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Expected Centre of Mass of the Random Kodaira Embedding

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Abstract
Let $X \subset \mathbb{P}^{N-1}$ be a smooth projective variety. To each $g \in SL(N, \mathbb{C})$ which induces the embedding $g \cdot X \subset \mathbb{P}^{N-1}$ given by the ambient linear action we can associate a matrix $\bar{\mu}_X(g)$ called the centre of mass, which depends nonlinearly on g. With respect to the probability measure on $SL(N, \mathbb{C})$ induced by the Haar measure and the Gaussian unitary ensemble, we prove that the expectation of the centre of mass is a constant multiple of the identity matrix for any smooth projective variety.

Keywords Kodaira embedding · Random matrices

Mathematics Subject Classification 53C55 · 60B20

1 Introduction and the Statement of the Main Result

Let X be a complex smooth projective variety, and $\iota : X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee) \cong \mathbb{P}^{N-1}$ be the Kodaira embedding defined with respect to a very ample line bundle L on X, where $N := \dim H^0(X, L)$. There is a natural $SL(N, \mathbb{C})$-action on the Kodaira embedding $\iota \mapsto g \cdot \iota$ given by the ambient linear action $SL(N, \mathbb{C}) \curvearrowright \mathbb{P}^{N-1}$. For each $g \in SL(N, \mathbb{C})$ we can define an $N \times N$ hermitian matrix $\bar{\mu}_X(g)$, called the centre of mass of the embedding $g \cdot \iota : X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee)$ (see Sect. 2.2 for more details). This plays an important role in Kähler geometry, and depends on $g \in SL(N, \mathbb{C})$ in a highly nonlinear manner. For example, when the automorphism group of (X, L) is discrete, there exists $g \in SL(N, \mathbb{C})$ such that $\bar{\mu}_X(g)$ is a constant multiple of the

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identity matrix if and only if the embedding $\iota : X \hookrightarrow \mathbb{P}^{N-1}$ is Chow stable [1,2], which is an important yet subtle algebro-geometric property of $X \subset \mathbb{P}^{N-1}$.

The following seems to be a natural question to ask.

Problem 1 Let $\mathrm{d}\sigma$ be a probability measure on $SL(N, \mathbb{C})$. Compute the expectation

$$\mathbb{E}[\bar{\mu}_X(g)] = \int_{g \in SL(N,\mathbb{C})} \bar{\mu}_X(g) \mathrm{d}\sigma.$$

In spite of its apparent simplicity, this is a nontrivial problem since $\bar{\mu}_X(g)$ depends nonlinearly on g. The main result of this paper is the following.

Theorem 1 *Let $X \subset \mathbb{P}^{N-1}$ be a smooth projective variety. With respect to the probability measure on $SL(N, \mathbb{C})$ defined by the Haar measure on $SU(N)$ and an absolutely continuous unitarily invariant measure of finite volume on $\mathcal{B} := SL(N, \mathbb{C})/SU(N)$ via the natural fibration structure, $\mathbb{E}[\bar{\mu}_X(g)]$ is a constant multiple of the identity matrix.*

See Sect. 2.1 for the details of the measure on $SL(N, \mathbb{C})$ as stated in the above, defined by the fibration $SU(N) \to SL(N, \mathbb{C}) \to \mathcal{B}$; it is also discussed therein that the measure on $SL(N, \mathbb{C})$ induced by the Gaussian unitary ensemble on $\mathcal{B}$ (Example 1) satisfies all the properties stated in the theorem. We also note that the absolute continuity of the measure on $\mathcal{B}$ is meant to be with respect to the Haar measure on $\mathcal{B}$.

The study of Kähler and Fubini–Study metrics in connection to the probability theory, such as the random matrix theory, has been an active area of research. There are works e.g. [3–7] by Berman, and [8–15] by Ferrari, Flurin, Klevtsov, Song, Zelditch. On the other hand, probabilistic aspects of the centre of mass $\bar{\mu}_X(g)$ does not seem to have been actively investigated in the aforementioned works, which is the focus of the present paper.

As pointed out in the above, whether $\bar{\mu}_X(g)$ itself is a constant multiple of the identity matrix depends on the Chow stability of $X \subset \mathbb{P}^{N-1}$ by the result of Luo [1] and Zhang [2]. Such subtleties disappear, however, when we take the average over $g \in SL(N, \mathbb{C})$ as in Theorem 1.

