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A Kirchhoff Type Equation in $\mathbb{R}^N$ Involving the fractional (p, q)-Laplacian

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Abstract

In this paper, we deal with the following class of fractional (p, q)-Laplacian Kirchhoff type problem:

$$\begin{cases} \left(1+[u]_{s,p}^{p}\right)(-\Delta)_{p}^{s}u+\left(1+[u]_{s,q}^{q}\right)(-\Delta)_{q}^{s}u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=f(u) \ \text{ in } \mathbb{R}^N, \\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $s \in (0, 1)$, $1 < p < q < \frac{N}{s} < 2q$, $(-\Delta)_t^s$, with $t \in \{p, q\}$, is the fractional t-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous potential such that $\inf_{\partial\Lambda} V > \inf_{\Lambda} V$ for some bounded open set $\Lambda \subset \mathbb{R}^N$, and $f : \mathbb{R} \to \mathbb{R}$ is a superlinear continuous nonlinearity with subcritical growth at infinity. By combining the method of Nehari manifold, a penalization technique, and the Lusternik–Schnirelman category theory, we study the multiplicity and concentration properties of solutions for the above problem when $\varepsilon \to 0$.

Keywords Fractional $(p,\ q)$-Laplacian problem · Kirchhoff type problem · Penalization technique · Lusternik–Schnirelman theory

Mathematics Subject Classification 35A15 · 35J62 · 35Q55 · 55M30

1 Introduction

In this paper, we investigate the multiplicity and concentration phenomenon of solutions for the following fractional (p, q)-Laplacian Kirchhoff type problem:

$$\begin{cases} \left(1+[u]_{s,p}^{p}\right)(-\Delta)_{p}^{s}u+\left(1+[u]_{s,q}^{q}\right)(-\Delta)_{q}^{s}u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=f(u) \ \text{ in } \mathbb{R}^N \\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $\varepsilon > 0$ is a small parameter, $s \in (0, 1)$, $1 < p < q < \frac{N}{s} < 2q$, $V : \mathbb{R}^N \to \mathbb{R}$ is a bounded and continuous potential fulfilling the following conditions [18]:

(V_1) there exists $V_0 > 0$ such that $V_0 = \inf_{x\in\mathbb{R}^N} V(x)$,
(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad 0 \in M = \{x \in \Lambda : V(x) = V_0\},$$

and $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity such that $f(t) = 0$ for $t \leq 0$ and satisfying the following hypotheses:

(f_1) $\lim_{|t|\to 0} \frac{|f(t)|}{|t|^{2p-1}} = 0$,

(f_2) there exists $\nu \in (2q, q_s^*)$ such that $\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{\nu-1}} = 0$, where $q_s^* = \frac{Nq}{N-sq}$,

(f_3) there exists $\vartheta \in (2q, \nu)$ such that $0 < \vartheta F(t) = \vartheta \int_0^t f(\tau)\,d\tau \leq tf(t)$ for all $t > 0$,

(f_4) the map $t \mapsto \frac{f(t)}{t^{2q-1}}$ is increasing in $(0, \infty)$.

The symbol $(-\Delta)_t^s$, with $t \in \{p, q\}$, stands for the fractional t-Laplacian operator defined, up to a normalization constant depending on N, s and t, by setting

$$(-\Delta)_t^s u(x) = 2 \lim_{r\to 0} \int_{\mathbb{R}^N\setminus B_r(x)} \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x - y|^{N+st}} dy \quad (x \in \mathbb{R}^N),$$

for any function $u : \mathbb{R}^N \to \mathbb{R}$ sufficiently smooth. We recall that the recent years have seen a surge of interest in nonlocal and fractional problems involving the fractional t-Laplacian operator because of the presence of two features: the nonlinearity of the operator and its nonlocal character. For this reason, several existence, multiplicity and regularity results have been established by many authors; see for instance [4, 8, 10, 20, 24, 28, 38].

When $s = 1$, the study of (1.1) is strictly related to the following (p, q)-Laplacian equation

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x, u) \text{ in } \mathbb{R}^N,$$

which comes from a general reaction–diffusion system

$$u_t = \text{div}(D(u)\nabla u) + c(x, u) \text{ where } D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}.$$

This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration, $\text{div}(D(u)\nabla u)$ corresponds to the diffusion with diffusion coefficient $D(u)$, and the reaction term $c(x, u)$ relates to source and loss processes. Typically, in chemical and biological applications, the reaction term

$c(x, u)$ is a polynomial of u with variable coefficients; see [17]. Some classical results for (p, q)-Laplacian problems in bounded or unbounded domains can be found in [2, 22, 26, 27, 31, 33, 34] and the references therein. We also mention [15, 30] in which the authors discussed Kirchhoff type problems with the (p, q)-Laplacian operator $-\Delta_p - \Delta_q$.

For what concerns the nonlocal framework, only few papers studied fractional (p, q)-Laplacian problems. Such problems involve the sum of two nonlocal nonlinear operators with different scaling properties and so some nontrivial additional technical difficulties arise with respect to the local case $s = 1$ and $p \neq q$, and the fractional case $s \in (0, 1)$ and $p = q$.

In [16], the authors obtained existence, nonexistence, and multiplicity of solutions for a subcritical fractional (p, q)-Laplacian problem. In [5], the author proved an existence result for a critical fractional (p, q)-Laplacian problem, by using a concentration-compactness lemma and the mountain pass theorem. Multiplicity results for a class of fractional (p, q)-Laplacian problems in bounded domains and with critical nonlinearities have been established in [12]. The multiplicity of concentrating solutions for a fractional (p, q)-Laplacian problem of Schrödinger type has been recently demonstrated in [11]. For other contributions devoted to this class of problems, we refer to [1, 7, 9, 12, 25, 29].

To our knowledge, no results for Kirchhoff type problems driven by the fractional (p, q)-Laplacian operator $(-\Delta)^s_p + (-\Delta)^s_q$ appear in the current literature. Particularly motivated by this fact and the above-mentioned works, in this paper, we examine the multiplicity and concentration properties of solutions for (1.1). More precisely, our main result can be stated as follows:

Theorem 1.1 *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold. Then, for any $\delta > 0$ such that*

$$M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1.1) has at least $cat_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^N$ is a global maximum point of u_ε, then

$$\lim_{\varepsilon \to 0} V(\varepsilon x_\varepsilon) = V_0.$$

The proof of Theorem 1.1 is based on the generalized Nehari manifold method, a penalization technique, and the Lusternik–Schnirelman category theory. Firstly, inspired by [18], we modify the nonlinearity f in a suitable way and we consider an auxiliary problem whose advantage with respect to (1.1) is that the corresponding energy functional $\mathcal{J}_\varepsilon$ possesses a mountain pass geometry [3]. Moreover, an accurate analysis allows us to verify that $\mathcal{J}_\varepsilon$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$ ($(PS)_c$ condition for short). Secondly, since we are interested in providing a multiplicity result for (1.1), and our nonlinearity f is only continuous, we implement the barycenter machinery and adapt some abstract critical point results found in [36]. This kind of argument also appears in [23] to analyze a Schrödinger–Kirchhoff elliptic equation, in [6] to handle various fractional Laplacian elliptic problems, and in [11] to deal with a fractional (p, q)-Schrödinger equation. However, with respect to [6, 11, 23], the mixture of Kirchhoff terms and two different nonhomogeneous nonlocal operators makes the study of (1.1) rather tough and an appropriate investigation will be done to circumvent some significant technical complications; see for instance the proofs of Lemmas 2.4, 2.5, 2.7 and Theorem 3.1. Finally, we show that the solutions of the modified problem are solutions to (1.1) for $\varepsilon > 0$ small enough, by using a Moser type iteration [32] and the Hölder regularity result in [11]. As far as we know, this is the first time that the penalization approach and the Lusternik–Schnirelman category theory are combined to treat fractional (p, q)-Laplacian problems like (1.1).

The paper is organized as follows. In Sect. 2, we collect some basic results for fractional Sobolev spaces and we introduce the modified problem. In Sect. 3, we tackle the limiting Kirchhoff problem. In Sect. 4, we present a multiplicity result for the modified problem. The last section is dedicated to the proof of Theorem 1.1.

2 The Modified Problem

2.1 Notations and Some Useful Lemmas

Let $p \in [1, \infty]$ and $A \subset \mathbb{R}^N$ be a measurable set. We will denote by $|\cdot|_{L^p(A)}$ the norm in $L^p(A)$, and we will simply use the notation $|\cdot|_p$ when $A = \mathbb{R}^N$.
Let $s \in (0, 1)$, $p \in (1, \infty)$ and $N > sp$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy < \infty \right\},$$

which is a Banach space with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = (|u|_p^p + [u]_{s,p}^p)^{\frac{1}{p}}, \text{ where } [u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}}.$$

For $u, v \in W^{s,p}(\mathbb{R}^N)$, we put

$$\langle u, v \rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dxdy.$$

The following embeddings are well known in the literature.

Theorem 2.1 [19] *Let $s \in (0, 1)$, $p \in (1, \infty)$ and $N > sp$. Then, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, p_s^*]$ and compactly embedded in $L^t_{loc}(\mathbb{R}^N)$ for any $t \in [1, p_s^*)$.*

For the reader's convenience, we also recall some useful lemmas.

Lemma 2.1 [8] *Let $s \in (0,1)$, $p \in (1,\infty)$ and $N > sp$. Let $r \in [p, p_s^*)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^r dx = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p_s^*)$.

Lemma 2.2 [8] *Let $s \in (0,1)$, $t \in (1,\infty)$ and $N > st$. Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{s,t}(\mathbb{R}^N)$ be a bounded sequence in $W^{s,t}(\mathbb{R}^N)$, and let $\phi \in C^\infty(\mathbb{R}^N)$ be a function such that $0 \le \phi \le 1$ in $\mathbb{R}^N$, $\phi = 0$ in $B_1(0)$ and $\phi = 1$ in $B_2^c(0)$. For each $\rho > 0$ let $\phi_\rho(x) = \phi(\frac{x}{\rho})$. Then*

$$\lim_{\rho\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^t}{|x-y|^{N+st}} |u_n(x)|^t \, dxdy = 0.$$

Proof The proof of this result can be found in [8], but here we give a more direct proof. Using the definition of ϕ_ρ, polar coordinates and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W^{s,t}(\mathbb{R}^N)$, we can see that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^t}{|x-y|^{N+st}} |u_n(x)|^t \, dxdy \\
&= \int_{\mathbb{R}^N} \int_{|y-x|>\rho} \frac{|\phi_\rho(x) - \phi_\rho(y)|^t}{|x-y|^{N+st}} |u_n(x)|^t \, dxdy \\
&\quad + \int_{\mathbb{R}^N} \int_{|y-x|\le\rho} \frac{|\phi_\rho(x) - \phi_\rho(y)|^t}{|x-y|^{N+st}} |u_n(x)|^t \, dxdy \\
&\le C \int_{\mathbb{R}^N} |u_n(x)|^t \left(\int_{|y-x|>\rho} \frac{dy}{|x-y|^{N+st}} \right) dx \\
&\quad + \frac{C}{\rho^t} \int_{\mathbb{R}^N} |u_n(x)|^t \left(\int_{|y-x|\le\rho} \frac{dy}{|x-y|^{N+st-t}} \right) dx \\
&\le C \int_{\mathbb{R}^N} |u_n(x)|^t \left(\int_{|z|>\rho} \frac{dz}{|z|^{N+st}} \right) dx + \frac{C}{\rho^t} \int_{\mathbb{R}^N} |u_n(x)|^t \left(\int_{|z|\le\rho} \frac{dz}{|z|^{N+st-t}} \right) dx \\
&\le C \int_{\mathbb{R}^N} |u_n(x)|^t \, dx \left(\int_\rho^\infty \frac{dr}{r^{st+1}} \right) + \frac{C}{\rho^t} \int_{\mathbb{R}^N} |u_n(x)|^t \, dx \left(\int_0^\rho \frac{dr}{r^{st-t+1}} \right) \\
&\le \frac{C}{\rho^{st}} \int_{\mathbb{R}^N} |u_n(x)|^t \, dx + \frac{C}{\rho^t} \rho^{-st+t} \int_{\mathbb{R}^N} |u_n(x)|^t \, dx \le \frac{C}{\rho^{st}} \int_{\mathbb{R}^N} |u_n(x)|^t \, dx \le \frac{C}{\rho^{st}},
\end{aligned}$$

and letting first $n \to \infty$ and then $\rho \to \infty$, we get the thesis. □

Let $s \in (0,1)$ and $p, q \in (1,\infty)$. Consider the space

$$\mathcal{W} = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$$

endowed with the norm

$$\|u\|_{\mathcal{W}} = \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}.$$

Since $W^{s,r}(\mathbb{R}^N)$, with $r \in (1, \infty)$, is a separable reflexive Banach space (this can be proved by using the operator $T : W^{s,r}(\mathbb{R}^N) \to L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^{2N})$ defined by $Tu = (u, (u(x) - u(y))|x - y|^{-\frac{N}{r}-s})$ and arguing as in the proof of Proposition 8.1 in [13]), we obtain that $\mathcal{W}$ is also a separable reflexive Banach space.

For any $\varepsilon > 0$, we introduce the space

$$\mathbb{X}_\varepsilon = \left\{u \in \mathcal{W} : \int_{\mathbb{R}^N} V(\varepsilon x)\left(|u|^p + |u|^q\right)\, dx < \infty\right\}$$

equipped with the norm

$$\|u\|_{\mathbb{X}_\varepsilon} = \|u\|_{V_\varepsilon,p} + \|u\|_{V_\varepsilon,q},$$

where

$$\|u\|_{V_\varepsilon,t} = \left([u]^t_{s,t} + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t\, dx\right)^{\frac{1}{t}} \quad \text{for } t \in \{p, q\}.$$

2.2 The Penalization Approach

We adapt in a suitable way the del Pino–Felmer penalization approach [18] to attack (1.1). First, we observe that the map $t \mapsto \frac{f(t)}{t^{p-1}+t^{q-1}}$ is increasing in $(0, \infty)$. Indeed,

$$\frac{f(t)}{t^{p-1} + t^{q-1}} = \frac{f(t)}{t^{2q-1}} \frac{t^{2q-1}}{t^{p-1} + t^{q-1}}$$

and noting that $t \mapsto \frac{f(t)}{t^{2q-1}}$ is increasing in $(0, \infty)$ (by (f_4)), and that $t \mapsto \frac{t^{2q-1}}{t^{p-1}+t^{q-1}}$ is increasing in $(0, \infty)$ (because $2q > p$), we deduce the desired result.

Now, let us fix

$$K > \frac{q}{p}\left(\frac{\vartheta - p}{\vartheta - q}\right) > 1,$$

and let $a > 0$ be such that

$$f(a) = \frac{V_0}{K}(a^{p-1} + a^{q-1}).$$

We define

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \le a, \\ \dfrac{V_0}{K}(t^{p-1} + t^{q-1}) & \text{if } t > a, \end{cases}$$

and

$$g(x,t)=\begin{cases}\chi_\Lambda(x)f(t)+(1-\chi_\Lambda(x))\tilde f(t) & \text{if } t>0,\\ 0 & \text{if } t\le 0,\end{cases}$$

where χ_A denotes the characteristic function of $A\subset\mathbb{R}^N$. By (f_1)-(f_4), we infer that $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function that fulfills the following assumptions:

(g_1) $\displaystyle\lim_{t\to 0}\frac{g(x,t)}{t^{2p-1}}=0$ uniformly with respect to $x\in\mathbb{R}^N$,

(g_2) $g(x,t)\le f(t)$ for all $x\in\mathbb{R}^N$ and $t>0$,

(g_3) (i) $0<\vartheta G(x,t)\le g(x,t)t$ for all $x\in\Lambda$ and $t>0$, (ii) $0\le pG(x,t)\le g(x,t)t\le\frac{V_0}{K}(t^p+t^q)$ for all $x\in\Lambda^c$ and $t>0$,

(g_4) for each $x\in\Lambda$, the function $t\mapsto\dfrac{g(x,t)}{t^{p-1}+t^{q-1}}$ is increasing in $(0,\infty)$, and for each $x\in\Lambda^c$, the function $t\mapsto\dfrac{g(x,t)}{t^{p-1}+t^{q-1}}$ is increasing in $(0,a)$.

