 1\}$$

is called a *Hartogs domain with balanced m-dimensional fibers over the basis Ω*. Note that with $H(z, w) := |w| \exp(u(z))$ the D discussed before is a Hartogs domain with 1-dimensional balanced fibers over the base $\mathbb{D}$.

Recall that D is a domain of holomorphy iff Ω is a domain of holomorphy and $\log H$ is plurisubharmonic on $\Omega \times \mathbb{C}^m$ ([10], Proposition 2.2.31).

To formulate the main result in this context the following to H associated function is needed. Let H be as above. Put

$$\widehat{H}(z,w) := \sup\{|Q(z,w)|^{1/j} : \\ j \in \mathbb{N},\ Q(\zeta,\omega) = \sum_{|\alpha|=j} a_\alpha(\zeta)\omega^\alpha,\ (\zeta,\omega) \in \Omega \times \mathbb{C}^m,\ a_\alpha \in \mathcal{O}(\Omega),\ |Q|^{1/j} \le H\}.$$

Note that $(\log \widehat{H})^*$ is plurisubharmonic on $\Omega \times \mathbb{C}^m$ (recall that f^* denotes as usual the upper semicontinuous regularization of f). Therefore, if Ω is a domain of holomorphy, then $D^* := \{(z,w) \in \Omega \times \mathbb{C}^m : \widehat{H}^*(z,w) < 1\}$ is a domain of holomorphy.

Using Hartogs series it is easy to see that any bounded holomorphic function on D extends holomorphically to $D^* := \{(z,w) \in \Omega \times \mathbb{C}^m : \widehat{H}^*(z,w) < 1\}$. Thus, if $D(\Omega, H)$ is an $\mathcal{H}^\infty$-domain of holomorphy, then $H = \widehat{H}^*$. Conversely, the following is true (cf. [10], Theorem 4.1.70).

Theorem 2.2 *Let $D = D(\Omega, H)$ be as above. Assume that the basis Ω is an $\mathcal{H}^\infty$-domain of holomorphy or that $H(z,w) \underset{\Omega \ni w \to \partial\Omega}{\longrightarrow} \infty$ for all w with $\|w\| = 1$.*

If $H = \widehat{H}^*$, then D is an $\mathcal{H}^\infty$-domain of holomorphy.

Remark 2.3 Denote by $\mathbb{B}(0,r) := \{z \in \mathbb{C}^2 : \|z\| < r\}$ the two-dimensional ball with center at zero and radius r and $\mathbb{B} := \mathbb{B}(0,1)$. Put $\Omega := \mathbb{B} \setminus \overline{\mathbb{B}(0,1/2)}$. Then the following domain

$$D := \{(z,w) \in \Omega \times \mathbb{C} : |w| < 1\}$$

is a Hartogs domain with $H(z,w) = |w|$ whose basis is not an $\mathcal{H}^\infty$-domain of holomorphy. Moreover, also the second assumption in the above theorem is not fulfilled. Obviously, $H = \widehat{H}$ but D is not an $\mathcal{H}^\infty$-domain of holomorphy. Thus, at least one of the above two assumptions is needed for the correctness of the theorem.

Let now $D = D(\Omega, u) := \{(z,w) \in \Omega \times \mathbb{C} : |w| < \exp(-u(z))\}$, where $\Omega \subset \mathbb{C}^n$ is a domain and $u : \Omega \longrightarrow \mathbb{R}$ is upper semicontinuous on Ω; i.e. D is a Hartogs domain with 1-dimensional balanced fibers over Ω. Then (cf. [10], Corollary 4.1.71):

Corollary 2.4 *Let D as before and assume that Ω is an $\mathcal{H}^\infty$-domain of holomorphy or that $\lim_{\Omega \ni z \to \partial\Omega} u(z) = \infty$.*

Then the following properties are equivalent:

(i) *D is an $\mathcal{H}^\infty$-domain of holomorphy;*
(ii) $u = \Big(\sup\{1/j \log|f(z)| : j \in \mathbb{N},\ \log|f| \le ju\}\Big)^*$.

