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Nodal Multi-peak Standing Waves of Fourth-Order Schrödinger Equations with Mixed Dispersion

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Abstract

We consider the existence and concentration properties of standing waves for a fourth-order Schrödinger equation with mixed dispersion, which was introduced to regularize and stabilize solutions to the classical time-dependent Schrödinger equation. This leads to study multi-peak solutions to the following singularly perturbed fourth-order nonlinear Schrödinger equation

$$\varepsilon^4 \Delta^2 u - \beta \varepsilon^2 \Delta u + V(x)u = |u|^{p-2}u \text{ in } \mathbb{R}^N,\ u \in H^2(\mathbb{R}^N).$$

We first establish a local $W^{4,p}$-estimate for a class of fourth-order semilinear elliptic equations, which is a key to get the uniform and global L^∞-estimate of solutions to the considered singularly perturbed equation above. Next, under certain assumptions on β and the potential $V(x)$, we construct a family of sign-changing multi-peak solutions with a unique maximum (or minimum) point on each component. We prove that these solutions concentrate around any prescribed finite set of local minima (possibly degenerate) of the potential $V(x)$. Compared with the classical singularly perturbed Schrödinger equation, the presence of a fourth-order term in the problem above forces

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the development of new techniques to obtain qualitative properties of multi-peak solutions.

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1 Introduction and Main Result

In this paper, we study the existence and the concentration behavior of multi-peak solutions to the following singularly perturbed fourth-order nonlinear Schrödinger equation with mixed dispersion:

$$\varepsilon^4\Delta^2 u - \beta\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u \text{ in } \mathbb{R}^N,\ u \in H^2(\mathbb{R}^N), \tag{1.1}$$

where ε is a small positive parameter, $N \geq 5$, $2 < p < 2^* := 2N/(N-4)$, and the potential $V : \mathbb{R}^N \to \mathbb{R}$ satisfies:
(V_1) $V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ and $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$;
(V_2) there exist K mutually disjoint bounded domains Λ^k $(k = 1, 2, \dots, K)$ such that

$$m_k := \inf_{\Lambda^k} V < \min_{\partial\Lambda^k} V.$$

We set

$$\mathcal{M}^k := \{x \in \Lambda^k :\ V(x) = m_k\}.$$

This kind of hypothesis was first introduced by del Pino and Felmer [1] and Gui [2]. Without loss of generality, we may assume that dist $(\Lambda^{k_1}, \Lambda^{k_2}) > 0$ for each $k_1 \neq k_2$, $1 \leq k_1, k_2 \leq K$; this can be achieved by making Λ^k smaller if necessary. Moreover, denoting $m := \max_{1\leq k\leq K} m_k$, we also assume that $\beta \geq 2m^{1/2}$.

Problem (1.1) arises from seeking standing waves for the following time-dependent fourth-order Schrödinger equation

$$i\partial_t\psi - \gamma\Delta^2\psi + \mu\Delta\psi + |\psi|^{p-2}\psi = 0, \quad \psi(0,x) = \psi_0(x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.2}$$

which was introduced by Karpman [3] to regularize and stabilize the solutions of classical Schrödinger equations. Locally well-posedness of the Cauchy problem (1.2) in $H^2(\mathbb{R}^N)$ if $2 < p < 2^*$ was proved in [4]. We also refer the reader to [5–7] for globally well-posedness and scattering, and [8,9] concerning the existence of finite-time blow up solutions, stability on instability of standing wave solutions to (1.2).

As it is shown in the above papers [8,9], the added defocusing fourth-order dispersion term ($\gamma > 0$ is small enough) clearly helps to stabilize the standing waves of problem (1.2). The effect of the fourth-order dispersion term (focusing or defocusing) depends on whether it is small or large compared with the Laplacian; see [9, Sect. 6] for details. Thus, it is a natural question to consider the asymptotic behavior of standing waves of problem (1.2) as $\gamma, \mu \to 0^+$ (this might depend on their comparison). This is the main purpose of the present paper.

When the fourth-order dispersion term in (1.1) vanishes, it becomes the following form of classical singularly perturbed Schrödinger equations, that is,

$$-\varepsilon^2 \Delta u + V(x)u = f(u),\ x \in \mathbb{R}^N,\ N \geq 1. \tag{1.3}$$

Floer and Weinstein [10] considered (1.3) in one dimension case, where $f(u) = u^3$, $V \in L^\infty(\mathbb{R})$ with $\inf_{\mathbb{R}} V > 0$. They constructed a single-peak solution concentrating around any given non-degenerate critical point of $V(x)$. Next, this result was extended by Oh [11] in higher dimensions when $f(u) = u^{p-1}(2 < p < \frac{2N}{N-2})$ and the potential V belongs to a Kato class. Furthermore, Oh [12] proved the existence of multi-peak solutions concentrating around any finite subsets of the non-degenerate critical points of V. The arguments developed in [10–12] are mainly based on a Lyapunov–Schmidt reduction which requires the uniqueness and non-degeneracy of ground state solutions to the following "limiting equation"

$$\begin{cases} -\Delta u + mu = u^{p-1} \text{ in } \mathbb{R}^N,\ m > 0, \\ u > 0,\ u \in H^1(\mathbb{R}^N),\ u(0) = \max\limits_{x\in\mathbb{R}^N} u(x),\ \Big(2 < p < \dfrac{2N}{N-2}\Big). \end{cases} \tag{1.4}$$

Namely, there exists a unique positive radially symmetric solution $u \in H^1(\mathbb{R}^N)$ to (1.4) and the kernel of the operator $Lw = -\Delta w + w - (p-1)u^{p-2}w$ in $H^1(\mathbb{R}^N)$ is spanned by $\{u_{x_1}, \dots, u_{x_N}\}$. However, the uniqueness and non-degeneracy of ground state solutions to "limiting problem"

$$\Delta^2 u - \beta\Delta u + \alpha u = |u|^{p-2}u \text{ in } \mathbb{R}^N, u \in H^2(\mathbb{R}^N), \tag{$E_{\beta,\alpha}$}$$

corresponding to problem (1.1) are, in general, difficult to check. These properties were partially proved by Bonheure et al. [8] only for the case $2 < p < 2 + \frac{2}{N}$. Notice that in this present paper, we are in a wider range $2 < p < \frac{2N}{N-4}$.

On the other hand, Rabinowitz [13] used the mountain pass theorem to show that (1.3) possesses a positive ground state solution for $\varepsilon > 0$ small under the conditions:
(V_3) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$.
We also refer to Wang [14] who proved that the positive ground state solutions to (1.3) obtained in [13] must concentrate at global minima of V as $\varepsilon \to 0$. del Pino and Felmer [15] studied (1.3) with the conditions on V replaced by
(V_4) $\inf_{x\in\mathbb{R}^N} V(x) > 0$;

(V_5) There is a bounded domain Λ such that

$$\inf_{\Lambda} V < \min_{\partial\Lambda} V.$$

They proved that (1.3) possesses a positive bound state solution for $\varepsilon > 0$ small which concentrates around the local minima of V in Λ as $\varepsilon \to 0$. del Pino and Felmer [1], Gui [2] obtained multi-peak solutions to (1.3) which exhibit concentration at any prescribed finite set of local minima, possibly degenerate, of the potential by gluing localized solutions due to Coti Zelati and Rabinowitz [16, Proposition 3.4].

Although there are many works dealing with singularly perturbed Schrödinger equations (1.3), just a few works can be found dealing with biharmonic semilinear equations. Among them we shall just mention [17]. Pimenta and Soares [17] studied the following biharmonic Schrödinger equation

$$\varepsilon^4 \Delta^2 u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \; u \in H^2(\mathbb{R}^N). \tag{1.5}$$

They developed the methods in [13,14] to obtain a family of solutions to (1.5) which concentrates around the global minima of V as $\varepsilon \to 0$, where f is of subcritical growth.

To the best of our knowledge, the existence and concentration behavior of multi-peak solutions to (1.1) has not ever been studied. It is worth pointing out that for the fourth-order nonlinear Schrödinger equation (1.1), some of the methods used in the literature have to be deeply modified. We first refer to the impossibility of splitting $u = u^+ - u^-$ in $H^2(\mathbb{R}^N)$, which leads that the classical Nash–Moser type iteration technique fails. Next, we point out the lack of a general maximum principle for the operator Δ^2 causes much trouble in finding multi-peak solutions to problem (1.1). On the other hand, since for each $\varepsilon > 0$ fixed, the limit $\lim_{|x|\to\infty} V(\varepsilon x)$ may not exist (even if the limit exists, $V(\varepsilon x)$ may not necessarily converge uniformly for $\varepsilon > 0$ small as $|x| \to \infty$), the common method in [18] for dealing with the decay of solutions to the biharmonic equations can not be applied. This implies that the classical global penalization method due to Byeon and Wang [19], which highly relies on the uniform exponential decay of solutions to (1.1), cannot be used directly. As we shall see later, the above two aspects prevent us from using variational method in a standard way.

Our main result is stated in what follows.

Theorem 1.1 *Assume that the potential V satisfies (V_1), (V_2), $N \geq 5$ and $\beta \geq 2m^{1/2}$. For any two positive integers K_1, K_2 with $K_1 + K_2 = K$, there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, (1.1) possesses a sign-changing bound state solution $u_\varepsilon \in H^2(\mathbb{R}^N) \cap C^4(\mathbb{R}^N)$. Moreover, for each $1 \leq i \leq K_1$, $1 \leq j \leq K_2$, u_ε possesses exactly one maximum point $x_\varepsilon^{p(i)}$ in $\Lambda^{p(i)}$ and one minimum point $x_\varepsilon^{q(j)}$ in $\Lambda^{q(j)}$ satisfying*

$$\lim_{\varepsilon\to 0} dist\,(x_\varepsilon^{p(i)}, \mathcal{M}^{p(i)}) = 0 \; and \; \lim_{\varepsilon\to 0} dist\,(x_\varepsilon^{q(j)}, \mathcal{M}^{q(j)}) = 0,$$

where $\{p(1), \ldots, p(K_1), q(1), \ldots, q(K_2)\}$ is a rearrangement of $\{1, 2, \ldots, K\}$.

To complete this section, we sketch our proof. First, we need to consider the "limiting problem" ($E_{\beta,\alpha}$) with $\alpha, \beta > 0$ and $\beta \geq 2\alpha^{1/2}$. Whether the positive (or negative) solution to ($E_{\beta,\alpha}$) is unique or not is unknown. Nevertheless we can prove that the set of positive (or negative) ground state solutions to ($E_{\beta,\alpha}$) satisfies some compactness properties (Proposition 2.2). This is crucial for finding multi-peak solutions which are close to a set of prescribed functions. More precisely, we search for a solution of (1.1) which consists essentially of K disjoints parts, each part being close to a ground state solution of the "limiting equation" ($E_{\beta,\alpha}$) associated to the corresponding set $\mathcal{M}^k$.

To study (1.1), we work with the following equivalent equation

$$\Delta^2 v - \beta\Delta v + V(\varepsilon x)v = |v|^{p-2}v \text{ in } \mathbb{R}^N,\ v \in H^2(\mathbb{R}^N). \tag{1.6}$$

The corresponding energy functional to (1.6) is

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta v|^2 + \frac{1}{2}\beta\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)v^2 - \frac{1}{p}\int_{\mathbb{R}^N} |v|^p,\ \ v \in H_\varepsilon,$$

where H_ε is a class of weighted Sobolev spaces defined as follows:

$$H_\varepsilon := \left\{ v \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)v^2 < \infty \right\}.$$

Unlike [13], where the minimum of $V(x)$ is global, the mountain pass theorem can be used globally, here in the present paper, the condition (V_2) is local, we need to use a penalization method introduced in [1,2,15], which helps us to overcome the difficulty caused by the non-compactness due to the unboundedness of the domain $\mathbb{R}^N$. For this purpose, we shall modify the functional I_ε. Following [1,2,15], we define auxiliary functionals J_ε, $J_\varepsilon^k (k = 1, \ldots, K)$, respectively (see Sect. 3 for details). It will be shown that this type of penalization will force the concentration phenomena to occur inside $\Lambda = \cup_{k=1}^K \Lambda^k$ (Lemma 3.4).

In order to get a critical point v_ε of J_ε, we use a version of quantitative deformation lemma (Lemma 3.7) to construct a special convergent Palais-Smale sequence of J_ε for $\varepsilon > 0$ small. To prove that v_ε is indeed a solution to the original problem (1.6), we need to exhibit a uniform decay of v_ε at infinity. For this purpose, we establish a local $W^{4,p}$-estimate and a global L^∞-estimate of the solutions to the fourth-order semilinear elliptic equations (Proposition 2.3).

This paper is organized as follows, in Sect. 2, we give some preliminary results. In Sect. 3, we prove the main result Theorem 1.1.

2 Auxiliary Results

The "limiting problem" to (1.1) is

$$\Delta^2 u - \beta\Delta u + \alpha u = |u|^{p-2}u \text{ in } \mathbb{R}^N,\ u \in H^2(\mathbb{R}^N), \tag{$E_{\beta,\alpha}$}$$

where $\alpha, \beta > 0$ and $\beta \geq 2\alpha^{1/2}$. The functional corresponding to $(E_{\beta,\alpha})$ is defined as

$$I_{\beta,\alpha}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 + \frac{1}{2}\beta \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\alpha \int_{\mathbb{R}^N} |u|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p,\ u \in H^2(\mathbb{R}^N),$$

where

$$H^2(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N), \Delta u \in L^2(\mathbb{R}^N)\}$$

endowed with the equivalent norm

$$\|u\|_{H^2(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\Delta u|^2 + \int_{\mathbb{R}^N} |u|^2\right)^{1/2}.$$

Denoting $c_{\beta,\alpha}$ the ground state level of $(E_{\beta,\alpha})$, that is

$$c_{\beta,\alpha} := \inf_{u \in \mathcal{G}_{\beta,\alpha}} I_{\beta,\alpha}(u),$$

where $\mathcal{G}_{\beta,\alpha} := \{u \in H^2(\mathbb{R}^N)\backslash\{0\} : I'_{\beta,\alpha}(u) = 0\}$. Arguing as in [13,20], we see that

$$\begin{aligned} c_{\beta,\alpha} &= \inf_{\gamma \in \Gamma_{\beta,\alpha}} \max_{t \in [0,1]} I_{\beta,\alpha}(\gamma(t)) = \inf_{u \in H^2(\mathbb{R}^N)\backslash\{0\}} \sup_{t>0} I_{\beta,\alpha}(tu) \\ &= \inf_{u \in \mathcal{N}_{\beta,\alpha}} I_{\beta,\alpha}(u) > 0, \end{aligned} \tag{2.1}$$

where the set of paths is defined as

$$\Gamma_{\beta,\alpha} := \left\{\gamma \in C([0,1], H^2(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I_{\beta,\alpha}(\gamma(1)) < 0\right\} \tag{2.2}$$

and $\mathcal{N}_{\beta,\alpha}$ is the Nehari manifold defined by

$$\mathcal{N}_{\beta,\alpha} := \{u \in H^2(\mathbb{R}^N)\backslash\{0\} : \langle I'_{\beta,\alpha}(u), u\rangle = 0\}.$$

The following result on the ground state solutions of $(E_{\beta,\alpha})$ was proved in [21].