While the main point of Theorem 1 is that $\mathbb{E}[\bar{\mu}_X(g)]$ is a constant multiple of the identity for any smooth projective variety, it implies in particular that the expectation $\mathbb{E}[\bar{\mu}_X(g)]$ keeps being a constant multiple of the identity for the embedding $X \hookrightarrow \mathbb{P}(H^0(X, L^{\otimes k})^\vee)$ for any higher exponent $k \gg 1$. This may be interesting in the study of the large N behaviour of random Kähler metrics, initiated by Ferrari–Klevtsov–Zelditch [10]. One may hope, for example, that $\mathbb{E}[\bar{\mu}_X(g)]$ keeps being a multiple of the identity for $k \gg 1$ gives a nontrivial constraint to the large N asymptotic behaviour of their theory.

We also note that we can prove the following unitary version of Theorem 1, although the proof (given in §2.3) is much easier.

Theorem 2 *Let $X \subset \mathbb{P}^{N-1}$ be a smooth projective variety. With respect to the Haar measure $\mathrm{d}\sigma_{SU}$ on $SU(N)$, the expectation*

$$\mathbb{E}_{SU}[\bar{\mu}_X(u)] := \int_{u \in SU(N)} \bar{\mu}_X(u) \mathrm{d}\sigma_{SU}$$

of the centre of mass of $X \subset \mathbb{P}^{N-1}$ *is a constant multiple of the identity matrix.*

We can also define a variant $\bar{\mu}_{X,\nu}$ of the centre of mass, as in Definition 4, by fixing a volume form $d\nu$ on X. It turns out that Theorems 1 and 2 easily extend to this variant, as explained in Remarks 5 and 6, essentially because $\bar{\mu}_{X,\nu}(g)$ depends on $g \in SL(N, \mathbb{C})$ in a much less nonlinear manner than $\bar{\mu}_X(g)$. The author is grateful to the anonymous referee for suggesting this point to him.

Remark 1 Although we shall only treat $SL(N, \mathbb{C})$ and $SU(N)$ throughout this paper, the determinant one condition does not play any significant role. We can run exactly the same argument for $GL(N, \mathbb{C})$ and $U(N)$ to get the same results, in fact with a slightly simpler proof.

2 Preliminaries

2.1 Random Matrices

Our aim is to define a class of probability measures on $SL(N, \mathbb{C})$ which has some good properties as in the statement of Theorem 1. The precise description of such measures is given in Definition 1, but that needs to be accompanied by a review of some elementary results in the theory of random matrices; the details can be found e.g. in [16–19] or any other standard textbooks on random matrices.

Let $\mathcal{B} := SL(N, \mathbb{C})/SU(N)$ be the left coset space, which can be naturally identified with the set of all positive definite hermitian matrices (of determinant one) on $\mathbb{C}^N$, which gives $SL(N, \mathbb{C})$ a natural structure of a principal $SU(N)$-bundle

$$\begin{array}{ccc} SU(N) & \longrightarrow & SL(N, \mathbb{C}) \\ & & \Big\downarrow \pi \\ & & \mathcal{B} \end{array}$$

by the projection

$$\pi : SL(N, \mathbb{C}) \ni g \mapsto gg^* \in \mathcal{B}, \tag{1}$$

where g^* stands for the hermitian conjugate of g with respect to the hermitian form represented by the identity matrix on $\mathbb{C}^N$. Throughout, we shall write e for the identity in $SL(N, \mathbb{C})$ or $SU(N)$.

Definition 1 We set our notational convention, and the definition of the measure $\mathrm{d}\sigma$ on $SL(N, \mathbb{C})$, as follows.

– We write $\mathrm{d}\sigma_{SU}$ for the Haar measure on $SU(N)$ of unit volume.

– We fix a measure $\mathrm{d}\sigma_B$ on $\mathcal{B}$, and assume that $d\sigma_B$ is absolutely continuous, unitarily invariant, and of finite volume.
– Given a measure $\mathrm{d}\sigma_B$ on $\mathcal{B}$ and $\mathrm{d}\sigma_{SU}$ on $SU(N)$, the measure defined on $SL(N,\mathbb{C})$ via the fibration structure (1) is denoted by $\mathrm{d}\sigma$.

