The Journal of Geometric Analysis (2022) 32:115
https://doi.org/10.1007/s12220-022-00871-w

Bounds on the First Betti Number: An Approach via Schatten Norm Estimates on Semigroup Differences

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Received: 25 February 2019 / Accepted: 13 December 2021 / Published online: 29 January 2022

Abstract
We derive new estimates for the first Betti number of compact Riemannian manifolds. Our approach relies on the Birman–Schwinger principle and Schatten norm estimates for semigroup differences. In contrast to previous works we do not require any a priori ultracontractivity estimates and we provide bounds which explicitly depend on suitable integral norms of the Ricci tensor.

Keywords Betti number · Semigroup · Ricci tensor

Mathematics Subject Classification 53C20 · 58J50 · 58J35

1 Introduction

The aim of this paper is to give estimates for the first Betti number $b_1(M)$ of a compact Riemannian manifold M. In particular, we show that $b_1(M)$ is small, or even zero, if the Ricci curvature of M is 'mostly positive'.

A starting point for our analysis is the paper [9], where a criterion is formulated under which $b_1(M) = 0$. The results of [9] were later generalized and put into a more quantitative form in [26] and [4], respectively (for some related work see also the literature cited in these two papers). The new feature of our main results can easily be explained: Roughly speaking, we employ methods from Functional Analysis and Operator Theory that originated in Mathematical Physics. This complements the standard approach using the Hodge theorem to identify the first Betti number with the dimension of the kernel of the Laplacian acting on 1-forms, and then to deduce bounds

Dedicated to the memory of Johannes F. Brasche.

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on $b_1(M)$ from trace norm estimates of the corresponding heat semigroup. However, the method we use in the present paper is different: In case that the Ricci curvature is mostly positive, an alternative, and obviously better, estimate can be obtained by means of the Birman–Schwinger principle, see Sect. 2 and Proposition 2.4. In particular, instead of estimating the trace norm of the semigroup itself, it suffices to bound the trace norm (or a more general Schatten norm) of an appropriate semigroup difference.

We derive such trace norm bounds by well-established factorization principles; however the abstract results we obtain in Proposition 2.6 and Theorems 2.8 and 2.10 are new and of independent interest. The application of these abstract trace norm bounds to the geometric setting is then straightforward, as we show in Sect. 3. Our main result, Theorem 3.3, gives a very satisfactory estimate: For any compact Riemannian manifold M of dimension n and for $\rho_0 > 0$ and $t_0 > 0$, we have

$$b_1(M) \leq 4n\rho_0^{-2} \, \|(\text{Ric.} - \rho_0)_-\|_{2,\text{HS}}^2 \left\| e^{-t_0(\Delta+\rho)} \right\|_{2,\infty}^2. \tag{1}$$

A detailed explanation of the notation used in this inequality will be given below. Let us just note that here the L^2-norm of the part of the Ricci tensor Ric below the positive threshold ρ_0 is used to control that Ric is indeed mostly positive. The ultracontractivity term on the right still depends on geometric data via the function ρ, which maps every point of the manifold to the lowest eigenvalue of the Ricci tensor. In Corollary 3.4, we will exemplify how this part of the estimate can be controlled in geometric terms.

As far as we can say, the above bound on $b_1(M)$ is the first containing a norm of the matrix-valued map $M \ni x \mapsto (\text{Ric}_x - \rho_0)_-$ in an explicit and multiplicative way. All previous works on this matter were formulated given assumptions on the function ρ only and offered right-hand sides of a much more complicated nature. Some more on this topic will be discussed in Sect. 3.

2 The Birman–Schwinger Principle and Hilbert–Schmidt Norm Estimates

Let us consider two selfadjoint operators H, H' on a Hilbert space $\mathcal{H}$, such that $H \geq 0$ and $H' \geq \rho_0$ for some $\rho_0 > 0$. Then both operators generate strongly continuous (even analytic) semigroups, denoted by $(e^{-tH}; t \geq 0)$ and $(e^{-tH'}; t \geq 0)$, respectively, and in the following we assume that for some $t_0 > 0$ the semigroup difference

$$D_{t_0} := e^{-t_0 H} - e^{-t_0 H'}$$

is compact. We can think of H and H' as being 'small' perturbations of each other, the smallness being reflected in the compactness of D_{t_0}.

Remark 2.1 As an example the reader should have in mind the case where $\mathcal{H}$ is a (vector-valued) L^2-space and $H' = H + V$ is a potential perturbation. Here the compactness of D_{t_0} will follow from a suitable smallness assumption on V. This case will be discussed in more detail in the second part of this section.

From the spectral (mapping) theorem it follows that the spectra of generator and semigroup are related by

$$\sigma(e^{-tH}) \setminus \{0\} = \{e^{-t\lambda} : \lambda \in \sigma(H)\},$$

and that the same identity is valid for the essential spectra σ_{ess} and discrete spectra σ_d as well. Moreover, for every $\lambda \in \mathbb{R}$ we have

$$\ker(H - \lambda) = \ker(e^{-tH} - e^{-t\lambda}),$$

so that also the multiplicities of the corresponding eigenvalues $\lambda_0 \in \sigma_d(H)$ and $e^{-t\lambda_0} \in \sigma_d(e^{-tH})$ coincide.

Together with Weyl's classical theorem on the invariance of the essential spectrum under compact perturbations [30], the above facts and assumptions imply that

$$\sigma_{ess}(H) = \sigma_{ess}(H') \subset [\rho_0, \infty)$$

and that the spectrum of H in $[0, \rho_0)$ is purely discrete. In particular, this shows that the dimension of the kernel of H is always finite (since either 0 is in the resolvent set or it is an eigenvalue of finite multiplicity).