Proposition 2.1 ([21], Theorem 1) *Assume that $\alpha > 0$, $\beta \geq 2\alpha^{1/2}$, $N \geq 5$ and $2 < p < 2^* := 2N/(N-4)$, then $(E_{\beta,\alpha})$ has a nontrivial ground state solution and any ground state solution of $(E_{\beta,\alpha})$ does not change sign, is radially symmetric around some point and strictly decreasing.*

Letting $S^+_{\beta,\alpha}$(or $S^-_{\beta,\alpha}$) the set of positive (or negative) ground state solutions U(or V) of $(E_{\beta,\alpha})$ satisfying $U(0) = \max_{x \in \mathbb{R}^N} U(x)$(or $V(0) = \min_{x \in \mathbb{R}^N} V(x)$), we obtain the following compactness of $S^+_{\beta,\alpha}$ (or $S^-_{\beta,\alpha}$).

Proposition 2.2 *Assume that $\alpha > 0$, $\beta \geq 2\alpha^{1/2}$, $N \geq 5$, then $S^+_{\beta,\alpha}$ and $S^-_{\beta,\alpha}$ are compact in $H^2(\mathbb{R}^N)$.*

Proof For any $U \in S^+_{\beta,\alpha}$,

$$\begin{aligned} c_{\beta,\alpha} &= I_{\beta,\alpha}(U) - \frac{1}{p}\left\langle I'_{\beta,\alpha}(U), U\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_{\mathbb{R}^N} |\Delta U|^2 + \beta\int_{\mathbb{R}^N} |\nabla U|^2 + \alpha\int_{\mathbb{R}^N} |U|^2\right), \end{aligned}$$

thus $S^+_{\beta,\alpha}$ is bounded in $H^2(\mathbb{R}^N)$.

For any sequence $\{U_n\}_{k=1}^{\infty} \subset S^+_{\beta,\alpha}$, up to a subsequence, we may assume that there is a $U_0 \in H^2(\mathbb{R}^N)$ such that

$$U_n \rightharpoonup U_0 \text{ in } H^2(\mathbb{R}^N) \tag{2.3}$$

and U_0 satisfies $(E_{\beta,\alpha})$. Next, we claim that there exist a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ and $R > 0$, $\beta_0 > 0$ such that

$$\int_{B_R(x_n)} |U_n|^2 \geq \beta_0. \tag{2.4}$$

Otherwise, by the vanishing theorem (see [22, Lemma I.1]), it follows that

$$\int_{\mathbb{R}^N} |U_n|^q \to 0 \text{ as } n \to \infty \text{ for } 2 < q < 2^*. \tag{2.5}$$

(2.5) and $\left\langle I'_{\beta,\alpha}(U_n), U_n\right\rangle = 0$ imply that $\|U_n\|_{H^2(\mathbb{R}^N)} = o(1)$ which contradicts the fact that $I_{\beta,\alpha}(U_n) = c_{\beta,\alpha} > 0$, thus (2.4) holds. In view of Proposition 2.2, U_n is radially symmetric around 0 and strictly radially decreasing, we see from (2.4) that,

$$\int_{B_R(0)} |U_n|^2 \geq \beta_0. \tag{2.6}$$

(2.3) and (2.6) imply that U_0 is nontrivial, then

$$\begin{aligned} c_{\beta,\alpha} &\leq I_{\beta,\alpha}(U_0) - \frac{1}{p}\left\langle I'_{\beta,\alpha}(U_0), U_0\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_{\mathbb{R}^N} |\Delta U_0|^2 + \beta\int_{\mathbb{R}^N} |\nabla U_0|^2 + \alpha\int_{\mathbb{R}^N} |U_0|^2\right) \\ &\leq \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_{\mathbb{R}^N} |\Delta U_n|^2 + \beta\int_{\mathbb{R}^N} |\nabla U_n|^2 + \alpha\int_{\mathbb{R}^N} |U_n|^2\right) \\ &= \lim_{n\to\infty}\left(I_{\beta,\alpha}(U_n) - \frac{1}{p}\left\langle I'_{\beta,\alpha}(U_n), U_n\right\rangle\right) = c_{\beta,\alpha}, \end{aligned} \tag{2.7}$$

by (2.3) and (2.7), we obtain $U_n \to U_0$ in $H^2(\mathbb{R}^N)$. This completes the proof that $S^+_{\beta,\alpha}$ is compact in $H^2(\mathbb{R}^N)$. Similarly, we also see that $S^-_{\beta,\alpha}$ is compact in $H^2(\mathbb{R}^N)$. □

For $u \in L^1(\mathbb{R}^N)$, we define its Fourier transform $\mathcal{F}u = \hat{u}$ by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\mathrm{d}x$$

and its inverse Fourier transform $\mathcal{F}^{-1}u$ by

$$\mathcal{F}^{-1}u(x) = \check{u}(x) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i\xi\cdot x} u(\xi)d\xi.$$

We recall that the fundamental solutions to the Helmholtz equation are solutions to

$$-\Delta\mathcal{K}_\mu + \mu\mathcal{K}_\mu = \delta(0), \tag{2.8}$$

where $\mu \in \mathbb{C}$, $y \in \mathbb{R}^N$ and $\delta(0)$ stands for the Dirac mass centered at 0. Of course, $\mathcal{K}_\mu$ is not uniquely determined, but in the following, we always choose those which satisfy nice integrability condition, namely, we require that $\mathcal{K}_\mu \in L^1(\mathbb{R}^N)$. Fixing a $c_0 > 0$ small such that $\beta^2 - 4c_0 > 0$ and $c_0 < \inf_{\mathbb{R}^N} V(x)$. Arguing as the Example 1 in Sect. 4.3.1. of [23], we see that

$$\mathcal{K}_{\lambda_i} := \frac{1}{(2\pi)^{N/2}}\left(\frac{1}{|\xi|^2+\lambda_i}\right)^\vee = \frac{1}{(4\pi)^{N/2}}\int_0^{+\infty} \frac{e^{-\lambda_i t - \frac{|x|^2}{4t}}}{t^{N/2}}\mathrm{d}t \quad (x \neq 0),$$

where $\mathcal{K}_{\lambda_i} (i = 1, 2)$ are the fundamental solutions to (2.8) with

$$\lambda_1 = \frac{\beta - \sqrt{\beta^2 - 4c_0}}{2} \text{ and } \lambda_2 = \frac{\beta + \sqrt{\beta^2 - 4c_0}}{2}.$$

Here, we observe that $\mathcal{K}_{\lambda_i} \in L^1(\mathbb{R}^N)$ is radially symmetric, non-negative, non-increasing in $r = |x|$ and it decays exponentially at infinity. Moreover, it is smooth in $\mathbb{R}^N\backslash\{0\}$. Next, we denote by $\mathcal{K}$ the fundamental solution to the operator $\Delta^2 - \beta\Delta + c_0 Id$, that is,

$$\Delta^2\mathcal{K} - \beta\Delta\mathcal{K} + c_0\mathcal{K} = \delta(0). \tag{2.9}$$

Taking the Fourier transform in (2.9), we get

$$\begin{aligned}\mathcal{K} &= \frac{1}{(2\pi)^{N/2}}\left(\frac{1}{|\xi|^4+\beta|\xi|^2+c_0}\right)^\vee \\ &= \frac{1}{\sqrt{\beta^2-4c_0}}\frac{1}{(2\pi)^{N/2}}\left(\frac{1}{|\xi|^2+\lambda_1} - \frac{1}{|\xi|^2+\lambda_2}\right)^\vee \\ &= \frac{1}{\sqrt{\beta^2-4c_0}}\left(\mathcal{K}_{\lambda_1} - \mathcal{K}_{\lambda_2}\right).\end{aligned}$$

Moreover, we see that $0 \le \mathcal{K} \in L^1(\mathbb{R}^N)$.

The following local $W^{4,p}$-estimate for fourth-order semilinear elliptic equations with mixed dispersion is a key to get the uniform and global L^∞-estimate of the solutions to (1.1) and the proof is standard. Since we have not found a local $W^{4,p}$-estimate suitable for fourth-order semilinear elliptic equations with mixed dispersion, for readers' convenience, we give a detailed proof.

Proposition 2.3 *Let $h \in L^p(\mathbb{R}^N)$, $1 < p < \infty$ and let $u := \mathcal{K} * h$. Then $u \in W^{4,p}(\mathbb{R}^N)$,*

$$\Delta^2 u - \beta \Delta u + c_0 u = h \; a.e. \; \mathbb{R}^N \tag{2.10}$$

and for any $x \in \mathbb{R}^N$,

$$\|u\|_{W^{4,p}(B_1(x))} \le C\Big(\|h\|_{L^p(B_2(x))} + \|u\|_{L^p(B_2(x))}\Big), \tag{2.11}$$

where $C > 0$ depends only on N and p.

Proof Let us deal first with the case $p = 2$. If $h \in C_c^\infty(\mathbb{R}^N)$, since $\mathcal{K} \in L^1(\mathbb{R}^N)$, we see from dominated convergence theorem that

$$\begin{aligned} u := \mathcal{K} * h &= \frac{1}{\sqrt{\beta^2 - 4c_0}} \left(\mathcal{K}_{\lambda_1} * h - \mathcal{K}_{\lambda_2} * h\right) \\ &:= \frac{1}{\sqrt{\beta^2 - 4c_0}} (g_{\lambda_1} - g_{\lambda_2}) \in C^\infty(\mathbb{R}^N). \end{aligned}$$

We claim that, u satisfies (2.10) in classical sense. To see this, for each $i = 1, 2$, fixing $\delta > 0$, then

$$\begin{aligned} -\Delta g_{\lambda_i} + \lambda_i g_{\lambda_i} &= \int_{B_\delta(0)} \mathcal{K}_{\lambda_i}(y)(-\Delta_x h(x-y) + \lambda_i h(x-y))\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N \setminus B_\delta(0)} \mathcal{K}_{\lambda_i}(y)(-\Delta_x h(x-y) + \lambda_i h(x-y))\mathrm{d}y \\ &= (I) + (II). \end{aligned} \tag{2.12}$$

We see that

$$|(I)| \le C\Big(\|h\|_{L^\infty(\mathbb{R}^N)} + \|\nabla^2 h\|_{L^\infty(\mathbb{R}^N)}\Big) \int_{B_\delta(0)} \mathcal{K}_{\lambda_i}(y)\mathrm{d}y = o(1) \text{ as } \delta \to 0. \tag{2.13}$$

An integration by parts yields

$$\int_{\mathbb{R}^N \setminus B_\delta(0)} \mathcal{K}_{\lambda_i}(y)\Delta_x h(x-y)\mathrm{d}y$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N \setminus B_\delta(0)} \mathcal{K}_{\lambda_i}(y) \Delta_y h(x-y) \mathrm{d}y \\
&= -\int_{\mathbb{R}^N \setminus B_\delta(0)} \nabla \mathcal{K}_{\lambda_i}(y) \nabla_y h(x-y) \mathrm{d}y + \int_{\partial B_\delta(0)} \mathcal{K}_{\lambda_i}(y) \frac{\partial h}{\partial \nu}(x-y) \mathrm{d}S(y) \\
&= (II)_1 + (II)_2,
\end{aligned} \tag{2.14}
$$

where ν denoting the inward pointing unit normal along $\partial B_\delta(0)$. Noting that

$$
\begin{aligned}
|(II)_2| &\leq \|\nabla h\|_{L^\infty(\mathbb{R}^N)} \int_{\partial B_\delta(0)} \mathcal{K}_{\lambda_i}(y) \mathrm{d}S(y) \\
&\leq C \int_{\partial B_\delta(0)} \left(\int_0^{+\infty} \frac{e^{-\lambda_i t - \frac{\delta^2}{4t}}}{t^{N/2}} \mathrm{d}t \right) \mathrm{d}S(y) \\
&\leq C \delta^{N-1} \int_0^{+\infty} \frac{e^{-\frac{\delta^2}{4t}}}{t^{N/2}} \mathrm{d}t \\
&\overset{t'=t/\delta^2}{=} C\delta \int_0^{+\infty} \frac{e^{-\frac{1}{4t'}}}{(t')^{N/2}} \mathrm{d}t' \leq C\delta.
\end{aligned} \tag{2.15}
$$

We continue by integrating by parts once again in the term $(II)_1$ to get that

$$
(II)_1 = \int_{\mathbb{R}^N \setminus B_\delta(0)} \Delta \mathcal{K}_{\lambda_i}(y) h(x-y) \mathrm{d}y - \int_{\partial B_\delta(0)} \frac{\partial \mathcal{K}_{\lambda_i}}{\partial \nu}(y) h(x-y) \mathrm{d}S(y). \tag{2.16}
$$

Since

$$
\nabla \mathcal{K}_{\lambda_i}(y) = \frac{1}{(4\pi)^{N/2}} \int_0^{+\infty} \frac{e^{-\lambda_i t - \frac{|y|^2}{4t}}}{t^{N/2}} \left(-\frac{y}{2t} \right) \mathrm{d}t \quad (y \neq 0)
$$

and $\nu = -y/|y| = -y/\delta$ on $\partial B_\delta(0)$, consequently,

$$
\begin{aligned}
\frac{\mathcal{K}_{\lambda_i}}{\partial \nu}(y) &= \nabla \mathcal{K}_{\lambda_i}(y) \cdot \nu \\
&= \frac{1}{2(4\pi)^{N/2}} \int_0^{+\infty} \frac{e^{-\lambda_i t - \frac{\delta^2}{4t}}}{t^{\frac{N}{2}+1}} \delta \mathrm{d}t \\
&\overset{t'=t/\delta^2}{=} \frac{1}{2(4\pi)^{N/2} \delta^{N-1}} \int_0^{+\infty} \frac{e^{-\lambda_i \delta^2 t' - \frac{1}{4t'}}}{(t')^{\frac{N}{2}+1}} \mathrm{d}t'
\end{aligned}
$$

on $\partial B_\delta(0)$. Hence we get

$$
\begin{aligned}
&\lim_{\delta \to 0} \int_{\partial B_\delta(0)} \frac{\partial \mathcal{K}_{\lambda_i}}{\partial \nu}(y) h(x-y) \mathrm{d}S(y) \\
&= \lim_{\delta \to 0} \frac{1}{2(4\pi)^{N/2}} \left(\int_0^{+\infty} \frac{e^{-\lambda_i \delta^2 t' - \frac{1}{4t'}}}{(t')^{\frac{N}{2}+1}} \mathrm{d}t' \right) \frac{1}{\delta^{N-1}} \int_{\partial B_\delta(x)} h(y) \mathrm{d}S(y)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2(4\pi)^{N/2}} \left(\int_0^{+\infty} \frac{e^{-\frac{1}{4t'}}}{(t')^{\frac{N}{2}+1}} \mathrm{d}t' \right) S_N h(x) \\
&\overset{t=1/4t'}{=} \frac{1}{2\pi^{N/2}} \left(\int_0^{+\infty} e^{-t} t^{\frac{N}{2}-1} \mathrm{d}t \right) S_N h(x) \\
&= \frac{1}{2\pi^{N/2}} \Gamma\left(\frac{N}{2}\right) S_N h(x) = h(x),
\end{aligned} \tag{2.17}
$$

where S_N is the surface area of the sphere $\partial B_1(0)$ in $\mathbb{R}^N$. Since $-\Delta\mathcal{K}_{\lambda_i} + \lambda_i\mathcal{K}_{\lambda_i} = 0$ away from 0, plugging (2.13)–(2.17) into (2.12), we see that,

$$
-\Delta g_{\lambda_i} + \lambda_i g_{\lambda_i} = h,
$$

then

$$
\begin{aligned}
&\Delta^2 u - \beta\Delta u + c_0 u \\
&\quad = \frac{1}{\sqrt{\beta^2 - 4c_0}} \big((-\Delta + \lambda_2 Id)(-\Delta + \lambda_1 Id) g_{\lambda_1} \\
&\qquad -(-\Delta + \lambda_1 Id)(-\Delta + \lambda_2 Id) g_{\lambda_2}\big) \\
&\quad = \frac{1}{\sqrt{\beta^2 - 4c_0}} (\lambda_2 - \lambda_1)\, h = h,
\end{aligned}
$$

this proves the claim. Consequently, for any ball $B_R(0)$,

$$
\int_{B_R(0)} (\Delta^2 u - \beta\Delta u + c_0 u)^2 = \int_{B_R(0)} h^2. \tag{2.18}
$$

integrating by parts, we obtain

$$
\int_{B_R(0)} \Delta^2 u \cdot \Delta u = -\int_{B_R(0)} |\nabla(\Delta u)|^2 + \int_{\partial B_R(0)} \frac{\partial \Delta u}{\partial \nu'} \Delta u, \tag{2.19}
$$

$$
\int_{B_R(0)} \Delta u \cdot u = -\int_{B_R(0)} |\nabla u|^2 + \int_{\partial B_R(0)} \frac{\partial u}{\partial \nu'} u, \tag{2.20}
$$

and

$$
\int_{B_R(0)} \Delta^2 u \cdot u = \int_{B_R(0)} |\Delta u|^2 - \int_{\partial B_R(0)} \frac{\partial u}{\partial \nu'} \Delta u + \int_{\partial B_R(0)} \frac{\partial \Delta u}{\partial \nu'} u, \tag{2.21}
$$

where ν' is the outward pointing unit normal vector field along $\partial B_R(0)$. We assume that $\mathrm{supp} h \subset B_{R_0}(0)$, for $R > 2R_0$, $x \in \partial B_R(0)$, similar to the argument in (2.15), we see that for $k \in \mathbb{N}$,

$$
|D^k u| \le C \int_{B_{R_0}(0)} |D^k \mathcal{K}(x-y)| \cdot |h(y)| \mathrm{d}y \le C/R^{N-2+k}.
$$

