Nonlinear Nonhomogeneous Obstacle Problems with Multivalued Convection Term

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Abstract

In this paper, a nonlinear elliptic obstacle problem is studied. The nonlinear nonhomogeneous partial differential operator generalizes the notions of p -Laplacian while on the right hand side we have a multivalued convection term (i.e., a multivalued reaction term may depend also on the gradient of the solution). The main result of the paper provides existence of the solutions as well as bondedness and closedness of the set of weak solutions of the problem, under quite general assumptions on the data. The main tool of the paper is the surjectivity theorem for multivalued functions given by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone one.

Keywords Nonhomogeneous partial differential operator · Multivalued convection term · Generalized pseudomonotone operators · Surjectivity theorem · Existence

Mathematics Subject Classification 35J20 · 35J25 · 35J60

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous elliptic problem with a multivalued convection term and under obstacle condition

$$\begin{cases} -\operatorname{div} a(x, \nabla u) \in f(x, u, \nabla u) & \text{in } \Omega, \\ u(x) \leq \Phi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous, monotone with respect to the second variable and satisfies particular other growth conditions to be described later, reaction term $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}}$ is multivalued and depends on the gradient of the solution (which makes the problem nonvariational) and obstacle $\Phi: \Omega \rightarrow [0, +\infty]$ is a given function.

As the setting is general enough and hypotheses are mild and natural, we incorporate in our framework many differential operators, such as the p -Laplacian, the (p, q) -Laplacian (i.e., the sum of a p -Laplacian and a q -Laplacian) and the generalized p -mean curvature differential operator. The precise conditions set on the data will be formulated in Sect. 3.

For the nonlinear elliptic problems with gradient dependence we refer to the following papers: Averna-Motreanu-Tornatore [1], Bai [2], Bai-Gasiński-Papageorgiou [3], Faraci-Motreanu-Puglisi [4], Gasiński-Papageorgiou [5,6], Gasiński-Winkert [7], Motreanu-Motreanu-Moussaoui [8], Guarnotta-Marano-Motreanu [9], Papageorgiou-Rădulescu-Repovš [10], Faraci-Puglisi [11], Figueiredo-Madeira [12], Papageorgiou-Rădulescu-Repovš [13], Tanaka [14], Guarnotta-Marano [15], Liu-Motreanu-Zeng [16], Marano-Winkert [17], Araujo-Faria [18], Bai-Papageorgiou-Zeng [19]. None of the above papers deals with multivalued or obstacle problems. To the best of our knowledge, this is the first paper combining all these phenomena in one problem. The main tool in the proof of the existence result for problem (1.1) will be the surjectivity result due to Le [20] for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping.

Since (1.1) is an obstacle problem, the appropriate set in which we are looking for its solutions is the following one

$$\left\{ u \in W_0^{1,p}(\Omega) \mid u(x) \leq \Phi(x) \text{ for a. a. } x \in \Omega \right\}$$

with a given obstacle function $\Phi: \Omega \rightarrow \overline{\mathbb{R}}_+ = [0, \infty]$. When $\Phi \equiv +\infty$, problem (1.1) becomes the following nonlinear problem with multivalued convection term

$$\begin{cases} -\operatorname{div} a(x, \nabla u) \in f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see e.g., [5,10,21,22]). In addition, when f is a single-valued function, problem (1.1) reduces to

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f(x, u, \nabla u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

(see e.g., [1,8,10]).