Remark 2.2 For our main application we will consider the case where $H = \Delta^1$ denotes the Hodge-Laplacian on the Riemannian manifold M, in which case $\dim\ker(H)$ coincides with the first Betti number $b_1(M)$.

Our next goal is to obtain an upper bound on $\dim\ker(H)$ given some more restrictive assumptions on D_t. To this end, we need to introduce the Schatten-von Neumann classes $\mathcal{S}_p$, $p > 0$, which consist of all compact operators K on $\mathcal{H}$ whose sequence of singular numbers $(s_n(K))$ is in $l^p(\mathbb{N})$. One can define a (quasi-) norm on $\mathcal{S}_p$ by setting $\|K\|_{\mathcal{S}_p} := \|(s_n(K))\|_p$. It is well known that $(\mathcal{S}_p, \|.\|_{\mathcal{S}_p})$ is a (quasi-) Banach space and a two sided ideal in the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, see, e.g., [10].

Remark 2.3 Operators in $\mathcal{S}_1$ and $\mathcal{S}_2$ are usually called *trace class* and *Hilbert–Schmidt* operators, respectively. We will follow this practice in the present article and will also use the more suggestive notation

$$\|K\|_{\mathrm{tr}} := \|K\|_{\mathcal{S}_1} \quad \text{and} \quad \|K\|_{\mathrm{HS}} := \|K\|_{\mathcal{S}_2}.$$

For later purposes we recall that the Hilbert–Schmidt norm can be computed as

$$\|K\|_{\mathrm{HS}}^2 = \sum_{\alpha \in A} \|K\varphi_\alpha\|^2,$$

where $(\varphi_\alpha)_{\alpha \in A}$ is any orthonormal basis of $\mathcal{H}$.

To obtain bounds on $\dim\ker(H)$ we will rely on the so-called 'Birman–Schwinger principle'. This principle was first stated in the context of Schrödinger operators in [2, 27] and originally referred to the following fact: Under suitable assumptions on the potential V a negative number λ is a discrete eigenvalue of $-\Delta + V$ if and only if 1 is in the spectrum of the compact integral operator $(\lambda + \Delta)^{-1}V$. The first part of the following proposition can be regarded as an abstract version of this principle.

Proposition 2.4 *Let $H \geq 0$, $H' \geq \rho_0 > 0$ and $D_t := e^{-tH} - e^{-tH'}$, $t > 0$, be defined as above. Then the following holds:*

(i) $\ker(H) = \ker((I - e^{-tH'})^{-1}D_t - I)$, *where $I \in \mathcal{L}(\mathcal{H})$ denotes the identity.*
(ii) *If $D_{t_0} \in \mathcal{S}_p$ for some $p > 0$ and $t_0 > 0$, then*

$$\dim\ker(H) \leq \|(I - e^{-t_0 H'})^{-1}D_{t_0}\|_{\mathcal{S}_p}^p. \tag{2}$$

In particular,

$$\dim\ker(H) \leq (1 - e^{-\rho_0 t_0})^{-p}\|D_{t_0}\|_{\mathcal{S}_p}^p. \tag{3}$$

We note that $(I - e^{-tH'})$ is indeed invertible, since $\sigma(e^{-tH'}) \subset [0, e^{-\rho_0 t}]$. In a different setting, the idea to use Schatten norm bounds on semigroup differences to obtain bounds on eigenvalues has also been used in [15,Remark 2.6] (see also [6]).

Proof of Proposition 2.4 (i) We have

$$\ker(H) = \ker(e^{-tH} - I) = \ker((I - e^{-tH'})^{-1}D_t - I).$$

Here the first equality follows (as stated above) from the spectral theorem (functional calculus) and the second can be proved as follows:

$$e^{-tH}f = f \Leftrightarrow \quad D_t f = (I - e^{-tH'})f \quad \Leftrightarrow \quad (I - e^{-tH'})^{-1}D_t f = f.$$

(ii) Let $\lambda_1(K), \lambda_2(K), \ldots$ denote the sequence of eigenvalues of the compact operator $K := (I - e^{-t_0 H'})^{-1}D_{t_0}$, each eigenvalue being counted according to its algebraic multiplicity. By part (i) we know that at least $N := \dim\ker(H)$ of these eigenvalues are equal to 1, so clearly

$$N \leq \sum_n |\lambda_n(K)|^p.$$

Another classical result of Weyl [31] asserts that

$$\sum_n |\lambda_n(K)|^p \leq \sum_n s_n(K)^p = \|K\|_{\mathcal{S}_p}^p,$$

which concludes the proof of (2). Finally, (3) follows from (2) using the estimate

$$\|(I - e^{-t_0 H'})^{-1}D_{t_0}\|_{\mathcal{S}_p} \leq \|(I - e^{-t_0 H'})^{-1}\|\|D_{t_0}\|_{\mathcal{S}_p} \leq (1 - e^{-\rho_0 t_0})^{-1}\|D_{t_0}\|_{\mathcal{S}_p}. \quad \square$$

Our ultimate goal is to apply the previous proposition to estimate the first Betti number of Riemannian manifolds. To this end, as a first step we now provide suitable $\mathcal{S}_p$-norm estimates for semigroup differences on L^2-spaces. Actually, in the present paper we will restrict ourselves to the case of Hilbert–Schmidt norm estimates, since, in addition to being easier to handle, these are particularly well-suited for the applications we have in mind.