Define $\widehat{u} := \sup\{1/j \log|f(z)| : j \in \mathbb{N},\ \log|f| \le ju\}$. Note that if $H(z,w) := |w|e^{-\log u}$, then $\widehat{H}^*(z,w) = |w|e^{-\log \widehat{u}^*}$ on $\Omega \times \mathbb{C}$.

Compare the above condition with the example given at the beginning of this section.

Note that the usual Hartogs triangle

$$D = \{(z,w) \in \mathbb{D}_* \times \mathbb{C} : |w| < |z| < 1\}$$

is a Hartogs domain with balanced 1-dimensional fibers over $\mathbb{D}_*$; i.e. $D = D(\mathbb{D}_*, \log|z|)$. Note that its base is not a $\mathcal{H}^\infty$-domain of holomorphy and $\lim_{\mathbb{D}_* \ni z \to \partial\mathbb{D}} u(z) = \infty$ is not fulfilled. Nevertheless, D is an $\mathcal{H}^\infty$-domain of holomorphy and both properties (i) and (ii) are true. Moreover, it is not $\mathcal{H}^\infty$-convex. *Note that it seems not to be known which weaker assumptions could also give the correctness of the corollary.*

Remark 2.5 It is important to mention that in [16] a somehow weaker result is presented. For example, take the following Hartogs domain with one dimensional fibers over $\mathbb{C}$:

$$D := \{(z, w) \in \mathbb{C} \times \mathbb{C} : |w| < \exp(-\sqrt{2} \log|z|)\}.$$

Then any bounded holomorphic function on D is constant, i.e. it extends holomorphically to the domain $\mathbb{C} \times \mathbb{C}$ (use power series expansion or show $\widehat{u} = -\infty$), while using the results in [16] one gets no holomorphic extension.

This kind of phenomenon does not occur if the function u in the definition of a Hartogs domain with one-dimensional fibers is everywhere positive.

2.2 Reinhardt Domains

Let $X \subset \mathbb{R}^n$ be a non empty convex domain. Then $\mathbf{E}(X)$ denotes the vector subspace of $\mathbb{R}^n$ satisfying $X + \mathbf{E}(X) = X$ such that every vector subspace $F \subset \mathbb{R}^n$ with $X + F = X$ satisfies $\dim F \leq \dim \mathbf{E}(X)$. Note that $\mathbf{E}(\mathbf{X})$ is well defined. For example, if X is bounded, then $\mathbf{E}(X) = \{0\}$. A vector subspace F is called to be of *rational type*, if F is generated by $F \cap \mathbb{Q}^n$. And X is called of *rational type* if $\mathbf{E}(X)$ is of this type.

Example 2.6 Let $X := \{(x_1, x_2) \in \mathbb{R}^2 : \mu x_1 < x_2 < 2 + \mu x_1\}$, $\mu \in \mathbb{R}$. Obviously, X is convex. Then $\mathbf{E}(X) = \mathbb{R}(1, \mu)$. Hence, X is of rational type iff $\mu \in \mathbb{Q}$.

Recall that a Reinhardt domain D of holomorphy has the property that $\log D$ is a convex domain. Thus one use the former notation and say that D is of *rational type* if $\log D$ is of rational type.

Theorem 2.7 *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain of holomorphy. Then: D is an $\mathcal{H}^\infty$-domain of holomorphy iff D is fat and of rational type (cf.* [8]*).*

In particular, any fat bounded Reinhardt domain of holomorphy is an $\mathcal{H}^\infty$-domain of holomorphy.

For example, the so-called Hartogs triangle $D := \{z \in \mathbb{C}^2 : |z_2| < |z_1| < 1\}$ is an $\mathcal{H}^\infty$-domain of holomorphy. Note that this D is not $\mathcal{H}^\infty$-convex in contrast to the plane situation. But surprisingly, the following converse result is true.