Let us introduce the auxiliary problem

$$\begin{cases}\left(1+[u]_{s,p}^p\right)(-\Delta)_p^s u+\left(1+[u]_{s,q}^q\right)(-\Delta)_q^s u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=g(\varepsilon x,u) & \text{in } \mathbb{R}^N,\\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N),\quad u>0 \text{ in } \mathbb{R}^N.\end{cases}\tag{2.1}$$

We stress that if u_ε is a solution to (2.1) such that $u_\varepsilon(x)\le a$ for all $x\in\Lambda_\varepsilon^c$, where $\Lambda_\varepsilon=\{x\in\mathbb{R}^N : \varepsilon x\in\Lambda\}$, then u_ε is also a solution to (1.1). Then we consider the functional $\mathcal{J}_\varepsilon:\mathbb{X}_\varepsilon\to\mathbb{R}$ associated with (2.1), that is

$$\mathcal{J}_\varepsilon(u)=\frac{1}{p}\|u\|_{V_\varepsilon,p}^p+\frac{1}{2p}[u]_{s,p}^{2p}+\frac{1}{q}\|u\|_{V_\varepsilon,q}^q+\frac{1}{2q}[u]_{s,q}^{2q}-\int_{\mathbb{R}^N}G(\varepsilon x,u)\,dx.$$

Clearly, $\mathcal{J}_\varepsilon\in C^1(\mathbb{X}_\varepsilon,\mathbb{R})$ and it holds

$$\begin{aligned}\langle\mathcal{J}_\varepsilon'(u),\varphi\rangle&=(1+[u]_{s,p}^p)\langle u,\varphi\rangle_{s,p}+(1+[u]_{s,q}^q)\langle u,\varphi\rangle_{s,q}\\&\quad+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^{p-2}u\,\varphi\,dx+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^{q-2}u\,\varphi\,dx\\&\quad-\int_{\mathbb{R}^N}g(\varepsilon x,u)\varphi\,dx\end{aligned}$$

for any $u,\varphi\in\mathbb{X}_\varepsilon$. We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold associated with $\mathcal{J}_\varepsilon$, namely

$$\mathcal{N}_\varepsilon=\{u\in\mathbb{X}_\varepsilon:\langle\mathcal{J}_\varepsilon'(u),u\rangle=0\},$$

and we set

$$c_\varepsilon=\inf_{u\in\mathcal{N}_\varepsilon}\mathcal{J}_\varepsilon(u).$$

Let $\mathbb{X}_\varepsilon^+$ be the open set given by

$$\mathbb{X}_\varepsilon^+ = \{u \in \mathbb{X}_\varepsilon : |\mathrm{supp}(u^+) \cap \Lambda_\varepsilon| > 0\},$$

and $\mathbb{S}_\varepsilon^+ = \mathbb{S}_\varepsilon \cap \mathbb{X}_\varepsilon^+$, where $\mathbb{S}_\varepsilon = \{u \in \mathbb{X}_\varepsilon : \|u\|_{\mathbb{X}_\varepsilon} = 1\}$ is the unit sphere in $\mathbb{X}_\varepsilon$. Note that $\mathbb{S}_\varepsilon^+$ is an incomplete $C^{1,1}$-manifold of codimension one. Hence, $\mathbb{X}_\varepsilon = T_u\mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$ for all $u \in \mathbb{S}_\varepsilon^+$, where

$$\begin{aligned} T_u\mathbb{S}_\varepsilon^+ = \Big\{v \in \mathbb{X}_\varepsilon : (1+[u]_{s,p}^p)\langle u, v\rangle_{s,p} + (1+[u]_{s,q}^q)\langle u, v\rangle_{s,q} \\ + \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^{p-2}uv + |u|^{q-2}uv)\,dx = 0\Big\}. \end{aligned}$$

The next lemma ensures that $\mathcal{J}_\varepsilon$ possesses a mountain pass geometry [3].

Lemma 2.3 *The functional $\mathcal{J}_\varepsilon$ satisfies the following properties:*

(i) *There exist $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ for any $u \in \mathbb{X}_\varepsilon$ with $\|u\|_{\mathbb{X}_\varepsilon} = \rho$.*
(ii) *There exists $e \in \mathbb{X}_\varepsilon$ such that $\|e\|_{\mathbb{X}_\varepsilon} > \rho$ and $\mathcal{J}_\varepsilon(e) < 0$.*

Proof (i) Pick $\zeta \in (0, V_0)$. From (g_1), (g_2), (f_1), and (f_2), we can find $C_\zeta > 0$ such that

$$|g(x,t)| \leq \zeta|t|^{p-1} + C_\zeta|t|^{\nu-1} \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Taking into account the above estimate and applying Theorem 2.1, we have

$$\begin{aligned} \mathcal{J}_\varepsilon(u) &\geq \frac{1}{p}\|u\|_{V_\varepsilon,p}^p + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q - \frac{\zeta}{p}|u|_p^p - \frac{C_\zeta}{\nu}|u|_\nu^\nu \\ &\geq C_1\|u\|_{V_\varepsilon,p}^p + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q - \frac{C_\zeta}{\nu}|u|_\nu^\nu. \end{aligned}$$

Choosing $\|u\|_{\mathbb{X}_\varepsilon} = \rho \in (0,1)$ and recalling that $1 < p < q$, we get $\|u\|_{V_\varepsilon,p} < 1$ and thus $\|u\|_{V_\varepsilon,p}^p \geq \|u\|_{V_\varepsilon,p}^q$. Using

$$a^t + b^t \geq C_t(a+b)^t \quad \text{for all } a, b \geq 0 \text{ and } t > 1,$$

and Theorem 2.1, we can see that

$$\mathcal{J}_\varepsilon(u) \geq C_2\|u\|_{\mathbb{X}_\varepsilon}^q - \frac{C_\zeta}{\nu}|u|_\nu^\nu \geq C_2\|u\|_{\mathbb{X}_\varepsilon}^q - C_3\|u\|_{\mathbb{X}_\varepsilon}^\nu.$$

Since $\nu > q$, there exists $\alpha > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ for any $u \in \mathbb{X}_\varepsilon$ with $\|u\|_{\mathbb{X}_\varepsilon} = \rho$.
(ii) It follows from (f_3) that, for some constants $A, B > 0$,

$$F(t) \geq At^\vartheta - B \quad \text{for all } t > 0.$$

Then, for all $u \in \mathbb{X}_\varepsilon^+$ and $t > 0$, we obtain

$$\begin{aligned}\mathcal{J}_\varepsilon(tu) \leq \frac{t^p}{p}\|u\|_{V_\varepsilon,p}^p + \frac{t^{2p}}{2p}[u]_{s,p}^{2p} + \frac{t^q}{q}\|u\|_{V_\varepsilon,q}^q + \frac{t^{2q}}{2q}[u]_{s,q}^{2q} \\ - At^\vartheta \int_{\Lambda_\varepsilon} (u^+)^\vartheta \, dx + B|\mathrm{supp}(u^+) \cap \Lambda_\varepsilon|\end{aligned}$$

which combined with the fact that $\vartheta > 2q > 2p$ implies that $\mathcal{J}_\varepsilon(tu) \to -\infty$ as $t \to \infty$. Hence, for large $t > 1$, we can take $e = tu$ such that $\|e\|_{\mathbb{X}_\varepsilon} > \rho$ and $\mathcal{J}_\varepsilon(e) < 0$. □

In view of Lemma 2.3, we can define the minimax level

$$c'_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \mathcal{J}_\varepsilon(\gamma(t)) \quad \text{where} \quad \Gamma_\varepsilon = \{\gamma \in C([0,1], \mathbb{X}_\varepsilon) : \gamma(0) = 0 \text{ and } \mathcal{J}_\varepsilon(\gamma(1)) < 0\}.$$

Exploiting a version of the mountain pass theorem without the Palais–Smale condition (see [37]), we can find a Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ at the level c'_ε ($(PS)_{c'_\varepsilon}$ sequence for short).

Remark 2.1 We may always assume that any $(PS)_c$ sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ of $\mathcal{J}_\varepsilon$ is nonnegative. Indeed, noting that $\langle \mathcal{J}'_\varepsilon(u_n), u_n^- \rangle = o_n(1)$, where $u_n^- = \min\{u_n, 0\}$, and using $g(\varepsilon\cdot, t) = 0$ for $t \leq 0$, we have

$$\begin{aligned}&(1 + [u_n]_{s,p}^p) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}}(u_n^-(x) - u_n^-(y))\, dxdy \\ &+ (1 + [u_n]_{s,q}^q) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))}{|x-y|^{N+sq}}(u_n^-(x) - u_n^-(y))\, dxdy \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n)\, u_n^- \, dx = o_n(1).\end{aligned}$$

Recalling that

$$|x-y|^{t-2}(x-y)(x^- - y^-) \geq |x^- - y^-|^t \quad \text{for all } x, y \in \mathbb{R} \text{ and } t > 1, \tag{2.2}$$

we arrive at

$$\|u_n^-\|_{V_\varepsilon,p}^p + \|u_n^-\|_{V_\varepsilon,q}^q = o_n(1),$$

that is $u_n^- \to 0$ in $\mathbb{X}_\varepsilon$. Moreover, $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$. Since $[u_n]_{s,t}^t = [u_n^+]_{s,t}^t + o_n(1)$ and $\|u_n\|_{V_\varepsilon,t} = \|u_n^+\|_{V_\varepsilon,t} + o_n(1)$ for $t \in \{p, q\}$, we can easily deduce that $\mathcal{J}_\varepsilon(u_n) = \mathcal{J}_\varepsilon(u_n^+) + o_n(1)$ and $\mathcal{J}'_\varepsilon(u_n) = \mathcal{J}'_\varepsilon(u_n^+) + o_n(1)$. Therefore, $\mathcal{J}_\varepsilon(u_n^+) \to c$ and $\mathcal{J}'_\varepsilon(u_n^+) \to 0$.

The next two results are very important because they allow us to overcome the nondifferentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}_\varepsilon^+$.

Lemma 2.4 *Assume that* (V_1)*-*(V_2) *and* (f_1)*-*(f_4) *hold. Then we have the following properties:*

(i) *For each* $u \in \mathbb{X}_\varepsilon^+$*, let* $h_u : \mathbb{R}^+ \to \mathbb{R}$ *be defined by* $h_u(t) = \mathcal{J}_\varepsilon(tu)$*. Then, there is a unique* $t_u > 0$ *such that*

$$h_u'(t) > 0 \text{ for all } t \in (0, t_u),$$
$$h_u'(t) < 0 \text{ for all } t \in (t_u, \infty).$$

(ii) *There exists* $\tau > 0$*, independent of* u*, such that* $t_u \geq \tau$ *for any* $u \in \mathbb{S}_\varepsilon^+$*. Moreover, for each compact set* $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$*, there is a constant* $C_\mathbb{K} > 0$ *such that* $t_u \leq C_\mathbb{K}$ *for any* $u \in \mathbb{K}$*.*

(iii) *The map* $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ *given by* $\hat{m}_\varepsilon(u) = t_u u$ *is continuous and* $m_\varepsilon = \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ *is a homeomorphism between* $\mathbb{S}_\varepsilon^+$ *and* $\mathcal{N}_\varepsilon$*. Moreover,* $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}}$*.*

(iv) *If there is a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ *such that* $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$*, then* $\|m_\varepsilon(u_n)\|_{\mathbb{X}_\varepsilon} \to \infty$ *and* $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$*.*

Proof (i) From the proof of Lemma 2.3, we derive that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Then there exists a global maximum point $t_u > 0$ for h_u in $[0, \infty)$ such that $h_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. We claim that $t_u > 0$ is the unique number such that $h_u'(t_u) = 0$. Arguing by contradiction, we assume that there exists $t_1 > t_2 > 0$ such that $h_u'(t_1) = h_u'(t_2) = 0$, or equivalently

$$t_1^{p-1}\|u\|_{V_\varepsilon,p}^p + t_1^{2p-1}[u]_{s,p}^{2p} + t_1^{q-1}\|u\|_{V_\varepsilon,q}^q + t_1^{2q-1}[u]_{s,q}^{2q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_1 u)u\,dx,$$
$$t_2^{p-1}\|u\|_{V_\varepsilon,p}^p + t_2^{2p-1}[u]_{s,p}^{2p} + t_2^{q-1}\|u\|_{V_\varepsilon,q}^q + t_2^{2q-1}[u]_{s,q}^{2q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_2 u)u\,dx.$$

Hence,

$$\frac{\|u\|_{V_\varepsilon,p}^p}{t_1^{2q-p}} + \frac{\|u\|_{V_\varepsilon,q}^q}{t_1^q} + \frac{[u]_{s,p}^{2p}}{t_1^{2q-2p}} + [u]_{s,q}^{2q} = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} u^{2q}\,dx$$

and

$$\frac{\|u\|_{V_\varepsilon,p}^p}{t_2^{2q-p}} + \frac{\|u\|_{V_\varepsilon,q}^q}{t_2^q} + \frac{[u]_{s,p}^{2p}}{t_2^{2q-2p}} + [u]_{s,q}^{2q} = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}} u^{2q}\,dx.$$

Using the definition of g, (g_4) and (f_4), we have

$$\left(\frac{1}{t_1^{2q-p}} - \frac{1}{t_2^{2q-p}}\right)\|u\|_{V_\varepsilon,p}^p + \left(\frac{1}{t_1^q} - \frac{1}{t_2^q}\right)\|u\|_{V_\varepsilon,q}^q + \left(\frac{1}{t_1^{2q-2p}} - \frac{1}{t_2^{2q-2p}}\right)[u]_{s,p}^{2p}$$
$$= \int_{\mathbb{R}^N}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}}\right]u^{2q}\,dx$$

$$
\begin{aligned}
&\geq \int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}\left[\frac{g(\varepsilon x,t_1u)}{(t_1u)^{2q-1}}-\frac{g(\varepsilon x,t_2u)}{(t_2u)^{2q-1}}\right]u^{2q}dx\\
q\quad &+\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}\left[\frac{g(\varepsilon x,t_1u)}{(t_1u)^{2q-1}}-\frac{g(\varepsilon x,t_2u)}{(t_2u)^{2q-1}}\right]u^{2q}dx\\
&+\int_{\Lambda_\varepsilon^c\cap\{t_1u<a\}}\left[\frac{g(\varepsilon x,t_1u)}{(t_1u)^{2q-1}}-\frac{g(\varepsilon x,t_2u)}{(t_2u)^{2q-1}}\right]u^{2q}dx\\
&\geq \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}\left[\left(\frac{1}{(t_1u)^{2q-p}}-\frac{1}{(t_2u)^{2q-p}}\right)+\left(\frac{1}{(t_1u)^{q}}-\frac{1}{(t_2u)^{q}}\right)\right]u^{2q}dx\\
&+\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}\left[\frac{V_0}{K}\left(\frac{1}{(t_1u)^{2q-p}}+\frac{1}{(t_1u)^{q}}\right)-\frac{f(t_2u)}{(t_2u)^{2q-1}}\right]u^{2q}dx.
\end{aligned}
$$

Multiplying both sides by $\frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}<0$ (recall that $2q>p$ and $t_1>t_2$), we get

$$
\begin{aligned}
&\|u\|_{V_\varepsilon,p}^p+\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\|u\|_{V_\varepsilon,q}^q\\
&=\|u\|_{V_\varepsilon,p}^p+\frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\frac{t_2^q-t_1^q}{(t_1t_2)^q}\|u\|_{V_\varepsilon,q}^q\\
&\leq \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^p dx+\frac{V_0}{K}\frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\frac{t_2^q-t_1^q}{(t_2t_1)^q}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^q dx\\
&+\frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}\left[\frac{V_0}{K}\left(\frac{1}{(t_1u)^{2q-p}}+\frac{1}{(t_1u)^{q}}\right)-\frac{f(t_2u)}{(t_2u)^{2q-1}}\right]u^{2q}dx\\
&\leq \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^p dx+\frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^q dx\\
&+\frac{V_0}{K}\frac{t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^p\,dx+\frac{V_0}{K}\frac{t_1^{q-p}t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^q\,dx\\
&-\frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}\frac{f(t_2u)}{(t_2u)^{2q-1}}u^{2q}dx\\
&\leq \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^p dx+\frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^q dx\\
&+\frac{V_0}{K}\frac{t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^p\,dx+\frac{V_0}{K}\frac{t_1^{q-p}t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^q\,dx\\
&-\frac{V_0}{K}\frac{t_1^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^p\,dx-\frac{V_0}{K}\frac{t_1^{2q-p}t_2^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^q\,dx\\
&=\frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^p dx+\frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^q dx\\
&+\frac{V_0}{K}\frac{t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^p\,dx+\frac{V_0}{K}\frac{t_1^{q-p}t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\leq a<t_1u\}}u^q\,dx
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{V_0}{K}\frac{t_1^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^p\,dx-\frac{V_0}{K}\frac{t_1^{2q-p}t_2^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^q\,dx\\
&\le \frac{V_0}{K}\int_{\Lambda_\varepsilon^c} u^p\,dx+\frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\int_{\Lambda_\varepsilon^c} u^q\,dx\\
&\le \frac{1}{K}\|u\|_{V_\varepsilon,p}^p+\frac{1}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\|u\|_{V_\varepsilon,q}^q,
\end{aligned}
$$

where we used the fact that (f_4) and our choice of the constant a produce

$$
\begin{aligned}
\frac{f(t_2u)}{(t_2u)^{2q-1}}&=\frac{f(t_2u)}{(t_2u)^{p-1}+(t_2u)^{q-1}}\,\frac{(t_2u)^{p-1}+(t_2u)^{q-1}}{(t_2u)^{2q-1}}\\
&\le\frac{f(a)}{a^{p-1}+a^{q-1}}\,\frac{(t_2u)^{p-1}+(t_2u)^{q-1}}{(t_2u)^{2q-1}}\\
&=\frac{V_0}{K}\left(\frac{1}{(t_2u)^{2q-p}}+\frac{1}{(t_2u)^q}\right)\quad\text{in } \Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}.
\end{aligned}
$$

Therefore,

$$
\left(1-\frac{1}{K}\right)\left[\|u\|_{V_\varepsilon,p}^p+\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q-t_1^q)\|u\|_{V_\varepsilon,q}^q\right]\le 0,
$$

which is inconsistent with $u\neq 0$ and $K>1$.