Given any measure $\mathrm{d}\sigma$ on $SL(N,\mathbb{C})$ as defined above, it is immediate that $\mathrm{d}\sigma$ is of finite volume (see also Lemma 1). Henceforth without loss of generality we shall assume

$$\int_{SL(N,\mathbb{C})} \mathrm{d}\sigma = 1 \tag{2}$$

by scaling, i.e. $\mathrm{d}\sigma$ is a probability measure on $SL(N,\mathbb{C})$.

We have a more explicit formula for $\mathrm{d}\sigma$, which follows immediately from the above definition.

Lemma 1 *Suppose that* $\mathrm{d}\sigma$ *is a probability measure on* $SL(N,\mathbb{C})$ *defined as in Definition 1. If* $\phi : SL(N,\mathbb{C}) \to \mathbb{R}$ *is a bounded measurable function, we have*

$$\int_{SL(N,\mathbb{C})} \phi(g)\mathrm{d}\sigma(g) = \frac{1}{\mathrm{Vol}(\mathcal{B})}\int_{\mathcal{B}} \mathrm{d}\sigma_B(hh^*)\int_{\pi^{-1}(hh^*)} \phi(hu)\mathrm{d}\sigma_{SU}(u),$$

where $\mathrm{Vol}(\mathcal{B}) := \int_{\mathcal{B}} \mathrm{d}\sigma_B$ *is the volume of* $\mathcal{B}$ *with respect to* $\mathrm{d}\sigma_B$, *and* $h \in SL(N,\mathbb{C})$ *is a hermitian matrix such that* $\pi(g) = hh^*$.

We now recall some basic facts on the Euclidean volume form (or its associated Lebesgue measure) on the $N \times N$ hermitian matrices (not necessarily positive definite or of determinant one), induced by the natural Euclidean metric. By unitarily diagonalising a hermitian matrix $\tilde{H}$ as $\tilde{H} = u^{-1}\Lambda u$ for $u \in U(N)$ and $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_N) \in \mathbb{R}^N$, we can write (see e.g. [20, §2], [17, Chap. 5], [18, Chap. 2])

$$\mathrm{d}\tilde{H} = \Delta^2(\lambda)\prod_{i=1}^{N}\mathrm{d}\lambda_i\ \mathrm{d}\sigma_U,$$

where $\mathrm{d}\sigma_U$ is the Haar measure on $U(N)$ and $\Delta^2(\lambda)$ is the square of the Vandermonde determinant

$$\Delta(\lambda) := \prod_{1\le i\neq j\le N}(\lambda_i - \lambda_j).$$

We consider the volume form on $\mathcal{B}$, which consists of positive definite hermitian matrices H of determinant one, induced by the Euclidean metric as above. Setting $\lambda_N = \prod_{i=1}^{N-1}\lambda_i^{-1}$ and carrying out the computation exactly as in [20, Sect. 2], we find

$$\mathrm{d}H = \Delta^2(\lambda)\gamma(\lambda)\prod_{i=1}^{N-1}d\lambda_i\ \mathrm{d}\sigma_{SU}, \tag{3}$$

for some smooth positive function $\gamma \in C^\infty(\mathbb{R}_{>0}^{N-1}, \mathbb{R}_{>0})$ on the $(N-1)$-fold direct product of positive real numbers $\mathbb{R}_{>0}^{N-1}$. The notation $\delta(\log\det\tilde{H})\mathrm{d}\tilde{H}$ using the delta function is also used e.g. in [10, Sect. 4.1] to denote dH as in (3).

Now returning to our original setting, we note that the measure $d\sigma_B$ on $\mathcal{B}$ being absolutely continuous means that we can write

$$\mathrm{d}\sigma_B = \rho(H)\mathrm{d}H, \tag{4}$$

where $\mathrm{d}H$ is as defined in (3) and $\rho : \mathcal{B} \to [0, +\infty)$ is a measurable function (called the Radon–Nikodym density) which is known to exist by the Radon–Nikodym theorem. Moreover, $\mathrm{d}\sigma_B$ being of finite volume implies