Theorem 2.8 *Any $\mathcal{H}^\infty$-convex Reinhardt domain $D \subset \mathbb{C}^n$ is an $\mathcal{H}^\infty$-domain of holomorphy.*

Proof [1] By assumption, D is a domain of holomorphy. Therefore, $D = \operatorname{int}\overline{D} \setminus \bigcup_{j:V_j\cap D=\varnothing} V_j$, where $V_k := \{z \in \mathbb{C}^n : z_k = 0\}$, $k = 1, \dots, n$.

Suppose now that D is not an $\mathcal{H}^\infty$-domain of holomorphy.

Case 1. D is not fat: Then there exists a V_j with $V_j \cap D = \varnothing$ and $V_j \cap \operatorname{int}\overline{D} \neq \varnothing$. W.l.o.g. let $j = n$. Then there is a point $b = (b_1, \dots, b_{n-1}, 0) \in \operatorname{int}\overline{D}$. Thus, for a small positive r, $\mathbb{P}(b, r) \subset \operatorname{int}\overline{D}$, where $\mathbb{P}(b, r) := \{z \in \mathbb{C}^n : |z_j - b_j| < r,\ j = 1, \dots, n\}$. Then one finds a point $a = (a_1, \dots, a_{n-1}, 0)$ with $a_1 \cdots a_{n-1} \neq 0$ such that $\mathbb{P}(a, r') \subset \operatorname{int}\overline{D}$. In particular, $\{a_1\} \times \cdots \times \{a_{n-1}\} \times \mathbb{D}_*(0, r') \subset D$. Using Riemann's theorem we see that the $\mathcal{H}^\infty$-hull of the compact set $K := \{a_1\} \times \cdots \times \{a_{n-1}\} \times \partial\mathbb{D}_*(0, r'/2) \subset D$ is not relatively compact in D; a contradiction.

Case 2. D is not of rational type, but fat: Let $X := \log D$, $\widehat{X} := \log \widehat{D}$, $r := n - \dim E(\widehat{X})$. Obviously, $1 \le r \le n - 1$. By [8] there exists a basis $\alpha^1, \dots, \alpha^r \in \mathbb{Z}^n$ of $E(\widehat{X})^\perp$ and a n-circled domain $\Omega \subset \mathbb{C}^r$, $\Phi(D \setminus V_0) \subset \Omega$, such that any bounded holomorphic function $f \in \mathcal{H}^\infty(D)$ can be written as $f(z) = F \circ \Phi(z)$, $z \in D \setminus V_0$, where $F \in \mathcal{H}^\infty(\Omega)$.

Let $K := \{a\}$, where $a \in D \cap \mathbb{R}^n_{>0}$. It suffices to prove that the $\mathcal{H}^\infty(D)$-hull of K is not relatively compact in D. Thus, it suffices to show that the set $Q := \{z \in D \setminus V_0 : \Phi(z) = \Phi(a)\}$ is not relatively compact in D.

Suppose that $Q \subset D$. Then $E(X) \supset \{x \in \mathbb{R}^n : \langle \alpha_j, x\rangle = 0,\ j = 1, \dots, r\} = E(\widehat{X}) \supset E(X)$ and consequently, $E(X) = E(\widehat{X})$. In particular, D is of the rational type contradicting the assumption.

Now, let $b \in Q \setminus D$, $b \notin V_0$. Since X is convex there exists a point $c \in Q \cap \partial D$ such that $\{a^{1-t}c^t \in D : 0 \le t < 1\}$, which finishes the proof. □

For Reinhardt domains D of holomorphy which are not $\mathcal{H}^\infty$-domains of holomorphy, there is even a way to calculate their $\mathcal{H}^\infty$-envelope of holomorphy, i.e. the $\mathcal{H}^\infty$-domain of holomorphy $\widehat{D}$, $D \subset \widehat{D}$, such that every bounded holomorphic function f on D is the restriction of an $\widehat{f} \in \mathcal{H}^\infty(\widehat{D})$. Therefore, any Reinhardt domain of holomorphy allows a univalent (schlicht) $\mathcal{H}^\infty$envelope of holomorphy (see [8]).