From now on we fix a σ-finite measure space $(X, \mathcal{F}, m)$ and consider the Hilbert space $L^2(X) = L^2(X; \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, as well as the vector-valued version $L^2(X; \mathbb{K}^n)$ for $n \in \mathbb{N}$.

Remark 2.5 Note that for most of what we show in the following, we could replace $\mathbb{K}^n$ by a $\mathbb{K}$-Hilbert space $\mathcal{K}$.

As usual, inner products and norms in the vector- and scalar-valued L^2-spaces will be denoted by the same symbols $(\cdot \mid \cdot)$ and $\|.\|_2$, respectively. In particular, for $f = (f_1, \dots, f_n) \in L^2(X; \mathbb{K}^n)$ we have $\|f\|_2^2 = \sum_{k=1}^n \|f_k\|_2^2$. Also, we use the same symbol $|.|$ to denote the absolute value on $\mathbb{K}$ as well as the Euclidean norm on $\mathbb{K}^n$.

As above we study a nonnegative selfadjoint operator H in $L^2(X; \mathbb{K}^n)$ and its semigroup $(e^{-tH}; t \geq 0)$. In the present setting the semigroup differences we are concerned with are given by

$$e^{-tH} - e^{-t(H+V)},$$

where V is a multiplication operator in the following sense:

$$V : X \to \mathbb{K}^{n \times n} \text{ is measurable}$$

and for every $x \in X$ the matrix $V(x)$ is nonnegative and hermitian (respectively symmetric). The entries of this matrix are written as $V_{ij}(x)$. In the following we use the shorthand notation

$$V \succeq 0, \tag{4}$$

to indicate that V satisfies the above conditions. We note that

$$V \succeq 0 \quad \Rightarrow \quad V \geq 0,$$

i.e., V is also a nonnegative operator in $L^2(X; \mathbb{K}^n)$.

The operator $H + V$ (corresponding to H' in our above considerations) denotes the form sum of the two operators. Note that in this generality, the corresponding form need not be densely defined. In this case, we obtain a selfadjoint operator in the subspace $\mathcal{H}_0 = \overline{\mathcal{D}(H^{1/2}) \cap \mathcal{D}(V^{1/2})}$ and extend the corresponding semigroup by 0 to $\mathcal{H}_0^{\perp}$. However, in most of the applications we have in mind, such complications will not arise since V is bounded.

In the following, we use $\|.\|_{\mathrm{HS}}$ to denote the Hilbert–Schmidt norm of operators on different spaces. Moreover, for V as above we set

$$\|V\|_{2,\mathrm{HS}}^2 := \int_X \|V(x)\|_{\mathrm{HS}}^2 \mathrm{d}m(x) \\ = \sum_{i,j=1}^n \|V_{ij}\|_2^2. \tag{5}$$

We write $V \in L^2(X; \mathbb{K}^{n\times n})$ provided $\|V\|_{2,\mathrm{HS}}$ is finite. Note that in this case V is a bounded operator from $L^\infty(X; \mathbb{K}^n)$ to $L^2(X; \mathbb{K}^n)$.

Before presenting our Hilbert–Schmidt norm estimates on the above semigroup difference, we single out one important step from their proof. Here and in the sequel we use the standard shorthand notation

$$\|T\|_{p,q} := \|T\|_{\mathcal{L}(L^p(X;\mathbb{K}^n), L^q(X;\mathbb{K}^n))}.$$

Proposition 2.6 *Assume that $T \in \mathcal{L}(L^2(X; \mathbb{K}^n), L^\infty(X; \mathbb{K}^n))$ and $V \in L^2(X; \mathbb{K}^{n\times n})$. Then the operator $VT \in \mathcal{L}(L^2(X; \mathbb{K}^n))$ is Hilbert–Schmidt and*

$$\|VT\|_{\mathrm{HS}} \le \sqrt{n} \cdot \|V\|_{2,\mathrm{HS}} \cdot \|T\|_{2,\infty}.$$

The proof of this proposition relies on the following result from [7] (Corollary 2 in that paper), which provides an estimate on the trace norm of certain operators on scalar-valued functions: If $A \in \mathcal{L}(L^1(X), L^2(X))$, $B \in \mathcal{L}(L^2(X), L^1(X))$ and if there exists $\Phi \in L^1(X)$ such that $|Bf| \le \Phi$ for every f in the unit ball of $L^2(X)$, then

$$\|AB\|_{\mathrm{tr}} \le \|A\|_{1,2}\|\Phi\|_1. \tag{6}$$

Proof of Proposition 2.6 We consider the scalar case $n = 1$ first: Here, using the identity

$$\|VT\|_{\mathrm{HS}}^2 = \|T^*|V|^2 T\|_{\mathrm{tr}},$$

we can apply the aforementioned Corollary from [7] with $A = T^*|_{L^1}$, $B = |V|^2 T$ and $\Phi := |V|^2\|T\|_{2,\infty}$. Indeed, since L^1 is isometrically embedded in $(L^\infty)'$, we have $A \in \mathcal{L}(L^1(X), L^2(X))$ and $\|A\|_{1,2} \le \|T^*\|_{(L^\infty)',L^2} = \|T\|_{2,\infty}$. Moreover, it is easily seen that $B \in \mathcal{L}(L^2(X), L^1(X))$, that $\Phi \in L^1(X)$ with $\|\Phi\|_1 = \|V\|_2^2\|T\|_{2,\infty}$ and that $|Bf| \le \Phi$ for f in the unit ball of $L^2(X)$. Hence, as desired, we obtain from (6) that

$$\|VT\|_{\mathrm{HS}}^2 = \|T^*|V|^2 T\|_{\mathrm{tr}} \le \|T\|_{2,\infty}^2\|V\|_2^2.$$