Letting $R \to \infty$ in (2.18)–(2.21), we get

$$\|u\|_{H^4(\mathbb{R}^N)} \le C\|h\|_{L^2(\mathbb{R}^N)}. \tag{2.22}$$

Fixing $1 \le i, j, k, l \le N$, we define the linear operator $T : C_c^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ by

$$Th := D_{ijkl}(\mathcal{K} * h).$$

Since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, by approximation and (2.22), we see that T can be uniquely extended as a bounded linear operator from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. By the classical Calderon–Zygmund decomposition and Marcinkiewicz interpolation theorem (see [24, Theorem 9.9]), we see that for $1 < p < \infty$,

$$\|Th\|_{L^p(\mathbb{R}^N)} \le C\|h\|_{L^p(\mathbb{R}^N)},$$

where $C > 0$ depends only on N and p. Moreover, since $\mathcal{K} \in L^1(\mathbb{R}^N)$, by Young's inequality for convolution, we have

$$\|\mathcal{K} * h\|_{L^p(\mathbb{R}^N)} \le \|\mathcal{K}\|_{L^1(\mathbb{R}^N)}\|h\|_{L^p(\mathbb{R}^N)}.$$

Hence

$$\|u\|_{W^{4,p}(\mathbb{R}^N)} \le C\|h\|_{L^p(\mathbb{R}^N)}. \tag{2.23}$$

For any $1 < s_1 < s_2 < 2$, we define the cut-off function $0 \le \eta \le 1$ such that $\eta = 1$ on $B_{s_1}(x)$, $\eta = 0$ on $\mathbb{R}^N \backslash B_{s_2}(x)$ and $|D^k\eta| \le C/(s_2 - s_1)^k$, $k \in \mathbb{N}$. Letting $v = \eta u$, then v satisfies

$$\Delta^2 v - \beta\Delta v + c_0 v = \bar{h},$$

where

$$\bar{h} = \eta h + 4\nabla\eta\nabla(\Delta u) + 6\Delta\eta\Delta u + 4\nabla(\Delta\eta)\nabla u + \Delta^2\eta u - 2\beta\nabla\eta\nabla u - \beta\Delta\eta u.$$

From (2.23) and the fact that $1 < s_1 < s_2 < 2$, we obtain

$$\|u\|_{W^{4,p}(B_{s_1}(x))} \le C\left(\|h\|_{L^p(B_{s_2}(x))} + \sum_{k=0}^{3}\frac{1}{(s_2 - s_1)^{4-k}}\|D^k u\|_{L^p(B_{s_2}(x))}\right).$$

By the interpolation inequality in Sobolev spaces (see [24, Theorem 7.28]), we see that

$$\|u\|_{W^{4,p}(B_{s_1}(x))} \le \frac{1}{2}\|u\|_{W^{4,p}(B_{s_2}(x))} + \frac{C}{(s_2-s_1)^4}\|u\|_{L^p(B_{s_2}(x))} + C\|h\|_{L^p(B_{s_2}(x))}. \tag{2.24}$$

Letting $t_0 = 1$ and $t_{i+1} = t_i + (1-\tau)\tau^i$, where $0 < \tau < 1$ to be fixed later, by (2.24),

$$\|u\|_{W^{4,p}(B_{t_i}(x))} \le \frac{1}{2}\|u\|_{W^{4,p}(B_{t_{i+1}}(x))} + \frac{C}{(1-\tau)^4\tau^{4i}}\|u\|_{L^p(B_{t_{i+1}}(x))} + C\|h\|_{L^p(B_{t_{i+1}}(x))}. \tag{2.25}$$

Iterating (2.25) for n times, we have

$$\|u\|_{W^{4,p}(B_1(x))} \le \frac{1}{2^n}\|u\|_{W^{4,p}(B_{t_n}(x))} + C\left[\frac{1}{(1-\tau)^4}\|u\|_{L^p(B_{t_n}(x))} + \|h\|_{L^p(B_{t_n}(x))}\right]\sum_{i=0}^{n-1}\frac{1}{2^i}\tau^{-4i}.$$

Choosing $\tau > 0$ such that $\frac{1}{2}\tau^{-4} < 1$ and letting $n \to \infty$, we get (2.11). □

3 The Singularly Perturbed Problem

Problem (1.1) can be rewritten as

$$\Delta^2 v - \beta\Delta v + V(\varepsilon x)v = |v|^{p-2}v \text{ in } \mathbb{R}^N,\ v \in H^2(\mathbb{R}^N). \tag{3.1}$$

The corresponding energy functional to (3.1) is

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\Delta v|^2 + \frac{1}{2}\beta\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N}V(\varepsilon x)v^2 - \frac{1}{p}\int_{\mathbb{R}^N}|v|^p,\ v \in H_\varepsilon,$$

where H_ε be a class of weighted Sobolev space as follows:

$$\left\{v \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)v^2 < \infty\right\}$$

and the norm of the space H_ε is denoted by

$$\|v\|_{H_\varepsilon} := \left(\int_{\mathbb{R}^N}|\Delta v|^2 + \int_{\mathbb{R}^N}V(\varepsilon x)v^2\right)^{1/2}.$$

Moreover, we see that H_ε is equivalent to $H^2(\mathbb{R}^N)$ owing to $0 < V_0 \le V \in L^\infty(\mathbb{R}^N)$. It will be convenient to consider mutually disjoint open set $\widetilde{\Lambda}^k$ compactly containing Λ^k satisfying $V(x) > \inf_{\xi\in\Lambda^k} V(\xi)$ for all $x \in \overline{\widetilde{\Lambda}^k}\backslash\Lambda^k$. We assume that dist $(\widetilde{\Lambda^{k_1}}, \widetilde{\Lambda^{k_2}}) > 0$ for $k_1 \neq k_2$, this can be achieved by making Λ^k smaller if necessary. From now on, we define $\Lambda = \cup_{k=1}^K \Lambda^k$, $\widetilde{\Lambda} = \cup_{k=1}^K \widetilde{\Lambda^k}$ and $\mathcal{M} = \cup_{k=1}^K \mathcal{M}^k$. Letting V_0 be as in (V_1) and choosing $a > 0$ such that $a^{p-2} < \frac{1}{l_0} V_0$ with $l_0 > \frac{p}{p-2}$. Following [1,2,15] with minor modification, we define the truncated function

$$g_\varepsilon(x,u) := \chi(\varepsilon x)|u|^{p-2}u + (1-\chi(\varepsilon x))\min\{|u|^{p-2}, a^{p-2}\}u$$

and

$$g_\varepsilon^k(x,u) := \chi^k(\varepsilon x)|u|^{p-2}u + (1-\chi^k(\varepsilon x))\min\{|u|^{p-2}, a^{p-2}\}u \ (1 \leq k \leq K),$$

respectively, where $0 \leq \chi^k(x) \leq 1$ is a smooth function such that $\chi^k(x) = 1$ on Λ^k, $\chi^k(x) = 0$ on $\mathbb{R}^N \backslash \widetilde{\Lambda^k}$ and $\chi(x) := \sum_{k=1}^K \chi^k(x)$. Moreover, we set

$$G_\varepsilon(x,u) := \int_0^u g_\varepsilon(x,\tau)d\tau \text{ and } G_\varepsilon^k(x,u) := \int_0^u g_\varepsilon^k(x,\tau)d\tau$$

accordingly. Finally, the penalized functionals $J_\varepsilon, J_\varepsilon^k (k = 1, \ldots, K)$ on H_ε are defined as

$$J_\varepsilon(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\Delta v|^2 + \frac{1}{2}\beta \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)v^2 - \int_{\mathbb{R}^N} G_\varepsilon(x,v)$$

and

$$J_\varepsilon^k(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\Delta v|^2 + \frac{1}{2}\beta \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)v^2 - \int_{\mathbb{R}^N} G_\varepsilon^k(x,v).$$

As we shall see, this type of modification will act as a penalization to force the concentration phenomena to occur inside Λ. It is standard to see that the functionals $J_\varepsilon, J_\varepsilon^k (k = 1, \ldots, K)$ are in $C^1(H_\varepsilon, \mathbb{R})$. To find solutions to (3.1) which concentrate around $\mathcal{M}$ as $\varepsilon \to 0$, we shall search critical points v_ε of J_ε for which $g_\varepsilon(x, v_\varepsilon) = |v_\varepsilon|^{p-2}v_\varepsilon$. The following lemma says that $J_\varepsilon, J_\varepsilon^k (k = 1, \ldots, K)$ satisfy Palais Smale condition and can be proved as Lemma 1.1 of [15], we omit the proof.

Lemma 3.1 *For each $\varepsilon > 0$ fixed, letting $\{u_n\}_{n=1}^\infty$ be a sequence in H_ε such that $J_\varepsilon(u_n)$(or $J_\varepsilon^k(u_n)$) is bounded and $J_\varepsilon'(u_n)$(or $\left(J_\varepsilon^k\right)'(u_n)$)$\to 0$, then $\{u_n\}_{n=1}^\infty$ has a convergent subsequence in H_ε.*

Defining $S^+_{\beta,m_{p(i)}}$(or $S^-_{\beta,m_{q(j)}}$) by the set of positive (or negative) ground state solutions $U(V)$ to $(E_{\beta,m_{p(i)}})$(or $(E_{\beta,m_{q(j)}})$) satisfying $U(0) = \max_{x\in\mathbb{R}^N} U(x)$(or $V(0) = \min_{x\in\mathbb{R}^N} V(x)$) and

$$\delta_0 := \frac{1}{10}\min\left\{\text{dist}\{\mathcal{M}, \mathbb{R}^N\backslash\Lambda\}, \min_{k_1\neq k_2} \text{dist}(\widetilde{\Lambda^{k_1}}, \widetilde{\Lambda^{k_2}})\right\},$$

we fix a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi(x) = 1$ for $|x| \le \delta_0$, $\varphi(x) = 0$ for $|x| \ge 2\delta_0$, $|\nabla\varphi| \le C/\delta_0$ and $|\Delta\varphi| \le C/(\delta_0)^2$. For $\varepsilon > 0$ small, we will find a solution of (3.1) near the set

$$X_\varepsilon := \Big\{ \sum_{i=1}^{K_1} \varphi(\varepsilon x - \bar{z}^i) U^i(x - (\bar{z}^i/\varepsilon)) + \sum_{j=1}^{K_2} \varphi(\varepsilon x - \tilde{z}^j) V^j(x - (\tilde{z}^j/\varepsilon)) \\ : \bar{z}^i \in (\mathcal{M}^{p(i)})^{\delta_0}, \tilde{z}^j \in (\mathcal{M}^{q(j)})^{\delta_0} \text{ and } U^i \in S^+_{\beta, m_{p(i)}}, V^j \in S^-_{\beta, m_{q(j)}} \Big\}.$$

where $(\mathcal{M}^k)^{\delta_0} := \big\{ y \in \mathbb{R}^N : \inf_{z \in \mathcal{M}^k} |y - z| \le \delta_0 \big\}$. Similarly, for $A \subset H_\varepsilon$, we use the notation

$$A^a := \big\{ u \in H_\varepsilon : \inf_{v \in A} \|u - v\|_{H_\varepsilon} \le a \big\}.$$

For each $1 \le i \le K_1$, $1 \le j \le K_2$, letting U_*^i(or V_*^j) a positive (or negative) ground state solution of $(E_{\beta, m_{p(i)}})$(or $(E_{\beta, m_{q(j)}})$), then there is a $S_i > 0$(or $T_j > 0$) such that $I_{\beta, m_{p(i)}}(S_i U_*^i) < -1$(or $I_{\beta, m_{q(j)}}(T_j V_*^j) < -1$). Moreover, we choose $z_*^k \in \mathcal{M}^k$ for $1 \le k \le K$. We define

$$U^i_{\varepsilon, \bar{s}}(x) := \varphi(\varepsilon x - z_*^{p(i)}) \bar{s} U_*^i(x - (z_*^{p(i)}/\varepsilon)),\ V^j_{\varepsilon, \bar{t}}(x) \\ := \varphi(\varepsilon x - z_*^{q(j)}) \bar{t} V_*^j(x - (z_*^{q(j)}/\varepsilon)) \tag{3.2}$$

for each $\varepsilon > 0$ and $\bar{s}, \bar{t} > 0$. Noting that $\mathrm{supp} U^i_{\varepsilon, \bar{s}} \subset \Lambda^{p(i)}/\varepsilon$ and $\mathrm{supp} V^j_{\varepsilon, \bar{t}} \subset \Lambda^{q(j)}/\varepsilon$, direct calculations show that for each $1 \le i \le K_1$,

$$J_\varepsilon^{p(i)}(U^i_{\varepsilon, S_i}) = I_\varepsilon(U^i_{\varepsilon, S_i}) = I_{\beta, m_{p(i)}}(S_i U_*^i) + o(1) < -1 + o(1) < -\frac{1}{2} \tag{3.3}$$

for $\varepsilon > 0$ small. Similarly, we also see that for each $1 \le j \le K_2$,

$$J_\varepsilon^{q(j)}(V^j_{\varepsilon, T_j}) < -\frac{1}{2} \tag{3.4}$$

for $\varepsilon > 0$ small. We define

$$\tilde{c}_\varepsilon := \max_{(s,t) \in [0,1]^K} J_\varepsilon(\gamma_\varepsilon(s,t)),$$

where

$$\gamma_\varepsilon(s,t) := \sum_{i=1}^{K_1} U^i_{\varepsilon, s_i S_i} + \sum_{j=1}^{K_2} V^j_{\varepsilon, t_j T_j} \tag{3.5}$$

for $(s,t) := (s_1, \ldots, s_{K_1}, t_1, \ldots, t_{K_2}) \in [0,1]^K$, we have the following estimates:

Lemma 3.2 *(i)* $\lim_{\varepsilon\to 0} \tilde{c}_\varepsilon = \sum_{k=1}^{K} c_{\beta,m_k}$*;*