Instead of giving details how to get the $\mathcal{H}^\infty$-envelope two examples are presented.

Example 2.9 (a) Let

$$D := \{z \in \mathbb{C}^3 : |z_1|^{\alpha_1} \cdot |z_2|^{\alpha_2} \cdot |z_3|^{\alpha_3} < 1,\ |z_1|^{1-\alpha_1} \cdot |z_2|^{1-\alpha_2} \cdot |z_3|^{1-\alpha_3} < 1\},$$

where $0 < \alpha_j < 1$, $\alpha_1 \neq \alpha_2$, and $\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2} \notin \mathbb{Q}$. Note D is a fat Reinhardt domain of holomorphy which is not of rational type. Hence D is not an $\mathcal{H}^\infty$-domain of holomorphy. Moreover, its $\mathcal{H}^\infty$-envelope of holomorphy is given by $\widehat{D} = \{z \in \mathbb{C}^3 : |z_1z_2z_3| < 1\}$.

(b) Let

$$D := \{z \in \mathbb{C}^4 : |z_2|^{\sqrt{2}}|z_3| < 1, |z_1z_2z_3z_4| < 1, |z_1|^{\sqrt{2}}|z_4| < 1\}.$$

Then D is not an $\mathcal{H}^\infty$-domain of holomorphy. Its $\mathcal{H}^\infty$-envelope of holomorphy is given as $\widehat{D} := \{z \in \mathbb{C}^4 : |z_1z_2|^{\sqrt{2}}|z_3z_4| < 1, |z_1z_2z_3z_4| < 1\}$.

[1] Since this result seems to be a new one, we add its proof.

Remark 2.10 Recall that Reinhardt domains of holomorphy have a univalent $\mathcal{H}^\infty$-envelope of holomorphy. In general there are domains in $\mathbb{C}^n$ whose $\mathcal{H}^\infty$-envelope of holomorphy is infinitely many sheeted (see [2] and [4]); in particular, their $\mathcal{H}^\infty$-envelope is not schlicht. The construction of these examples is based on the Sibony example (see 2.1). It seems there are no criteria known which guaranties for a domain to have a schlicht $\mathcal{H}^\infty$-envelope of holomorphy.

2.3 Balanced Domains

A domain $D \subset \mathbb{C}^n$ is called to be *balanced* (or equivalently *complete circular*) if whenever $z \in D$ and $\lambda \in \mathbb{D}$, then $\lambda z \in D$. Such a D may be also described as $D = \{z \in \mathbb{C}^n : h(z) < 1\}$, where $h : \mathbb{C}^n \longrightarrow [0, \infty)$ is a upper semicontinuous function satisfying $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$. This function is known as the *Minkowski* function for D and it is uniquely determined.

Moreover, it is well known that D is a domain of holomorphy iff its Minkowski function h is plurisubharmonic iff $\log h$ is plurisubharmonic ([10], Proposition 2.2.31).

Note that a complete Reinhardt domain is in particular balanced and its Minkowski function is even everywhere continuous. Then the following is true:

Theorem 2.11 *Let $D = \{z \in \mathbb{C}^n : h(z) < 1\}$ is a bounded balanced domain of holomorphy with its Minkowski function h. Then D is $\mathcal{H}^\infty$-convex. Even more is true, namely, D is convex with respect to the family of homogeneous polynomials. In particular, any bounded balanced $\mathcal{H}^\infty$-domain of holomorphy is $\mathcal{H}^\infty$-convex*

Note that the following unbounded balanced domain of holomorphy $D := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2| < 1\}$ is not $\mathcal{H}^\infty$-convex.