The vector-valued case is now readily deduced from the scalar one. First, we diagonalize: Since $(V(x))_{x\in X}$ is a measurable family of hermitian (symmetric) matrices in

$\mathbb{K}^{n\times n}$, we know from [1,Corollary 2] that there exists a measurable family $(U(x))_{x\in X}$ of unitary (orthogonal) matrices in $\mathbb{K}^{n\times n}$ such that for every $x \in X$ the matrix

$$\Lambda(x) := U(x)^* V(x) U(x)$$

is diagonal. Now for $x \in X$ and $f \in L^2(X; \mathbb{K}^n)$ we define the unitary operator U on $L^2(X; \mathbb{K}^n)$ by

$$(Uf)(x) := U(x) f(x),$$

and we define the operator Λ in $L^2(X; \mathbb{K}^n)$ by

$$(\Lambda f)(x) := \Lambda(x) f(x), \qquad \mathcal{D}(\Lambda) = \{f \in L^2(X; \mathbb{K}^n) : Uf \in \mathcal{D}(V)\}.$$

Finally, let us define

$$T_{ij} : L^2(X) \to L^\infty(X), \quad T_{ij} f := (U^* T(f e_i) | e_j), \qquad i, j \in \{1, \dots, n\},$$

where $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{K}^n$. For later purposes we note that for $f \in L^2(X)$ we have

$$\begin{aligned} \|T_{ij} f\|_\infty &= \operatorname{esssup}_{x\in X} \left|\left(U^*(x)[T(f e_i)](x) | e_j\right)\right| \le \operatorname{esssup}_{x\in X} |T(f e_i)(x)| \\ &\le \|T\|_{2,\infty} \|f e_i\|_2 = \|T\|_{2,\infty} \|f\|_2. \end{aligned} \tag{7}$$

Now we compute the Hilbert–Schmidt norm of the operator VT with respect to the orthonormal basis $(\varphi_\alpha \otimes e_i)_{a\in A, i=1,\dots,n}$, where $(\varphi_\alpha)_{\alpha\in A}$ is some ONB of $L^2(X)$:

$$\begin{aligned} \|VT\|_{\mathrm{HS}}^2 &= \|U\Lambda U^* T\|_{\mathrm{HS}}^2 = \|\Lambda U^* T\|_{\mathrm{HS}}^2 \\ &= \sum_{\alpha\in A} \sum_{i=1}^n \|\Lambda U^* T(\varphi_\alpha \otimes e_i)\|_2^2 = \sum_{\alpha\in A} \sum_{i=1}^n \sum_{j=1}^n \|(\Lambda U^* T(\varphi_\alpha \otimes e_i) | e_j)\|_2^2 \\ &= \sum_{\alpha\in A} \sum_{i=1}^n \sum_{j=1}^n \left\|\Lambda_{jj} T_{ij} \varphi_\alpha\right\|_2^2 = \sum_{i=1}^n \sum_{j=1}^n \left\|\Lambda_{jj} T_{ij}\right\|_{\mathrm{HS}}^2. \end{aligned}$$

From the scalar case and (7) we obtain that

$$\|\Lambda_{jj} T_{ij}\|_{\mathrm{HS}} \le \|\Lambda_{jj}\|_2 \|T_{ij}\|_{2,\infty} \le \|\Lambda_{jj}\|_2 \|T\|_{2,\infty}.$$

Hence, the previous identities imply that

$$\|VT\|_{\mathrm{HS}}^2 \le n \|T\|_{2,\infty}^2 \sum_{j=1}^n \|\Lambda_{jj}\|_2^2 = n \|T\|_{2,\infty}^2 \|\Lambda\|_{2,\mathrm{HS}}^2 = n \|T\|_{2,\infty}^2 \|V\|_{2,\mathrm{HS}}^2.$$

This concludes the proof of the proposition. □

In the following we present several estimates on the Hilbert–Schmidt norm of the difference of the semigroups e^{-tH} and $e^{-t(H+V)}$, which are defined as above. We begin with two results for the case of bounded V, which will be used later on in our estimate on the first Betti number of compact manifolds. After that, we extend one of these results to unbounded potentials. While unbounded potentials will not play a role in the present article, we include them for completeness and because they might well become important in future work (for instance, when considering the case of non-compact manifolds).

Remark 2.7 Hilbert–Schmidt norm estimates for semigroup differences of operators in $L^2(X)$ have been studied in a variety of contexts and the literature on the subject is extensive. For an overview and many references we refer to the monograph [8], which studies such estimates for generators of Feller semigroups. However, as far as we can say, the following results are the first estimates concerning operators on vector-valued functions. Moreover, in the stated generality, we think that they might even be new in the scalar case.

The natural setting of our first result is when the involved semigroups $(T(t))_{t\geq 0}$ are *ultracontractive*, i.e., for every $t > 0$ they map $L^2(X;\mathbb{K}^n)$ to $L^\infty(X;\mathbb{K}^n)$ and $\|T(t)\|_{2,\infty} < \infty$. However, we need this additional property for one t_0 only.