Moreover, for problems with double phase operators (which include the case for the (p, q) -Laplacian) and multivalued terms (and also dealing with obstacle problems), we refer to Zeng-Gasiński-Winkert-Bai [23–25]. Finally, we mention that Mingione-Rădulescu [26] provided an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $1 \leq r \leq \infty$. In what follows, we denote by $L^r(\Omega)$ and $L^r(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$. Moreover, $W_0^{1,r}(\Omega)$ stands for the Sobolev space endowed with the norm

$$\|u\| := \|\nabla u\|_r \text{ for all } u \in W_0^{1,r}(\Omega).$$

Let us now consider the eigenvalue problem for the r -Laplacian with homogeneous Dirichlet boundary condition and $1 < r < \infty$ which is defined by

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \quad (2.1)$$

A number $\lambda \in \mathbb{R}$ is an eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$ if problem (2.1) has a nontrivial solution $u \in W_0^{1,r}(\Omega)$ which is called an eigenfunction corresponding to the eigenvalue λ . We denote by σ_r the set of eigenvalues of $(-\Delta_r, W_0^{1,r}(\Omega))$. From Lê [27] we know that the set σ_r has a smallest element $\lambda_{1,r}$ which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,r} = \inf \left\{ \frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}. \quad (2.2)$$

For $s > 1$ we denote by $s' = \frac{s}{s-1}$ its conjugate, the inner product in $\mathbb{R}^N$ is denoted by $\cdot$ and the Euclidean norm of $\mathbb{R}^N$ by $|\cdot|$. Moreover, $\mathbb{R}_+ = [0, +\infty)$ and the Lebesgue measure of a set A in $\mathbb{R}^N$ is denoted by $|A|_N$.

As for the data a of the problem (1.1) we assume that

$H(a)_0$: $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a function such that $a(x, \xi) = a_0(x, |\xi|)\xi$ with $a_0 \in C(\overline{\Omega} \times \mathbb{R}_+)$ for all $\xi \in \mathbb{R}^N$ and with $a_0(x, t) > 0$ for all $x \in \overline{\Omega}$, all $t > 0$.

For the regularity of a_0 as well as its behavior at zero, we will assume the following $H(a)_1$: $a_0 \in C^1(\overline{\Omega} \times (0, \infty))$, $t \mapsto ta_0(x, t)$ is strictly increasing in $(0, \infty)$, $\lim_{t \rightarrow 0^+} ta_0(x, t) = 0$ for all $x \in \overline{\Omega}$ and

$$\lim_{t \rightarrow 0^+} \frac{ta'_0(x, t)}{a_0(x, t)} = c > -1 \text{ for all } x \in \overline{\Omega};$$

For the growth assumptions on a , we will exploit a function $\vartheta \in C^1(0, \infty)$ satisfying

$$0 < a_1 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq a_2 \quad \text{and} \quad a_3 t^{p-1} \leq \vartheta(t) \leq a_4 (t^{q-1} + t^{p-1}) \quad (2.3)$$

for all $t > 0$, with some constants $a_1, a_2, a_3, a_4 > 0$ and for $1 < q < p < \infty$. Now we can state the growth of a as follows

$H(a)_2$: there exists $a_5 > 0$ such that

$$|\nabla_\xi a(x, \xi)| \leq a_5 \frac{\vartheta(|\xi|)}{|\xi|} \text{ for all } x \in \overline{\Omega}, \xi \in \mathbb{R}^N \setminus \{0\};$$

The above defined function ϑ will be also exploited in the next assumption guaranteeing coercivity-like behavior of the function a

$$H(a)_3: \nabla_\xi a(x, \xi) y \cdot y \geq \frac{\vartheta(|\xi|)}{|\xi|} |y|^2 \text{ for all } x \in \overline{\Omega}, \xi \in \mathbb{R}^N \setminus \{0\} \text{ and } y \in \mathbb{R}^N.$$

Under the above hypotheses, we can state the following lemma in which we will summarize some properties of the function $a: \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ (see e.g., Bai-Gasiński-Papageorgiou [3, Lemma 2.2]).