Example 2.12 Let $(a_j)_j \subset \mathbb{T} := \partial\mathbb{D}$ be a dense subset in $\mathbb{T}$ and let $(\alpha_j)_j \subset (0, \infty)$ with $\sum_{j=1}^\infty \alpha_j = 1$. Put $u(z_1, z_2) := \sum_{j=1}^\infty \alpha_j \log|z_1 - a_j z_2|$ and $h(z) := \exp u(z) + \max\{|z_1|, |z_2|\}$. Then h is positive homogeneous and plurisubharmonic; thus $D := \{z \in \mathbb{C}^2 : h(z) < 1\}$ is a bounded pseudoconvex balanced domain. As above, then D is $\mathcal{H}^\infty$-convex. the set of points at which h is not continuous, is of measure zero. Therefore, D is a fat domain. Nevertheless, D is not an $\mathcal{H}^\infty$-domain of holomorphy.

Note that also here there is no equivalence between the notions of $\mathcal{H}^\infty$-convex and to be an $\mathcal{H}^\infty$-domain of holomorphy.

For an arbitrary domain $D \subset \mathbb{C}^n$ put

$$\widehat{\overline{D}} := \{z \in \mathbb{C}^n : |Q(z)| \leq \|Q\|_D \text{ for every homogeneous polynomial } Q\}.$$

If D is a balanced domain, then

$$\widehat{\overline{D}} := \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_D \text{ for every polynomial } p\};$$

i.e. $\widehat{\overline{D}}$ is the *polynomial convex hull* of $\overline{D}$.

The following characterization of a balanced domain to be an $\mathcal{H}^\infty$-domain of holomorphy is due to Siciak ([17]).

Theorem 2.13 *Let $D = \{z \in \mathbb{C}^n : h(z) < 1\}$ be a bounded balanced domain of holomorphy. Then the following properties are equivalent:*

(a) *D is an $\mathcal{H}^\infty$-domain of holomorphy;*
(b) *$D = \operatorname{int} \widehat{\overline{D}}$ (in particular, D is fat).*
Moreover, if D is bounded, then (a)*, resp.* (b)*, is equivalent to:*
(c) *h is plurisubharmonic and the set $\{z \in \mathbb{C}^n : h$ is not continuous at $z\}$ is pluripolar.*

The proof of the former theorem is based on two deep results, namely the theorem of Bedford–Taylor and a modification of Josefson's theorem due to Siciak.

Remark 2.14 (1) In the case of unbounded balanced domains of holomorphy, (c) does not necessarily imply (a) of the above theorem; take, for example, the unbounded Reinhardt domain $D_0 := \{z \in \mathbb{C}^2 : |z_1||z_2|^{\sqrt{2}} < 1\}$. Its Minkowski function is given as $h(z) = (|z_1||z_2|^{\sqrt{2}})^{\frac{1}{1+\sqrt{2}}}$ and it is obviously plurisubharmonic and continuous. But any bounded holomorphic function on D is constant; therefore, D is not an $\mathcal{H}^\infty$-domain of holomorphy.

(2) Obviously, if $D \subset \mathbb{C}^n$ is an arbitrary fat domain satisfying that $\overline{D}$ is polynomially convex, then $D = \operatorname{int} \widehat{\overline{D}}$. In particular, every balanced domain $D \subset \mathbb{C}^n$ which is fat and whose closure is polynomially convex is an $\mathcal{H}^\infty$-domain of holomorphy. The converse remains true for a bounded balanced domain D of holomorphy in $\mathbb{C}^2$, i.e. $D = \operatorname{int} \widehat{\overline{D}}$ iff D is fat and $\overline{D}$ is polynomially convex. The proof of this result (see [18]) is based on the following description of a subharmonic function u with $a + \log(1 + |z|) \le u(z) \le b + \log(1 + |z|)$, $z \in \mathbb{C}$:

$$u_*(z) = \sup\{|p(z)|^{1/k} : p \text{ a polynomial of degree } k, \text{ such that } u \le \tfrac{1}{k} \log |p|\},$$

where u_* denotes the lower semicontinuous regularization of u.