(ii) Fix $u\in\mathbb{S}_\varepsilon^+$. By (i), there exists $t_u>0$ such that $h_u'(t_u)=0$, that is

$$
t_u^{p-1}\|u\|_{V_\varepsilon,p}^p+t_u^{q-1}\|u\|_{V_\varepsilon,q}^q+t_u^{2p-1}[u]_{s,p}^{2p}+t_u^{2q-1}[u]_{s,q}^{2q}=\int_{\mathbb{R}^N} g(\varepsilon x,t_uu)\,u\,dx.
$$

Pick $\xi>0$. From (g_1)-(g_2) and Theorem 2.1, we derive

$$
\begin{aligned}
t_u^{p-1}\|u\|_{V_\varepsilon,p}^p+t_u^{q-1}\|u\|_{V_\varepsilon,q}^q\le\int_{\mathbb{R}^3} g(\varepsilon x,t_uu)\,u\,dx&\le\xi t_u^{p-1}\|u\|_{V_\varepsilon,p}^p\\
&\quad+C_\xi t_u^{\nu-1}\|u\|_{V_\varepsilon,q}^\nu.
\end{aligned}
$$

Choosing $\xi>0$ sufficiently small, we have

$$
Ct_u^{p-1}\|u\|_{V_\varepsilon,p}^p+t_u^{q-1}\|u\|_{V_\varepsilon,q}^q\le Ct_u^{\nu-1}\|u\|_{V_\varepsilon,q}^\nu\le Ct_u^{\nu-1}.
$$

Now, if $t_u\le 1$, then $t_u^{q-1}\le t_u^{p-1}$, and using the facts that $1=\|u\|_{\mathbb{X}_\varepsilon}\ge\|u\|_{V_\varepsilon,p}$ and that $q>p$ imply that $\|u\|_{V_\varepsilon,p}^p\ge\|u\|_{V_\varepsilon,p}^q$, we get

$$
\begin{aligned}
Ct_u^{q-1}&=Ct_u^{q-1}\|u\|_{\mathbb{X}_\varepsilon}^q\le t_u^{q-1}(C\|u\|_{V_\varepsilon,p}^q+\|u\|_{V_\varepsilon,q}^q)\le t_u^{q-1}(C\|u\|_{V_\varepsilon,p}^p+\|u\|_{V_\varepsilon,q}^q)\\
&\le Ct_u^{\nu-1}.
\end{aligned}
$$

Since $\nu > q$, there exists $\tau > 0$, independent of u, such that $t_u \geq \tau$.

When $t_u > 1$, then $t_u^{q-1} > t_u^{p-1}$, and observing that $1 = \|u\|_{\mathbb{X}_\varepsilon} \geq \|u\|_{V_\varepsilon,p}$ and that $q > p$ yield $\|u\|_{V_\varepsilon,p}^p \geq \|u\|_{V_\varepsilon,p}^q$, we obtain

$$
\begin{aligned}
Ct_u^{p-1} &= Ct_u^{p-1}\|u\|_{\mathbb{X}_\varepsilon}^q \leq t_u^{p-1}(C\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \leq t_u^{p-1}(C\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \\
&\leq Ct_u^{\nu-1}.
\end{aligned}
$$

As $\nu > q > p$, we can find $\tau > 0$, independent of u, such that $t_u \geq \tau$.

Now, let $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ be a compact set, and suppose, by contradiction, that there exists $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ such that $t_n = t_{u_n} \to \infty$. Since $\mathbb{K}$ is compact, there is $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. By the proof of (ii) of Lemma 2.3, we see that

$$
\mathcal{J}_\varepsilon(t_n u_n) \to -\infty. \tag{2.3}
$$

On the other hand, if $v \in \mathcal{N}_\varepsilon$, by $\langle \mathcal{J}_\varepsilon'(v), v\rangle = 0$ and (g_3), we get

$$
\mathcal{J}_\varepsilon(v) = \mathcal{J}_\varepsilon(v) - \frac{1}{\vartheta}\langle \mathcal{J}_\varepsilon'(v), v\rangle \geq \tilde{C}(\|v\|_{V_\varepsilon,p}^p + \|v\|_{V_\varepsilon,q}^q).
$$

Taking $v_n = t_{u_n}u_n \in \mathcal{N}_\varepsilon$ in the above inequality, we arrive at

$$
\mathcal{J}_\varepsilon(t_n u_n) \geq \tilde{C}(\|v_n\|_{V_\varepsilon,p}^p + \|v_n\|_{V_\varepsilon,q}^q).
$$

Since $\|v_n\|_{\mathbb{X}_\varepsilon} = t_n \to \infty$ and $\|v_n\|_{\mathbb{X}_\varepsilon} = \|v_n\|_{V_\varepsilon,p} + \|v_n\|_{V_\varepsilon,q}$, we can use (2.3) to reach a contradiction.

(iii) First we note that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (i), for each $u \in \mathbb{X}_\varepsilon^+$, there is a unique $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in \mathbb{X}_\varepsilon^+$. Otherwise, we would have

$$
|\operatorname{supp}(u^+) \cap \Lambda_\varepsilon| = 0,
$$

and by (g_3)-(ii) we infer that

$$
\begin{aligned}
\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q &\leq \int_{\mathbb{R}^N} g(\varepsilon x, u)\, u\, dx = \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u)\, u\, dx + \int_{\Lambda_\varepsilon} g(\varepsilon x, u)\, u\, dx \\
&= \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u^+)\, u^+\, dx \\
&\leq \frac{1}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u|^p + |u|^q) dx \\
&\leq \frac{1}{K}(\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q)
\end{aligned}
$$

which gives a contradiction because $K > 1$ and $u \neq 0$. Consequently, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} \in \mathbb{S}_\varepsilon^+$ is well defined and continuous. Since

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{\mathbb{X}_\varepsilon}} = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} = u \quad \text{for all } u \in \mathbb{S}_\varepsilon^+,$$

we deduce that m_ε is a bijection. Now we prove that $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon^+$ and $u \in \mathbb{X}_\varepsilon^+$ be such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. By (ii), there exists $t_0 > 0$ such that $t_n = t_{\frac{u_n}{\|u_n\|_{\mathbb{X}_\varepsilon}}} \to t_0$. Using $t_n \frac{u_n}{\|u_n\|_{\mathbb{X}_\varepsilon}} \in \mathcal{N}_\varepsilon$, that is

$$t_n^p \frac{\|u_n\|_{V_\varepsilon,p}^p}{\|u_n\|_{\mathbb{X}_\varepsilon}^p} + t_n^q \frac{\|u_n\|_{V_\varepsilon,q}^q}{\|u_n\|_{\mathbb{X}_\varepsilon}^q} + t_n^{2p} \frac{[u_n]_{s,p}^{2p}}{\|u_n\|_{\mathbb{X}_\varepsilon}^{2p}} + t_n^{2q} \frac{[u_n]_{s,q}^{2q}}{\|u_n\|_{\mathbb{X}_\varepsilon}^{2q}} = \int_{\mathbb{R}^N} g\left(\varepsilon x, t_n \frac{u_n}{\|u_n\|_{\mathbb{X}_\varepsilon}}\right) t_n \frac{u_n}{\|u_n\|_{\mathbb{X}_\varepsilon}}\, dx,$$

and letting $n \to \infty$ we find

$$t_0^p \frac{\|u\|_{V_\varepsilon,p}^p}{\|u\|_{\mathbb{X}_\varepsilon}^p} + t_0^q \frac{\|u\|_{V_\varepsilon,q}^q}{\|u\|_{\mathbb{X}_\varepsilon}^q} + t_0^{2p} \frac{[u]_{s,p}^{2p}}{\|u\|_{\mathbb{X}_\varepsilon}^{2p}} + t_0^{2q} \frac{[u]_{s,q}^{2q}}{\|u\|_{\mathbb{X}_\varepsilon}^{2q}} = \int_{\mathbb{R}^N} g\left(\varepsilon x, t_0 \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}}\right) t_0 \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}}\, dx,$$

which implies that $t_0 \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} \in \mathcal{N}_\varepsilon$. From (i), $t_{\frac{u}{\|u\|_{\mathbb{X}_\varepsilon}}} = t_0$ and this assures that $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in $\mathbb{X}_\varepsilon^+$. Therefore, $\hat{m}_\varepsilon$ and m_ε are continuous functions.
(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be a sequence such that $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Then, for each $v \in \partial\mathbb{S}_\varepsilon^+$ and $n \in \mathbb{N}$, we have $u_n^+ \leq |u_n - v|$ a.e. in Λ_ε. Hence, by (V_1), (V_2) and Theorem 2.1, we can see that for each $r \in [p, q_s^*]$, there exists $C_r > 0$ such that

$$\begin{aligned} |u_n^+|_{L^r(\Lambda_\varepsilon)} &\leq \inf_{v\in\partial\mathbb{S}_\varepsilon^+} |u_n - v|_{L^r(\Lambda_\varepsilon)} \\ &\leq C_r \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_{\mathbb{X}_\varepsilon} \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Combining (g_1), (g_2), (g_3)-(ii) and $q > p$, we get, for all $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\, dx &= \int_{\Lambda_\varepsilon^c} G(\varepsilon x, tu_n)\, dx + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)\, dx \\ &\leq \frac{V_0}{Kp} \int_{\Lambda_\varepsilon^c} (t^p|u_n|^p + t^q|u_n|^q) dx + \int_{\Lambda_\varepsilon} F(tu_n)\, dx \\ &\leq \frac{t^p}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\, dx + \frac{t^q}{Kp} \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx \\ &\quad + C_1 t^p \int_{\Lambda_\varepsilon} (u_n^+)^p dx + C_2 t^\nu \int_{\Lambda_\varepsilon} (u_n^+)^\nu dx \end{aligned}$$

$$\leq \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,dx + \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,dx + C'_p t^p \operatorname{dist}(u_n, \partial\mathbb{S}^+_\varepsilon)^p + C'_\nu t^\nu \operatorname{dist}(u_n, \partial\mathbb{S}^+_\varepsilon)^\nu.$$

Thus, for all $t > 0$,

$$\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\,dx \leq \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,dx + \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,dx + o_n(1). \tag{2.4}$$

Now, we recall that $K > \frac{q}{p} > 1$, and that $1 = \|u_n\|_{\mathbb{X}_\varepsilon} \geq \|u_n\|_{V_\varepsilon,p}$ implies $\|u_n\|^p_{V_\varepsilon,p} \geq \|u_n\|^q_{V_\varepsilon,p}$. Then, for all $t > 1$, we obtain

$$\begin{aligned}
&\frac{t^p}{p}\|u_n\|^p_{V_\varepsilon,p} + \frac{t^q}{q}\|u_n\|^q_{V_\varepsilon,q} - \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,dx - \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,dx \\
&= \frac{t^p}{p}[u_n]^p_{s,p} + t^p\left(\frac{1}{p} - \frac{1}{Kp}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,dx \\
&\quad + \frac{t^q}{q}[u_n]^q_{s,q} + t^q\left(\frac{1}{q} - \frac{1}{Kp}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,dx \\
&\geq C_1 t^p \|u_n\|^p_{V_\varepsilon,p} + C_2 t^q \|u_n\|^q_{V_\varepsilon,q} \\
&\geq C_1 t^p \|u_n\|^q_{V_\varepsilon,p} + C_2 t^q \|u_n\|^q_{V_\varepsilon,q} \\
&\geq C_1 t^p \|u_n\|^q_{V_\varepsilon,p} + C_2 t^p \|u_n\|^q_{V_\varepsilon,q} \\
&\geq C_3 t^p (\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^q = C_3 t^p.
\end{aligned} \tag{2.5}$$

By using the definition of $m_\varepsilon(u_n)$, (2.4) and (2.5), we have

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) &\geq \liminf_{n\to\infty} \mathcal{J}_\varepsilon(tu_n) \\
&\geq \liminf_{n\to\infty}\left[\frac{t^p}{p}\|u_n\|^p_{V_\varepsilon,p} + \frac{t^q}{q}\|u_n\|^q_{V_\varepsilon,q} \right.\\
&\quad \left. - \int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\,dx\right] \geq C_3 t^p \quad \text{for all } t > 1.
\end{aligned}$$

Letting $t \to \infty$ we deduce that $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$ as $n \to \infty$. Furthermore, by the definition of $\mathcal{J}_\varepsilon$, we can see that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\frac{1}{p}\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p}(1 + \|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p}) + \frac{1}{q}\|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q}(1 + \|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q}) \\
&\geq \frac{1}{p}\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p} + \frac{1}{2p}[m_\varepsilon(u_n)]^{2p}_{s,p} + \frac{1}{q}\|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q} + \frac{1}{2q}[m_\varepsilon(u_n)]^{2q}_{s,q} \\
&\geq \mathcal{J}_\varepsilon(m_\varepsilon(u_n))
\end{aligned}$$

and this yields $\|m_\varepsilon(u_n)\|_{\mathbb{X}_\varepsilon} \to \infty$ as $n \to \infty$. □

Remark 2.2 There exists $\kappa > 0$, independent of ε, such that $\|u\|_{\mathbb{X}_\varepsilon} \geq \kappa$ for all $u \in \mathcal{N}_\varepsilon$. Indeed, if $u \in \mathcal{N}_\varepsilon$, we can use (g_1), (g_2) and Theorem 2.1 to see that

$$\begin{aligned}\|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} &\leq \int_{\mathbb{R}^N} g(\varepsilon x, u)\, u\, dx \\ &\leq \zeta |u|^p_p + C_\zeta |u|^{q^*_s}_{q^*_s} \\ &\leq \frac{\zeta}{V_0} \|u\|^p_{V_\varepsilon,p} + C'_\zeta \|u\|^{q^*_s}_{V_\varepsilon,q}.\end{aligned}$$

Choosing $\zeta \in (0, V_0)$, we get $\|u\|_{V_\varepsilon,q} \geq \kappa = (C'_\zeta)^{-\frac{1}{q^*_s - q}}$ and thus $\|u\|_{\mathbb{X}_\varepsilon} \geq \|u\|_{V_\varepsilon,q} \geq \kappa$.

Now we define the maps

$$\hat{\psi}_\varepsilon : \mathbb{X}^+_\varepsilon \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}^+_\varepsilon \to \mathbb{R},$$

by setting $\hat{\psi}_\varepsilon(u) = \mathcal{J}_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon = \hat{\psi}_\varepsilon|_{\mathbb{S}^+_\varepsilon}$. From Lemma 2.4 and arguing as in the proofs of Proposition 9 and Corollary 10 in [36], we may obtain the result below.