Lemma 2.1 *If hypotheses $H(a)_0$ – $H(a)_3$ hold, then:*

- (i) $a \in C(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}); \mathbb{R}^N)$ and for all $x \in \overline{\Omega}$ the map $\xi \rightarrow a(x, \xi)$ is continuous, strictly monotone (and so maximal monotone);
- (ii) there exists $a_6 > 0$, such that

$$|a(x, \xi)| \leq a_6 (1 + |\xi|^{p-1}) \text{ for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N;$$

- (iii) $a(x, \xi) \cdot \xi \geq \frac{a_3}{p-1} |\xi|^p$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

The nonlinear operator $A: W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*$ defined by

$$\langle A(u), \varphi \rangle = \int_{\Omega} (a(x, \nabla u), \nabla \varphi)_{\mathbb{R}^N} dx \text{ for all } u, \varphi \in W_0^{1,p}(\Omega), \quad (2.4)$$

possesses the following useful properties (see e.g., Gasiński-Papageorgiou [28]).

Proposition 2.2 *If hypotheses $H(a)_0$ – $H(a)_3$ hold, then the operator A defined by (2.4) is bounded, monotone, continuous, hence maximal monotone and of type (S_+) .*

In the following example we indicate some operators fitting in our framework.

Example 2.3 In what follows for simplicity, we drop the dependence of the operators a on x . All the following maps satisfy hypotheses $H(a)_0$ – $H(a)_3$:

- (i) If $a(\xi) = |\xi|^{p-2}\xi$ with $1 < p < \infty$, then the corresponding operator is the classical p -Laplacian

$$\Delta_p u = \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) \text{ for all } u \in W^{1,p}(\Omega).$$

- (ii) If $a(\xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$ with $1 < q < p < \infty$ and $\mu > 0$ then the corresponding operator is the so called weighted (p, q) -Laplacian defined by $\Delta_p u + \mu \Delta_q u$ for all $u \in W^{1,p}(\Omega)$.

- (iii) If $a(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}} \xi$ with $1 < p < \infty$, then the corresponding operator represents the generalized p -mean curvature differential operator defined by

$$\operatorname{div} \left[(1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right] \text{ for all } u \in W^{1,p}(\Omega).$$

- (iv) If $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$ with $2 < p < +\infty$, then the corresponding operator takes the form

$$\Delta_p u + \operatorname{div} \left(\frac{|\nabla u|^{p-2} \nabla u}{1 + |\nabla u|^p} \right) \text{ for all } u \in W_0^{1,p}(\Omega),$$

which arises in various problems of plasticity.

Next, let us recall the notions of pseudomonotonicity and generalized pseudomonotonicity for multivalued operators (see e.g., Gasiński-Papageorgiou [29, Definition 1.4.8]) which will be useful in the sequel.

Definition 2.4 Let X be a real reflexive Banach space. The operator $A: X \rightarrow 2^{X^*}$ is called

- (a) pseudomonotone if the following conditions hold:

- (i) the set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$.
- (ii) A is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^* .
- (iii) if $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ in X and $u_n^* \in A(u_n)$ are such that

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0,$$

then to each element $v \in X$, there exists $u^*(v) \in A(u)$ with

$$\langle u^*(v), u - v \rangle_{X^* \times X} \leq \liminf_{n \rightarrow \infty} \langle u_n^*, u_n - v \rangle_{X^* \times X}.$$

- (b) generalized pseudomonotone if the following holds: Let $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ with $u_n^* \in A(u_n)$. If $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* and

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0,$$

then the element u^* lies in $A(u)$ and

$$\langle u_n^*, u_n \rangle_{X^* \times X} \rightarrow \langle u^*, u \rangle_{X^* \times X}.$$

It is not difficult to see that every pseudomonotone operator is generalized pseudomonotone, see e.g., Carl-Le-Motreanu [30, Proposition 2.122] or Gasiński-Papageorgiou [29, Proposition 1.4.11]. However, under the additional assumption of boundedness, we obtain the converse statement, see e.g., Carl-Le-Motreanu [30, Proposition 2.123] or Gasiński-Papageorgiou [29, Proposition 1.4.12].