Theorem 2.8 *Let $H \geq 0$ be a selfadjoint operator in $L^2(X;\mathbb{K}^n)$ and $V \in L^2 \cap L^\infty(X;\mathbb{K}^{n\times n})$. Suppose there is a $t_0 > 0$ such that $e^{-t_0 H}, e^{-t_0(H+V)} \in \mathcal{L}(L^2(X;\mathbb{K}^n), L^\infty(X;\mathbb{K}^n))$. Then we have*

$$\|e^{-2t_0 H} - e^{-2t_0(H+V)}\|_{\mathrm{HS}} \leq \sqrt{n}\,\|V\|_{2,\mathrm{HS}}\left(\|e^{-t_0 H}\|_{2,\infty} + \|e^{-t_0(H+V)}\|_{2,\infty}\right)\cdot \cdot\int_0^{t_0}\|e^{-s(H+V)}\|_{2,2}\,\mathrm{d}s. \tag{8}$$

Proof By the Duhamel principle, see [21], formula (1.8) on page 78, we obtain that

$$\begin{aligned} e^{-2t_0 H} - e^{-2t_0(H+V)} &= \int_0^{2t_0} e^{-(2t_0-s)(H+V)} V e^{-sH}\,\mathrm{d}s \\ &= \int_0^{t_0} e^{-(2t_0-s)(H+V)} V e^{-sH}\,\mathrm{d}s + \int_{t_0}^{2t_0} e^{-(2t_0-s)(H+V)} V e^{-sH}\,\mathrm{d}s \end{aligned}$$

and hence

$$\begin{aligned} \|e^{-2t_0 H} - e^{-2t_0(H+V)}\|_{\mathrm{HS}} \leq &\int_0^{t_0} \|e^{-(2t_0-s)(H+V)} V e^{-sH}\|_{\mathrm{HS}}\,\mathrm{d}s \\ &+ \int_{t_0}^{2t_0} \|e^{-(2t_0-s)(H+V)} V e^{-sH}\|_{\mathrm{HS}}\,\mathrm{d}s. \end{aligned}$$

We estimate the integrals separately but with the same idea and start with the second one: With a change of variables and an application of Proposition 2.6 (with $T = e^{-t_0 H}$) we obtain

$$\begin{aligned}
\int_{t_0}^{2t_0} \|e^{-(2t_0-s)(H+V)} V e^{-sH}\|_{\mathrm{HS}}\, \mathrm{d}s &= \int_0^{t_0} \|e^{-(t_0-r)(H+V)} V e^{-(t_0+r)H}\|_{\mathrm{HS}}\, \mathrm{d}r \\
&\leq \int_0^{t_0} \|e^{-(t_0-r)(H+V)}\|_{2,2} \|V e^{-t_0 H}\|_{\mathrm{HS}} \|e^{-rH}\|_{2,2}\, \mathrm{d}r \\
&\leq \sqrt{n}\|V\|_{2,\mathrm{HS}} \|e^{-t_0 H}\|_{2,\infty} \int_0^{t_0} \|e^{-(t_0-r)(H+V)}\|_{2,2} \|e^{-rH}\|_{2,2}\, \mathrm{d}r \\
&\leq \sqrt{n}\|V\|_{2,\mathrm{HS}} \|e^{-t_0 H}\|_{2,\infty} \int_0^{t_0} \|e^{-s(H+V)}\|_{2,2} \mathrm{d}s.
\end{aligned}$$

In the last inequality we used a change of variables and the fact that for all $t \geq 0$ we have $\|e^{-tH}\|_{2,2} \leq 1$. Finally, for the first integral we use that $\|A\|_{\mathrm{HS}} = \|A^*\|_{\mathrm{HS}}$ and obtain in a similar fashion that

$$\begin{aligned}
\int_0^{t_0} \|e^{-(2t_0-s)(H+V)} V e^{-sH}\|_{\mathrm{HS}}\, \mathrm{d}s &= \int_0^{t_0} \|e^{-sH} V e^{-(2t_0-s)(H+V)}\|_{\mathrm{HS}}\, \mathrm{d}s \\
&\leq \sqrt{n}\|V\|_{2,\mathrm{HS}} \|e^{-t_0(H+V)}\|_{2,\infty} \int_0^{t_0} \|e^{-sH}\|_{2,2} \|e^{-(t_0-s)(H+V)}\|_{2,2}\, \mathrm{d}s \\
&\leq \sqrt{n}\|V\|_{2,\mathrm{HS}} \|e^{-t_0(H+V)}\|_{2,\infty} \int_0^{t_0} \|e^{-s(H+V)}\|_{2,2}\, \mathrm{d}s.
\end{aligned}$$

This concludes the proof. □

A slightly unpleasant feature of the previous theorem is the fact that we need to assume that also the perturbed semigroup is ultracontractive, which might be difficult to check. In the scalar case, however, this does usually not pose a problem, since in this case the boundedness of $e^{-t_0(H+V)} : L^2(X) \to L^\infty(X)$ automatically follows from the corresponding boundedness of $e^{-t_0 H}$ provided the last semigroup is positivity preserving. Also, we just note that here the additional assumption that V is bounded is not necessary.

To overcome the described problem in the vector-valued case, we now make the additional assumption that there is a dominating semigroup on the scalar L^2-space. This requires some terminology:

Let H_0 be a selfadjoint lower-semibounded operator in $L^2(X)$. We say that its semigroup $(e^{-tH_0})_{t\geq 0}$ **dominates** $(e^{-tH})_{t\geq 0}$ if the following relation is satisfied for all $t > 0$:

$$|e^{-tH} f|(x) \leq e^{-tH_0}|f|(x), \quad x \in X,\ f \in L^2(X, \mathbb{K}^n).$$

This implies $(e^{-tH_0})_{t\geq 0}$ is positivity preserving, i.e., for all $t \geq 0$: $e^{-tH_0} f \geq 0$ if $f \geq 0$.