Proposition 2.1 *Assume that* (V_1)-(V_2) *and* (f_1)-(f_4) *hold. Then we have the following properties:*

(a) $\hat{\psi}_\varepsilon \in C^1(\mathbb{X}^+_\varepsilon, \mathbb{R})$ *and*

$$\langle \hat{\psi}'_\varepsilon(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_{\mathbb{X}_\varepsilon}}{\|u\|_{\mathbb{X}_\varepsilon}} \langle \mathcal{J}'_\varepsilon(\hat{m}_\varepsilon(u)), v\rangle \quad \text{for all } u \in \mathbb{X}^+_\varepsilon \text{ and } v \in \mathbb{X}_\varepsilon.$$

(b) $\psi_\varepsilon \in C^1(\mathbb{S}^+_\varepsilon, \mathbb{R})$ *and*

$$\langle \psi'_\varepsilon(u), v\rangle = \|m_\varepsilon(u)\|_{\mathbb{X}_\varepsilon} \langle \mathcal{J}'_\varepsilon(m_\varepsilon(u)), v\rangle \quad \text{for all } v \in T_u\mathbb{S}^+_\varepsilon.$$

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(PS)_c$ *sequence for* ψ_ε, *then* $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_c$ *sequence for* $\mathcal{J}_\varepsilon$. *If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_c$ *sequence for* $\mathcal{J}_\varepsilon$, *then* $\{m^{-1}_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_c$ *sequence for* ψ_ε.

(d) u *is a critical point of* ψ_ε *if and only if* $m_\varepsilon(u)$ *is a critical point for* $\mathcal{J}_\varepsilon$. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}^+_\varepsilon} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u).$$

Remark 2.3 As in [36], we have the following minimax characterization of the infimum of $\mathcal{J}_\varepsilon$ over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u) = \inf_{u\in\mathbb{X}^+_\varepsilon} \max_{t>0} \mathcal{J}_\varepsilon(tu) = \inf_{u\in\mathbb{S}^+_\varepsilon} \max_{t>0} \mathcal{J}_\varepsilon(tu).$$

Moreover, arguing as in [37], we can prove that $c_\varepsilon = c'_\varepsilon$.

In the remainder of this section, we check that the modified functional satisfies the Palais–Smale condition. We start by showing the boundedness of Palais–Smale sequences.

Lemma 2.5 *Let $c \in \mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$.*

Proof Using (g_3), $q > p$ and $\vartheta > 2q$, we see that

$$\begin{aligned}
C_0(1 + \|u_n\|_{\mathbb{X}_\varepsilon}) &\geq \mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle \\
&= \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\|u_n\|^p_{V_\varepsilon,p} + \left(\frac{1}{2p} - \frac{1}{\vartheta}\right)[u_n]^{2p}_{s,p} \\
&\quad + \left(\frac{1}{q} - \frac{1}{\vartheta}\right)\|u_n\|^q_{V_\varepsilon,q} + \left(\frac{1}{2q} - \frac{1}{\vartheta}\right)[u_n]^{2q}_{s,q} \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon^c}[g(\varepsilon x, u_n)u_n - \vartheta G(\varepsilon x, u_n)]\,dx \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon}[g(\varepsilon x, u_n)u_n - \vartheta G(\varepsilon x, u_n)]\,dx \\
&\geq \left(\frac{1}{q} - \frac{1}{\vartheta}\right)[\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}] \\
&\quad - \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\frac{1}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u_n|^p + |u_n|^q)\,dx \\
&\geq \left[\left(\frac{1}{q} - \frac{1}{\vartheta}\right) - \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\frac{1}{K}\right](\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) \\
&= \tilde{C}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}),
\end{aligned} \tag{2.6}$$

where $\tilde{C} = \left[\left(\frac{1}{q} - \frac{1}{\vartheta}\right) - \left(\frac{1}{p} - \frac{1}{\vartheta}\right)\frac{1}{K}\right] > 0$ since $K > \left(\frac{\vartheta-p}{\vartheta-q}\right)\frac{q}{p}$. Suppose, by contradiction, that $\|u_n\|_{\mathbb{X}_\varepsilon} \to \infty$. Then we discuss the following cases:

Case 1 $\|u_n\|_{V_\varepsilon,p} \to \infty$ and $\|u_n\|_{V_\varepsilon,q} \to \infty$.

For n large, we get $\|u_n\|^{q-p}_{V_\varepsilon,q} \geq 1$, that is $\|u_n\|^q_{V_\varepsilon,q} \geq \|u_n\|^p_{V_\varepsilon,q}$. Therefore, from (2.6),

$$\begin{aligned}
C_0(1 + \|u_n\|_{\mathbb{X}_\varepsilon}) &\geq \tilde{C}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^p_{V_\varepsilon,q}) \geq C_1(\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^p \\
&= C_1\|u_n\|^p_{\mathbb{X}_\varepsilon}
\end{aligned}$$

which is a contradiction.

Case 2 $\|u_n\|_{V_\varepsilon,p} \to \infty$ and $\|u_n\|_{V_\varepsilon,q}$ is bounded.

We have

$$C_0(1 + \|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q}) = C_0(1 + \|u_n\|_{\mathbb{X}_\varepsilon}) \geq \tilde{C}\|u_n\|^p_{V_\varepsilon,p}$$

and thus

$$C_0\left(\frac{1}{\|u_n\|^p_{V_\varepsilon,p}}+\frac{1}{\|u_n\|^{p-1}_{V_\varepsilon,p}}+\frac{\|u_n\|_{V_\varepsilon,q}}{\|u_n\|^p_{V_\varepsilon,p}}\right)\geq \tilde{C}.$$

Since $p>1$ and letting $n\to\infty$, we find $0<\tilde{C}\leq 0$, that is a contradiction.
Case 3 $\|u_n\|_{V_\varepsilon,q}\to\infty$ and $\|u_n\|_{V_\varepsilon,p}$ is bounded.
This case is similar to the case 2, so we skip the details.
In conclusion, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$. □

Lemma 2.6 *Let $c\in\mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. Then for any $\eta>0$ there exists $R=R(\eta)>0$ such that*

$$\limsup_{n\to\infty}\int_{B^c_R(0)}\left(\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}+\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\,dy\right.$$
$$\left.+V(\varepsilon x)(|u_n|^p+|u_n|^q)\right)dx<\eta. \tag{2.7}$$

Proof Let $\psi\in C^\infty(\mathbb{R}^N)$ be such that $0\leq\psi\leq 1$, $\psi=0$ in $B_{\frac{1}{2}}(0)$, $\psi_R=1$ in $B^c_1(0)$, and $|\nabla\psi|_\infty\leq C$, for some $C>0$. For $R>0$, define $\psi_R(x)=\psi(\frac{x}{R})$. Then, $0\leq\psi_R\leq 1$, $\psi_R=0$ in $B_{\frac{R}{2}}(0)$, $\psi_R=1$ in $B^c_R(0)$, and $|\nabla\psi_R|_\infty\leq\frac{C}{R}$ with $C>0$ independent of R. Since $\{\psi_R u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$, it holds $\langle\mathcal{J}'_\varepsilon(u_n),\psi_R u_n\rangle=o_n(1)$, that is

$$\begin{aligned}
&(1+[u_n]^p_{s,p})\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\psi_R(x)\,dxdy+(1+[u_n]^q_{s,q})\\
&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\psi_R(x)\,dxdy\\
&\quad+\int_{\mathbb{R}^N}V(\varepsilon x)|u_n|^p\psi_R\,dx+\int_{\mathbb{R}^N}V(\varepsilon x)|u_n|^q\psi_R\,dx\\
&=o_n(1)+\int_{\mathbb{R}^N}g(\varepsilon x,u_n)\psi_R u_n\,dx\\
&\quad-(1+[u_n]^p_{s,p})\iint_{\mathbb{R}^{2N}}\\
&\quad\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))}{|x-y|^{N+sp}}u_n(y)\,dxdy\\
&\quad-(1+[u_n]^q_{s,q})\iint_{\mathbb{R}^{2N}}\\
&\quad\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))}{|x-y|^{N+sq}}u_n(y)\,dxdy.
\end{aligned}$$

Pick $R > 0$ such that $\Lambda_\varepsilon \subset B_{\frac{R}{2}}(0)$. By the definition of ψ_R and using (g_3)-(ii), we obtain that

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \psi_R(x)\, dxdy + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^q}{|x-y|^{N+sq}} \psi_R(x)\, dxdy \\
&\quad + \left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^p + |u_n|^q)\psi_R\, dx \\
&\leq o_n(1) - (1 + [u_n]_{s,p}^p) \\
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{N+sp}} u_n(y)\, dxdy \\
&\quad - (1 + [u_n]_{s,q}^q) \\
&\qquad \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{N+sq}} u_n(y)\, dxdy.
\end{aligned}
\tag{2.8}
$$

Now, from the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$, we get, for $t \in \{p, q\}$,

$$
\begin{aligned}
&\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))(\psi_R(x) - \psi_R(y))}{|x-y|^{N+st}} u_n(y)\, dxdy \right| \\
&\quad \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_R(x) - \psi_R(y)|^t}{|x-y|^{N+st}} |u_n(y)|^t\, dxdy \right)^{\frac{1}{t}}.
\end{aligned}
\tag{2.9}
$$

An inspection of the proof of Lemma 2.2 shows that, for $t \in \{p, q\}$,

$$
\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|\psi_R(x) - \psi_R(y)|^t}{|x-y|^{N+st}} |u_n(y)|^t\, dxdy \leq \frac{C}{R^{st}}. \tag{2.10}
$$

Combining (2.8), (2.9) and (2.10), and recalling the definition of ψ_R, for some $C > 0$, we can take $R = R(\eta) > (\frac{C}{\eta})^{\frac{1}{s}}$ so that (2.7) is satisfied. □

Since we are working with a Kirchhoff type problem, the next lemma will be fundamental to obtain the strong convergence of bounded Palais–Smale sequences.

Lemma 2.7 *Let $c \in \mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. Let $R > 0$. Then*

$$\begin{aligned}
&\lim_{n\to\infty} \int_{B_R(0)} \\
&\left\{ \int_{\mathbb{R}^N} \left[\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} + \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} \right] dy + V(\varepsilon x)(|u_n|^p + |u_n|^q) \right\} dx \\
&= \int_{B_R(0)} \\
&\left\{ \int_{\mathbb{R}^N} \left[\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} + \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \right] dy + V(\varepsilon x)(|u|^p + |u|^q) \right\} dx.
\end{aligned} \tag{2.11}$$

Proof Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $B_1(0)$, $\eta = 0$ in $B_2^c(0)$ and $|\nabla \eta|_\infty \le 2$. For $\rho > 0$, put $\eta_\rho(x) = \eta(\frac{x}{\rho})$. Then $0 \le \eta_\rho \le 1$, $\eta = 1$ in $B_\rho(0)$, $\eta = 0$ in $B_{2\rho}^c(0)$ and $|\nabla \eta|_\infty \le \frac{2}{\rho}$. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$ (by Lemma 2.5), we may suppose that $[u_n]_{s,p}^p \to \ell_p$ and $[u_n]_{s,q}^q \to \ell_q$ as $n \to \infty$.

Fix $R > 0$ and take $\rho > R$. We recall the following well-known elementary inequalities [35]: for any $\xi, \eta \in \mathbb{R}^N$ we have

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \ge c_1 |\xi - \eta|^r \quad \text{for } r \ge 2, \tag{2.12}$$

$$(|\xi| + |\eta|)^{2-r} [(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta)] \ge c_2 |\xi - \eta|^2 \quad \text{for } 1 < r < 2, \tag{2.13}$$

for some constants $c_1, c_2 > 0$. Note that, when $1 < r < 2$, using (2.13) and the elementary inequality

$$(|\xi| + |\eta|)^r \le 2^{r-1}(|\xi|^r + |\eta|^r) \quad \text{for all } \xi, \eta \in \mathbb{R}^N,$$

we deduce that there exists $c_3 > 0$ such that, for any $\xi, \eta \in \mathbb{R}^N$, the following relation is satisfied

$$(|\xi|^r + |\eta|^r)^{\frac{2-r}{2}} \left[(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \right]^{\frac{r}{2}} \ge c_3 |\xi - \eta|^r \quad \text{for } 1 < r < 2. \tag{2.14}$$

For $t \in \{p, q\}$ and $n \in \mathbb{N}$, we set

$$\begin{aligned}
A_n^t(x) = (1 + [u_n]_{s,t}^t) \\
&\int_{\mathbb{R}^N} \left[\frac{|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y))}{|x-y|^{N+st}} - \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}} \right] \times \\
&\quad \times [(u_n(x) - u_n(y)) - (u(x) - u(y))]\, dy \\
&+ V(\varepsilon x)(|u_n(x)|^{t-2}u_n(x) - |u(x)|^{t-2}u(x))(u_n(x) - u(x)).
\end{aligned}$$

Note that, for $t \in \{p, q\}$ and $n \in \mathbb{N}$, we have

$$
\begin{aligned}
0 &\le \int_{B_R(0)} A_n^t(x)\,dx = \int_{B_R(0)} A_n^t(x)\eta_\rho(x)\,dx \\
&\le (1+[u_n]_{s,t}^t) \\
&\iint_{\mathbb{R}^{2N}} \left[\frac{|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y))}{|x-y|^{N+st}} - \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}}\right]\times \\
&\quad \times [(u_n(x)-u_n(y))-(u(x)-u(y))]\eta_\rho(x)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^{t-2}u_n - |u|^{t-2}u)(u_n-u)\eta_\rho\,dx \\
&= (1+[u_n]_{s,t}^t)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^t}{|x-y|^{N+st}}\eta_\rho(x)\,dxdy + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^t\eta_\rho\,dx \\
&\quad + (1+[u_n]_{s,t}^t)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^t}{|x-y|^{N+st}}\eta_\rho(x)\,dxdy + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t\eta_\rho\,dx \\
&\quad - \Bigg[(1+[u_n]_{s,t}^t)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{t-2}}{|x-y|^{N+st}}(u_n(x)-u_n(y))(u(x) \\
&\quad -u(y))\eta_\rho(x)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{t-2}u_n u\eta_\rho\,dx\Bigg] \\
&\quad - \Bigg[(1+[u_n]_{s,t}^t)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}}{|x-y|^{N+st}}(u_n(x) \\
&\quad -u_n(y))(u(x)-u(y))\eta_\rho(x)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{t-2}uu_n\eta_\rho\,dx\Bigg].
\end{aligned}
$$

Define

$$
\begin{aligned}
I_{n,\rho}^1 &= (1+[u_n]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\eta_\rho(x)\,dxdy + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\eta_\rho\,dx \\
&\quad + (1+[u_n]_{s,q}^q)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\eta_\rho(x)\,dxdy + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\eta_\rho\,dx \\
&\quad - \int_{\mathbb{R}^N} g(\varepsilon x, u_n)u_n\eta_\rho\,dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{n,\rho}^2 &= (1+[u_n]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\eta_\rho(x)\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p\eta_\rho\,dx \\
&\quad - (1+[u_n]_{s,p}^p)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+sp}}(u(x)
\end{aligned}
$$

$$
\begin{aligned}
&- u(y))(u_n(x) - u_n(y))\eta_\rho(x)\,dxdy \\
&- \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}uu_n\eta_\rho\,dx \\
&+ (1 + [u_n]_{s,q}^q) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^q}{|x - y|^{N+sq}} \eta_\rho(x)\,dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)|u|^q\eta_\rho\,dx \\
&- (1 + [u_n]_{s,q}^q) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}}{|x - y|^{N+sq}} (u(x) - u(y))(u_n(x) \\
&- u_n(y))\eta_\rho(x)\,dxdy \\
&- \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{q-2}uu_n\eta_\rho\,dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{n,\rho}^3 = &(1 + [u_n]_{s,p}^p) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}}{|x - y|^{N+sp}} (u_n(x) \\
&- u_n(y))(u(x) - u(y))\eta_\rho(x)\,dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{p-2}u_n u\eta_\rho\,dx \\
&+ (1 + [u_n]_{s,q}^q) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}}{|x - y|^{N+sq}} (u_n(x) - u_n(y))(u(x) \\
&- u(y))\eta_\rho(x)\,dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{q-2}u_n u\eta_\rho\,dx \\
&- \int_{\mathbb{R}^N} g(\varepsilon x, u_n)u\eta_\rho\,dx,
\end{aligned}
$$

and

$$
I_{n,\rho}^4 = \int_{\mathbb{R}^N} g(\varepsilon x, u_n)(u_n - u)\eta_\rho\,dx.
$$