(3) On the other hand, there is a fat bounded balanced domain $D \subset \mathbb{C}^3$ of holomorphy with $D = \operatorname{int} \widehat{\overline{D}}$, but its closure $\overline{D}$ is not polynomially convex (see [6]). Let $(a_j)_j \subset \mathbb{T}$ be dense in $\mathbb{T}$ and $(\alpha_j)_j \subset (0, \infty)$. For $z \in \mathbb{C}^3$ put

$$h(z) := \exp\Big(\sum_{j=1}^{\infty} \alpha_j \max\{\log |z_1 - a_j z_2|, \log |z_3|\}\Big) + \max\{|z_1|, |z_2|, |z_3|\}.$$

Then $D := \{z \in \mathbb{C}^3 : h(z) < 1\}$ is a bounded fat $\mathcal{H}^\infty$-domain of holomorphy with $A := \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \{0\} \subset \widehat{\overline{D}}$, but $A \not\subset \overline{D}$, e.g. $(1, 0, 0) \notin \overline{D}$.

2.4 Smoothly Bounded Domains

This survey is finished with a short discussion on smooth pseudoconvex domains. The important results here are due to Catlin (see [5]) and Hakim–Sibony (see [7]).

Theorem 2.15 *Let $D \subset \mathbb{C}^n$ be a bounded domain of holomorphy with $\mathcal{C}^\infty$-smooth boundary. Then D is an $\mathcal{H}^\infty$-domain and it is $\mathcal{H}^\infty$-convex.*

The proof of the first result is based on regularity results due to J.J. Kohn while the proof of the second assertion uses good plurisubharmonic exhaustion functions and the Jensen measure.

Remark 2.16 (a) In fact, for such domains even more is true; namely, they are $\mathcal{A}^\infty$-domains of holomorphy and $\mathcal{A}^\infty$-convex. Recall that

$$\mathcal{A}^\infty := \{f \in \mathcal{O}(D) : \text{all derivatives of } f \text{ are continuous on } \overline{D}\}.$$

(b) It seems to be unknown what happens in the case the boundary is only assumed to be $\mathcal{C}^k$-smooth.

2.5 Miscellanea

In [3], general open sets which are given as sublevel sets of certain plurisubharmonic functions are discussed. Let

$$u \in L(\mathbb{C}^n) := \{v \in \mathcal{PSH}(\mathbb{C}^n) : \exists_{r \in \mathbb{R}} : v(z) \leq r + \log(1 + \|z\|), z \in \mathbb{C}^n\}.$$

Put $D_u := \{z \in \mathbb{C}^n : u(z) < 0\}$. Note that balanced domains of holomorphy are examples of such D_u. Then:

Theorem 2.17 *Let D_u be as above and, in addition, bounded. If the set of points where u is not continuous is pluripolar, then every connected component of D_u is an $\mathcal{H}^\infty$-domain of holomorphy.*

Remark 2.18 Observe that there is a function $u \in L(\mathbb{C}^n)$ with

$$\inf\{u(z) - \log(1 + \|z\|)\} > -\infty$$

whose set of discontinuity is not pluripolar, but nevertheless D_u is an $\mathcal{H}^\infty$-domain of holomorphy. Note that D_u is bounded. Compare Theorem 2.13

3 The Carathéodory Metric

3.1 The Carathéodory Topology

If $\mathcal{H}^\infty(D)$ separates the points of D, then $\mathfrak{c}_D$ is in fact a distance and $(D, \mathfrak{c}_D)$ is a metric space. Therefore, $\mathfrak{c}_D$ induces a topology top $\mathfrak{c}$ on D, the $\mathfrak{c}$-topology on D. Using the formulas for the ball it is easy to see that in case of a bounded domain $D \subset \mathbb{C}^n$ this $\mathfrak{c}_D$-topology is the same as the standard Euclidean topology top of D. Moreover, (see [16]):

Theorem 3.1 *If $D \subset \mathbb{C}$ is $\mathfrak{c}$-hyperbolic, then its $\mathfrak{c}$-topology coincides with the standard Euclidean topology induced on D.*

In higher dimensions, this result does not remain to be true (see [11]).