Proposition 2.5 *Let X be a real reflexive Banach space and assume that $A: X \rightarrow 2^{X^*}$ satisfies the following conditions:*

- (i) *for each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^* .*
- (ii) *$A: X \rightarrow 2^{X^*}$ is bounded.*
- (iii) *if $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if*

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0,$$

then $u^ \in A(u)$ and*

$$\langle u_n^*, u_n \rangle_{X^* \times X} \rightarrow \langle u^*, u \rangle_{X^* \times X}.$$

Then the operator $A: X \rightarrow 2^{X^}$ is pseudomonotone.*

Finally, we will state the following surjectivity theorem for multivalued mappings which are defined as the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping. The following theorem can be found in Le [20, Theorem 2.2]. We use the notation $B_R(0) := \{u \in X : \|u\|_X < R\}$.

Theorem 2.6 *Let X be a real reflexive Banach space, let $F: D(F) \subset X \rightarrow 2^{X^*}$ be a maximal monotone operator, let $G: D(G) = X \rightarrow 2^{X^*}$ be a bounded multivalued pseudomonotone operator and let $L \in X^*$. Assume that there exist $u_0 \in X$ and $R \geq \|u_0\|_X$ such that $D(F) \cap B_R(0) \neq \emptyset$ and*

$$\langle \xi + \eta - L, u - u_0 \rangle_{X^* \times X} > 0$$

for all $u \in D(F)$ with $\|u\|_X = R$, all $\xi \in F(u)$ and all $\eta \in G(u)$. Then the inclusion

$$F(u) + G(u) \ni L$$

has a solution in $D(F)$.

3 Main Results

Let us start this section with the assumption of the multivalued convection term $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}}$ which will be needed in the existence result for problem (1.1). First of them provides general information on the regularity of f .

$H(f)_0$: $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}}$ has nonempty, compact and convex values; for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the multivalued mapping $x \mapsto f(x, s, \xi)$ has a measurable selection; for almost all $x \in \Omega$, the multivalued mapping $(s, \xi) \mapsto f(x, s, \xi)$ is upper semicontinuous.

Next two assumptions provide the growth conditions on f . In what follows by p^* we denote the critical exponent corresponding to p , namely

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases};$$

$H(f)_1$: there exists $\alpha \in L^{\frac{q_1}{q_1-1}}(\Omega)$, $e_1, e_2 \geq 0$ and $1 < q_1 < p^*$ such that

$$|\eta| \leq e_1 |\xi|^p \frac{q_1-1}{q_1} + e_2 |s|^{q_1-1} + \alpha(x)$$

for all $\eta \in f(x, s, \xi)$, for a.a. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

$H(f)_2$: there exist $w \in L^1_+(\Omega)$ and $b_1, b_2 \geq 0$ are such that

$$b_1 + b_2 \lambda_{1,p}^{-1} < \frac{a_3}{p-1},$$

[see (2.3) for the definition of a_3 and (2.2) for the definition of $\lambda_{1,p}$] and

$$\eta s \leq b_1 |\xi|^p + b_2 |s|^p + w(x)$$

for all $\eta \in f(x, s, \xi)$, for a.a. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

We can provide an explicit example of a function f satisfying the above hypotheses.

Example 3.1 For the simplicity we drop the x -dependence. Let $1 < p < \infty$, $g: \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous function and $h: \mathbb{R} \rightarrow \mathbb{R}$ a locally Lipschitz function such that there exists constants $e_1, e_2 > 0$ satisfying

- (i) $|g(\xi)| \leq e_1 |\xi|^{p-1}$ for all $\xi \in \mathbb{R}^N$;
- (ii) $\max_{\xi \in \partial h(s)} |\xi| \leq e_2 |s|^{p-1}$ for all $s \in \mathbb{R}$,

where ∂h stands for the generalized (Clarke) subdifferential of h . Let $a_3 > (e_2 + \frac{1}{p})\lambda_{1,p}^{-1} + \frac{e_1^{p'}}{p'}$. Then, it is straightforward to check that the function $f(s, \xi) = g(\xi) + \partial h(s)$ satisfies hypotheses $H(f)_0$ – $H(f)_2$ with respect to the weighted p -Laplacian and $q_1 = p$, namely, $a(\xi) = \frac{a_3}{p-1} |\xi|^{p-2} \xi$ for all $\xi \in \mathbb{R}^N$.