Then it holds

$$
0 \le \int_{B_R(0)} (A_n^p(x) + A_n^q(x))\,dx \le |I_{n,\rho}^1| + |I_{n,\rho}^2| + |I_{n,\rho}^3| + |I_{n,\rho}^4|. \tag{2.15}
$$

Since

$$
\begin{aligned}
I_{n,\rho}^1 = \langle \mathcal{J}_\varepsilon'(u_n), u_n\eta_\rho\rangle - \Big[\Big(1 + [u_n]_{s,p}^p\Big) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}}{|x - y|^{N+sp}} \\
(u_n(x) - u_n(y))(\eta_\rho(x) - \eta_\rho(y))u_n(y)\,dxdy
\end{aligned}
$$

$$
+ \left(1 + [u_n]_{s,q}^q\right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}}{|x-y|^{N+sq}} (u_n(x) - u_n(y))(\eta_\rho(x) - \eta_\rho(y)) u_n(y)\, dxdy \Big]
$$

and $\{u_n \eta_\rho\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$, we see that $\langle \mathcal{J}_\varepsilon'(u_n), u_n\eta_\rho\rangle = o_n(1)$. Using the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$, we have

$$
\begin{aligned}
&\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{t-2}}{|x-y|^{N+st}} (u_n(x) - u_n(y))(\eta_\rho(x) - \eta_\rho(y)) u_n(y)\, dxdy \right| \\
&\leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\eta_\rho(x) - \eta_\rho(y)|^t}{|x-y|^{N+st}} |u_n(y)|^t \, dxdy \right)^{\frac{1}{t}} \quad \text{for } t \in \{p, q\},
\end{aligned}
$$

which combined with Lemma in 2.2 (applied with $\phi_\rho = 1 - \eta_\rho$) yields

$$
\begin{aligned}
&\lim_{\rho\to\infty} \limsup_{n\to\infty} \\
&\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{t-2}}{|x-y|^{N+st}} (u_n(x) - u_n(y))(\eta_\rho(x) - \eta_\rho(y)) u_n(y)\, dxdy \right| = 0 \\
&\text{for } t \in \{p, q\}.
\end{aligned}
$$

Consequently, recalling that $[u_n]_{s,t}^t \to \ell_t$ for $t \in \{p, q\}$, we get

$$
\lim_{\rho\to\infty} \left[\limsup_{n\to\infty} \left| I_{n,\rho}^1 \right| \right] = 0. \tag{2.16}
$$

We also observe that

$$
\begin{aligned}
I_{n,\rho}^3 = \langle \mathcal{J}_\varepsilon'(u_n), u\eta_\rho\rangle - \Big[&\left(1 + [u_n]_{s,p}^p\right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}}{|x-y|^{N+sp}} \\
&(u_n(x) - u_n(y))(\eta_\rho(x) - \eta_\rho(y)) u(y)\, dxdy \\
&+ \left(1 + [u_n]_{s,q}^q\right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}}{|x-y|^{N+sq}} (u_n(x) \\
&- u_n(y))(\eta_\rho(x) - \eta_\rho(y)) u(y)\, dxdy \Big],
\end{aligned}
$$

and using $\langle \mathcal{J}_\varepsilon'(u_n), u\eta_\rho\rangle = o_n(1)$, we can argue as before to achieve that

$$
\lim_{\rho\to\infty} \left[\limsup_{n\to\infty} |I_{n,\rho}^3| \right] = 0. \tag{2.17}
$$

Next we prove that

$$
\lim_{\rho\to\infty} \left[\limsup_{n\to\infty} |I_{n,\rho}^2| \right] = 0. \tag{2.18}
$$

From the weak convergence, we have

$$\int_{\mathbb{R}^N} V(\varepsilon x)|u|^{t-2}u(u_n-u)\eta_\rho\,dx = o_n(1) \quad \text{for } t\in\{p,q\}.$$

Notice that, for $t\in\{p,q\}$,

$$\begin{aligned}
&\left(1+[u_n]_{s,t}^t\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}}{|x-y|^{N+st}}(u(x)-u(y))\\
&[(u_n-u)(x)-(u_n-u)(y)]\eta_\rho(x)\,dxdy\\
&=\left(1+[u_n]_{s,t}^t\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}}{|x-y|^{N+st}}(u(x)-u(y))\\
&[(u_n-u)(x)-(u_n-u)(y)](\eta_\rho(x)-1)\,dxdy\\
&\quad+\left(1+[u_n]_{s,t}^t\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}}{|x-y|^{N+st}}(u(x)\\
&\quad-u(y))[(u_n-u)(x)-(u_n-u)(y)]\,dxdy.
\end{aligned}$$

By $u_n\rightharpoonup u$ in $\mathbb{X}_\varepsilon$ and $[u_n]_{s,t}^t\to\ell_t$ for $t\in\{p,q\}$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\left(1+[u_n]_{s,t}^t\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}}{|x-y|^{N+st}}(u(x)-u(y))\\
&[(u_n-u)(x)-(u_n-u)(y)]\,dxdy=0 \quad \text{for } t\in\{p,q\}.
\end{aligned}$$

On the other hand, using the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$ and applying the Hölder inequality, we see that

$$\begin{aligned}
&\Bigg|\left(1+[u_n]_{s,t}^t\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}}{|x-y|^{N+st}}(u(x)-u(y))\\
&[(u_n-u)(x)-(u_n-u)(y)](\eta_\rho(x)-1)\,dxdy\Bigg|\\
&\le(1+C)[u_n-u]_{s,t}\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^t}{|x-y|^{N+st}}|\eta_\rho(x)-1|^{\frac{t}{t-1}}\,dxdy\right)^{\frac{t-1}{t}}\\
&\le C\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^t}{|x-y|^{N+st}}|\eta_\rho(x)-1|^{\frac{t}{t-1}}\,dxdy\right)^{\frac{t-1}{t}} \quad \text{for } t\in\{p,q\}.
\end{aligned}$$

Since $\eta_\rho\to 1$ a.e. in $\mathbb{R}^N$ as $\rho\to\infty$ and $u\in W^{s,t}(\mathbb{R}^N)$, it follows from the dominated convergence theorem that

$$\lim_{\rho\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^t}{|x-y|^{N+st}}|\eta_\rho(x)-1|^{\frac{t}{t-1}}\,dxdy=0 \quad \text{for } t\in\{p,q\}.$$

The validity of (2.18) is now an immediate consequence of the definition of $I_{n,\rho}^2$ and of the above relations.

Finally, exploiting $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [p, q_s^*)$ and the growth assumptions on g, we obtain

$$\lim_{n\to\infty} |I^4_{n,\rho}| = 0 \quad \text{for any } \rho > R. \tag{2.19}$$

Combining (2.15) with (2.16)–(2.19), we find

$$\lim_{n\to\infty} \int_{B_R(0)} (A_n^p(x) + A_n^q(x))\, dx = 0,$$

whence

$$\begin{aligned}
&\lim_{n\to\infty} \Big\{ (1 + [u_n]^t_{s,t}) \\
&\int_{B_R(0)} \Big[\int_{\mathbb{R}^N} \\
&\left(\frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))}{|x-y|^{N+st}} - \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x-y|^{N+st}} \right) \times \\
&\times ((u_n(x) - u_n(y)) - (u(x) - u(y)))\, dy \Big] dx \\
&\quad + \int_{B_R(0)} V(\varepsilon x) \left(|u_n|^{t-2}u_n - |u|^{t-2}u \right)(u_n - u)\, dx \Big\} = 0 \quad \text{for } t \in \{p, q\}.
\end{aligned} \tag{2.20}$$

Assume first that $t \geq 2$. Using (2.12), the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$ and (2.20), we get

$$\begin{aligned}
0 &\leq \int_{B_R(0)} \left[\int_{\mathbb{R}^N} \frac{|(u_n - u)(x) - (u_n - u)(y)|^t}{|x-y|^{N+st}}\, dy \right] dx \\
&\leq C \int_{B_R(0)} \Big[\int_{\mathbb{R}^N} \Big(\frac{|u_n(x) - u_n(y)|^{t-2}(u_n(x) - u_n(y))}{|x-y|^{N+st}} \\
&\quad - \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x-y|^{N+st}} \Big) \times \\
&\quad \times ((u_n(x) - u_n(y)) - (u(x) - u(y)))\, dy \Big] dx = o_n(1).
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
0 &\leq \int_{B_R(0)} V(\varepsilon x)|u_n - u|^t\, dx \leq C \int_{B_R(0)} \\
&\quad V(\varepsilon x) \left(|u_n|^{t-2}u_n - |u|^{t-2}u \right)(u_n - u)\, dx = o_n(1).
\end{aligned}$$

Suppose now that $1 < t < 2$. From (2.14), the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$, Hölder's inequality, and (2.20), we derive

$$\begin{aligned}
&\int_{B_R(0)} \left[\int_{\mathbb{R}^N} \frac{|(u_n-u)(x)-(u_n-u)(y)|^t}{|x-y|^{N+st}}\,dy\right] dx \\
&\leq C\Big([u_n]_{s,t}^t+[u]_{s,t}^t\Big)^{\frac{2-t}{2}} \\
&\quad \Big\{\int_{B_R(0)}\Big[\int_{\mathbb{R}^N} \\
&\quad \left(\frac{|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y))}{|x-y|^{N+st}}-\frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}}\right)\times \\
&\quad \times((u_n(x)-u_n(y))-(u(x)-u(y)))\,dy\Big]\,dx\Big\}^{\frac{t}{2}} \\
&\leq C\Big\{\int_{B_R(0)}\Big[\int_{\mathbb{R}^N} \\
&\quad \left(\frac{|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y))}{|x-y|^{N+st}}-\frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}}\right)\times \\
&\quad \times((u_n(x)-u_n(y))-(u(x)-u(y)))\,dy\Big]\,dx\Big\}^{\frac{t}{2}} = o_n(1).
\end{aligned}$$

Analogously,

$$\begin{aligned}
0 &\leq \int_{B_R(0)} V(\varepsilon x)|u_n-u|^t\,dx \\
&\leq C\left[\int_{B_R(0)} V(\varepsilon x)\left(|u_n|^{t-2}u_n-|u|^{t-2}u\right)(u_n-u)\,dx\right]^{\frac{t}{2}} = o_n(1).
\end{aligned}$$

Consequently, for $t\in\{p,q\}$,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{B_R(0)}\left[\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^t}{|x-y|^{N+st}}\,dy+V(\varepsilon x)|u_n|^t\right]dx \\
&=\int_{B_R(0)}\left[\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^t}{|x-y|^{N+st}}\,dy+V(\varepsilon x)|u|^t\right]dx
\end{aligned}$$

which implies (2.11). This completes the proof. □

Now we are ready to prove the following compactness result.

Lemma 2.8 *$\mathcal{J}_\varepsilon$ satisfies the $(PS)_c$ condition at any level $c\in\mathbb{R}$.*

Proof Let $c\in\mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. By Lemma 2.5, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$. Up to a subsequence, we may suppose that $u_n\rightharpoonup u$ in $\mathbb{X}_\varepsilon$ and $u_n\to u$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r\in[1,q_s^*)$. In view of Lemma 2.6, for each $\eta > 0$, there exists $R = R(\eta) > (\frac{C}{\eta})^{\frac{1}{s}}$, with $C > 0$ independent of η, such that (2.11) holds. This fact combined with Lemma 2.7 yields

$$\begin{aligned}
\|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} &\leq \liminf_{n\to\infty}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) \\
&\leq \limsup_{n\to\infty}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) \\
&= \limsup_{n\to\infty}\Big\{\int_{B_R(0)} \\
&\quad \Big[\int_{\mathbb{R}^N}\left(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} + \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\right)dy \\
&\quad +V(\varepsilon x)(|u_n|^p + |u_n|^q)\Big]\,dx \\
&\quad + \int_{B_R^c(0)}\Big[\int_{\mathbb{R}^N}\left(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} + \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\right)dy \\
&\quad +V(\varepsilon x)(|u_n|^p + |u_n|^q)\Big]\,dx\Big\} \\
&= \int_{B_R(0)}\Big[\int_{\mathbb{R}^N}\left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} + \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\right)dy \\
&\quad +V(\varepsilon x)(|u|^p + |u|^q)\Big]\,dx \\
&\quad + \limsup_{n\to\infty}\Big\{\int_{B_R^c(0)} \\
&\quad \Big[\int_{\mathbb{R}^N}\left(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} + \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\right)dy \\
&\quad +V(\varepsilon x)(|u_n|^p + |u_n|^q)\Big]\,dx\Big\} \\
&< \int_{B_R(0)}\Big[\int_{\mathbb{R}^N}\left(\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} + \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\right)dy \\
&\quad +V(\varepsilon x)(|u|^p + |u|^q)\Big]\,dx + \eta.
\end{aligned}$$

Letting $\eta \to 0$, we have $R \to \infty$ and then

$$\begin{aligned}
\|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} &\leq \liminf_{n\to\infty}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) \\
&\leq \limsup_{n\to\infty}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) \\
&\leq \|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q},
\end{aligned}$$

whence

$$\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q} = \|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} + o_n(1). \tag{2.21}$$

Since the Brezis–Lieb lemma [14] gives

$$\|u_n - u\|^p_{V_\varepsilon,p} = \|u_n\|^p_{V_\varepsilon,p} - \|u\|^p_{V_\varepsilon,p} + o_n(1) \text{ and } \|u_n - u\|^q_{V_\varepsilon,q} = \|u_n\|^q_{V_\varepsilon,q} - \|u\|^q_{V_\varepsilon,q} + o_n(1),$$

we infer that

$$\|u_n - u\|_{V_\varepsilon,p}^p + \|u_n - u\|_{V_\varepsilon,q}^q = o_n(1).$$

This last fact implies that $u_n \to u$ in $\mathbb{X}_\varepsilon$ as $n \to \infty$. □

Corollary 2.1 *The functional ψ_ε satisfies the $(PS)_c$ condition on $\mathbb{S}_\varepsilon^+$ at any level $c \in \mathbb{R}$.*

Proof Let $c \in \mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be a $(PS)_c$ sequence for ψ_ε. Hence,

$$\psi_\varepsilon(u_n) \to c \quad \text{and} \quad \psi_\varepsilon'(u_n) \to 0 \text{ in } (T_{u_n}\mathbb{S}_\varepsilon^+)'.$$

By Proposition 2.1-(c), we know that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ is a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. Then, by Lemma 2.8, we deduce that $\mathcal{J}_\varepsilon$ satisfies the $(PS)_c$ condition in $\mathbb{X}_\varepsilon$, and thus there exists $u \in \mathbb{S}_\varepsilon^+$ such that, up to a subsequence,

$$m_\varepsilon(u_n) \to m_\varepsilon(u) \text{ in } \mathbb{X}_\varepsilon.$$

By Lemma 2.4-(iii), we conclude that $u_n \to u$ in $\mathbb{S}_\varepsilon^+$. □

We conclude this section by establishing an existence result for (2.1).