Theorem 3.2 *For any $n > 2$ there exists a $\mathfrak{c}$-hyperbolic domain $D \subset \mathbb{C}^n$ such that* $\operatorname{top}\mathfrak{c} \subsetneq \operatorname{top} D$.

The proof relies heavily on the Remmert embedding theorem. For the dimension $n = 2$ an example like the one in the former theorem is, so far, not known.

3.2 $\mathfrak{c}$-Completeness

Let $D \subset \mathbb{C}^n$. D is said to be $\mathfrak{c}$-*finitely compact* if it is $\mathfrak{c}$-hyperbolic and all $\mathfrak{c}$-balls with center in D and finite radius are relatively compact subsets in D. Moreover, D is called to be $\mathfrak{c}$-*complete* if D is $\mathfrak{c}$-hyperbolic and any $\mathfrak{c}$-Cauchy sequence converges in the Euclidean norm to a point in D.

Obviously, if D is $\mathfrak{c}$-finitely compact, then it is $\mathfrak{c}$-complete. Note that for certain domains (for example an annulus) $\mathfrak{c}$ is not an inner metric. Therefore the theorem of Rinow cannot be applied. Nevertheless, in [15, 16] and [14] it has been shown that the converse is true for plane domains.

Theorem 3.3 *Let $D \subset \mathbb{C}$ be a $\mathfrak{c}$-hyperbolic domain. Then D is $\mathfrak{c}$-complete iff it is $\mathfrak{c}$-finitely compact.*

The proof needs rather difficult one-dimensional results.

So far it is still an *open problem* whether the above equivalence remains true in higher dimensions. Nevertheless, for certain classes of domains this equivalence holds (see [19]).

Theorem 3.4 *Let $D \subset \mathbb{C}^n$ be a $\mathfrak{c}$-hyperbolic Reinhardt domain of holomorphy. Then D is $\mathfrak{c}$-finitely compact iff D is $\mathfrak{c}$-complete.*

Remark 3.5 It should be mentioned that Theorem 3.3, in general, fails to hold for one-dimensional complex spaces (see [12]).

Finally, the following remarks put the completeness properties of $\mathfrak{c}$ in relation with properties of $\mathcal{H}^\infty$.

Remark 3.6 (a) Let $D \subset \mathbb{C}^n$ be $\mathfrak{c}$-hyperbolic. Then D is $\mathfrak{c}$-finitely compact iff for any point $a \in D$ and any sequence $(a_j)_{j\in\mathbb{N}} \subset D$ without accumulation points in D there exists $f \in \mathcal{O}(D, \mathbb{D})$ with $f(a) = 0$ and $\sup_j |f(a_j)| = 1$.

(b) Therefore, any $\mathfrak{c}$-finitely compact domain is an $\mathcal{H}^\infty$-domain of holomorphy and $\mathcal{H}^\infty$-convex.

(c) Any $\mathfrak{c}$-complete domain is an $\mathcal{H}^\infty$-domain of holomorphy; in particular, it is a domain of holomorphy.

(d) It should be emphasized that so far it is still unclear, whether $\mathfrak{c}$-completeness also implies $\mathcal{H}^\infty$-convexity.

(e) Applying the criterium of (a) shows, for example, that strongly pseudoconvex domains are $\mathfrak{c}$-finitely compact. Moreover, any $\mathbb{C}$-convex bounded domain is $\mathfrak{c}$-finitely compact. On the other side it is still open whether a bounded domain of holomorphy with a $\mathcal{C}^\infty$-boundary is $\mathfrak{c}$-finitely compact.

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