Let K be a subset of $W_0^{1,p}(\Omega)$ defined by

$$K := \left\{ u \in W_0^{1,p}(\Omega) \mid u(x) \leq \Phi(x) \text{ for a.a. } x \in \Omega \right\}, \quad (3.1)$$

where

$$\Phi: \Omega \rightarrow [0, +\infty] \text{ is a function.} \quad (3.2)$$

It is obvious that the set K is a nonempty, closed and convex subset of $W_0^{1,p}(\Omega)$.

Remark 3.2 From (3.2) it is clear that $0 \in K$.

The weak solutions for problem (1.1) are understood in the following sense.

Definition 3.3 We say that $u \in K$ is a weak solution of problem (1.1) if there exists $\eta \in L^{\frac{q_1}{q_1-1}}(\Omega)$ such that $\eta(x) \in f(x, u(x), \nabla u(x))$ for a.a. $x \in \Omega$ and

$$\int_{\Omega} (a(x, \nabla u), \nabla(v - u))_{\mathbb{R}^N} dx = \int_{\Omega} \eta(x)(v(x) - u(x)) dx \quad \text{for all } v \in K,$$

where K is given by (3.1).

The main result of this paper, providing existence of solutions as well as the properties of the solution set, is stated as the next theorem.

Theorem 3.4 Assume that hypotheses $H(a)_0$ – $H(a)_3$, $H(f)_0$ – $H(f)_2$ and (3.2) hold. Then the set of solutions of problem (1.1), $\mathcal{S}$, is nonempty, bounded and closed.

Proof The proof of the theorem is divided into three steps.

Step 1. $\mathcal{S} \neq \emptyset$ (i.e., problem (1.1) is solvable).

Consider the embedding operator $i: W_0^{1,p}(\Omega) \rightarrow L^{q_1}(\Omega)$ and denote by $i^*: L^{q'_1}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*$ its adjoint operator. As $1 < q_1 < p^*$, the embedding operator i is compact and so is i^* . Moreover, by virtue of hypotheses $H(f)_0$ and $H(f)_1$, the Nemytskij operator $\tilde{N}_f: W_0^{1,p}(\Omega) \subset L^{q_1}(\Omega) \rightarrow 2^{L^{q'_1}(\Omega)}$ associated to the multivalued mapping f :

$$\tilde{N}_f(u) := \left\{ \eta \in L^{q'_1}(\Omega) \mid \eta(x) \in f(x, u(x), \nabla u(x)) \text{ for a.a. } x \in \Omega \right\}$$

for all $u \in W_0^{1,p}(\Omega)$ is well-defined.

Let $N_f := i^* \circ \tilde{N}_f: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)^*}$ and introduce the indicator function $I_K: W_0^{1,p}(\Omega) \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ of K , by

$$I_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that $u \in K$ is a weak solution of problem (1.1) (see Definition 3.3), if and only if u solves the following inequality:

$$\langle A(u) - \eta, v - u \rangle + I_K(v) - I_K(u) \geq 0 \quad \text{for all } v \in W_0^{1,p}(\Omega), \quad (3.3)$$

with some $\eta \in N_f(u)$, where $A: W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*$ is given by (2.4) and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W_0^{1,p}(\Omega)^*$ and $W_0^{1,p}(\Omega)$.