(ii) $\lim_{\varepsilon\to 0} \max_{(s,t)\in\partial[0,1]^K} J_\varepsilon(\gamma_\varepsilon(s,t)) \le \sum_{k=1}^{K} c_{\beta,m_k} - \sigma,$
where $0 < \sigma < \min\{c_{\beta,m_k} : k = 1, 2, \ldots, K\}$ *is a fixed number.*

Proof Since for each $1 \le k_1, k_2 \le K$ with $k_1 \ne k_2$, $\Lambda^{k_1} \cap \Lambda^{k_2} = \emptyset$ and $\mathrm{supp} U^i_{\varepsilon,s_i S_i} \subset \Lambda^{p(i)}/\varepsilon$, $\mathrm{supp} V^j_{\varepsilon,t_j T_j} \subset \Lambda^{q(j)}/\varepsilon$, we see that

$$\begin{aligned}\tilde{c}_\varepsilon &= \sum_{i=1}^{K_1} \max_{s_i\in[0,1]} J_\varepsilon^{p(i)}(U^i_{\varepsilon,s_i S_i}) + \sum_{j=1}^{K_2} \max_{t_j\in[0,1]} J_\varepsilon^{q(j)}(V^j_{\varepsilon,t_j T_j})\\ &= \sum_{i=1}^{K_1} \max_{s_i\in[0,1]} I_{\beta,m_{p(i)}}(s_i S_i U^i_*) + \sum_{j=1}^{K_2} \max_{t_j\in[0,1]} I_{\beta,m_{q(j)}}(t_j T_j V^j_*) + o(1)\\ &= \sum_{k=1}^{K} c_{\beta,m_k} + o(1),\end{aligned}$$

(i) holds. Moreover, by (3.3) and (3.4), (ii) is obvious. □

Letting

$$c^k_\varepsilon := \inf_{\gamma\in\Gamma^k_\varepsilon} \max_{r\in[0,1]} J^k_\varepsilon(\gamma(r)),$$

where

$$\begin{aligned}\Gamma^k_\varepsilon := \{&\gamma(r) \in C([0,1], H_\varepsilon) : \gamma(0) = 0 \text{ and } \gamma(1) = U^i_{\varepsilon,S_i} \text{ if } k = p(i), i = 1, \ldots K_1\\ &\text{or } \gamma(1) = V^j_{\varepsilon,T_j} \text{ if } k = q(j), j = 1, \ldots K_2\}.\end{aligned}$$

We have the following estimates:

Lemma 3.3 *For each* $1 \le k \le K$*,*

$$\lim_{\varepsilon\to 0} c^k_\varepsilon = c_{\beta,m_k}.$$

Proof For each $1 \le k \le K$, the upper estimate of the form

$$\overline{\lim_{\varepsilon\to 0}}\, c^k_\varepsilon \le c_{\beta,m_k} \tag{3.6}$$

follows immediately from the use of a test path constructed as in the proof of Lemma 3.2 (i).

On the other hand, we see from Lemma 3.1 that J^k_ε satisfies Palais Smale condition on H_ε. By (3.3) and (3.4), the mountain pass theorem implies that for $\varepsilon > 0$ small, c^k_ε

is a critical value for J_ε^k. Letting w_ε^k be an associated critical point. Using the definition of g_ε^k and (3.6), we see that for $\varepsilon > 0$ small,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\Delta w_\varepsilon^k|^2 + \beta \int_{\mathbb{R}^N} |\nabla w_\varepsilon^k|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|w_\varepsilon^k|^2 \\
&\le C + 2\int_{\mathbb{R}^N} G_\varepsilon^k(x, w_\varepsilon^k) \\
&\le C + \frac{2}{p}\int_{\mathbb{R}^N} \chi^k(\varepsilon x)|w_\varepsilon^k|^p + a^{p-2}\int_{\mathbb{R}^N} (1-\chi^k(\varepsilon x))|w_\varepsilon^k|^2 \\
&\le C + \frac{2}{p}\int_{\mathbb{R}^N} g_\varepsilon^k(x, w_\varepsilon^k)w_\varepsilon^k + \frac{1}{l_0}\int_{\mathbb{R}^N} V(\varepsilon x)|w_\varepsilon^k|^2,
\end{aligned}$$

combining with $\left\langle (J_\varepsilon^k)'(w_\varepsilon^k), w_\varepsilon^k \right\rangle = 0$, we obtain

$$\left(\frac{p-2}{p} - \frac{1}{l_0}\right)\left(\int_{\mathbb{R}^N} |\Delta w_\varepsilon^k|^2 + \beta\int_{\mathbb{R}^N} |\nabla w_\varepsilon^k|^2 + \int_{\mathbb{R}^N} V(\varepsilon x)|w_\varepsilon^k|^2\right) \le C \quad (3.7)$$

for $\varepsilon > 0$ small.

For any sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \to 0$, we claim that, up to a subsequence, $\exists$ $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $R > 0$, $\beta_0 > 0$ such that

$$\int_{B_R(y_n)} \left|w_{\varepsilon_n}^k\right|^2 \ge \beta_0. \quad (3.8)$$

Otherwise, by vanishing theorem (see [22, Lemma I.1]), it follows that

$$\int_{\mathbb{R}^N} \left|w_{\varepsilon_n}^k\right|^q \to 0$$

as $n \to \infty$ for all $2 < q < 2^*$. Combining $\left\langle (J_\varepsilon^k)'(w_\varepsilon^k), w_\varepsilon^k \right\rangle = 0$ and the definition of g_ε^k, we see that $\left\| w_{\varepsilon_n}^k \right\|_{H_{\varepsilon_n}} = o(1)$, which contradicts $J_\varepsilon^k(w_\varepsilon^k) = c_\varepsilon^k \ge c_{\beta, V_0} > 0$.

Moreover, we also have

$$\operatorname{dist}(\varepsilon_n y_n, \widetilde{\Lambda^k}) \le \varepsilon_n R. \quad (3.9)$$

Indeed, for any $\delta > 0$ fixed, we define a smooth cut-off function $0 \le \psi(x) \le 1$ such that $\psi(x) = 0$ for $x \in \widetilde{\Lambda^k}$, $\psi(x) = 1$ for $x \in \mathbb{R}^N \backslash (\widetilde{\Lambda^k})^\delta$, $|\nabla\psi| \le C/\delta$ and $|\Delta\psi| \le C/\delta^2$. Using $\left\langle (J_{\varepsilon_n}^k)'(w_{\varepsilon_n}^k), w_{\varepsilon_n}^k \psi(\varepsilon_n x) \right\rangle = 0$, the definition of g_ε^k and the fact that $\operatorname{supp}\psi(\varepsilon_n x) \cap (\widetilde{\Lambda^k}/\varepsilon) = \emptyset$, we get

$$\begin{aligned}
&\left(1 - \frac{1}{l_0}\right) V_0 \int_{\mathbb{R}^N} |w_{\varepsilon_n}^k|^2 \psi(\varepsilon_n x) \\
&\le \left(1 - \frac{1}{l_0}\right)\int_{\mathbb{R}^N} V(\varepsilon_n x)|w_{\varepsilon_n}^k|^2 \psi(\varepsilon_n x)
\end{aligned}$$

$$\leq -2\int_{\mathbb{R}^N} \Delta w_{\varepsilon_n}^k (\nabla w_{\varepsilon_n}^k \cdot \nabla\psi(\varepsilon_n x)) - \int_{\mathbb{R}^N} \Delta w_{\varepsilon_n}^k w_{\varepsilon_n}^k \Delta\psi(\varepsilon_n x)$$
$$-\beta\int_{\mathbb{R}^N} w_{\varepsilon_n}^k (\nabla w_{\varepsilon_n}^k \cdot \nabla\psi(\varepsilon_n x))$$
$$\leq \frac{C}{\delta}\varepsilon_n + \frac{C}{\delta^2}\varepsilon_n^2.$$

If there is a subsequence, still denote it by $\{\varepsilon_n\}_{n=1}^{\infty}$, such that $B_R(y_n)\cap((\widetilde{\Lambda^k})^\delta/\varepsilon_n) = \emptyset$, then

$$\int_{B_R(y_n)} |w_{\varepsilon_n}^k|^2 \leq \frac{C}{\delta}\varepsilon_n + \frac{C}{\delta^2}\varepsilon_n^2,$$

which contradicts (3.8). Thus, for $\varepsilon_n > 0$ small, $B_R(y_n) \cap ((\widetilde{\Lambda^k})^\delta/\varepsilon_n) \neq \emptyset$, which means that $\operatorname{dist}(\varepsilon_n y_n, \widetilde{\Lambda^k}) \leq \varepsilon_n R + \delta$. Letting $\delta \to 0^+$, we obtain (3.9).

Letting $v_{\varepsilon_n}^k := w_{\varepsilon_n}^k(x + y_n)$, by (3.7), (3.8) and (3.9), we see that, up to a subsequence, $\varepsilon_n y_n \to y^k \in \overline{\widetilde{\Lambda^k}}$, $v_{\varepsilon_n}^k \rightharpoonup v^k$ in $H^2(\mathbb{R}^N)$, where v^k is a nontrivial solution of

$$\Delta^2 u - \beta\Delta u + V(y^k)u = g^k(u), \tag{3.10}$$

where

$$g^k(u) = \chi^k(y^k)|u|^{p-2}u + (1 - \chi^k(y^k))\min\{|u|^{p-2}, a^{p-2}\}u.$$

We denote

$$h_n := \frac{1}{2}\left(|\Delta v_{\varepsilon_n}^k|^2 + \beta|\nabla v_{\varepsilon_n}^k|^2 + V(\varepsilon_n x + \varepsilon_n y_n)|v_{\varepsilon_n}^k|^2\right) - G_{\varepsilon_n}^k(x + y_n, v_{\varepsilon_n}^k).$$

Standard argument shows that $v_{\varepsilon_n}^k \to v^k$ in $H^2_{\mathrm{loc}}(\mathbb{R}^N)$. Thus, for each $R > 0$ fixed,

$$\lim_{n\to\infty}\int_{B_R(0)} h_n = \frac{1}{2}\int_{B_R(0)}\left(|\Delta v^k|^2 + \beta|\nabla v^k|^2 + V(y^k)|v^k|^2\right) - \int_{B_R(0)} G^k(v^k), \tag{3.11}$$

where $G^k(u) := \int_0^u g^k(s)\mathrm{d}s$. Letting $0 \leq \varphi_R \leq 1$ be a smooth cut-off function such that $\varphi_R = 0$ on $B_{R-1}(0)$, $\varphi_R = 1$ on $\mathbb{R}^N\backslash B_R(0)$, $|\nabla\varphi_R| \leq C$ and $|\Delta\varphi_R| \leq C$. Choosing $\varphi_R v_{\varepsilon_n}^k$ as a test function for

$$\Delta^2 v_{\varepsilon_n}^k - \beta\Delta v_{\varepsilon_n}^k + V(\varepsilon_n x + \varepsilon_n y_n)v_{\varepsilon_n}^k = g_{\varepsilon_n}^k(x + y_n, v_{\varepsilon_n}^k)$$

to get

$$E_n + 2\int_{\mathbb{R}^N\setminus B_R(0)} h_n + \int_{\mathbb{R}^N\setminus B_R(0)} 2G^k_{\varepsilon_n}(x+y_n, v^k_{\varepsilon_n}) - g^k_{\varepsilon_n}(x+y_n, v^k_{\varepsilon_n})v^k_{\varepsilon_n} = 0, \tag{3.12}$$

where

$$\begin{aligned} E_n = &\int_{B_R(0)\setminus B_{R-1}(0)} \Delta v^k_{\varepsilon_n}\Delta(\varphi_R v^k_{\varepsilon_n}) - \beta\int_{B_R(0)\setminus B_{R-1}(0)} \nabla v^k_{\varepsilon_n}\nabla(\varphi_R v^k_{\varepsilon_n}) \\ &+ \int_{B_R(0)\setminus B_{R-1}(0)} V(\varepsilon_n x + \varepsilon_n y_n)|v^k_{\varepsilon_n}|^2\varphi_R \\ &- \int_{B_R(0)\setminus B_{R-1}(0)} g^k_{\varepsilon_n}(x+y_n, v^k_{\varepsilon_n})v^k_{\varepsilon_n}\varphi_R. \end{aligned}$$

The fact that $v^k_{\varepsilon_n} \to v^k$ in $H^2_{\text{loc}}(\mathbb{R}^N)$ and $v^k \in H^2(\mathbb{R}^N)$ imply that for any $\delta > 0$, $\exists R > 0$ such that $\overline{\lim}_{n\to\infty} |E_n| \le \delta$. On the other hand, the definition of g^k_ε gives that $2G^k_{\varepsilon_n}(x+y_n, v^k_{\varepsilon_n}) - g^k_{\varepsilon_n}(x+y_n, v^k_{\varepsilon_n})v^k_{\varepsilon_n} \le 0$. Using this in (3.12) and combining with (3.11), we have $\underline{\lim}_{n\to\infty} J^k_{\varepsilon_n}(w^k_{\varepsilon_n}) \ge J^k(v^k)$, where J^k is the corresponding functional to (3.10). Since $V(y^k) \ge m_k$ and $G^k(v^k) \le \frac{1}{p}|v^k|^p$, we have $J^k(v^k) \ge c_{\beta,m_k}$. The arbitrariness of $\{\varepsilon_n\}_{n=1}^\infty$ implies that $\underline{\lim}_{\varepsilon\to 0} c^k_\varepsilon \ge c_{\beta,m_k}$. This finishes the proof. □

The following lemma is a key for the proof of Theorem 1.1:

Lemma 3.4 *For each $d_0 > 0$ small and $\{\varepsilon_n\}_{n=1}^\infty$, $\{u_{\varepsilon_n}\}_{n=1}^\infty$ satisfying*

$$\lim_{n\to\infty} \varepsilon_n = 0, u_{\varepsilon_n} \in X^{d_0}_{\varepsilon_n}, \lim_{n\to\infty} J_{\varepsilon_n}(u_{\varepsilon_n}) \le \sum_{k=1}^K c_{\beta,m_k} \text{ and } \lim_{n\to\infty} \|J'_{\varepsilon_n}(u_{\varepsilon_n})\|_{(H_{\varepsilon_n})^{-1}} = 0,$$

there exists, up to a subsequence, $\{y^{p(i)}_{\varepsilon_n}\}_{n=1}^\infty \subset \mathbb{R}^N$, $z^{p(i)} \in \mathcal{M}^{p(i)}$, $U^i \in S^+_{\beta,m_{p(i)}}$ $(1 \le i \le K_1)$ and $\{y^{q(j)}_{\varepsilon_n}\}_{n=1}^\infty \subset \mathbb{R}^N$, $z^{q(j)} \in \mathcal{M}^{q(j)}$, $V^j \in S^-_{\beta,m_{q(j)}}$ $(1 \le j \le K_2)$ such that

$$\lim_{n\to\infty} |\varepsilon_n y^{p(i)}_{\varepsilon_n} - z^{p(i)}| = 0, \ \lim_{n\to\infty} |\varepsilon_n y^{q(j)}_{\varepsilon_n} - z^{q(j)}| = 0$$

and

$$\begin{aligned} \lim_{n\to\infty} \Bigg\| u_{\varepsilon_n} &- \sum_{i=1}^{K_1} \varphi(\varepsilon_n x - \varepsilon_n y^{p(i)}_{\varepsilon_n})U^i(x - y^{p(i)}_{\varepsilon_n}) \\ &- \sum_{j=1}^{K_2} \varphi(\varepsilon_n x - \varepsilon_n y^{q(j)}_{\varepsilon_n})V^j(x - y^{q(j)}_{\varepsilon_n}) \Bigg\|_{H_{\varepsilon_n}} = 0. \end{aligned}$$

Proof For notational simplicity, we write ε for ε_n and still use ε after taking a subsequence. By the definition of $X^{d_0}_\varepsilon$ and the compactness of $S^+_{\beta,m_{p(i)}}$, $S^-_{\beta,m_{q(j)}}$ and

$(\mathcal{M}^k)^{\delta_0}$, we see that there exist $\bar{W}^i \in S^+_{\beta,m_{p(i)}}$, $\tilde{W}^j \in S^-_{\beta,m_{q(j)}}$, $\{z_\varepsilon^{p(i)}\}_{\varepsilon>0} \subset (\mathcal{M}^{p(i)})^{\delta_0}$, $\{z_\varepsilon^{q(j)}\}_{\varepsilon>0} \subset (\mathcal{M}^{q(j)})^{\delta_0}$ such that for $\varepsilon > 0$ small and $1 \le i \le K_1$, $1 \le j \le K_2$,

$$\Bigg\| u_\varepsilon - \sum_{i=1}^{K_1} \varphi(\varepsilon x - z_\varepsilon^{p(i)}) \bar{W}^i (x - (z_\varepsilon^{p(i)}/\varepsilon)) - \sum_{j=1}^{K_2} \varphi(\varepsilon x - z_\varepsilon^{q(j)}) \tilde{W}^j (x - (z_\varepsilon^{q(j)}/\varepsilon)) \Bigg\|_{H_\varepsilon} \le 2d_0 \tag{3.13}$$

and

$$z_\varepsilon^{p(i)} \to z^{p(i)} \in (\mathcal{M}^{p(i)})^{\delta_0} \text{ and } z_\varepsilon^{q(j)} \to z^{q(j)} \in (\mathcal{M}^{q(j)})^{\delta_0} \text{ as } \varepsilon \to 0. \tag{3.14}$$

Step 1: We claim that

$$\lim_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 = 0, \tag{3.15}$$

where $A_\varepsilon = \cup_{k=1}^K (B_{3\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon) \backslash B_{\delta_0/2\varepsilon}(z_\varepsilon^k/\varepsilon))$.