Remark 2.9 Domination of semigroups has a long history and its beginnings are intimately related to the situation we have in mind, the semigroups of the Laplace–Beltrami operator on functions and the semigroup of the Hodge–Laplacian on 1-forms. See [16, 28] for early results. In [17] the form characterization of domination was used to show that the semigroup of the Bochner-Laplacian on a Riemannian manifold is in fact dominated by the semigroup of the Laplace-Beltrami operator. Furthermore, it was shown that under suitable conditions, the Hodge-deRham Laplacian is dominated by a Schrödinger operator generated by the Laplace-Beltrami plus a suitable potential depending on Ricci curvature, as we will use later. Especially in Riemannian geometry, this fact has been used extensively to study geometric and topological properties of manifolds as well as properties of the semigroup and corresponding heat kernel of generalized Schrödinger operators on vector bundles, see [5, 9, 12–14, 22–25] and the references therein. For a recent survey, see Sect. 2 in [26]; for an abstract point of view and a more thorough discussion of the literature on semigroup domination, see the recent [19].

Theorem 2.10 *Assume that $H \geq 0$ is a selfadjoint operator in $L^2(X;\mathbb{K}^n)$ and that $V \in L^2(X;\mathbb{K}^{n\times n})$ with $V \succeq 0$. Moreover, let H_0 be selfadjoint and lower-semibounded in $L^2(X)$ such that $(e^{-tH_0})_{t\geq 0}$ dominates $(e^{-tH})_{t\geq 0}$ and in addition $e^{-t_0H_0} \in \mathcal{L}(L^2(X), L^\infty(X))$ for some $t_0 > 0$. Then*

$$\|e^{-2t_0H} - e^{-2t_0(H+V)}\|_{\mathrm{HS}} \leq 2\sqrt{n}\|V\|_{2,\mathrm{HS}}\|e^{-t_0H_0}\|_{2,\infty} \cdot t_0. \tag{9}$$

If, furthermore, V is bounded, also the following estimate holds true:

$$\|e^{-2t_0H} - e^{-2t_0(H+V)}\|_{\mathrm{HS}} \leq 2\sqrt{n}\|V\|_{2,\mathrm{HS}}\|e^{-t_0H_0}\|_{2,\infty}\int_0^{t_0}\|e^{-s(H+V)}\|_{2,2}\,\mathrm{d}s. \tag{10}$$

Remark 2.11 Given the above assumptions on H and V, the operator $H+V$ is non-negative. In our later applications, this operator will even be positive. Taking this into account, it is good to observe that if $H+V \geq \rho_0$ for some $\rho_0 \geq 0$, then the spectral theorem implies

$$\int_0^{t_0}\|e^{-s(H+V)}\|_{2,2}\,\mathrm{d}s \leq \int_0^{t_0} e^{-\rho_0 s}\mathrm{d}s = \begin{cases} t_0, & \rho_0 = 0, \\ \frac{1}{\rho_0}\left(1-e^{-t_0\rho_0}\right), & \rho_0 > 0. \end{cases} \tag{11}$$

Proof of Theorem 2.10 We first treat the case that V is bounded and use the fact that $V \succeq 0$ (i.e., $V(x) \geq 0$ for all $x \in X$), which implies that for $t > 0$ arbitrary the operator norm of the matrix $e^{-tV(x)} : \mathbb{K}^n \to \mathbb{K}^n$ is at most 1. This implies the following estimate for $f \in L^2(X;\mathbb{K}^n)$ and $x \in X$:

$$|e^{-tV}f|(x) = |e^{-tV(x)}f(x)| \leq \|e^{-tV(x)}\||f(x)| \leq |f(x)|.$$

Moreover, since e^{-tH_0} is positivity preserving, we can use the previous estimate and the domination property to obtain

$$|e^{-tH}e^{-tV}f|(x) \leq e^{-tH_0}|e^{-tV}f|(x) \leq e^{-tH_0}|f|(x).$$

By induction, we thus see that for all $k \in \mathbb{N}$ we have the pointwise inequality

$$|\left(e^{-\frac{t}{k}H}e^{-\frac{t}{k}V}\right)^k f| \leq \left(e^{-\frac{t}{k}H_0}\right)^k |f| = e^{-tH_0}|f|, \quad f \in L^2(X;\mathbb{K}^n).$$

The Trotter product formula and the assumption that $e^{-t_0 H_0} : L^2 \to L^\infty$ is bounded imply that

$$\|e^{-t_0(H+V)}\|_{2,\infty} \leq \|e^{-t_0 H_0}\|_{2,\infty}. \tag{12}$$

In particular, we also obtain

$$\|e^{-t_0 H}\|_{2,\infty} \leq \|e^{-t_0 H_0}\|_{2,\infty}. \tag{13}$$

Hence, the assumptions of Theorem 2.8 are satisfied and (10) immediately follows from (8), using the above estimates (12) and (13). Thus, we have established estimate (10) for bounded potentials.