Next, let us consider the multivalued operator $\mathcal{A}: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)^*}$ defined by

$$\mathcal{A}(u) = A(u) - N_f(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Now we can reformulate problem (3.3) in the following equivalently way:
Find $u \in K$ such that

$$\mathcal{A}(u) + \partial I_K(u) \ni 0, \quad (3.4)$$

where the notation ∂I_K stands for the subdifferential of I_K in the sense of convex analysis.

In order to prove that problem (3.4) has at least one weak solution, we will apply the surjectivity result for multivalued pseudomonotone operators (see Theorem 2.6). Let $u \in W_0^{1,p}(\Omega)$ and $\eta \in N_f(u)$ be arbitrary. By condition $H(f)_1$, we have

$$\begin{aligned} \|\eta\|_{W_0^{1,p}(\Omega)^*}^{q'_1} &\leq \|i^* \|\xi\|_{L^{q'_1}(\Omega)}^{q'_1} = \|i^* \|\xi\|_{L^{q'_1}(\Omega)}^{q'_1} \int_{\Omega} |\xi(x)|^{q'_1} dx \\ &\leq C_0 \int_{\Omega} (e_1 |\nabla u(x)|^{p \frac{q_1-1}{q_1}} + e_2 |u(x)|^{q_1-1} + \alpha(x))^{q'_1} dx \\ &\leq C_1 (\|\nabla u\|_p^{q_1} + \|u\|_{q_1}^{q_1} + \|\alpha\|_{q'_1}^{q'_1}), \end{aligned} \quad (3.5)$$

for some $C_0, C_1 > 0$, where $\xi \in \tilde{N}_f(u)$ is such that $\eta = i^* \xi$. Remembering that $1 < q_1 < p^*$ and using Proposition 2.2 (or Lemma 2.1(ii)) we get that $\mathcal{A}: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)^*}$ is a bounded mapping.

Next, using Proposition 2.5, we will prove that $\mathcal{A}$ is a pseudomonotone operator. By hypotheses on f , it is clear that $\mathcal{A}(u)$ is nonempty, closed and convex subset of $W^{1,p}(\Omega)^*$ for all $u \in W_0^{1,p}(\Omega)$. Moreover, as we just showed, $\mathcal{A}$ is a bounded

mapping. So, it is enough to show that $\mathcal{A}$ is a generalized pseudomonotone operator (see Proposition 2.5).

Let $\{u_n\} \subset W_0^{1,p}(\Omega)$, $\{u_n^*\} \subset W_0^{1,p}(\Omega)^*$ and $u \in W_0^{1,p}(\Omega)$ be such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), u_n^* \rightharpoonup u^* \text{ in } W_0^{1,p}(\Omega)^*, \quad (3.6)$$

$$u_n^* \in \mathcal{A}(u_n) \text{ for all } n \in \mathbb{N}, \quad (3.7)$$

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0. \quad (3.8)$$

So, for each $n \in \mathbb{N}$, we are able to find an element $\xi_n \in \tilde{N}_f(u_n)$ such that $u_n^* = A(u_n) - i^* \xi_n$. Because the embedding $W_0^{1,p}(\Omega) \rightarrow L^{q_1}(\Omega)$ is compact, from (3.6), we get that $u_n \rightarrow u$ in $L^{q_1}(\Omega)$. On the other hand, by virtue of (3.5), we have that the sequence $\{\xi_n\}$ is bounded in $L^{q_1'}(\Omega)$. Therefore, by (3.8) we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \\ &= \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle - \limsup_{n \rightarrow \infty} \langle \xi_n, u_n - u \rangle_{L^{q_1}(\Omega)} \\ &\leq \limsup_{n \rightarrow \infty} \langle A(u_n) - i^* \xi_n, u_n - u \rangle \\ &= \limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0. \end{aligned} \quad (3.9)$$