Assuming on the contrary that there exists $r > 0$ such that

$$\underline{\lim}_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$ such that for $\varepsilon > 0$ small,

$$\int_{B_1(y_\varepsilon)} |u_\varepsilon|^2 \ge r > 0. \tag{3.16}$$

Letting $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, up to a subsequence, there exists $v \in H^2(\mathbb{R}^N)\backslash\{0\}$ such that $v_\varepsilon \rightharpoonup v$ in $H^2(\mathbb{R}^N)$ and $\varepsilon y_\varepsilon \to x_0 \in \overline{\cup_{k=1}^K (B_{3\delta_0}(z^k) \backslash B_{\delta_0/2}(z^k))} \in \mathcal{M}^{4\delta_0} \in \Lambda$. Moreover, we see that v satisfies $(E_{\beta,V(x_0)})$. Since

$$\begin{aligned} c_{\beta,V(x_0)} &\le I_{\beta,V(x_0)}(v) - \frac{1}{p}\left\langle I'_{\beta,V(x_0)}(v), v\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_{\mathbb{R}^N} |\Delta v|^2 + \beta \int_{\mathbb{R}^N} |\nabla v|^2 + V(x_0) \int_{\mathbb{R}^N} |v|^2\right) \end{aligned}$$

then for $R > 0$ large,

$$\begin{aligned} &\underline{\lim}_{\varepsilon\to 0} \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_{B_R(y_\varepsilon)} |\Delta u_\varepsilon|^2 + \beta \int_{B_R(y_\varepsilon)} |\nabla u_\varepsilon|^2 + V(x_0) \int_{B_R(y_\varepsilon)} |u_\varepsilon|^2\right) \\ &= \underline{\lim}_{\varepsilon\to 0} \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_{B_R(0)} |\Delta v_\varepsilon|^2 + \beta \int_{B_R(0)} |\nabla v_\varepsilon|^2 + V(x_0) \int_{B_R(0)} |v_\varepsilon|^2\right) \end{aligned}$$

$$\geq \left(\frac{1}{2}-\frac{1}{p}\right)\left(\int_{B_R(0)}|\Delta v|^2+\beta\int_{B_R(0)}|\nabla v|^2+V(x_0)\int_{B_R(0)}|v|^2\right)\geq \frac{1}{2}c_{\beta,V(x_0)}$$
$$>0. \tag{3.17}$$

On the other hand, by (3.13) and Sobolev's imbedding theorem, we have

$$\begin{aligned}
&\int_{B_R(y_\varepsilon)}|\Delta u_\varepsilon|^2+\beta\int_{B_R(y_\varepsilon)}|\nabla u_\varepsilon|^2+V(x_0)\int_{B_R(y_\varepsilon)}|u_\varepsilon|^2\\
&\leq C\sum_{i=1}^{K_1}\int_{B_R(y_\varepsilon-(z_\varepsilon^{p(i)}/\varepsilon))}|\Delta\bar{W}^i|^2+|\nabla\bar{W}^i|^2+|\bar{W}^i|^2\\
&\quad+C\sum_{j=1}^{K_2}\int_{B_R(y_\varepsilon-(z_\varepsilon^{q(j)}/\varepsilon))}|\Delta\tilde{W}^j|^2+|\nabla\tilde{W}^j|^2+|\tilde{W}^j|^2+Cd_0+o(1)\\
&=Cd_0+o(1),
\end{aligned}$$

where $o(1)\to 0$ as $\varepsilon\to 0$ and we have used the fact that $|y_\varepsilon-(z_\varepsilon^k/\varepsilon)|\geq \delta_0/2\varepsilon$. This leads to a contradiction for d_0 small. Hence, (3.15) holds.

Since

$$\sup_{y\in A_\varepsilon}\int_{B_1(y)}|u_\varepsilon|^2\geq \sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|\eta_\varepsilon u_\varepsilon|^2,$$

where $\eta_\varepsilon\in C_c^\infty(\mathbb{R}^N,[0,1])$ such that $\eta_\varepsilon(x)=1$ for $x\in\cup_{k=1}^K(B_{(3\delta_0/\varepsilon)-2}(z_\varepsilon^k/\varepsilon)\backslash B_{(\delta_0/2\varepsilon)+2}(z_\varepsilon^k/\varepsilon))$, $\mathrm{supp}\eta_\varepsilon\subset\cup_{k=1}^K(B_{(3\delta_0/\varepsilon)-1}(z_\varepsilon^k/\varepsilon)\backslash B_{(\delta_0/2\varepsilon)+1}(z_\varepsilon^k/\varepsilon))$, $|\nabla\eta_\varepsilon|\leq C$ and $|\Delta\eta_\varepsilon|\leq C$. By (3.15) and the boundedness of $\{\eta_\varepsilon u_\varepsilon\}_{\varepsilon>0}$ in $H^2(\mathbb{R}^N)$, we derive from vanishing theorem (see [22, Lemma I.1]) that for $2<q<2^*$,

$$\lim_{\varepsilon\to 0}\int_{\overset{K}{\underset{k=1}{\cup}}(B_{2\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon)\backslash B_{\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon))}|u_\varepsilon|^q\to 0. \tag{3.18}$$

Step 2: Let $u_{\varepsilon,1}(x):=\sum_{k=1}^K u_{\varepsilon,1}^k(x):=\sum_{k=1}^K\varphi(\varepsilon x-z_\varepsilon^k)u_\varepsilon(x)$, $u_{\varepsilon,2}(x):=u_\varepsilon(x)-u_{\varepsilon,1}(x)$, by (3.18), we see that

$$\int_{\mathbb{R}^N}|\Delta u_\varepsilon|^2\geq\int_{\mathbb{R}^N}|\Delta u_{\varepsilon,1}|^2+\int_{\mathbb{R}^N}|\Delta u_{\varepsilon,2}|^2+o(1), \tag{3.19}$$
$$\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\geq\int_{\mathbb{R}^N}|\nabla u_{\varepsilon,1}|^2+\int_{\mathbb{R}^N}|\nabla u_{\varepsilon,2}|^2+o(1), \tag{3.20}$$
$$\int_{\mathbb{R}^N}V(\varepsilon x)|u_\varepsilon|^2\geq\int_{\mathbb{R}^N}V(\varepsilon x)|u_{\varepsilon,1}|^2+\int_{\mathbb{R}^N}V(\varepsilon x)|u_{\varepsilon,2}|^2, \tag{3.21}$$
$$\int_{\mathbb{R}^N}G_\varepsilon(x,u_\varepsilon)=\int_{\mathbb{R}^N}G_\varepsilon(x,u_{\varepsilon,1})+\int_{\mathbb{R}^N}G_\varepsilon(x,u_{\varepsilon,2})+o(1), \tag{3.22}$$

From (3.19)–(3.22), we infer that

$$J_\varepsilon(u_\varepsilon) \geq J_\varepsilon(u_{\varepsilon,1}) + J_\varepsilon(u_{\varepsilon,2}) + o(1). \tag{3.23}$$

By (3.13), it follows that

$$\begin{aligned}
&\left\|u_{\varepsilon,2}\right\|_{H_\varepsilon} \\
&\leq \Bigg\| u_{\varepsilon,1} - \sum_{i=1}^{K_1} \varphi(\varepsilon x - z_\varepsilon^{p(i)}) \bar{W}^i(x - (z_\varepsilon^{p(i)}/\varepsilon)) \\
&\quad - \sum_{j=1}^{K_2} \varphi(\varepsilon x - z_\varepsilon^{q(j)}) \tilde{W}^j(x - (z_\varepsilon^{q(j)}/\varepsilon)) \Bigg\|_{H_\varepsilon} + 2d_0 \\
&\leq \left\|u_{\varepsilon,2}\right\|_{H_\varepsilon\left(\overset{K}{\underset{k=1}{\cup}} B_{2\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon)\right)} + 4d_0 \\
&\leq C\|u_\varepsilon\|_{H_\varepsilon\left(\overset{K}{\underset{k=1}{\cup}} \left(B_{2\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon) \backslash B_{\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon)\right)\right)} + 4d_0 \\
&\leq C\sum_{i=1}^{K_1} \left\|\varphi(\varepsilon x - z_\varepsilon^{p(i)}) \bar{W}^i(x - (z_\varepsilon^{p(i)}/\varepsilon))\right\|_{H^2(B_{2\delta_0/\varepsilon}(z_\varepsilon^{p(i)}/\varepsilon) \backslash B_{\delta_0/\varepsilon}(z_\varepsilon^{p(i)}/\varepsilon))} \\
&\quad + C\sum_{j=1}^{K_2} \left\|\varphi(\varepsilon x - z_\varepsilon^{q(j)}) \tilde{W}^j(x - (z_\varepsilon^{p(i)}/\varepsilon))\right\|_{H^2(B_{2\delta_0/\varepsilon}(z_\varepsilon^{q(j)}/\varepsilon) \backslash B_{\delta_0/\varepsilon}(z_\varepsilon^{q(j)}/\varepsilon))} + Cd_0 \\
&\leq C\sum_{i=1}^{K_1} \left\|\bar{W}^i\right\|_{H^2(B_{2\delta_0/\varepsilon}(0) \backslash B_{\delta_0/\varepsilon}(0))} + C\sum_{j=1}^{K_2} \left\|\tilde{W}^j\right\|_{H^2(B_{2\delta_0/\varepsilon}(0) \backslash B_{\delta_0/\varepsilon}(0))} \\
&\quad + Cd_0 = Cd_0 + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Thus, $\overline{\lim_{\varepsilon \to 0}} \|u_{\varepsilon,2}\|_{H_\varepsilon} \leq Cd_0$.

On the other hand, since $\langle J'_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle \to 0$ as $\varepsilon \to 0$, we deduce from (3.18) and Sobolev's imbedding theorem that

$$\|u_{\varepsilon,2}\|_{H_\varepsilon}^2 \leq C\|u_{\varepsilon,2}\|_{H_\varepsilon}^p + o(1).$$

Choosing $d_0 > 0$ small, we see that $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$, by (3.23),

$$J_\varepsilon(u_\varepsilon) \geq J_\varepsilon(u_{\varepsilon,1}) + o(1). \tag{3.24}$$

Step 3: For each $1 \leq k \leq K$, letting $\tilde{w}_\varepsilon^k(x) := u_{\varepsilon,1}^k(x + (z_\varepsilon^k/\varepsilon)) := \varphi(\varepsilon x)u_\varepsilon(x + (z_\varepsilon^k/\varepsilon))$, up to a subsequence, as $\varepsilon \to 0$, $\exists \tilde{w}^k \in H^2(\mathbb{R}^N)$ such that $\tilde{w}_\varepsilon^k \rightharpoonup \tilde{w}^k$ in $H^2(\mathbb{R}^N)$. Next, we claim that

$$\tilde{w}_\varepsilon^k \to \tilde{w}^k \text{ in } L^q(\mathbb{R}^N) \text{ for } q \in (2, 2^*). \tag{3.25}$$

If not, by vanishing theorem (see [22, Lemma I.1]), $\exists r > 0$ such that

$$\varlimsup_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^N} \int_{B_1(x)} |\tilde{w}_\varepsilon^k - \tilde{w}^k|^2 = 2r > 0,$$

then for $\varepsilon > 0$ small, $\exists x_\varepsilon^k \in \mathbb{R}^N$ such that

$$\int_{B_1(x_\varepsilon^k)} |\tilde{w}_\varepsilon^k - \tilde{w}^k|^2 \geq r > 0. \tag{3.26}$$

There are two cases:
Case 1: $\{x_\varepsilon^k\}_{\varepsilon>0}$ is bounded, that is, $|x_\varepsilon^k| \leq R_k$ for some $R_k > 0$, then for $\varepsilon > 0$ small,

$$\int_{B_{R_k+1}(0)} |\tilde{w}_\varepsilon^k - \tilde{w}^k|^2 \geq r > 0,$$

which contradicts that $\tilde{w}_\varepsilon^k \to \tilde{w}^k$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.
Case 2: $\{x_\varepsilon^k\}_{\varepsilon>0}$ is unbounded, by (3.26),

$$\varlimsup_{\varepsilon\to 0} \int_{B_1(x_\varepsilon^k)} |\varphi(\varepsilon x)u_\varepsilon(x + (z_\varepsilon^k/\varepsilon))|^2 \geq r > 0. \tag{3.27}$$

Since $\varphi(x) = 0$ for $|x| \geq 2\delta_0$, we see that $|x_\varepsilon^k| \leq 3\delta_0/\varepsilon$ for $\varepsilon > 0$ small. Moreover, we see that $|x_\varepsilon^k| \leq \delta_0/2\varepsilon$ for $\varepsilon > 0$ small. If not, $x_\varepsilon^k \in B_{3\delta_0/\varepsilon}(0)\backslash B_{\delta_0/2\varepsilon}(0)$, by (3.15),

$$\begin{aligned}
&\varlimsup_{\varepsilon\to 0} \int_{B_1(x_\varepsilon^k)} |\varphi(\varepsilon x)u_\varepsilon(x + (z_\varepsilon^k/\varepsilon))|^2 \\
&\leq \varlimsup_{\varepsilon\to 0} \sup_{z\in B_{3\delta_0/\varepsilon}(0)\backslash B_{\delta_0/2\varepsilon}(0)} \int_{B_1(z)} |u_\varepsilon(x + (z_\varepsilon^k/\varepsilon))|^2 \\
&= \varlimsup_{\varepsilon\to 0} \sup_{y\in B_{3\delta_0/\varepsilon}(z_\varepsilon^k/\varepsilon)\backslash B_{\delta_0/2\varepsilon}(z_\varepsilon^k/\varepsilon)} \int_{B_1(y)} |u_\varepsilon|^2 \\
&\leq \varlimsup_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 = 0,
\end{aligned}$$

which contradicts (3.27). Up to a subsequence, $\varepsilon x_\varepsilon^k \to x^k \in \overline{B_{\delta_0/2}(0)}$ and $\bar{w}_\varepsilon^k(x) := \tilde{w}_\varepsilon^k(x + x_\varepsilon^k) \rightharpoonup \bar{w}^k$ in $H^2(\mathbb{R}^N)$, by (3.27), $\bar{w}^k \neq 0$ and satisfies $(E_{\beta,V(z^k+x^k)})$. Arguing as in **Step 1**, we get a contradiction for $d_0 > 0$ small. (3.25) follows.