For general $V \in L^2(X;\mathbb{K}^n)$ let us introduce

$$V^{(k)}(x) := 1_{\{x\in X:\|V(x)\|\leq k\}}(x)\cdot V(x), \quad k \in \mathbb{N}.$$

Then $V \succeq V^{(k)} \succeq 0$ and $V^{(k)} \to V$ pointwise for $k \to \infty$. Moreover,

$$H + V^{(k)} \overset{\text{srs}}{\to} H + V \quad \text{for } k \to \infty,$$

by monotone form convergence, see, e.g., [29]; here 'srs' refers to convergence in the strong resolvent sense, so that we have strong convergence of semigroups as well:

$$e^{-t(H+V^{(k)})} \overset{\text{s}}{\to} e^{-t(H+V)} \quad \text{for } k \to \infty. \tag{14}$$

Since each $V^{(k)}$ is bounded and $\|V^{(k)}\|_{2,\text{HS}} \leq \|V\|_{2,\text{HS}}$, we know from the first part of the proof, also using (11), that for all $k \in \mathbb{N}$

$$\|e^{-2t_0 H} - e^{-2t_0(H+V^{(k)})}\|_{\text{HS}} \leq 2\sqrt{n}\|V\|_{2,\text{HS}}\|e^{-t_0 H_0}\|_{2,\infty}\cdot t_0.$$

Noting that this bound is uniform in k, we can use the fact that the Hilbert–Schmidt operators $(\mathcal{S}_2, \|.\|_{\text{HS}})$ are a Hilbert space, and hence the corresponding unit ball is weakly compact, to see that there exists a subsequence $(k_l)_{l\in\mathbb{N}}$ and $T \in \mathcal{S}_2$ such that

$$e^{-2t_0 H} - e^{-2t_0(H+V^{(k_l)})} \to T \quad \text{for } l \to \infty \tag{15}$$

weakly, and

$$\|T\|_{\text{HS}} \leq \liminf_l \|e^{-2t_0 H} - e^{-2t_0(H+V^{(k_l)})}\|_{\text{HS}} \leq 2\sqrt{n}\|V\|_{2,\text{HS}}\|e^{-t_0 H_0}\|_{2,\infty}\cdot t_0.$$

(14) shows that $T = e^{-2t_0 H} - e^{-2t_0(H+V)}$ and hence the estimate (9) follows. □

3 Bounds on the First Betti Number

Let us now introduce the set-up to which we apply the results of the preceding section. We fix a compact n-dimensional Riemannian manifold (M, g) and use the Riemannian volume element to define the corresponding L^2-spaces.

Next, we introduce the Laplace–Beltrami operator

$$\Delta = \delta d \geq 0,$$

which, according to our sign convention, is a non-negative selfadjoint operator in $L^2(M)$. Moreover, we consider the Hodge–Laplacian

$$\Delta^1 = \delta d + d\delta \geq 0$$

acting on 1-forms, so that it is a non-negative selfadjoint operator in the Hilbert space $L^2(M; \Omega^1)$ of square integrable sections of the cotangent bundle. We will frequently identify $L^2(M; \Omega^1) = L^2(M; \mathbb{R}^n)$.

The Weitzenböck formula gives that

$$\Delta^1 = \nabla^*\nabla + \text{Ric},$$

where the latter term is a matrix-valued potential $M \ni x \mapsto \text{Ric}_x$, and Ric_x is the Ricci tensor interpreted as an endomorphism of the cotangent space $\Omega^1_x(M) := (T_x M)^*$. The former term, $\nabla^*\nabla$, is the rough or Bochner–Laplacian. Note that Ric_x is given by a symmetric matrix with entries varying smoothly in x so that

$$\text{Ric} \in L^\infty(M; \mathbb{R}^{n\times n}) \subset L^2(M; \mathbb{R}^{n\times n})$$

and we can substitute Ric for V of the preceding section. It follows from the Hodge theorem, see [18], Thm. 2.2.1, that we can identify the first real cohomology group with the space of harmonic 1-forms,

$$H^1(M) \simeq \text{Ker}(\Delta^1),$$

so that the first Betti number equals

$$b_1(M) = \dim\left(\text{Ker}(\Delta^1)\right).$$

In view of what we studied in the previous section, it is therefore natural to consider $H := \Delta^1$. The appropriate comparison operator H' is constructed next. We fix $\rho_0 > 0$ and consider the symmetric matrix $\text{Ric}_x - \rho_0$, where we omit the identity matrix in the last expression. We can decompose

$$\text{Ric}_x - \rho_0 = (\text{Ric}_x - \rho_0)_+ - (\text{Ric}_x - \rho_0)_-,$$

where $(\mathrm{Ric}_x - \rho_0)_+$ is non-negative definit, $(\mathrm{Ric}_x - \rho_0)_-$ is positive definite and their product is 0. This decomposition is measurable in x, since the involved matrices are continuous as functions of x.

We put

$$H' := \Delta^1 + (\mathrm{Ric}_x - \rho_0)_-. \tag{16}$$

Note that

$$V := (\mathrm{Ric} - \rho_0)_- \succeq 0$$

in the sense of the previous section. Moreover

$$\begin{aligned} H' &= \Delta^1 + (\mathrm{Ric} - \rho_0)_- \\ &= \nabla^*\nabla + \mathrm{Ric} - \rho_0 + \rho_0 + (\mathrm{Ric} - \rho_0)_- \\ &= \nabla^*\nabla + (\mathrm{Ric} - \rho_0)_+ + \rho_0 \\ &\geq \rho_0. \end{aligned}$$

As a first application of our abstract results we note that Proposition 2.4 readily implies:

Corollary 3.1 *For $\rho_0 > 0$, $t_0 > 0$ and $p \geq 1$:*

$$b_1(M) \leq \left(1 - e^{-2\rho_0 t_0}\right)^{-p} \left\| e^{-2t_0\Delta^1} - e^{-2t_0(\Delta^1 + (\mathrm{Ric} - \rho_0)_-)} \right\|_{\mathcal{S}_p}^p. \tag{17}$$

Note that the semigroups involved consist of trace class operators, so that the RHS of (17) is finite for all values of the parameters involved.