This fact together with (3.6) and the (S_+) -property of A (see Proposition 2.2), imply that $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ and by the continuity of A (see Lemma 2.1(i)), we have

$$\langle u_n^*, u_n \rangle \rightarrow \langle u^*, u \rangle \quad \text{and} \quad A(u_n) \rightarrow A(u) \quad \text{in } W_0^{1,p}(\Omega)^*.$$

As $\xi_n \in \tilde{N}_f(u_n)$, we have

$$\xi_n(x) \in f(x, u_n(x), \nabla u_n(x)) \quad \text{for a.a. } x \in \Omega.$$

Estimate (3.5) and convergence (3.6) imply that the sequence $\{\xi_n\}$ is bounded in $L^{q_1'}(\Omega)$. Passing to a subsequence if necessary, we may assume that

$$\xi_n \rightharpoonup \xi \text{ in } L^{q_1'}(\Omega) \quad (3.10)$$

for some $\xi \in L^{q_1'}(\Omega)$. Recall that $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$, so, passing to a subsequence if necessary, we have

$$u_n(x) \rightarrow u(x) \text{ and } \nabla u_n(x) \rightarrow \nabla u(x) \text{ as } n \rightarrow \infty \text{ for a.e. } x \in \Omega. \quad (3.11)$$

Since $\mathbb{R} \times \mathbb{R}^n \ni (s, w) \mapsto f(x, s, w) \subset \mathbb{R}^N$ is upper semicontinuous and has nonempty closed convex values (see hypotheses $H(f)_0$), it follows from Theorem 7.2.2 of Aubin and Frankowska [31, p. 273] that

$$\xi(x) \in f(x, u(x), \nabla u(x)) \quad \text{for a.a. } x \in \Omega.$$

Thus $\xi \in \tilde{N}_f(u)$ and $i^*\xi \in N_f(u)$. Therefore, we obtain that

$$u^* = A(u) - i^*\xi \in \mathcal{A}(u),$$

which implies that $\mathcal{A}$ is generalized pseudomonotone.

Because $\mathcal{A}$ is a bounded operator with nonempty, closed and convex values, by Proposition 2.5 we conclude that $\mathcal{A}$ is a pseudomonotone operator.

Next, we will show the existence of a constant $R > 0$ such that

$$\langle u^* + \eta, u \rangle > 0 \quad (3.12)$$

for all $u^* \in \mathcal{A}(u)$, all $\eta \in \partial I_K(u)$ and all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = R$.

For any $u^* \in \mathcal{A}(u)$, we can find $\xi \in \tilde{N}_f(u)$ such that $u^* = A(u) - i^*\xi$. As $0 \in K$ (see Lemma 2.1(iii)), we have

$$\begin{aligned} \langle u^* + \eta, u \rangle &\geq \int_{\Omega} (a(x, \nabla u), \nabla u)_{\mathbb{R}^N} dx - \int_{\Omega} \xi(x)u(x) dx + I_K(u) - I_K(0) \\ &\geq \frac{a_3}{p-1} \|\nabla u\|_p^p - \int_{\Omega} \xi(x)u(x) dx + I_K(u). \end{aligned} \quad (3.13)$$

Because $I_K : W_0^{1,p}(\Omega) \rightarrow \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, we can apply Proposition 1.10 in Brézis [32], and obtain that

$$I_K(v) \geq -\alpha_K \|v\| - \beta_K \quad \text{for all } v \in W_0^{1,p}(\Omega), \quad (3.14)$$

for some $\alpha_K, \beta_K > 0$. Moreover, by hypothesis $H(f)_2$, we have that

$$\int_{\Omega} \xi(x)u(x) dx \leq b_1 \|\nabla u\|_p^p + b_2 \|u\|_p^p + \|w\|_1. \quad (3.15)$$

Using (3.14), (3.15) and the inequality

$$\|u\|_p^p \leq \lambda_{1,p}^{-1} \|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