Similar to the argument in Lemma 3.2(i), we have $J_\varepsilon(u_{\varepsilon,1}) = \sum_{k=1}^K J_\varepsilon(u_{\varepsilon,1}^k(x))$. Recalling that for each $1 \leq k \leq K$, $z_\varepsilon^k \to z^k$ and $\tilde{w}_\varepsilon^k(x) = u_{\varepsilon,1}^k(x + (z_\varepsilon^k/\varepsilon))$, by (3.24) and (3.25), we obtain

$$\sum_{k=1}^K I_{\beta,V(z^k)}(\tilde{w}^k) \leq \sum_{k=1}^K c_{\beta,m_k}. \tag{3.28}$$

For any $\psi \in C_c^\infty(\mathbb{R}^N)$, letting $\psi(x-(z_\varepsilon^k/\varepsilon))$ as a test function for $J'_\varepsilon(u_\varepsilon)$. Since for $\varepsilon>0$ small, $\text{supp}\psi(x-(z_\varepsilon^k/\varepsilon)) \subset \Lambda/\varepsilon$, we see that $\tilde{w}^k$ is a solution of $(E_{\beta,V(z^k)})$. Moreover, thanks to (3.25) and $\langle J'_\varepsilon(u_\varepsilon), u^k_{\varepsilon,1}\rangle \to 0$, $\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\Delta\tilde{w}^k|^2 + \beta\int_{\mathbb{R}^N} |\nabla\tilde{w}^k|^2 + \int_{\mathbb{R}^N} V(z^k)|\tilde{w}^k|^2 \\
&\le \varliminf_{\varepsilon\to 0}\left[\int_{\mathbb{R}^N} |\Delta\tilde{w}_\varepsilon^k|^2 + \beta\int_{\mathbb{R}^N} |\nabla\tilde{w}_\varepsilon^k|^2 + \int_{\mathbb{R}^N} V(\varepsilon x+z_\varepsilon^k)|\nabla\tilde{w}_\varepsilon^k|^2\right] \\
&= \varliminf_{\varepsilon\to 0}\int_{\mathbb{R}^N} |\tilde{w}_\varepsilon^k|^p = \int_{\mathbb{R}^N} |\tilde{w}^k|^p = \int_{\mathbb{R}^N} |\Delta\tilde{w}^k|^2 + \beta\int_{\mathbb{R}^N} |\nabla\tilde{w}^k|^2 + \int_{\mathbb{R}^N} V(z^k)|\tilde{w}^k|^2,
\end{aligned}$$

then as $\varepsilon \to 0$,

$$\begin{cases}
\int_{\mathbb{R}^N} |\Delta\tilde{w}_\varepsilon^k|^2 \to \int_{\mathbb{R}^N} |\Delta\tilde{w}^k|^2, \\
\int_{\mathbb{R}^N} |\nabla\tilde{w}_\varepsilon^k|^2 \to \int_{\mathbb{R}^N} |\nabla\tilde{w}^k|^2, \\
\int_{\mathbb{R}^N} V(\varepsilon x+z_\varepsilon^k)|\tilde{w}_\varepsilon^k|^2 \to \int_{\mathbb{R}^N} V(z^k)|\tilde{w}^k|^2.
\end{cases} \tag{3.29}$$

By (3.13), (3.25) and $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$, we see that $\tilde{w}^k \neq 0$ for $d_0>0$ small. Thus

$$I_{\beta,V(z^k)}(\tilde{w}^k) \ge c_{\beta,V(z^k)}. \tag{3.30}$$

Since $z^k \in (\mathcal{M}^k)^{\delta_0} \subset \Lambda^k$, (3.28) and (3.30) imply that $V(z^k)=m_k$, $z^k \in \mathcal{M}^k$ and $I_{\beta,m_k}(\tilde{w}^k) = c_{\beta,m_k}$. Moreover

$$m_k\int_{\mathbb{R}^N} |\tilde{w}_\varepsilon^k|^2 \le \int_{\mathbb{R}^N} V(\varepsilon x+z_\varepsilon^k)|\tilde{w}_\varepsilon^k|^2,$$

by (3.29), $\tilde{w}_\varepsilon^k \to \tilde{w}^k$ in $H^2(\mathbb{R}^N)$. At this point, it is clear that for $d_0>0$ small and each $1\le i\le K_1$, $1\le j\le K_2$, $\exists U^i \in S_{\beta,m_{p(i)}}$, $V^j \in S_{\beta,m_{q(j)}}$ and $\bar{z}^{p(i)}, \bar{z}^{q(j)} \in \mathbb{R}^N$ such that $\tilde{w}^{p(i)}(x) = U^i(x-\bar{z}^{p(i)})$, $\tilde{w}^{q(j)}(x) = V^j(x-\bar{z}^{q(j)})$. Therefore, as $\varepsilon\to 0$,

$$\begin{aligned}
&\Bigg\| u_\varepsilon - \sum_{i=1}^{K_1}\varphi(\varepsilon x-(z_\varepsilon^{p(i)}+\varepsilon\bar{z}^{p(i)}))U^i(x-((z_\varepsilon^{p(i)}/\varepsilon)+\bar{z}^{p(i)})) \\
&-\sum_{j=1}^{K_2}\varphi(\varepsilon x-(z_\varepsilon^{q(j)}+\varepsilon\bar{z}^{q(j)}))V^j(x-((z_\varepsilon^{q(j)}/\varepsilon)+\bar{z}^{q(j)}))\Bigg\|_{H_\varepsilon} \to 0.
\end{aligned}$$

This completes the proof. □

We define $J_\varepsilon^\alpha \subset H_\varepsilon$ by

$$J_\varepsilon^\alpha := \{u \in H_\varepsilon : J_\varepsilon(u) \le \alpha\}.$$

Lemma 3.5 *Letting d_0 be the number given in Lemma 3.4, then for any $d \in (0, d_0)$, there exist $\varepsilon_d > 0$, $\rho_d > 0$ and $\omega_d > 0$ such that*

$$\|J'_\varepsilon(u)\|_{(H_\varepsilon)^{-1}} \geq \omega_d$$

for all $u \in J_\varepsilon^{\sum_{k=1}^K c_{m_{\beta,k}}+\rho_d} \cap (X_\varepsilon^{d_0}\backslash X_\varepsilon^d)$ with $\varepsilon \in (0, \varepsilon_d)$.

Proof Assuming on the contrary that, there exist $d \in (0, d_0)$, $\{\varepsilon_n\}_{n=1}^\infty$, $\{\rho_n\}_{n=1}^\infty$ with $\varepsilon_n, \rho_n \to 0$ and $u_n \in J_{\varepsilon_n}^{\sum_{k=1}^K c_{m_{\beta,k}}+\rho_n} \cap (X_{\varepsilon_n}^{d_0}\backslash X_{\varepsilon_n}^d)$ such that

$$\|J'_{\varepsilon_n}(u_n)\|_{(H_{\varepsilon_n})^{-1}} \to 0 \text{ as } n \to \infty.$$

By Lemma 3.4, for each $1 \leq i \leq K_1$, $1 \leq j \leq K_2$, we find $\{y_n^{p(i)}\}_{n=1}^\infty, \{y_n^{q(j)}\}_{n=1}^\infty \subset \mathbb{R}^N$, $z^{p(i)} \in \mathcal{M}^{p(i)}$, $z^{q(j)} \in \mathcal{M}^{q(j)}$, $U^i \in S_{\beta,m_{p(i)}}$, $V^j \in S_{\beta,m_{q(j)}}$ such that

$$\lim_{n\to\infty} |\varepsilon_n y_n^{p(i)} - z^{p(i)}| = 0, \quad \lim_{n\to\infty} |\varepsilon_n y_n^{q(j)} - z^{q(j)}| = 0$$

and

$$\lim_{n\to\infty} \Bigg\| u_n - \sum_{i=1}^{K_1} \varphi(\varepsilon_n x - \varepsilon_n y_n^{p(i)}) U^i(x - y_n^{p(i)}) - \sum_{j=1}^{K_2} \varphi(\varepsilon_n x - \varepsilon_n y_n^{q(j)}) V^j(x - y_n^{q(j)}) \Bigg\|_{H_{\varepsilon_n}} = 0,$$

which gives that $u_n \in X_{\varepsilon_n}^d$ for large n. This contradicts that $u_n \notin X_{\varepsilon_n}^d$. □

Lemma 3.6 *There exists $T_0 > 0$ with the following property: for any $\delta > 0$ small, there exist $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J_\varepsilon(\gamma_\varepsilon(s,t)) \geq \sum_{k=1}^K c_{\beta,m_k} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_\varepsilon(s,t) \in X_\varepsilon^{T_0\delta}$, where $\gamma_\varepsilon(s,t)$ has been mentioned in (3.5).*

Proof First, there is a $T_0 > 0$ such that for each $1 \leq k \leq K$ and $u \in H^2(\mathbb{R}^N)$,

$$\|\varphi(\varepsilon x - z_*^k) u(x - (z_*^k/\varepsilon))\|_{H_\varepsilon} \leq T_0 \|u(x)\|_{H^2(\mathbb{R}^N)}, \tag{3.31}$$

where $z_*^k \in \mathcal{M}^k$ has been mentioned in (3.2). We define

$$\alpha_\delta = \frac{1}{4} \min\Big\{ \sum_{k=1}^K c_{\beta,m_k} - \sum_{i=1}^{K_1} I_{\beta,m_{p(i)}}(s_i S_i U_*^i) - \sum_{j=1}^{K_2} I_{\beta,m_{q(j)}}(t_j T_j V_*^j) : s_i, t_j \in [0,1], \sum_{i=1}^{K_1} |s_i S_i - 1| \|U_*^i\|_{H^2(\mathbb{R}^N)} + \sum_{j=1}^{K_2} |t_j T_j - 1| \|V_*^i\|_{H^2(\mathbb{R}^N)} \geq \delta \Big\} > 0,$$

we have

$$\sum_{i=1}^{K_1} I_{\beta,m_{p(i)}}(s_iS_iU_*^i)+\sum_{j=1}^{K_2} I_{\beta,m_{q(j)}}(t_jT_jV_*^j)\geq\sum_{k=1}^{K}c_{\beta,m_k}-2\alpha_\delta \text{ implies}$$
$$\sum_{i=1}^{K_1}|s_iS_i-1|\left\|U_*^i\right\|_{H^2(\mathbb{R}^N)}+\sum_{j=1}^{K_2}|t_jT_j-1|\left\|V_*^i\right\|_{H^2(\mathbb{R}^N)}\leq\delta. \tag{3.32}$$

Similar to the proof of Lemma 3.2(i), we see that there exists an $\varepsilon_\delta>0$ such that

$$\max_{(s,t)\in[0,1]^K}\left|J_\varepsilon(\gamma_\varepsilon(s,t))-\sum_{i=1}^{K_1} I_{\beta,m_{p(i)}}(s_iS_iU_*^i)-\sum_{j=1}^{K_2} I_{\beta,m_{q(j)}}(t_jT_jV_*^j)\right|\leq\alpha_\delta \tag{3.33}$$

for all $\varepsilon\in(0,\varepsilon_\delta)$. Thus if $\varepsilon\in(0,\varepsilon_\delta)$ and $J_\varepsilon(\gamma_\varepsilon(s,t))\geq\sum_{k=1}^{K}c_{\beta,m_k}-\alpha_\delta$, by (3.32) and (3.33), we have

$$\sum_{i=1}^{K_1}|s_iS_i-1|\|U_*^i\|_{H^2(\mathbb{R}^N)}+\sum_{j=1}^{K_2}|t_jT_j-1|\|V_*^i\|_{H^2(\mathbb{R}^N)}\leq\delta,$$

by (3.31), we have

$$\begin{aligned}&\left\|\gamma_\varepsilon(s,t)-\sum_{i=1}^{K_1}\varphi(\varepsilon x-z_*^{p(i)})U_*^i(x-(z_*^{p(i)}/\varepsilon))\right.\\&\qquad\left.-\sum_{j=1}^{K_2}\varphi(\varepsilon x-z_*^{q(j)})V_*^j(x-(z_*^{q(j)}/\varepsilon))\right\|_{H_\varepsilon}\\&\leq\sum_{i=1}^{K_1}|s_iS_i-1|\left\|\varphi(\varepsilon x-z_*^{p(i)})U_*^i(x-(z_*^{p(i)}/\varepsilon))\right\|_{H_\varepsilon}\\&\quad+\sum_{j=1}^{K_2}|t_jT_j-1|\left\|\varphi(\varepsilon x-z_*^{q(j)})V_*^j(x-(z_*^{q(j)}/\varepsilon))\right\|_{H_\varepsilon}\\&\leq T_0\sum_{i=1}^{K_1}|s_iS_i-1|\left\|U_*^i\right\|_{H^2(\mathbb{R}^N)}+T_0\sum_{j=1}^{K_2}|t_jT_j-1|\left\|V_*^j\right\|_{H^2(\mathbb{R}^N)}\leq T_0\delta.\end{aligned}$$

Thus $\gamma_\varepsilon(s,t)\in X_\varepsilon^{T_0\delta}$. □

Choosing $\delta_1>0$ to ensure that $T_0\delta_1<d_0/4$, letting $\bar{\alpha}=\min\{\alpha_{\delta_1},\sigma\}$ and fixing $d=d_0/4:=d_1$ in Lemma 3.5. To prove the next lemma, we use the idea developed in [25]. However, for constructing multi-peak solutions, we give a proof which is

slightly different from the one given in [25], where only the single-peak solution was considered.

Lemma 3.7 $\exists\bar{\varepsilon} > 0$ *such that for each* $\varepsilon \in (0, \bar{\varepsilon}]$, *there exists a sequence* $\{v_{n,\varepsilon}\}_{n=1}^{\infty} \subset J_{\varepsilon}^{\tilde{c}_{\varepsilon}+\varepsilon} \cap X_{\varepsilon}^{d_0}$ *such that* $J'_{\varepsilon}(v_{n,\varepsilon}) \to 0$ *in* $(H_{\varepsilon})^{-1}$ *as* $n \to \infty$.

Proof Assuming on the contrary that there always exist $\varepsilon > 0$ small and $\gamma(\varepsilon) > 0$ such that

$$\|J'_{\varepsilon}(u)\|_{(H_{\varepsilon})^{-1}} \geq \gamma(\varepsilon) > 0 \tag{3.34}$$

for $u \in J_{\varepsilon}^{\tilde{c}_{\varepsilon}+\varepsilon} \cap X_{\varepsilon}^{d_0}$.