Theorem 2.2 *Assume that (V_1)–(V_2) and (f_1)–(f_4) hold. Then, for all $\varepsilon > 0$, there exists a positive ground state solution to (2.1).*

Proof In light of Lemmas 2.3 and 2.8, we can apply the mountain pass theorem [3] to see that for all $\varepsilon > 0$ there exists a nontrivial critical point $u_\varepsilon \in \mathbb{X}_\varepsilon$ of $\mathcal{J}_\varepsilon$. By Remark 2.3, we deduce that u_ε is a ground state solution to (2.1). Using $\langle \mathcal{J}_\varepsilon'(u_\varepsilon), u_\varepsilon^- \rangle = 0$, where $u^- = \min\{u, 0\}$, (V_1), $g(\cdot, t) = 0$ for $t \leq 0$ and (2.2), we have

$$C(\|u_\varepsilon^-\|_{W^{s,p}(\mathbb{R}^N)}^p + \|u_\varepsilon^-\|_{W^{s,q}(\mathbb{R}^N)}^q) \leq 0,$$

which gives $u_\varepsilon^- = 0$, that is $u_\varepsilon \geq 0$ in $\mathbb{R}^N$. Arguing as in the proof of Lemma 5.1 below (see also Lemma 4.1 and Theorem 2.2 in [11]), we obtain that $u_\varepsilon \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, and applying the strong maximum principle [7] we infer that $u_\varepsilon > 0$ in $\mathbb{R}^N$. □

3 The Limiting Kirchhoff Problem

Since we are interested in providing a multiplicity result for the auxiliary problem (2.1), it is important to analyze the limiting problem associated with (1.1), namely

$$\begin{cases} \left(1 + [u]_{s,p}^p\right)(-\Delta)_p^s u + \left(1 + [u]_{s,q}^q\right)(-\Delta)_q^s u + V_0(|u|^{p-2}u + |u|^{q-2}u) = f(u) \text{ in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

Let $\mathbb{Y}_{V_0} = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ equipped with the norm

$$\|u\|_{\mathbb{Y}_{V_0}} = \|u\|_{s,p} + \|u\|_{s,q},$$

where

$$\|u\|_{s,t} = \left([u]_{s,t}^t + V_0|u|_t^t\right)^{\frac{1}{t}} \quad \text{for } t \in \{p, q\}.$$

The energy functional $\mathcal{L}_{V_0} : \mathbb{Y}_{V_0} \to \mathbb{R}$ associated with (3.1) is given by

$$\mathcal{L}_{V_0}(u) = \frac{1}{p}\|u\|_{s,p}^p + \frac{1}{q}\|u\|_{s,q}^q + \frac{1}{2p}[u]_{s,p}^{2p} + \frac{1}{2q}[u]_{s,q}^{2q} - \int_{\mathbb{R}^N} F(u)\,dx.$$

Standard arguments show that $\mathcal{L}_{V_0} \in C^1(\mathbb{Y}_{V_0}, \mathbb{R})$ and that

$$\begin{aligned}\langle \mathcal{L}'_{V_0}(u), \varphi\rangle &= (1 + [u]_{s,p}^p)\langle u, \varphi\rangle_{s,p} + (1 + [u]_{s,q}^q)\langle u, \varphi\rangle_{s,q} \\ &\quad + V_0\left[\int_{\mathbb{R}^N} |u|^{p-2}u\,\varphi\,dx + \int_{\mathbb{R}^N} |u|^{q-2}u\,\varphi\,dx\right] - \int_{\mathbb{R}^N} f(u)\varphi\,dx\end{aligned}$$

for any $u, \varphi \in \mathbb{Y}_{V_0}$. We also consider the Nehari manifold $\mathcal{M}_{V_0}$ associated with $\mathcal{L}_{V_0}$, that is

$$\mathcal{M}_{V_0} = \{u \in \mathbb{Y}_{V_0} \setminus \{0\} : \langle \mathcal{L}'_{V_0}(u), u\rangle = 0\},$$

and we set $d_{V_0} = \inf_{u \in \mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u)$. Now we define

$$\mathbb{Y}_{V_0}^+ = \{u \in \mathbb{Y}_{V_0} : |\operatorname{supp}(u^+)| > 0\},$$

and $\mathbb{S}_{V_0}^+ = \mathbb{S}_{V_0} \cap \mathbb{Y}_{V_0}^+$, where $\mathbb{S}_{V_0}$ is the unit sphere of $\mathbb{Y}_{V_0}$. As in Sect. 2, $\mathbb{S}_{V_0}^+$ is an incomplete $C^{1,1}$-manifold of codimension one and contained in $\mathbb{Y}_{V_0}^+$. Thus, $\mathbb{Y}_{V_0} = T_u\mathbb{S}_{V_0}^+ \oplus \mathbb{R}u$ for each $u \in \mathbb{S}_{V_0}^+$, where

$$\begin{aligned}T_u\mathbb{S}_{V_0}^+ = \Big\{v \in \mathbb{Y}_{V_0} &: (1 + [u]_{s,p}^p)\langle u, v\rangle_{s,p} + (1 + [u]_{s,q}^q)\langle u, v\rangle_{s,q} \\ &+ V_0\int_{\mathbb{R}^N} (|u|^{p-2}uv + |u|^{q-2}uv)\,dx = 0\Big\}.\end{aligned}$$

In the sequel, we state without proofs the following results which can be obtained arguing as in Sect. 2.

Lemma 3.1 *Assume that (f_1)–(f_4) hold. Then we have the following properties:*

(i) *For each $u \in \mathbb{Y}_{V_0}^+$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{L}_{V_0}(tu)$. Then, there is a unique $t_u > 0$ such that*

$$\begin{aligned}&h'_u(t) > 0 \text{ for all } t \in (0, t_u),\\ &h'_u(t) < 0 \text{ for all } t \in (t_u, \infty).\end{aligned}$$

(ii) *There exists* $\tau > 0$*, independent of* u*, such that* $t_u \geq \tau$ *for any* $u \in \mathbb{S}^+_{V_0}$*. Moreover, for each compact set* $\mathbb{K} \subset \mathbb{S}^+_{V_0}$*, there is a constant* $C_\mathbb{K} > 0$ *such that* $t_u \leq C_\mathbb{K}$ *for any* $u \in \mathbb{K}$*.*

(iii) *The map* $\hat{m}_{V_0} : \mathbb{Y}^+_{V_0} \to \mathcal{M}_{V_0}$ *given by* $\hat{m}_{V_0}(u) = t_u u$ *is continuous and* $m_{V_0} = \hat{m}_{V_0}|_{\mathbb{S}^+_{V_0}}$ *is a homeomorphism between* $\mathbb{S}^+_{V_0}$ *and* $\mathcal{M}_{V_0}$*. Moreover,* $m^{-1}_{V_0}(u) = \frac{u}{\|u\|_{\mathbb{Y}_{V_0}}}$.

(iv) *If there is a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^+_{V_0}$ *such that* $\mathrm{dist}(u_n, \partial\mathbb{S}^+_{V_0}) \to 0$*, then* $\|m_{V_0}(u_n)\|_{\mathbb{Y}_{V_0}} \to \infty$ *and* $\mathcal{L}_{V_0}(m_{V_0}(u_n)) \to \infty$*.*

Let us consider the maps

$$\hat{\psi}_{V_0} : \mathbb{Y}^+_{V_0} \to \mathbb{R} \quad \text{and} \quad \psi_{V_0} : \mathbb{S}^+_{V_0} \to \mathbb{R},$$

defined by $\hat{\psi}_{V_0}(u) = \mathcal{L}_{V_0}(\hat{m}_{V_0}(u))$ and $\psi_{V_0} = \hat{\psi}_{V_0}|_{\mathbb{S}^+_{V_0}}$.

Proposition 3.1 *Assume that* (f_1)*-*(f_4) *hold. Then we have the following properties:*

(a) $\hat{\psi}_{V_0} \in C^1(\mathbb{Y}^+_{V_0}, \mathbb{R})$ *and*

$$\langle \hat{\psi}'_{V_0}(u), v\rangle = \frac{\|\hat{m}_{V_0}(u)\|_{\mathbb{Y}_{V_0}}}{\|u\|_{\mathbb{Y}_{V_0}}} \langle \mathcal{L}'_{V_0}(\hat{m}_{V_0}(u)), v\rangle \quad \textit{for all } u \in \mathbb{Y}^+_{V_0} \textit{ and } v \in \mathbb{Y}_{V_0}.$$

(b) $\psi_{V_0} \in C^1(\mathbb{S}^+_{V_0}, \mathbb{R})$ *and*

$$\langle \psi'_{V_0}(u), v\rangle = \|m_{V_0}(u)\|_{\mathbb{Y}_{V_0}} \langle \mathcal{L}'_{V_0}(m_{V_0}(u)), v\rangle \quad \textit{for all } v \in T_u\mathbb{S}^+_{V_0}.$$

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_{V_0}*, then* $\{m_{V_0}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* $\mathcal{L}_{V_0}$*. If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ *is a bounded* $(PS)_d$ *sequence for* $\mathcal{L}_{V_0}$*, then* $\{m^{-1}_{V_0}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_{V_0}*.*

(d) u *is a critical point of* ψ_{V_0} *if and only if* $m_{V_0}(u)$ *is a nontrivial critical point for* $\mathcal{L}_{V_0}$*. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}^+_{V_0}} \psi_{V_0}(u) = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u).$$

Remark 3.1 As in Sect. 2, we have the following minimax characterization of the infimum of $\mathcal{L}_{V_0}$ over $\mathcal{M}_{V_0}$:

$$0 < d_{V_0} = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u) = \inf_{u\in\mathbb{Y}^+_{V_0}} \max_{t>0} \mathcal{L}_{V_0}(tu) = \inf_{u\in\mathbb{S}^+_{V_0}} \max_{t>0} \mathcal{L}_{V_0}(tu).$$

The lemma below allows us to assume that the weak limit of a $(PS)_{d_{V_0}}$ sequence of $\mathcal{L}_{V_0}$ is nontrivial.

Lemma 3.2 *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ *be a* $(PS)_{d_{V_0}}$ *sequence for* $\mathcal{L}_{V_0}$ *such that* $u_n \rightharpoonup 0$ *in* $\mathbb{Y}_{V_0}$*. Then we have either*

(a) $u_n \to 0$ *in* $\mathbb{Y}_{V_0}$, *or*
(b) *there exists a sequence* $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^q \, dx \geq \beta.$$

Proof Suppose that (b) is false. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, we can use Lemma 2.1 to see that

$$u_n \to 0 \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for all } r \in (p, q_s^*).$$

Moreover, by (f_1) and (f_2), we have that

$$\int_{\mathbb{R}^N} f(u_n)u_n \, dx = o_n(1) \quad \text{as } n \to \infty.$$

Since $\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle = o_n(1)$, we get

$$\|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q \leq \int_{\mathbb{R}^N} f(u_n)u_n \, dx = o_n(1),$$

that is $\|u_n\|_{\mathbb{Y}_{V_0}} \to 0$ as $n \to \infty$. Then, (a) is true. □

Remark 3.2 As it has been mentioned earlier, if $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ is a $(PS)_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$ such that $u_n \rightharpoonup u$ in $\mathbb{Y}_{V_0}$, then we may assume that $u \neq 0$. Otherwise, if $u_n \rightharpoonup 0$ in $\mathbb{Y}_{V_0}$ and, if $u_n \not\to 0$ in $\mathbb{Y}_{V_0}$, it follows from Lemma 3.2 that there are $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^q \, dx \geq \beta.$$

Define $v_n(x) = u_n(x + y_n)$. Then, using the invariance of $\mathbb{R}^N$ by translation, we see that $\{v_n\}_{n\in\mathbb{N}}$ is a bounded $(PS)_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$ such that $v_n \rightharpoonup v$ in $\mathbb{Y}_{V_0}$ with $v \neq 0$.

In the following lemma, we obtain a positive ground state solution for the autonomous problem (3.1).

Theorem 3.1 *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ *be a* $(PS)_{d_{V_0}}$ *sequence of* $\mathcal{L}_{V_0}$*. Then there exists* $u \in \mathbb{Y}_{V_0} \setminus \{0\}$*, with* $u \geq 0$*, such that, up to a subsequence,* $u_n \to u$ *in* $\mathbb{Y}_{V_0}$*. Moreover,* u *is a positive ground state solution to* (3.1)*.*

Proof Proceeding as in the proof of Lemma 2.5, we can verify that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$. By passing to a subsequence if necessary, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } \mathbb{Y}_{V_0}, \\ &u_n \to u \quad \text{in } L^r_{loc}(\mathbb{R}^N) \quad \text{for all } r \in [1, p_s^*). \end{aligned} \tag{3.2}$$

From Remark 3.2, we may suppose that $u \neq 0$. Moreover, we may assume that $[u_n]_{s,p}^p \to t_1$ and $[u_n]_{s,q}^q \to t_2$. Our aim is to prove that $[u_n]_{s,t} \to [u]_{s,t}$ for $t \in \{p, q\}$. By Fatou's lemma, we know that $[u]_{s,p}^p \leq t_1$ and $[u]_{s,q}^q \leq t_2$. Now we show that $[u]_{s,p}^p = t_1$ and $[u]_{s,q}^q = t_2$. Assume, by contradiction, that $[u]_{s,p}^p < t_1$ and $[u]_{s,q}^q \leq t_2$. Since $\langle \mathcal{L}'_{V_0}(u_n), \varphi\rangle \to 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, and $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathbb{Y}_{V_0}$ (see [19]), we can deduce that

$$(1+t_1)[u]_{s,p}^p + (1+t_2)[u]_{s,q}^q + V_0(|u|_p^p + |u|_q^q) = \int_{\mathbb{R}^N} f(u)u\,dx.$$

Therefore,

$$\begin{aligned}&(1+[u]_{s,p}^p)[u]_{s,p}^p + (1+[u]_{s,q}^q)[u]_{s,q}^q + V_0(|u|_p^p + |u|_q^q) - \int_{\mathbb{R}^N} f(u)u\,dx \\ &\quad < (1+t_1)[u]_{s,p}^p + (1+t_2)[u]_{s,q}^q + V_0(|u|_p^p + |u|_q^q) - \int_{\mathbb{R}^N} f(u)u\,dx = 0,\end{aligned}$$

that is $\langle \mathcal{L}'_{V_0}(u), u\rangle < 0$. From (f_1) and (f_2), we have $\langle \mathcal{L}'_{V_0}(t_0u), t_0u\rangle > 0$ for some $0 < t_0 \ll 1$. Hence, there exists $\tau \in (t_0, 1)$ such that $\langle \mathcal{L}'_{V_0}(\tau u), \tau u\rangle = 0$. Combining this fact with the characterization of d_{V_0} and using the fact that $t \mapsto \frac{1}{2q} f(t)t - F(t)$ is increasing (thanks to (f_3) and (f_4)), by Fatou's lemma, we get

$$\begin{aligned}d_{V_0} \leq \mathcal{L}_{V_0}(\tau u) &= \mathcal{L}_{V_0}(\tau u) - \frac{1}{2q}\langle \mathcal{L}'_{V_0}(\tau u), \tau u\rangle \\ &< \mathcal{L}_{V_0}(u) - \frac{1}{2q}\langle \mathcal{L}'_{V_0}(u), u\rangle \\ &\leq \liminf_{n\to\infty}\left[\mathcal{L}_{V_0}(u_n) - \frac{1}{2q}\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle\right] = d_{V_0}\end{aligned}$$

and we arrive at a contradiction. Hence, $[u_n]_{s,t} \to [u]_{s,t}$ for $t \in \{p, q\}$, and we obtain $\mathcal{L}'_{V_0}(u) = 0$. Finally, we prove that u is positive in $\mathbb{R}^N$. Since $\langle \mathcal{L}'_{V_0}(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$, and $f(t) = 0$ for $t \leq 0$, we have

$$\|u^-\|_{s,p}^p + \|u^-\|_{s,q}^q \leq 0$$

which implies that $u^- = 0$, that is $u \geq 0$ in $\mathbb{R}^N$. Thus, $u \geq 0$ and $u \not\equiv 0$ in $\mathbb{R}^N$. Using a Moser iteration argument [32] (see the proof of Lemma 5.1 below), we obtain that $u \in L^\infty(\mathbb{R}^N)$. Since u solves

$$\alpha_u(-\Delta)_p^s u + \beta_u(-\Delta)_q^s u = -V_0(u^{p-1} + u^{q-1}) + f(u) \in L^\infty(\mathbb{R}^N),$$

where $\alpha_u = 1 + [u]_{s,p}^p$ and $\beta_u = 1 + [u]_{s,q}^q$ are bounded quantities, we can argue as in the proof of Theorem 2.2 in [11] to infer that $u \in C^{0,\alpha}(\mathbb{R}^N)$. In particular, $u(x) \to 0$ as $|x| \to \infty$. By using the strong maximum principle [7], we deduce that $u > 0$ in $\mathbb{R}^N$. □

The next lemma is a compactness result for the autonomous problem (3.1).

Lemma 3.3 *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ *be a sequence such that* $\mathcal{L}_{V_0}(u_n) \to d_{V_0}$. *Then,* $\{u_n\}_{n\in\mathbb{N}}$ *has a convergent subsequence in* $\mathbb{Y}_{V_0}$.