[(see (2.2)], in (3.13), we obtain

$$\begin{aligned} \langle u^* + \eta, u \rangle &\geq \frac{a_3}{p-1} \|\nabla u\|_p^p - b_1 \|\nabla u\|_p^p - b_2 \|u\|_p^p - \|w\|_1 - \alpha_K \|u\| - \beta_K \\ &\geq \left(\frac{a_3}{p-1} - b_1 - b_2 \lambda_{1,p}^{-1} \right) \|u\|^p - \|w\|_1 - \alpha_K \|u\| - \beta_K. \end{aligned}$$

As $b_1 + b_2\lambda_{1,p}^{-1} < \frac{a_3}{p-1}$ (see hypotheses $H(f)_2$), we can find $R_0 > 0$ large enough such that for all $R \geq R_0$ we have

$$\left(\frac{a_3}{p-1} - b_1 - b_2\lambda_{1,p}^{-1}\right) R^p - \|w\|_1 - \alpha_K R - \beta_K > 0.$$

Therefore, inequality (3.12) holds.

Because $\partial I_K : W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)^*}$ is a maximal monotone operator, we can apply Theorem 2.6 with $F = \partial I_K$, $G = \mathcal{A}$, $L = 0$, and conclude that inclusion (3.4) has at least one solution $u \in K$ which is a solution of (3.3) and so also a solution of (1.1) in the sense of Definition 3.3. Thus, $\mathcal{S} \neq \emptyset$.

Step 2. The set $\mathcal{S}$ is closed in $W_0^{1,p}(\Omega)$.

Let $\{u_n\} \subset \mathcal{S}$ be a sequence such that

$$u_n \rightarrow u \quad \text{in } W_0^{1,p}(\Omega), \quad (3.16)$$

for some $u \in W_0^{1,p}(\Omega)$. For each $n \in \mathbb{N}$, we can find $\xi_n \in \tilde{N}_f(u_n)$ such that

$$\langle A(u_n), v - u_n \rangle + \langle \xi_n, v - u_n \rangle_{L^{q_1}(\Omega)} + I_K(v) - I_K(u_n) \geq 0, \quad (3.17)$$

for all $v \in W_0^{1,p}(\Omega)$. By hypothesis $H(f)_1$ and (3.16) we know that $\{\xi_n\}$ is bounded in $L^{q_1'}(\Omega)$. So, passing to a subsequence if necessary, we may assume that

$$\xi_n \rightharpoonup \xi \quad \text{in } L^{q_1'}(\Omega).$$

As before, using Theorem 7.2.2 of Aubin and Frankowska [31, p. 273], we obtain that

$$\xi(x) \in f(x, u(x), \nabla u(x)) \quad \text{for a.a. } x \in \Omega,$$

i.e., $\xi \in \tilde{N}_f(u)$. Passing to the upper limit in (3.17) as $n \rightarrow \infty$ and using the lower semicontinuity of I_K we conclude that $u \in K$ is a solution of problem (1.1). Thus, the set $\mathcal{S}$ is closed.

Step 3. The set $\mathcal{S}$ is bounded.

If the set K is bounded the result is clearly true. So, let us assume that K be unbounded. Proceeding by contradiction, assume that the set $\mathcal{S}$ is unbounded, so we can find a sequence $\{u_n\} \subseteq \mathcal{S}$ such that

$$\|u_n\| \rightarrow +\infty. \quad (3.18)$$

Similarly as in (3.13), we show that

$$0 \geq \langle A(u_n) - i^* \xi_n, u_n \rangle$$

$$\geq \left(\frac{a_3}{p-1} - b_1 - b_2 \lambda_{1,p}^{-1} \right) \|u_n\|^p - \|w\|_1 - \alpha_K \|u_n\| - \beta_K,$$

for some $\xi_n \in \tilde{N}_f(u_n)$. This inequality together with (3.18) give a contradiction. Therefore, the set $\mathcal{S}$ is bounded. $\square$

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