Letting Y be a pseudo-gradient vector field for J'_{ε} in H_{ε}, that is, $H_{\varepsilon} \to H_{\varepsilon}$ is a locally Lipschitz continuous vector field such that for every $u \in H_{\varepsilon}$,

$$\|Y(u)\|_{H_{\varepsilon}} \leq 2\|J'_{\varepsilon}(u)\|_{(H_{\varepsilon})^{-1}}, \tag{3.35}$$

$$\langle J'_{\varepsilon}(u), Y(u)\rangle \geq \|J'_{\varepsilon}(u)\|^2_{(H_{\varepsilon})^{-1}}. \tag{3.36}$$

Letting ψ_1, ψ_2 be locally Lipschitz continuous functions in H_{ε} such that $0 \leq \psi_1, \psi_2 \leq 1$ and

$$\psi_1(u) = \begin{cases} 1, \ \sum\limits_{k=1}^{K} c_{\beta,m_k} - \dfrac{1}{2}\bar{\alpha} \leq J_{\varepsilon}(u) \leq \tilde{c}_{\varepsilon}, \\ 0, \ J_{\varepsilon}(u) \leq \sum\limits_{k=1}^{K} c_{\beta,m_k} - \bar{\alpha} \text{ or } \tilde{c}_{\varepsilon} + \varepsilon \leq J_{\varepsilon}(u), \end{cases}$$

$$\psi_2(u) = \begin{cases} 1, \ u \in X_{\varepsilon}^{3d_0/4}, \\ 0, \ u \notin X_{\varepsilon}^{d_0}. \end{cases}$$

Considering the following ordinary differential equations:

$$\begin{cases} \dfrac{d}{dr}\eta(r,u) = -\dfrac{Y(\eta(r,u))}{\|Y(\eta(r,u))\|_{H_{\varepsilon}}}\psi_1(\eta(r,u))\psi_2(\eta(r,u)), \\ \eta(0,u) = u. \end{cases} \tag{3.37}$$

By (3.35), (3.36) and (3.37), we have

$$\begin{aligned} &\frac{d}{dr}J_{\varepsilon}(\eta(r,u)) \\ &= \left\langle J'_{\varepsilon}(\eta(r,u)), \frac{d}{dr}\eta(r,u)\right\rangle \\ &= \left\langle J'_{\varepsilon}(\eta(r,u)), -\frac{Y(\eta(r,u))}{\|Y(\eta(r,u))\|_{H_{\varepsilon}}}\psi_1(\eta(r,u))\psi_2(\eta(r,u))\right\rangle \\ &\leq -\frac{\psi_1(\eta(r,u))\psi_2(\eta(r,u))}{\|Y(\eta(r,u))\|_{H_{\varepsilon}}}\|J'_{\varepsilon}(\eta(r,u))\|^2_{(H_{\varepsilon})^{-1}} \end{aligned}$$

$$\le -\frac{1}{2}\psi_1(\eta(r,u))\psi_2(\eta(r,u))\|J'_\varepsilon(\eta(r,u))\|_{(H_\varepsilon)^{-1}}$$

and combining with Lemma 3.2(i), Lemma 3.5, (3.34), (3.37) and the definition of ψ_1, ψ_2, it is standard to show that $\eta \in C([0,+\infty)\times H_\varepsilon, H_\varepsilon)$ and satisfies that for $\varepsilon > 0$ small,
(i) $\frac{d}{dr}J_\varepsilon(\eta(r,u)) \le 0$ for each $r \in [0,+\infty)$ and $u \in H_\varepsilon$;
(ii) $\frac{d}{dr}J_\varepsilon(\eta(r,u)) \le -\omega_{d_1}/2$ if $\eta(r,u) \in \overline{J_\varepsilon^{\tilde{c}_\varepsilon}\setminus J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\frac{1}{2}\bar{\alpha}}} \cap \overline{X_\varepsilon^{3d_0/4}\setminus X_\varepsilon^{d_0/4}}$;
(iii) $\frac{d}{dr}J_\varepsilon(\eta(r,u)) \le -\gamma(\varepsilon)/2$ if $\eta(r,u) \in \overline{J_\varepsilon^{\tilde{c}_\varepsilon}\setminus J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\frac{1}{2}\bar{\alpha}}} \cap X_\varepsilon^{3d_0/4}$;
(iv) $\eta(r,u) = u$ if $J_\varepsilon(u) \le \sum_{k=1}^K c_{\beta,m_k} - \bar{\alpha}$.

Setting $r_1 := \omega_{d_1}d_0/\gamma(\varepsilon)$ and $\xi_\varepsilon(s,t) := \eta(r_1,\gamma_\varepsilon(s,t))$, we have the following cases:

Case 1: $\gamma_\varepsilon(s,t) \in J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\bar{\alpha}}$. By (iv), we see that

$$\eta(r,\gamma_\varepsilon(s,t)) = \gamma_\varepsilon(s,t). \tag{3.38}$$

Case 2: $\gamma_\varepsilon(s,t) \notin J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\bar{\alpha}}$. By Lemma 3.6 and the definition of $\tilde{c}_\varepsilon$, we see that

$$\gamma_\varepsilon(s,t) \in \overline{J_\varepsilon^{\tilde{c}_\varepsilon}\setminus J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\bar{\alpha}}} \cap X_\varepsilon^{d_0/4}.$$

Moreover, we have

$$\eta(r,\gamma_\varepsilon(s,t)) \in X_\varepsilon^{d_0} \text{ for } r \in [0,r_1]. \tag{3.39}$$

Indeed, if not, $\exists r' \in [0,r_1]$ such that $\eta(r',\gamma_\varepsilon(s,t)) \notin X_\varepsilon^{d_0}$. Denote

$$r'' := \sup\left\{r \in [0,r'] : \eta(r,\gamma_\varepsilon(s,t)) \in X_\varepsilon^{d_0}\right\},$$

then by (3.37) and the definition of ψ_2, we see $\eta(r',\gamma_\varepsilon(s,t)) = \eta(r'',\gamma_\varepsilon(s,t)) \in X_\varepsilon^{d_0}$, which leads to a contradiction.

Next, we divide **Case 2** into the following three subcases:

Case 2.1: $\eta(r_1,\gamma_\varepsilon(s,t)) \in J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\frac{1}{2}\bar{\alpha}}$;

Case 2.2: $\eta(r_1,\gamma_\varepsilon(s,t)) \in J_\varepsilon^{\tilde{c}_\varepsilon}\setminus J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\frac{1}{2}\bar{\alpha}}$ and $\eta(r,\gamma_\varepsilon(s,t)) \notin X_\varepsilon^{3d_0/4}$ for some $r \in [0,r_1]$;

Case 2.3: $\eta(r_1,\gamma_\varepsilon(s,t)) \in J_\varepsilon^{\tilde{c}_\varepsilon}\setminus J_\varepsilon^{\sum_{k=1}^K c_{\beta,m_k}-\frac{1}{2}\bar{\alpha}}$ and $\eta(r,\gamma_\varepsilon(s,t)) \in X_\varepsilon^{3d_0/4}$ for all $r \in [0,r_1]$.

In **Case 2.2**, denote

$$r_2 := \inf\left\{r \in [0,r_1] : \eta(r,\gamma_\varepsilon(s,t)) \notin X_\varepsilon^{3d_0/4}\right\}$$

and

$$r_3 := \sup\left\{r \in [0, r_2] : \eta(r, \gamma_\varepsilon(s,t)) \in X_\varepsilon^{d_0/4}\right\},$$

then by (3.37), $r_2 - r_3 \geq \frac{1}{2}d_0$ and $\eta(r, \gamma_\varepsilon(s,t)) \in \overline{X_\varepsilon^{3d_0/4} \backslash X_\varepsilon^{d_0/4}}$ for each $r \in [r_3, r_2]$. By (i), (ii) and Lemma 3.2(i), we obtain

$$\begin{aligned}
&J_\varepsilon(\eta(r_1, \gamma_\varepsilon(s,t))) \\
&= J_\varepsilon(\gamma_\varepsilon(s,t)) + \int_0^{r_1} \frac{d}{dr} J_\varepsilon(\eta(r, \gamma_\varepsilon(s,t)))\mathrm{d}s \\
&\leq \tilde{c}_\varepsilon + \int_{r_3}^{r_2} \frac{d}{dr} J_\varepsilon(\eta(r, \gamma_\varepsilon(s,t)))\mathrm{d}s \\
&\leq \tilde{c}_\varepsilon - \frac{1}{4}\omega_{d_1} d_0 = \sum_{k=1}^{K} c_{\beta,m_k} - \frac{1}{4}\omega_{d_1} d_0 + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

In **Case 2.3**, by (iii) and the definition of r_1, we have

$$\begin{aligned}
J_\varepsilon(\eta(r_1, \gamma_\varepsilon(s,t))) &= J_\varepsilon(\gamma_\varepsilon(s,t)) + \int_0^{r_1} \frac{d}{dr} J_\varepsilon(\eta(r, \gamma_\varepsilon(s,t)))\mathrm{d}s \\
&\leq \tilde{c}_\varepsilon - \frac{1}{2}\omega_{d_1} d_0 = \sum_{k=1}^{K} c_{\beta,m_k} - \frac{1}{2}\omega_{d_1} d_0 + o(1).
\end{aligned}$$

To sum up, choosing $\bar{\mu} = \min\left\{\bar{\alpha}/2, \omega_{d_1} d_0/4\right\} > 0$, we see that, for $(s,t) \in [0,1]^K$,

$$J_\varepsilon(\xi_\varepsilon(s,t)) \leq \sum_{k=1}^{K} c_{\beta,m_k} - \bar{\mu} + o(1). \tag{3.40}$$

From (3.38) and (3.39), we have

$$\|\xi_\varepsilon(s,t)\|_{H_\varepsilon} \leq C \text{ for } \varepsilon > 0 \text{ small and } (s,t) \in [0,1]^K. \tag{3.41}$$

Letting $k_\varepsilon \in \mathbb{N}$ such that $k_\varepsilon^2 \leq \delta_0/(5\varepsilon)$, $k_\varepsilon \to \infty$,and putting

$$\tilde{A}_{j,\varepsilon} := (\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon + 5(j+1)k_\varepsilon} \backslash (\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon + 5jk_\varepsilon},\ j = 0, 1, \ldots, k_\varepsilon - 1.$$

By (3.41), we see that

$$\sum_{j=0}^{k_\varepsilon - 1} \int_{\tilde{A}_{j,\varepsilon}} |\Delta\xi_\varepsilon(s,t)|^2 + \beta|\nabla\xi_\varepsilon(s,t)|^2 + V(\varepsilon x)|\xi_\varepsilon(s,t)|^2 \leq C.$$

Thus, there exists a $j_\varepsilon \in \{0, 1, \ldots, k_\varepsilon - 1\}$ such that

$$\int_{\tilde{A}_{j_\varepsilon,\varepsilon}} |\Delta\xi_\varepsilon(s,t)|^2 + \beta|\nabla\xi_\varepsilon(s,t)|^2 + V(\varepsilon x)|\xi_\varepsilon(s,t)|^2 \leq C/k_\varepsilon \to 0 \tag{3.42}$$

uniformly for $(s,t) \in [0,1]^K$. Choosing cut-off functions $\zeta_{\varepsilon,1}$ and $\zeta_{\varepsilon,2}$ such that

$$\zeta_{\varepsilon,1}(x) = \begin{cases} 1, \ \text{if } x \in (\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon+(5j_\varepsilon+1)k_\varepsilon}, \\ 0, \ \text{if } x \in \mathbb{R}^N \backslash (\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon+(5j_\varepsilon+2)k_\varepsilon}, \end{cases}$$

$$\zeta_{\varepsilon,2}(x) = \begin{cases} 0, \ \text{if } x \in (\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon+(5j_\varepsilon+3)k_\varepsilon}, \\ 1, \ \text{if } x \in \mathbb{R}^N \backslash (\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon+(5j_\varepsilon+4)k_\varepsilon} \end{cases}$$

and $\xi_{\varepsilon,i}(s,t) := \zeta_{\varepsilon,i}\xi_\varepsilon(s,t)$, $i = 1, 2$. By (3.42), we have

$$\left\|\xi_\varepsilon(s,t) - \xi_{\varepsilon,1}(s,t) - \xi_{\varepsilon,2}(s,t)\right\|_{H_\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \tag{3.43}$$

uniformly for $(s,t) \in [0,1]^K$. (3.43) implies that

$$J_\varepsilon(\xi_\varepsilon(s,t)) \geq J_\varepsilon(\xi_{\varepsilon,1}(s,t)) + J_\varepsilon(\xi_{\varepsilon,2}(s,t)) + o(1). \tag{3.44}$$

In **Case 1**, by (3.38), $\xi_{\varepsilon,2}(s,t) = \zeta_{\varepsilon,2}\xi_\varepsilon(s,t) = 0$. In **Case 2**, by (3.39),

$$\left\|\xi_{\varepsilon,2}(s,t)\right\|_{H_\varepsilon} = \left\|\zeta_{\varepsilon,2}\xi_\varepsilon(s,t)\right\|_{H_\varepsilon} \leq C\|\xi_\varepsilon(s,t)\|_{H_\varepsilon(\mathbb{R}^N\backslash(\tilde{\Lambda}/\varepsilon)^{2\delta_0/\varepsilon})} \leq Cd_0.$$

Choosing $d_0 > 0$ small, we see from Sobolev's imbedding theorem that

$$J_\varepsilon(\xi_{\varepsilon,2}(s,t)) \geq \left\|\xi_{\varepsilon,2}(s,t)\right\|^2_{H_\varepsilon}\left(\frac{1}{2} - Cd_0^{p-2}\right) \geq 0.$$

No matter which case occurs, we always have

$$J_\varepsilon(\xi_\varepsilon(s,t)) \geq J_\varepsilon(\xi_{\varepsilon,1}(s,t)) + o(1). \tag{3.45}$$

Next, defining $\xi^k_{\varepsilon,1}(s,t)(x) = \xi_{\varepsilon,1}(s,t)(x)$ for $x \in (\widetilde{\Lambda^k}/\varepsilon)^{3\delta_0/\varepsilon}$, $\xi^k_{\varepsilon,1}(s,t)(x) = 0$ for $x \notin (\widetilde{\Lambda^k}/\varepsilon)^{3\delta_0/\varepsilon}$ for each $1 \leq k \leq K$. Arguing as in the proof of Lemma 3.2(i), we get

$$J_\varepsilon(\xi_{\varepsilon,1}(s,t)) \geq \sum_{k=1}^{K} J_\varepsilon(\xi^k_{\varepsilon,1}(s,t)) + o(1) = \sum_{k=1}^{K} J^k_\varepsilon(\xi^k_{\varepsilon,1}(s,t)) + o(1). \tag{3.46}$$

Next, we introduce some notations as in [16]. For $(s,t) \in [0,1]^K$, let

$$0_{s_i} = (s_1, .., s_{i-1}, 0, s_{i+1}, .., s_{K_1}, t_1, .., t_{K_2})$$

$$\text{and } 1_{s_i} = (s_1, ., s_{i-1}, 1, s_{i+1}, ., s_{K_1}, t_1, ., t_{K_2}).$$