We go on to specialize to $p = 2$, using Theorem 2.10. To this end we recall the following results from [17] that we mentioned already, where the authors use the opposite sign convention! We define

$$\rho(x) := \min \sigma(\mathrm{Ric}_x), \quad x \in M. \tag{18}$$

Proposition 3.2 *For the operators defined above, we have the following domination of the corresponding semigroups:*

(1) *For all $\omega \in L^2(M; \Omega^1)$ and $t \geq 0$:*

$$\left| e^{-t\nabla^*\nabla}\omega \right| \leq e^{-t\Delta}|\omega|.$$

(2) *For all $\omega \in L^2(M; \Omega^1)$ and $t \geq 0$:*

$$\left| e^{-t\Delta^1}\omega \right| \leq e^{-t(\Delta + \rho)}|\omega|.$$

Therefore we can apply the $p = 2$ case of Theorem 2.10, in particular (10) and (11), with the above bound (17) to obtain the following estimate, putting $H_0 := \Delta + \rho$:

Theorem 3.3 *For $\rho_0 > 0$ and $t_0 > 0$:*

$$b_1(M) \leq \frac{4n}{(\rho_0(1 + e^{-t_0\rho_0}))^2} \|(\text{Ric.} - \rho_0)_-\|_{2,\text{HS}}^2 \left\|e^{-t_0(\Delta+\rho)}\right\|_{2,\infty}^2. \tag{19}$$

We want to point out again that earlier results on Betti number bounds with negative curvature assumptions always depended implicitly on an a priori bound at least on the heat kernel or even a Sobolev constant. In contrast, our result shows that such a control is indeed not needed if we have bounds on certain norms of the perturbed heat semigroup for one $t_0 > 0$, which could be achieved using other techniques.

Let us provide at least one example showing how the above bound (19) can be made more explicit if further information on the heat kernel is available. Here we use an upper bound on the kernel going back to the celebrated paper [20].

Corollary 3.4 *Assume that $\text{Ric} \geq -K$, where $K \geq 0$ and denote by D the diameter of M. Then*

$$b_1(M) \leq c(n)\rho_0^{-2} \|(\text{Ric.} - \rho_0)_-\|_{2,\text{HS}}^2 \text{Vol}(M)^{-1} e^{\alpha(n)KD^2}, \tag{20}$$

where $c(n), \alpha(n)$ depend on n only.

Proof First note that

$$\left\|e^{-t_0(\Delta+\rho)}\right\|_{2,\infty}^2 \leq e^{2t_0K} \left\|e^{-t_0\Delta}\right\|_{2,\infty}^2$$

since $\rho \geq -K$. Denoting the heat kernel by p, we get

$$\left\|e^{-t_0\Delta}\right\|_{2,\infty}^2 = \text{esssup}_{x\in M} \|p(t_0; x, \cdot)\|_2^2.$$

We now infer the heat kernel estimate from Corollary 3.1 in [20], setting $t_0 = D^2$ so that the balls appearing equal M, whence

$$|p(t_0; x, \cdot)|^2 \leq c(n) \text{Vol}(M)^{-2} e^{\alpha(n)KD^2}.$$

Integrating this pointwise bound we arrive at

$$\|p(t_0; x, \cdot)\|_2^2 \leq c(n) \text{Vol}(M)^{-1} e^{\alpha(n)KD^2}.$$

Combined with the above observation and adapting the dimension dependent constants, estimate (19) proves the claim. □

Finally, let us mention that other results on heat kernel bounds can easily be applied in a similar manner to Theorem 3.3, instead of the very crude one we employed in the preceding corollary. We conclude the paper with the following list of examples, stating conditions on M where quantitative heat kernel bounds can be obtained. For more details, the reader should consult [3, 11, 22, 24, 26].

Example 3.5 The following assumptions admit quantitative bounds on the heat kernel:

(i) M is geodesically complete, of dimension at least two, and possesses a distance function r_ξ, $\xi \in M$, such that

$$\forall x, \xi \in M: \quad |\nabla r_\xi(x)| \le 1, \quad \Delta r_\xi(x) \ge 2n,$$

(ii) M is an n-dimensional minimal submanifold of $\mathbb{R}^N$, $N > n$, i.e., all its mean curvature vectors vanish identically on M,
(iii) M is a Cartan-Hadamard manifold, i.e., it is simply connected and its sectional curvature is non-positive everywhere,
(iv) M is an n-dimensional manifold of bounded geometry, i.e., its Ricci curvature is bounded below and its injectivity radius is positive,
(v) M is complete and of dimension $n \ge 2$, the injectivity radius is positive, and there exist an $r \in (0, \operatorname{diam}(M))$ and a $p > n/2$ such that the quantity

$$\sup_{x \in M} \frac{1}{\operatorname{Vol}(B(x,r))} \int_{B(x,r)} \rho_-^p \,\mathrm{dvol}$$

is small enough,
(vi) M is of dimension $n \ge 3$, has bounded diameter and Ricci curvature bounded below,
(vii) M is of dimension $n \ge 2$, has bounded diameter, and we have

$$\int_0^{\operatorname{diam} M^2} \|\mathrm{e}^{-t\Delta}\rho_-\|_\infty \mathrm{d}t \le \frac{1}{16n}.$$

Funding Open Access funding enabled and organized by Projekt DEAL.

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