Proof Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ and $\mathcal{L}_{V_0}(u_n) \to d_{V_0}$, it follows from Lemma 3.1-(iii), Proposition 3.1-(d) and the definition of d_{V_0} that

$$v_n = m_{V_0}^{-1}(u_n) = \frac{u_n}{\|u_n\|_{\mathbb{Y}_{V_0}}} \in \mathbb{S}_{V_0}^+ \quad \text{for all } n \in \mathbb{N},$$

and

$$\psi_{V_0}(v_n) = \mathcal{L}_{V_0}(u_n) \to d_{V_0} = \inf_{v\in\mathbb{S}_{V_0}^+} \psi_{V_0}(v).$$

Let us define $\mathcal{G} : \overline{\mathbb{S}}_{V_0}^+ \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{G}(u) = \begin{cases} \psi_{V_0}(u) & \text{if} u \in \mathbb{S}_{V_0}^+, \\ \infty & \text{if } u \in \partial\mathbb{S}_{V_0}^+. \end{cases}$$

We observe that the following properties hold:

- $(\overline{\mathbb{S}}_{V_0}^+, \delta_{V_0})$, where $\delta_{V_0}(u, v) = \|u - v\|_{\mathbb{Y}_{V_0}}$, is a complete metric space.
- $\mathcal{G} \in C(\overline{\mathbb{S}}_{V_0}^+, \mathbb{R} \cup \{\infty\})$, by Lemma 3.1-(iv).
- $\mathcal{G}$ is bounded below, by Proposition 3.1-(d).

By using the Ekeland variational principle [21], there exists $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_{V_0}^+$ such that $\{\hat{v}_n\}_{n\in\mathbb{N}}$ is a $(PS)_{d_{V_0}}$ sequence for ψ_{V_0} and $\|\hat{v}_n - v_n\|_{\mathbb{Y}_{V_0}} = o_n(1)$. Now the remainder of the proof follows from Proposition 3.1, Theorem 3.1, and arguing as in the proof of Corollary 2.1. □

We conclude this section by showing a useful relation between the minimax levels c_ε and d_{V_0}.

Lemma 3.4 *It holds* $\lim_{\varepsilon\to 0} c_\varepsilon = d_{V_0}$.

Proof For $\varepsilon > 0$, let $\omega_\varepsilon(x) = \psi_\varepsilon(x)\omega(x)$, where ω is a positive ground state of (3.1) (whose existence is guaranteed by Theorem 3.1), and $\psi_\varepsilon(x) = \psi(\varepsilon x)$ with $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$. For simplicity, we assume that $\text{supp}(\psi) \subset B_2 \subset \Lambda$. Using the dominated convergence theorem, we see that

$$\omega_\varepsilon \to \omega \quad \text{in } \mathcal{W} \quad \text{and} \quad \mathcal{L}_{V_0}(\omega_\varepsilon) \to \mathcal{L}_{V_0}(\omega) = d_{V_0} \tag{3.3}$$

as $\varepsilon \to 0$. Now, for each $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\mathcal{J}_\varepsilon(t_\varepsilon\omega_\varepsilon) = \max_{t\ge 0} \mathcal{J}_\varepsilon(t\omega_\varepsilon).$$

Therefore, $\langle \mathcal{J}'_\varepsilon(t_\varepsilon \omega_\varepsilon), \omega_\varepsilon \rangle = 0$ and this implies that

$$\begin{aligned} & t_\varepsilon^p [\omega_\varepsilon]_{s,p}^p + t_\varepsilon^{2p} [\omega_\varepsilon]_{s,p}^{2p} + t_\varepsilon^q [\omega_\varepsilon]_{s,q}^q \\ & + t_\varepsilon^{2q} [\omega_\varepsilon]_{s,q}^{2q} + t_\varepsilon^p \int_{\mathbb{R}^N} V(\varepsilon x) \omega_\varepsilon^p \, dx + t_\varepsilon^q \int_{\mathbb{R}^N} V(\varepsilon x) \omega_\varepsilon^q \, dx \\ & = \int_{\mathbb{R}^N} f(t_\varepsilon \omega_\varepsilon) t_\varepsilon \omega_\varepsilon \, dx. \end{aligned}$$

If $t_\varepsilon \to \infty$, then

$$\begin{aligned} & t_\varepsilon^{p-2q} [\omega_\varepsilon]_{s,p}^p + t_\varepsilon^{2p-2q} [\omega_\varepsilon]_{s,p}^{2p} \\ & + t^{-q} [\omega_\varepsilon]_{s,q}^q + [\omega_\varepsilon]_{s,q}^{2q} + t_\varepsilon^{p-2q} \int_{\mathbb{R}^N} V(\varepsilon x) \omega_\varepsilon^p \, dx + t^{-q} \int_{\mathbb{R}^N} V(\varepsilon x) \omega_\varepsilon^q \, dx \\ & = \int_{\mathbb{R}^N} \frac{f(t_\varepsilon \omega_\varepsilon)}{(t_\varepsilon \omega_\varepsilon)^{2q-1}} \omega_\varepsilon^{2q} \, dx, \end{aligned} \tag{3.4}$$

and using (3.3), $p < 2q$ and (f_3), we obtain that $[\omega]_{s,q}^{2q} = \infty$, which is impossible. Then, $t_\varepsilon \to t_0 \in [0, \infty)$. If $t_0 = 0$, using (f_1) and (f_2), we see that, for $\zeta \in (0, V_0)$, it holds

$$\left(1 - \frac{\zeta}{V_0}\right) \|\omega_\varepsilon\|_{V_{\varepsilon,p}}^p + t_\varepsilon^{q-p} \|\omega_\varepsilon\|_{V_{\varepsilon,q}}^q \le C_\zeta t_\varepsilon^{q-p} \|\omega_\varepsilon\|_{V_{\varepsilon,q}}^{q_s^*}.$$

This together with $q > p$ yields $\|\omega\|_{s,p}^p = 0$, that is a contradiction. Hence, $t_\varepsilon \to t_0 \in (0, \infty)$.

Taking the limit as $\varepsilon \to 0$ in (3.4), we get

$$\begin{aligned} & t_0^{p-2q} [\omega]_{s,p}^p + t_0^{2p-2q} [\omega]_{s,p}^{2p} + t_0^{-q} [\omega]_{s,q}^q + [\omega]_{s,q}^{2q} \\ & + t_0^{p-2q} \int_{\mathbb{R}^N} V_0 \omega^p \, dx + t_0^{-q} \int_{\mathbb{R}^N} V_0 \omega^q \, dx = \int_{\mathbb{R}^N} \frac{f(t_0 \omega)}{(t_0 \omega)^{2q-1}} \omega^{2q} \, dx, \end{aligned}$$

which combined with $2q > q > p$, (f_4) and $\omega \in \mathcal{M}_{V_0}$, implies that $t_0 = 1$.

Now, we note that

$$\begin{aligned} c_\varepsilon \le \max_{t \ge 0} \mathcal{J}_\varepsilon(t\omega_\varepsilon) = \mathcal{J}_\varepsilon(t_\varepsilon \omega_\varepsilon) = \mathcal{L}_{V_0}(t_\varepsilon \omega_\varepsilon) + \tfrac{t_\varepsilon^p}{p} \textstyle\int_{\mathbb{R}^N} (V(\varepsilon x) - V_0) \omega_\varepsilon^p \, dx \\ + \tfrac{t_\varepsilon^q}{q} \textstyle\int_{\mathbb{R}^N} (V(\varepsilon x) - V_0) \omega_\varepsilon^q \, dx. \end{aligned}$$

Since $V(\varepsilon \cdot)$ is bounded on the support of ω_ε, we can use the dominated convergence theorem, (3.3) and the above inequality to deduce that $\limsup_{\varepsilon \to 0} c_\varepsilon \le d_{V_0}$. By (V_1), we obtain that $\liminf_{\varepsilon \to 0} c_\varepsilon \ge d_{V_0}$, and thus $\lim_{\varepsilon \to 0} c_\varepsilon = d_{V_0}$. This completes the proof. □

4 A Multiplicity Result for (2.1)

In this section, we deal with the multiplicity of solutions to (2.1). Let $\delta > 0$ be such that

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\} \subset \Lambda,$$

and let $w \in \mathbb{Y}_{V_0}$ be a positive ground state solution to (3.1) (by virtue of Theorem 3.1).

Consider a nonincreasing function $\eta \in C^\infty([0, \infty), [0, 1])$ such that $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$, $\eta(t) = 0$ if $t \geq \delta$ and $|\eta'(t)| \leq c$ for some $c > 0$. For any $y \in M$, we define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ be given by

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y},$$

where $t_\varepsilon > 0$ satisfies

$$\max_{t \geq 0} \mathcal{J}_\varepsilon(t\Psi_{\varepsilon,y}) = \mathcal{J}_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 4.1 *The function Φ_ε has the following property:*

$$\lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \quad \textit{uniformly in } y \in M.$$

Proof Assume, by contradiction, that there exist $\delta_0 > 0$, $\{y_n\}_{n \in \mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - d_{V_0}| \geq \delta_0. \tag{4.1}$$

For each $n \in \mathbb{N}$ and for all $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$, we have $\varepsilon_n z \in B_\delta(0)$, and thus

$$\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda.$$

Using the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ and the fact that $G = F$ in $\Lambda \times \mathbb{R}$, we can write

$$\begin{aligned}\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = &\frac{t_{\varepsilon_n}^p}{p}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \frac{t_{\varepsilon_n}^{2p}}{2p}[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} + \frac{t_{\varepsilon_n}^q}{q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q \\ &+ \frac{t_{\varepsilon_n}^{2q}}{2q}[\Psi_{\varepsilon_n,y_n}]_{s,q}^{2q} - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\, dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{t_{\varepsilon_n}^{p}}{p}\left([\eta(|\varepsilon_n\cdot|)w]_{s,p}^{p} + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^p\,dz\right)\\
&\quad + \frac{t_{\varepsilon_n}^{2p}}{2p}[\eta(|\varepsilon_n\cdot|)w]_{s,p}^{2p}\\
&\quad + \frac{t_{\varepsilon_n}^{q}}{q}\left([\eta(|\varepsilon_n\cdot|)w]_{s,q}^{q} + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^q\,dz\right)\\
&\quad + \frac{t_{\varepsilon_n}^{2q}}{2q}[\eta(|\varepsilon_n\cdot|)w]_{s,q}^{2q}\\
&\quad - \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))\,dz.
\end{aligned} \tag{4.2}
$$

We claim that $t_{\varepsilon_n}\to 1$ as $n\to\infty$. We start by proving that $t_{\varepsilon_n}\to t_0\in[0,\infty)$. Since $\Phi_{\varepsilon_n}(y_n)\in\mathcal{N}_{\varepsilon_n}$ and $g=f$ on $\Lambda\times\mathbb{R}$, we have

$$
\begin{aligned}
&\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^{p} + \frac{1}{t_{\varepsilon}^{q}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^{q} + \frac{1}{t_{\varepsilon}^{2q-2p}}[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} + [\Psi_{\varepsilon_n,y_n}]_{s,q}^{2q}\\
&= \int_{\mathbb{R}^N}\left[\frac{f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))}{(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^{2q-1}}\right](\eta(|\varepsilon_n z|)w(z))^{2q}\,dz.
\end{aligned} \tag{4.3}
$$

Observing that $\eta(|x|)=1$ for $x\in B_{\frac{\delta}{2}}(0)$ and that $B_{\frac{\delta}{2}}(0)\subset B_{\frac{\delta}{\varepsilon_n}}(0)$ for all n large enough, the identity (4.3) yields

$$
\begin{aligned}
&\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^{p} + \frac{1}{t_{\varepsilon}^{q}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^{q} + \frac{1}{t_{\varepsilon}^{2q-2p}}[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} + [\Psi_{\varepsilon_n,y_n}]_{s,q}^{2q}\\
&\geq \int_{B_{\frac{\delta}{2}}(0)}\left[\frac{f(t_{\varepsilon_n}w(z))}{(t_{\varepsilon_n}w(z))^{2q-1}}\right]|w(z)|^{2q}\,dz,
\end{aligned}
$$

which together with (f_4) gives

$$
\begin{aligned}
&\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^{p} + \frac{1}{t_{\varepsilon}^{q}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^{q} + \frac{1}{t_{\varepsilon}^{2q-2p}}[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} + [\Psi_{\varepsilon_n,y_n}]_{s,q}^{2q}\\
&\geq \left[\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{2q-1}}\right]|w(\hat{z})|^{2q}|B_{\frac{\delta}{2}}(0)|,
\end{aligned} \tag{4.4}
$$

where

$$
w(\hat{z}) = \min_{z\in\bar{B}_{\frac{\delta}{2}}(0)} w(z) > 0
$$

(we recall that w is continuous and positive in $\mathbb{R}^N$). If $t_{\varepsilon_n}\to\infty$, the dominated convergence theorem results in

$$
\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},r}\to\|w\|_{s,r}\in(0,\infty)\quad\text{for all } r\in\{p,q\}, \tag{4.5}
$$

and recalling that $2q > q > p$, we also have

$$\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \frac{1}{t_\varepsilon^q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q + \frac{1}{t_\varepsilon^{2q-2p}}[\Psi_{\varepsilon_n,y_n}]_{s,p}^{2p} + [\Psi_{\varepsilon_n,y_n}]_{s,q}^{2q} \to [\omega]_{s,q}^{2q}. \tag{4.6}$$

On the other hand, by (f_3), we get

$$\lim_{n\to\infty}\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{2q-1}} = \infty. \tag{4.7}$$

Combining (4.4), (4.6) and (4.7), we achieve a contradiction. Consequently, $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and, up to a subsequence, we may assume that $t_{\varepsilon_n} \to t_0$ for some $t_0 \in [0, \infty)$. From (4.3), (4.5), (f_1), (f_2), we can see that $t_0 \in (0, \infty)$. Now we prove that $t_0 = 1$. Letting $n \to \infty$ in (4.3), and using (4.5) and the dominated convergence theorem, we have that

$$t_0^{p-2q}\|w\|_{s,p}^p + t_0^{2p-2q}[w]_{s,p}^{2p} + t_0^{-q}\|w\|_{s,q}^q + [w]_{s,q}^{2q} = \int_{\mathbb{R}^N}\frac{f(t_0w)}{(t_0w)^{2q-1}}\,w^{2q}\,dx.$$

Since $w \in \mathcal{M}_{V_0}$, it holds

$$\|w\|_{s,p}^p + \|w\|_{s,q}^q + [w]_{s,p}^{2p} + [w]_{s,q}^{2q} = \int_{\mathbb{R}^N} f(w)w\,dx,$$

Then we obtain

$$(t_0^{p-2q} - 1)\|w\|_{s,p}^p + (t_0^{2p-2q} - 1)[w]_{s,p}^{2p} + (t_0^{-q} - 1)[w]_{s,q}^{2q} = \int_{\mathbb{R}^N}\left[\frac{f(t_0w)}{(t_0w)^{2q-1}} - \frac{f(w)}{w^{2q-1}}\right] w^{2q}\,dx.$$

Using $2q > q > p$ and assumption (f_4), we conclude that $t_0 = 1$. Therefore, passing to the limit as $n \to \infty$ in (4.2), we deduce that

$$\lim_{n\to\infty}\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \mathcal{L}_{V_0}(w) = d_{V_0},$$

which contradicts (4.1). □

Let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by setting

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)(|u(x)|^p + |u(x)|^q)\, dx}{\displaystyle\int_{\mathbb{R}^N} (|u(x)|^p + |u(x)|^q)\, dx}.$$

Arguing as in the proof of Lemma 3.6 in [11], we can prove the following result.

Lemma 4.2 *The function β_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \textit{uniformly in } y \in M.$$

The next compactness result plays an important role in showing that the solutions of the modified problem are also solutions of the original one.