Similarly, we can also define 0_{t_j} and 1_{t_j}. We see from Lemma 3.2(ii) and (iv) in the proof of Lemma 3.7 that $\xi_\varepsilon(0_{s_i}) = \gamma_\varepsilon(0_{s_i})$, $\xi_\varepsilon(0_{t_j}) = \gamma_\varepsilon(0_{t_j})$ and $\xi_\varepsilon(1_{s_i}) = \gamma_\varepsilon(1_{s_i})$, $\xi_\varepsilon(1_{t_j}) = \gamma_\varepsilon(1_{t_j})$. By the definition of $\xi^k_{\varepsilon,1}(s,t)$, we see that $J^{p(i)}_\varepsilon(\xi^{p(i)}_{\varepsilon,1}(0_{s_i})) = J^{p(i)}_\varepsilon(0) = 0$, $J^{q(j)}_\varepsilon(\xi^{q(j)}_{\varepsilon,1}(0_{t_j})) = J^{q(j)}_\varepsilon(0) = 0$ and $J^{p(i)}_\varepsilon(\xi^{p(i)}_{\varepsilon,1}(1_{s_i})) = J^{p(i)}_\varepsilon(U^i_{\varepsilon,S_i}) < 0$, $J^{q(j)}_\varepsilon(\xi^{q(j)}_{\varepsilon,1}(1_{t_j})) = J^{q(j)}_\varepsilon(V^j_{\varepsilon,T_j}) < 0$ for $\varepsilon > 0$ small by (3.3) and (3.4). Using the celebrated gluing method due to Coti Zelati and Rabinowitz (see [16, Proposition 3.4]), there exists $(\bar{s}_\varepsilon, \bar{t}_\varepsilon) \in [0,1]^K$ such that

$$J^k_\varepsilon(\xi^k_{\varepsilon,1}(\bar{s}_\varepsilon, \bar{t}_\varepsilon)) \geq c^k_\varepsilon \text{ for each } 1 \leq k \leq K. \tag{3.47}$$

(3.45), (3.46), (3.47) and Lemma 3.3 yield

$$\max_{(s,t)\in[0,1]^K} J_\varepsilon(\xi_\varepsilon(s,t)) \geq \sum_{k=1}^K c_{\beta,m_k} + o(1),$$

which contradicts (3.40) for $\varepsilon > 0$ small. □

Proof of Theorem 1.1 By Lemma 3.7, $\exists\bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$, there exists a sequence $\{v_{n,\varepsilon}\}_{n=1}^\infty \subset J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon} \cap X_\varepsilon^{d_0}$ such that $J'_\varepsilon(v_{n,\varepsilon}) \to 0$ in $(H_\varepsilon)^{-1}$ as $n \to \infty$. By Lemma 3.1, $\exists v_\varepsilon \in J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon} \cap X_\varepsilon^{d_0}$ such that, up to a subsequence, $v_{n,\varepsilon} \to v_\varepsilon$ in H_ε and v_ε satisfies

$$\Delta^2 v_\varepsilon - \beta\Delta v_\varepsilon + V(\varepsilon x)v_\varepsilon = g_\varepsilon(x, v_\varepsilon) \text{ in } \mathbb{R}^N. \tag{3.48}$$

Since $c_{\beta,m_k} > 0 (1 \leq k \leq K)$, we see that $0 \notin X_\varepsilon^{d_0}$ for $d_0 > 0$ small. Thus $v_\varepsilon \neq 0$.

For any sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \to 0$, by Lemma 3.4, there exist, up to a subsequence, $\{y^{p(i)}_{\varepsilon_n}\}_{n=1}^\infty \subset \mathbb{R}^N$, $z^{p(i)} \in \mathcal{M}^{p(i)}$, $U^i \in S^+_{\beta,m_{p(i)}}$ $(1 \leq i \leq K_1)$ and $\{y^{q(j)}_{\varepsilon_n}\}_{n=1}^\infty \subset \mathbb{R}^N$, $z^{q(j)} \in \mathcal{M}^{q(j)}$, $V^j \in S^-_{\beta,m_{q(j)}}$ $(1 \leq j \leq K_2)$ such that as $n \to \infty$,

$$|\varepsilon_n y^{p(i)}_{\varepsilon_n} - z^{p(i)}| \to 0, \ |\varepsilon_n y^{q(j)}_{\varepsilon_n} - z^{q(j)}| \to 0 \tag{3.49}$$

and

$$\left\| v_{\varepsilon_n} - \sum_{i=1}^{K_1} \varphi(\varepsilon_n x - \varepsilon_n y^{p(i)}_{\varepsilon_n}) U^i(x - y^{p(i)}_{\varepsilon_n}) \right. \\ \left. - \sum_{j=1}^{K_2} \varphi(\varepsilon_n x - \varepsilon_n y^{q(j)}_{\varepsilon_n}) V^j(x - y^{q(j)}_{\varepsilon_n}) \right\|_{H_{\varepsilon_n}} \to 0. \tag{3.50}$$

For each $R > 0$, we have

$$\begin{aligned}
&\left\|v_{\varepsilon_n}\right\|_{L^2(\mathbb{R}^N\setminus \overset{K}{\underset{k=1}{\cup}} B_R(y^k_{\varepsilon_n}))}\\
&\leq \left\|v_{\varepsilon_n} - \sum_{i=1}^{K_1} \varphi(\varepsilon_n x - \varepsilon_n y^{p(i)}_{\varepsilon_n}) U^i(x - y^{p(i)}_{\varepsilon_n})\right.\\
&\quad \left. - \sum_{j=1}^{K_2} \varphi(\varepsilon_n x - \varepsilon_n y^{q(j)}_{\varepsilon_n}) V^j(x - y^{q(j)}_{\varepsilon_n})\right\|_{L^2(\mathbb{R}^N)}\\
&\quad + \sum_{i=1}^{K_1} \left\|U^i\right\|_{L^2(\mathbb{R}^N\setminus B_R(0))} + \sum_{j=1}^{K_2} \left\|V^j\right\|_{L^2(\mathbb{R}^N\setminus B_R(0))}. \qquad (3.51)
\end{aligned}$$

On the other hand, since $v_{\varepsilon_n} \in X^{d_0}_{\varepsilon_n}$, then v_{ε_n} is bounded in $H^2(\mathbb{R}^N)$. Writing (3.48) as

$$\Delta^2 v_{\varepsilon_n} - \beta\Delta v_{\varepsilon_n} + c_0 v_{\varepsilon_n} = (c_0 - V(\varepsilon_n x))v_{\varepsilon_n} + g_{\varepsilon_n}(x, v_{\varepsilon_n}) \text{ in } \mathbb{R}^N,$$

where $c_0 > 0$ has been mentioned in (2.9). Observing that $h_n := (c_0 - V(\varepsilon_n x))v_{\varepsilon_n} + g_{\varepsilon_n}(x, v_{\varepsilon_n}) \in L^q_{\text{loc}}(\mathbb{R}^N)$ for $1 \leq q \leq \frac{2N}{(N-4)(p-1)}$, we deduce from Sobolev's imbedding theorem and classical bootstrap technique based on the local $W^{4,p}$-estimates for fourth-order semilinear elliptic equations (Proposition 2.3) that $v_{\varepsilon_n} \in W^{4,q}_{\text{loc}}(\mathbb{R}^N)$ for every $q \geq 1$ with a uniform estimate on unit balls. Given $q > N/4$, by Morrey's inequality, we infer that $\{v_{\varepsilon_n}\}_{n=1}^{\infty}$ is bounded in $L^\infty(\mathbb{R}^N)$. Letting $p = N$ in (2.11), we see that for any $x \in \mathbb{R}^N$,

$$\left\|v_{\varepsilon_n}\right\|_{W^{4,N}(B_1(x))} \leq C\left(\|h_n\|_{L^N(B_2(x))} + \left\|v_{\varepsilon_n}\right\|_{L^N(B_2(x))}\right) \leq C\left\|v_{\varepsilon_n}\right\|^{2/N}_{L^2(B_2(x))},$$

by Morrey's inequality,

$$\left\|v_{\varepsilon_n}\right\|_{L^\infty(B_1(x))} \leq C\left\|v_{\varepsilon_n}\right\|^{2/N}_{L^2(B_2(x))}, \qquad (3.52)$$

where $C > 0$ depends only on N. We obtain from (3.50), (3.51) and (3.52) that for any $\delta > 0$, there exists $R_\delta > 0$ such that

$$|v_{\varepsilon_n}(x)| < \delta \text{ uniformly for } x \in \mathbb{R}^N\setminus \overset{K}{\underset{k=1}{\cup}} B_{R_\delta}(y^k_{\varepsilon_n}) \text{ and } \varepsilon_n > 0 \text{ small.} \qquad (3.53)$$

Choosing $\delta = a$ in (3.53), by (3.49), we have $\cup_{k=1}^K B_{R_a}(y^k_{\varepsilon_n}) \subset \Lambda/\varepsilon_n$ for $\varepsilon_n > 0$ small. Thus, we see from the definition of g_ε that v_{ε_n} is a solution to (3.1). Moreover, by Proposition 2.3, Morrey's inequality and Schauder's estimate, we see that $v_{\varepsilon_n} \in C^4(\mathbb{R}^N)$. Therefore $u_{\varepsilon_n}(x) := v_{\varepsilon_n}(x/\varepsilon_n)$ is a classical solution to the original problem (1.1) with ε replaced by ε_n.

Since $\{v_{\varepsilon_n}\}_{n=1}^{\infty}$ is bounded in $L^\infty(\mathbb{R}^N)$, by Proposition 2.3 and Morrey's inequality, we see that for each $1 \le i \le K_1$, $1 \le j \le K_2$, $\{v_{\varepsilon_n}(x + y_{\varepsilon_n}^{p(i)})\}_{n=1}^{\infty}$ and $\{v_{\varepsilon_n}(x + y_{\varepsilon_n}^{q(j)})\}_{n=1}^{\infty}$ is bounded in $C_{\text{loc}}^{3,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. It follows from Arzelá-Ascoli's theorem and (3.50) that,

$$v_{\varepsilon_n}(x + y_{\varepsilon_n}^{p(i)}) \to U^i(x) \text{ and } v_{\varepsilon_n}(x + y_{\varepsilon_n}^{q(j)}) \to V^j(x) \text{ in } C_{\text{loc}}^3(\mathbb{R}^N) \text{ as } n \to \infty. \tag{3.54}$$

In particular,

$$v_{\varepsilon_n}(y_{\varepsilon_n}^{p(i)}) \to U^i(0) > 0 \text{ and } v_{\varepsilon_n}(y_{\varepsilon_n}^{q(j)}) \to V^j(0) < 0 \text{ as } n \to \infty. \tag{3.55}$$

Letting $x_{\varepsilon_n}^{p(i)}$ (or $x_{\varepsilon_n}^{q(j)}$) be a maximum (or minimum) point of u_{ε_n} in $\overline{\Lambda^{p(i)}}$ (or $\overline{\Lambda^{q(j)}}$), we obtain from (3.55) that for $\varepsilon_n > 0$ small,

$$u_{\varepsilon_n}(x_{\varepsilon_n}^{p(i)}) = v_{\varepsilon_n}(x_{\varepsilon_n}^{p(i)}/\varepsilon_n) \ge v_{\varepsilon_n}(y_{\varepsilon_n}^{p(i)}) \ge \frac{U^i(0)}{2} > 0 \tag{3.56}$$

and

$$u_{\varepsilon_n}(x_{\varepsilon_n}^{q(j)}) = v_{\varepsilon_n}(x_{\varepsilon_n}^{q(j)}/\varepsilon_n) \le v_{\varepsilon_n}(y_{\varepsilon_n}^{q(j)}) \le \frac{V^j(0)}{2} < 0. \tag{3.57}$$

Given $\delta = \bar{\delta} := \min\left\{\{U^i(0)/2\}_{i=1}^{K_1} \cup \{-V^j(0)/2\}_{j=1}^{K_2}\right\}$ in (3.53), then there exists $R_{\bar{\delta}} > 0$ such that $|v_{\varepsilon_n}(x)| < \bar{\delta}$ for all $x \in \mathbb{R}^N \setminus \cup_{k=1}^{K} B_{R_{\bar{\delta}}}(y_{\varepsilon_n}^k)$. Recalling (3.49), we have

$$|(x_{\varepsilon_n}^k/\varepsilon_n) - y_{\varepsilon_n}^k| \le R_{\bar{\delta}}, \tag{3.58}$$

thus $x_{\varepsilon_n}^k \to z^k \in \mathcal{M}^k$ as $n \to \infty$.

We only need to prove the uniqueness of $x_{\varepsilon_n}^{p(i)}$ and $x_{\varepsilon_n}^{q(j)}$. For each $1 \le i \le K_1$, we assume on the contrary that, up to a subsequence, u_{ε_n} possesses at least two maximum points $x_{\varepsilon_n,l}^{p(i)}$ in $\Lambda^{p(i)}$ ($l = 1, 2$). By (3.58), for each $l = 1, 2$, after passing to a subsequence, $(x_{\varepsilon_n,l}^{p(i)}/\varepsilon_n) - y_{\varepsilon_n}^{p(i)} \to P_l \in \overline{B_{R_{\bar{\delta}}}(0)}$. Let $v_{\varepsilon_n,l}(x) = u_{\varepsilon_n}(\varepsilon_n x + x_{\varepsilon_n,l}^{p(i)})$, by (3.54), we see that

$$v_{\varepsilon_n,l}(x) \rightharpoonup U^i(x + P_l) \text{ in } H^2(\mathbb{R}^N) \text{ and } v_{\varepsilon_n,l}(x) \to U^i(x + P_l) \text{ in } C_{\text{loc}}^3(\mathbb{R}^N). \tag{3.59}$$

The function U^i has a unique local maximum point at zero, it is radially symmetric and strictly decreasing as Proposition 2.1 shows, then $P_l = 0$.

Next, we claim that

$$\Delta U^i(0) < 0. \tag{3.60}$$

Suppose not, we assume that $\Delta U^i(0) = 0$. Set $W^i := -\Delta U^i + \frac{\beta}{2}U^i$, we see that (U^i, W^i) satisfies

$$\begin{cases} -\Delta U^i + \dfrac{\beta}{2}U^i - W^i = 0, \\ -\Delta W^i + \dfrac{\beta}{2}W^i + \left(m_{p(i)} - \dfrac{\beta^2}{4}\right)U^i - |U^i|^{p-2}U^i = 0. \end{cases} \tag{3.61}$$

Since $U^i > 0$ and $\frac{\beta^2}{4} \geq m_{p(i)}$, by (3.61) and strong maximum principle, $W^i > 0$. In view of Theorem 1 in [26] or *proof of Theorem 1.1 continued* in [21], we see that U^i, W^i must be radially symmetric and strictly decreasing respect to zero. Let $\varphi(r) = U^i(r) - U^i(0)$ and $\psi(r) = W^i(r) - W^i(0)$, we compute

$$\begin{aligned} \Delta\varphi(r) = \Delta U^i(r) &= \frac{\beta}{2}(\varphi(r) + U^i(0)) - (\psi(r) + W^i(0)) \\ &= \frac{\beta}{2}\varphi(r) - \psi(r) + \Delta U^i(0), \end{aligned}$$

then

$$-\Delta\varphi(r) + \frac{\beta}{2}\varphi(r) = \psi(r) \leq 0.$$

By strong maximum principle, either $\varphi = 0$ or $\varphi < 0$, which is impossible. Hence, (3.60) holds. Therefore, we can choose $r_0 > 0$ such that $(U^i)''(r) < 0$ for $0 \leq r \leq r_0$. By (3.59) and [27, Lemma 4.2], we see that

$$\frac{|x^{p(i)}_{\varepsilon_n,1} - x^{p(i)}_{\varepsilon_n,2}|}{\varepsilon_n} \geq r_0 > 0,$$

which contradicts the fact that $(x^{p(i)}_{\varepsilon_n,l}/\varepsilon_n) - y^{p(i)}_{\varepsilon_n} \to P_l = 0$. This proves the uniqueness of $x^{p(i)}_{\varepsilon_n}$. The uniqueness of $x^{q(j)}_{\varepsilon_n}$ is similar, we omit it here.

Since $\{\varepsilon_n\}_{n=1}^{\infty}$ is arbitrary, we obtain all the results in Theorem 1.1. □

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