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CORRECTION

Correction to: A Weak Reverse Hölder Inequality for Caloric Measure

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In our paper [1], in which we presented a parabolic version of results of [2], the proof of Lemma 2.7 (the main lemma) was divided into several cases, one of which we had inadvertently failed to note and treat. We now rectify this omission. Fortunately, this will be a simple matter. On the other hand we advise the reader to have on hand a copy of the published version of the paper [1], as we will follow the notation already established there, and we shall frequently provide precise references to the text of the original manuscript.

We note that we make the convention here and in [1] that the time axis runs horizontally from left to right. We recall (see [1,Definition 1.9, p. 1535]) that we define a parabolic "cube" as follows:

$$Q_r = Q_r(X,t) := \{(Y,s) \in \mathbb{R}^n \times \mathbb{R} : |X_i - Y_i| < r,\ 1 \le i \le n,\ t - r^2 < s < t + r^2\}.$$

We shall refer to r as the "side length" of Q_r. The "back face" of Q_r is the "left hand face" of Q_r, i.e., the portion of ∂Q_r with time coordinate $T_{\min}(Q_r) = t - r^2$.

We recall that the parabolic length is $\|(X,t)\| := |X| + |t|^{1/2}$ (see [1,p. 1537]).

In the course of the argument in [1,Proof of Lemma 2.7], we have fixed a cube $Q_r(x_0,t_0)$, centered on the "quasi-lateral" boundary Σ (see [1,(1.13), p. 1536]), and we consider a particular cube $Q_{\widehat{r}_k}(x_0,t_0)$ concentric with $Q_r(x_0,t_0)$, for which $5r/4 < \widehat{r}_k < 3r/2$ (see [1,p. 1545]). For convenience in this note, we set $Q(k) := Q_{\widehat{r}_k}(x_0,t_0)$.

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As in [1,p. 1545], we set

$$S_k := \partial Q_{\widehat{r}_k}(x_0, t_0) \cap \Omega = \partial Q(k) \cap \Omega.$$

In addition, we let F_k denote the "back face" of $Q(k)$, as discussed above, and as in [1], we use the term "back face of S_k" to refer to $F_k \cap \Omega$. (To be completely honest, we explicitly stated in [1] only the meaning of the "back face of a cube Q", not the "back face of S_k", but our intended meaning for the latter term was the one stated here.)

In [1,Proof of Lemma 2.7], using the continuity of $\delta(Y, s) := \operatorname{dist}((Y, s), \Sigma)$, we had then considered the following 3 cases (the various parameters are discussed in [1,pp. 1544–1545]):

Case 1: There is a point (Y_0, s_0) on the back face of S_k such that $\delta(Y_0, s_0) = \dfrac{\epsilon'}{200} r$.

Or, in the event that there is no such point.

Case 2: For every point (Y, s) on the back face of S_k, we have $\delta(Y, s) > \epsilon' r/200$.
Case 3: For every (Y, s) on the back face of S_k, we have $\delta(Y, s) < \epsilon' r/200$.

However, these three cases do not exhaust all possible scenarios: in [1], we had neglected the possibility that $F_k \cap \Omega$ (i.e., the back face of S_k) could be empty. Thus, to cover all possible situations, we modify Case 3 as follows:

Case 3: Either $F_k \cap \Omega = \emptyset$, **or** for every (Y, s) on the back face of S_k, we have $\delta(Y, s) < \epsilon' r/200$.

Let us now explain how to treat this modified version of Case 3. We shall adjust somewhat our treatment of the original (incomplete) version of Case 3, to allow us to deal with the two scenarios of the modified Case 3 in a unified manner. In addition to being complete, this new approach will have the added virtue of slightly simplifying part of the argument (in particular, we can dispense with the decomposition of the region $\widehat{I_k}$ in the last paragraph on p. 1547 of [1], as well as Claim 2 and its proof on p. 1548).

Recall that $\widehat{r}_k$ is the "side length" of $Q(k)$. In the sequel, $a > 0$ will be a uniform constant depending only on dimension and the ADR and time backwards ADR constants (see [1,Definitions 1.20 and 1.22, and Remark 1.28]). Our goal is to show that there is a time value t_k satisfying

$$t_0 - (\widehat{r}_k)^2 \leq t_k < t_0 - (ar)^2 \tag{0.1}$$

and

$$t_k < t\,, \quad \forall\,(x, t) \in \Delta_r(x_0, t_0) := Q_r(x_0, t_0) \cap \Sigma \tag{0.2}$$

such that

$$1 \leq C\omega^{Y,s}(U_k) + C_\epsilon \omega^{Y,s}(F), \quad \forall\,(Y, s) \in S_k \cap \{t \geq t_k\} \tag{0.3}$$

(observe that if $t_k = t_0 - (\widehat{r}_k)^2$ then we are considering all of S_k), where ϵ is a parameter that has been fixed previously (see [1,p. 1544]), and where U_k is the "annular" region between a certain pair of concentric parabolic cubes and F is a certain subset of Σ (see [1,p. 1545] for precise descriptions, although for our purposes at this point one does not really need to concern oneself with exact definitions of these two sets). Once we have established (0.3) for all such (Y, s), we may follow the argument on p. 1549 (see also the prelude on the last paragraph of p. 1548) of [1] to reach the conclusion of Lemma 2.7. (We note that estimate (0.3) is already established in [1] in Cases 1 and 2.)

We first observe that if S_k is empty, then (0.3) is vacuous, and there is nothing to prove.