Lemma 4.3 *Let $\varepsilon_n \to 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $\mathbb{Y}_{V_0}$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}} = \{\varepsilon_n \tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n \to y_0 \in M$.*

Proof Since $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$, we can argue as in the proof of Lemma 2.5 to verify that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$. According to $d_{V_0} > 0$, $\|u_n\|_{\mathbb{X}_{\varepsilon_n}} \nrightarrow 0$. Then, proceeding as in the proof of Lemma 3.2, we obtain a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^q dx \geq \beta.$$

Set $v_n(x) = u_n(x+\tilde{y}_n)$. Thus, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, and, up to a subsequence, we may assume that $v_n \rightharpoonup v \not\equiv 0$ in $\mathbb{Y}_{V_0}$. Let $t_n \in (0,\infty)$ be such that $\tilde{v}_n = t_n v_n \in \mathcal{M}_{V_0}$, and set $y_n = \varepsilon_n \tilde{y}_n$. From the definition of d_{V_0}, $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$, (g_2) and $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$, we have

$$\begin{aligned}
d_{V_0} &\leq \mathcal{L}_{V_0}(\tilde{v}_n) \\
&\leq \frac{1}{p}[\tilde{v}_n]^p_{s,p} + \frac{1}{2p}[\tilde{v}_n]^{2p}_{s,p} + \frac{1}{q}[\tilde{v}_n]^q_{s,q} + \frac{1}{2q}[\tilde{v}_n]^{2q}_{s,q} \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n)\, dx \\
&\leq \frac{t_n^p}{p}[u_n]^p_{s,p} + \frac{t_n^{2p}}{2p}[u_n]^{2p}_{s,p} + \frac{t_n^q}{q}[u_n]^q_{s,q} + \frac{t_n^{2q}}{2q}[u_n]^{2q}_{s,q} \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon_n x)\left(\frac{t_n^p}{p}|u_n|^p + \frac{t_n^q}{q}|u_n|^q\right) dx \\
&\quad - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_n u_n)\, dx \\
&= \mathcal{J}_{\varepsilon_n}(t_n u_n) \leq \mathcal{J}_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1),
\end{aligned}$$

which implies that

$$\mathcal{L}_{V_0}(\tilde{v}_n) \to d_{V_0} \text{ and } \{\tilde{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}. \tag{4.8}$$

In particular, $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$ and, by extracting a subsequence if necessary, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{Y}_{V_0}$. Since $\{v_n\}_{n\in\mathbb{N}}$ and $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ are bounded in $\mathbb{Y}_{V_0}$, and $v_n \nrightarrow 0$ in $\mathbb{Y}_{V_0}$, we deduce that $\{t_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$. If $t_0 = 0$, then $\tilde{v}_n \to 0$ in $\mathbb{Y}_{V_0}$ (because $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$), and thus $\mathcal{L}_{V_0}(\tilde{v}_n) \to 0$, which contradicts $d_{V_0} > 0$. Hence, $t_0 \in (0, \infty)$. From the uniqueness of the weak limit, we see that $\tilde{v} = t_0 v \not\equiv 0$. This fact combined with Lemma 3.3 yields $\tilde{v}_n \to \tilde{v}$ in $\mathbb{Y}_{V_0}$, and so $v_n \to v$ in $\mathbb{Y}_{V_0}$. Furthermore,

$$\mathcal{L}_{V_0}(\tilde{v}) = d_{V_0} \text{ and } \langle \mathcal{L}'_{V_0}(\tilde{v}), \tilde{v}\rangle = 0.$$

In what follows, we show that $\{y_n\}_{n\in\mathbb{N}}$ admits a subsequence, still denoted by itself, such that $y_n \to y_0 \in M$. We begin by proving that $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$. Suppose, by contradiction, that there exists a subsequence of $\{y_n\}_{n\in\mathbb{N}}$, still denoted by itself, such that $|y_n| \to \infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. For n large enough, we may assume that $|y_n| > 2R$. Then, for each $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R.$$

Using $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$, a change of variable, the definition of g and the above relation, we have

$$\begin{aligned}\|v_n\|_{s,p}^p + \|v_n\|_{s,q}^q &\leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, v_n)v_n\,dx \\ &\leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(v_n)v_n\,dx + \int_{B^c_{R/\varepsilon_n}(0)} f(v_n)v_n\,dx.\end{aligned}$$

Since $v_n \to v$ in $\mathbb{Y}_{V_0}$ and $|B^c_{R/\varepsilon_n}(0)| \to 0$, it follows from the dominated convergence theorem that

$$\int_{B^c_{R/\varepsilon_n}(0)} f(v_n)v_n\,dx = o_n(1).$$

On the other hand, $\tilde{f}(v_n)v_n \leq \frac{V_0}{K}(|v_n|^p + |v_n|^q)$, and so

$$\|v_n\|_{s,p}^p + \|v_n\|_{s,q}^q \leq \frac{V_0}{K}\int_{B_{R/\varepsilon_n}(0)} (|v_n|^p + |v_n|^q)\,dx + o_n(1).$$

Consequently,

$$\left(1 - \frac{1}{K}\right)(\|v_n\|_{s,p}^p + \|v_n\|_{s,q}^q) \leq o_n(1),$$

and we reach a contradiction because $v_n \to v \not\equiv 0$ in $\mathbb{Y}_{V_0}$. Thus, $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$ and, up to a subsequence, we may assume that $y_n \to y_0 \in \mathbb{R}^N$. If $y_0 \notin \overline{\Lambda}$, then we can argue as before to get $v_n \to 0$ in $\mathbb{Y}_{V_0}$, that is a contradiction. Hence, $y_0 \in \overline{\Lambda}$. Let us note that if $V(y_0) = V_0$, then $y_0 \notin \partial\Lambda$ in view of $(V2)$. Therefore, it suffices to prove that $V(y_0) = V_0$ to deduce that $y_0 \in M$. To accomplish this, we assume, by contradiction, that $V(y_0) > V_0$. Using this fact, $\tilde{v}_n \to \tilde{v}$ in $\mathbb{Y}_{V_0}$, Fatou's lemma and the invariance of $\mathbb{R}^N$ by translation, we see that

$$\begin{aligned} d_{V_0} &= \mathcal{L}_{V_0}(\tilde{v}) \\ &< \liminf_{n\to\infty}\Big[\frac{1}{p}[\tilde{v}_n]^p_{s,p} + \frac{1}{2p}[\tilde{v}_n]^{2p}_{s,p} + \frac{1}{q}[\tilde{v}_n]^q_{s,q} + \frac{1}{2q}[\tilde{v}_n]^{2q}_{s,q} \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right)dx - \int_{\mathbb{R}^N} F(\tilde{v}_n)\,dx\Big] \\ &\le \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(t_n u_n) \le \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) = d_{V_0}, \end{aligned}$$

which is a contradiction. The proof is now complete. □

Let us define

$$\widetilde{\mathcal{N}}_\varepsilon = \left\{u \in \mathcal{N}_\varepsilon : \mathcal{J}_\varepsilon(u) \le d_{V_0} + \pi(\varepsilon)\right\},$$

where $\pi(\varepsilon) = \sup_{y\in M} |\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) - d_{V_0}| \to 0$ as $\varepsilon \to 0$, according to Lemma 4.1. By the definition of $\pi(\varepsilon)$, we have that, for all $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and thus $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$. Arguing as in the proof of Lemma 3.7 in [11], we deduce the following result.

Lemma 4.4 *For any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

We conclude the section by presenting a relation between the topology of M and the number of solutions of the modified problem (2.1). Since $\mathbb{S}^+_\varepsilon$ is not a complete metric space, we invoke the abstract category result in [36] to achieve our purpose.

Theorem 4.1 *Assume that (V_1)–(V_2) and (f_1)–(f_4) hold. Then, for any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\bar{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_\delta)$, problem (2.1) has at least $cat_{M_\delta}(M)$ positive solutions.*

Proof For each $\varepsilon > 0$, we define the map $\alpha_\varepsilon : M \to \mathbb{S}^+_\varepsilon$ by setting $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. By Lemma 4.1, we see that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y \in M.$$

Hence, there is a number $\hat{\varepsilon} > 0$ such that the set $\widetilde{\mathcal{S}}^+_\varepsilon = \{w \in \mathbb{S}^+_\varepsilon : \psi_\varepsilon(w) \le d_{V_0} + \pi(\varepsilon)\}$ is nonempty for all $\varepsilon \in (0, \hat{\varepsilon})$, since $\psi_\varepsilon(M) \subset \widetilde{\mathcal{S}}^+_\varepsilon$. Here $\pi(\varepsilon) = \sup_{y\in M} |\psi_\varepsilon(\alpha_\varepsilon(y)) -$

$d_{V_0}| \to 0$ as $\varepsilon \to 0$. From the above considerations, and taking into account Lemma 4.1, Lemma 2.4-(iii), Lemmas 4.4 and 4.2, we see that there exists $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon})$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \alpha_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined. According to Lemma 4.2, for $\varepsilon > 0$ small, we can write $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for $y \in M$, where $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$. Define $H(t, y) = y + (1-t)\theta(\varepsilon, y)$ for $(t, y) \in [0, 1] \times M$. Clearly, $H : [0, 1] \times M \to M_\delta$ is continuous, $H(0, y) = \beta_\varepsilon(\Phi_\varepsilon(y))$ and $H(1, y) = y$ for all $y \in M$. Then $H(t, y)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $id : M \to M_\delta$. This fact implies that

$$cat_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M) \geq cat_{M_\delta}(M). \tag{4.9}$$

It follows from Corollary 2.1, Lemma 3.4, and Theorem 27 in [36], with $c = c_\varepsilon \leq d_{V_0} + \pi(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, that Ψ_ε has at least $cat_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Therefore, by Proposition 2.1-(d) and (4.9), we conclude that $\mathcal{J}_\varepsilon$ admits at least $cat_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

5 Proof of Theorem 1.1

This section is devoted to the proof of the main result of this paper. The idea is to show that the solutions obtained in Theorem 4.1 satisfy, for $\varepsilon > 0$ small enough, the estimate $u_\varepsilon(x) \leq a$ for any $x \in \Lambda_\varepsilon^c$. This fact implies that these solutions are indeed solutions of the original problem (1.1). We start with the following lemma which plays a key role in studying the behavior of the maximum points of solutions to (1.1), whose proof is related to the Moser iteration method [32].

Lemma 5.1 *Let $\varepsilon_n \to 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a sequence of solutions to (2.1). Then $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$, and there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N)$ and for some $C > 0$ it holds*

$$|v_n|_\infty \leq C \quad \textit{for all } n \in \mathbb{N}.$$

Moreover,

$$v_n(x) \to 0 \ \textit{as} \ |x| \to \infty \ \textit{uniformly in} \ n \in \mathbb{N}. \tag{5.1}$$

Proof Since $\mathcal{J}_{\varepsilon_n}(u_n) \leq d_{V_0} + \pi(\varepsilon_n)$, with $\pi(\varepsilon_n) \to 0$ as $n \to \infty$, we can argue as at the beginning of the proof of Lemma 4.3 to deduce that $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then, using Lemma 4.3, we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \to v$ in $\mathbb{Y}_{V_0}$ for some $v \in \mathbb{Y}_{V_0} \setminus \{0\}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

Now we examine the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{R}^N)$. For each $n \in \mathbb{N}$ and $L > 0$, we define

$$\gamma(v_n) = v_n v_{n,L}^{q(\beta-1)} \in \mathbb{X}_\varepsilon,$$

where $v_{n,L} = \min\{v_n, L\}$, and $\beta > 1$ will be chosen later. Taking $\gamma(v_n)$ as test function in the problem solved by v_n, we have

$$\begin{aligned}
&(1+[v_n]_{s,p}^p) \\
&\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((v_n v_{n,L}^{q(\beta-1)})(x)-(v_n v_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sp}}\,dxdy \\
&\quad + (1+[v_n]_{s,q}^q) \\
&\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{q-2}(v_n(x)-v_n(y))((v_n v_{n,L}^{q(\beta-1)})(x)-(v_n v_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sq}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|v_n|^p v_{n,L}^{q(\beta-1)}\,dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|v_n|^q v_{n,L}^{q(\beta-1)}\,dx \\
&= \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n) v_n v_{n,L}^{q(\beta-1)}\,dx.
\end{aligned}$$

In light of the growth assumptions on g, we know that for all $\xi \in (0, V_0)$, there exists $C_\xi > 0$ such that

$$|g(x,t)| \le \xi|t|^{p-1} + C_\xi |t|^{q_s^*-1} \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

From the above facts and (V_1), we obtain

$$\begin{aligned}
&(1+[v_n]_{s,p}^p) \\
&\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((v_n v_{n,L}^{q(\beta-1)})(x)-(v_n v_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sp}}\,dxdy \\
&\quad + (1+[v_n]_{s,q}^q) \\
&\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{q-2}(v_n(x)-v_n(y))((v_n v_{n,L}^{q(\beta-1)})(x)-(v_n v_{n,L}^{q(\beta-1)})(y))}{|x-y|^{N+sq}}\,dxdy \\
&\le C \int_{\mathbb{R}^N} |v_n|^{q_s^*} v_{n,L}^{q(\beta-1)}\,dx. \qquad (5.2)
\end{aligned}$$

Observing that, for $t \in \{p, q\}$, $1 \le 1 + [v_n]_{s,t}^t \le C$ for all $n \in \mathbb{N}$, we can reproduce the Moser iteration argument carried out in the proof of Lemma 4.1 in [11] to derive that $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Since $\{v_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap \mathbb{Y}_{V_0}$, we can argue as in the proof of Theorem 2.2 in [11] to deduce that $\|v_n\|_{C^{0,\alpha}(\mathbb{R}^N)} \le C$ for all $n \in \mathbb{N}$. This fact combined with $v_n \to v$ in $\mathbb{Y}_{V_0}$ implies that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. The proof of Lemma 5.1 is complete. □

We now have all ingredients to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\delta > 0$ be a number satisfying $M_\delta \subset \Lambda$. We first show that there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (2.1), it holds

$$|u_\varepsilon|_{L^\infty(\Lambda_\varepsilon^c)} < a. \tag{5.3}$$

Assume, by contradiction, that there exists a subsequence $\varepsilon_n \to 0$, $u_n = u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $\mathcal{J}'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_n|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geq a. \tag{5.4}$$

As in the proof of Lemma 5.1, we can verify that $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then, applying Lemma 4.3, we obtain a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \to v$ in $\mathbb{Y}_{V_0}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

Pick $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$. Thus, $B_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$ for all $n \in \mathbb{N}$. Moreover, for any $y \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, we see that

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n}$$

for n large enough. For these values of n, we have

$$\Lambda_{\varepsilon_n}^c \subset B^c_{\frac{r}{\varepsilon_n}}(\tilde{y}_n).$$

Using (5.1), we can find $R > 0$ such that $v_n(x) < a$ for any $|x| \geq R$ and $n \in \mathbb{N}$, and so $u_n(x) < a$ for any $x \in B_R^c(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$,

$$\Lambda_{\varepsilon_n}^c \subset B^c_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset B_R^c(\tilde{y}_n).$$

Hence, $u_n(x) < a$ for any $x \in \Lambda_{\varepsilon_n}^c$ and $n \geq n_0$, which is in contrast with (5.4). This proves our claim.

Let $\bar{\varepsilon}_\delta > 0$ be given by Theorem 4.1 and set $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. Fix $\varepsilon \in (0, \varepsilon_\delta)$. Applying Theorem 4.1, we get at least $cat_{M_\delta}(M)$ positive solutions to (2.1). If u_ε denotes one of these solutions, we have that $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$, and using (5.3) and the definition of g, we deduce that u_ε is also a solution to (1.1). Consequently, (1.1) admits at least $cat_{M_\delta}(M)$ positive solutions.

Now we investigate the behavior of the maximum points of solutions to (1.1). Take $\varepsilon_n \to 0$ and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_{\varepsilon_n}$ of solutions to (1.1) as above. Let us observe that (g_1) implies that there exists $\sigma \in (0, a)$ such that

$$g(\varepsilon x, t)t \leq \frac{V_0}{K}(t^p + t^q) \quad \text{for } (x, t) \in \mathbb{R}^N \times [0, \sigma]. \tag{5.5}$$

Arguing as before, we can choose $R > 0$ such that

$$|u_n|_{L^\infty(B_R^c(\tilde{y}_n))} < \sigma. \tag{5.6}$$

Moreover, up to a subsequence, we may assume that

$$|u_n|_{L^\infty(B_R(\tilde{y}_n))} \geq \sigma. \tag{5.7}$$

Indeed, if (5.7) does not hold, then, in view of (5.6), we have that $|u_n|_\infty < \sigma$. Hence, using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and (5.5), we get

$$\|u_n\|^p_{V_{\varepsilon_n},p} + \|u_n\|^q_{V_{\varepsilon_n},q} \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n)u_n\,dx \leq \frac{V_0}{K}\int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q)\,dx$$

which leads to a contradiction. Therefore, (5.7) is satisfied.

Let $p_n \in \mathbb{R}^N$ be a global maximum point of u_n. Combining (5.6) and (5.7), we infer that $p_n = \tilde{y}_n + q_n$, for some $q_n \in B_R(0)$. Since $\varepsilon_n \tilde{y}_n \to y_0 \in M$ and $|q_n| < R$ for all $n \in \mathbb{N}$, we have that $\varepsilon_n p_n \to y_0$, and using the continuity of V we obtain

$$\lim_{n\to\infty} V(\varepsilon_n p_n) = V(y_0) = V_0.$$

The proof of Theorem 1.1 is now complete. □

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