Next, we assume that S_k is non-empty, and let $(Y, s) \in S_k$. We then have

$$\omega^{Y,s}(U_k) \gtrsim 1\,, \ \text{ if } \delta(Y, s) < \epsilon' r/200 \tag{0.4}$$

by [1,Remark 3.7] so that in particular (0.3) holds for such (Y, s). If all $(Y, s) \in S_k$ are as in (0.4), then we are done, and we may simply take $t_k = t_0 - (\widehat{r}_k)^2$.

Otherwise, suppose there is a point (Y, s) on S_k, with $\delta(Y, s) \geq \epsilon' r/200$. Set

$$\Delta_r(x_0, t_0) := Q_r(x_0, t_0) \cap \Sigma\,,$$

and recall that $Q(k)$ is concentric with $Q_r(x_0, t_0)$ and strictly contains it, in fact, as noted above, the "side length" $\widehat{r}_k$ of the former satisfies $5r/4 < \widehat{r}_k < 3r/2$. Define

$$T_1 := \inf\{t : \exists\, (x, t) \in \Delta_r(x_0, t_0)\}.$$

In particular, there is a point $(x_1, T_1) \in \Sigma$ for some x_1, since Σ is a closed set. Trivially $T_1 \leq t_0$. Moreover, by [1,Remark 1.28] (applied with (x_1, T_1) in place of (x_0, t_0)), and our observations about the relative size of Q_r and $Q(k)$, there is a point $(z, \tau) \in Q(k) \cap \Sigma$ with $\tau < T_1 - (ar)^2$, hence, more precisely,

$$t_0 - (\widehat{r}_k)^2 < \tau < T_1 - (ar)^2 \leq t_0 - (ar)^2\,, \tag{0.5}$$

where the first inequality holds by definition of $Q(k)$. Since $(z, \tau) \in \Sigma \subset \partial\Omega$, there exist points in Ω arbitrarily close to (z, τ), and thus in particular there is a point $(X_k, t_k) \in \Omega$ such that

$$\begin{aligned} |\tau - t_k| &\leq \|(z, \tau) - (X_k, t_k)\|^2 \\ &< \min\left\{\Big(\operatorname{dist}\big[(z, \tau), \partial Q(k)\big]\Big)^2,\ T_1 - (ar)^2 - \tau,\ (ar)^2,\ \tau - t_0 + (\widehat{r}_k)^2,\ \big(\epsilon' r/200\big)^2\right\}. \end{aligned} \tag{0.6}$$

Combining the latter estimate with (0.5) (and the definition of T_1), we see that (0.1) and (0.2) hold for the specified t_k, and that $(X_k, t_k) \in Q(k) \cap \Omega$.

It remains now to verify that (0.3) holds for this choice of t_k. We construct a parabolic rectangle $I(k)$ by sliding the back face of $Q(k)$ forward until we reach $t = t_k$, thus

$$\partial I(k) = \big(\partial Q(k) \cap \{t \geq t_k\}\big) \cup \big(\{t = t_k\} \cap Q(k)\big).$$

Define

$$S_k' := \partial I(k) \cap \Omega$$

so that

$$S_k \cap S_k' = S_k \cap \{t \geq t_k\},$$

and let F_k' denote the back face of $I(k)$. We refer to $F_k' \cap \Omega$ as the back face of S_k'. Observe that $(X_k, t_k) \in F_k'$ by construction, hence in particular $F_k' \cap \Omega$ is non-empty. Note also that by (0.6), in particular $\delta(X_k, t_k) < \epsilon' r/200$. Consider now $(Y, s) \in S_k' \cap S_k$, and observe that by (0.4), it is enough to treat the case that $\delta(Y, s) \geq \epsilon' r/200$. Since $F_k' \cap \Omega$ is non-empty and includes (X_k, t_k), by continuity of δ, there are two possibilities:

(i) $\delta(Y, s) < \epsilon' r/200$, for all $(Y, s) \in F_k' \cap \Omega$, or
(ii) $\exists\, (Y_0, s_0) \in F_k' \cap \Omega$ such that $\delta(Y_0, s_0) = \epsilon' r/200$.

In case (ii), S_k' and (Y_0, s_0) enjoy exactly the same properties as do their counterparts in the treatment of Case 2 (see [1,p. 1547]), and thus this case may be handled exactly like Case 2 in [1].

Now suppose that we are in case (i), and consider any point $(Y, s) \in S_k' \cap S_k$ such that $\delta(Y, s) \geq \epsilon' r/200$. Observe that in the scenario of case (i), every point $(Y, t_k) \in F_k'$ lies either in the complement of Ω, or else $\delta(Y, t_k) < \epsilon' r/200$. Hence, if $\delta(Y, s) \geq \epsilon' r/200$, then we may slide the time coordinate of (Y, s) backwards until we reach a point (Y, s'), with $t_k < s' \leq s$, such that $\delta(Y, s') = \epsilon' r/200$. Of course, we may suppose that s' is the largest time value less than or equal to s for which this happens. Then by [1,Remark 3.5],

$$\omega^{Y,s'}(F) \gtrsim 1,$$

and thus by Harnack's inequality, we also have $\omega^{Y,s}(F) \gtrsim 1$ so that (0.3) holds.

The proof is now complete.

References

1. Genschaw, A., Hofmann, S.: A weak reverse Hölder inequality for caloric measure. J. Geometr. Anal. **30**, 1530–1564 (2020)
2. Bennewitz, B., Lewis, J.L.: On weak reverse Hölder inequalities for nondoubling harmonic measures. Complex Var. Theory Appl. **49**(7–9), 571–582 (2004)