$$\int_{\mathcal{B}} \rho(H)\mathrm{d}H < +\infty. \tag{5}$$

Finally, $\mathrm{d}\sigma_B$ being unitarily invariant means that $d\sigma_B(H) = \mathrm{d}\sigma_B(uHu^{-1})$ for all $H \in \mathcal{B}$ and $u \in SU(N)$, which is equivalent to saying that $\rho(H)$ depends only on the eigenvalues $\lambda_1, \dots, \lambda_N$ of H (where $\lambda_N = \prod_{i=1}^{N-1}\lambda_i^{-1}$). By abuse of notation we also write $\rho(\lambda_1, \dots, \lambda_{N-1})$ for $\rho(H)$. With this notation, the finite volume condition (5) translates to

$$\int_{\mathbb{R}_{>0}^{N-1}} \rho(\lambda_1, \dots, \lambda_{N-1})\Delta^2(\lambda)\gamma(\lambda) \prod_{i=1}^{N-1} d\lambda_i < +\infty. \tag{6}$$

Example 1 An example of the measure as defined in Definition 1 can be given by the **Gaussian unitary ensemble** on $\mathcal{B}$ (or more precisely, the Gaussian unitary ensemble restricted to the set of positive definite hermitian forms $\mathcal{B}$) defined by the following Radon–Nikodym density

$$\rho(H) = \exp\left(-\frac{1}{2}\mathrm{tr}(H^2)\right).$$

Recalling (3), the Gaussian unitary ensemble $d\sigma_B$ can be written more explicitly as

$$\mathrm{d}\sigma_B = \mathrm{const.}\Delta^2(\lambda)\gamma(\lambda)\exp\left(-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2\right)\prod_{i=1}^{N-1}\mathrm{d}\lambda_i\ \mathrm{d}\sigma_{SU},$$

with $\lambda_N = \prod_{i=1}^{N-1}\lambda_i^{-1}$, up to an overall positive constant. With the Haar measure $\mathrm{d}\sigma_{SU}$ on the fibres of π, the Gaussian unitary ensemble defines a probability measure $\mathrm{d}\sigma$ on $SL(N, \mathbb{C})$ satisfying all the properties of Definition 1.

Remark 2 A well-known theorem [19, Chap. 2] in fact shows that, if $\rho(H)$ is absolutely continuous, unitarily invariant, and moreover the diagonal entries and the real and imaginary parts of the off-diagonal entries of H are statistically independent, $\rho(H)$ must be of the form $\exp(-(a\mathrm{tr}(H^2)+b\mathrm{tr}(H)+c))$ for some constants $a > 0, b, c \in \mathbb{R}$.

Example 2 Yet another example of the measure $d\sigma_B$ on $\mathcal{B}$ is given by the **heat kernel measure**, which is defined by the heat kernel on the homogeneous manifold $\mathcal{B} = SL(N, \mathbb{C})/SU(N)$. More explicitly, the heat kernel measure $d\sigma_{B,t}$, defined for each $t > 0$, can be written in terms of the Lebesgue measure dH on $\mathcal{B}$ and the eigenvalues $\lambda'_1, \ldots, \lambda'_N$ of $\log H$ (i.e. the $\mathbb{R}^N$-part of the polar coordinates on $\mathcal{B}$) as

$$d\sigma_{B,t} := \text{const.} \frac{\Delta(\lambda')}{\Delta(e^{\lambda'})} \exp\left(-\frac{1}{4t} \sum_{i=1}^{N} (\lambda'_i)^2 \right) dH,$$

up to an overall positive constant. The above measure satisfies all the properties in Definition 1 for each $t > 0$. See [21, Proposition 3.2] and [14, Sect. 3.1] for more details.

Remark 3 Klevtsov–Zelditch [13, Sect. 5] considered the measure $\exp(-\gamma S_\nu(H))dH$, where $\gamma > 0$ is a constant and S_ν is a certain functional defined on $\mathcal{B}$ with respect to a volume form ν on X, for the study of the partition function of some field theory. Interesting as it is, the unitary invariance $S_\nu(H) = S_\nu(uHu^{-1})$ (for all $u \in SU(N)$) does not seem to hold for S_ν, so Theorem 1 does not seem to apply to the case when we use $\exp(-\gamma S_\nu(H))dH$ as a measure on $\mathcal{B}$.

Remark 4 Note that the measure $d\sigma_B$ or $d\sigma$ as discussed in the above depends on the fixed hermitian form on $\mathbb{C}^N$, represented by the identity matrix. This corresponds to the choice of the reference basis $\{Z_i\}_{i=1}^N$ that we take to identify $H^0(X, L)$ with $\mathbb{C}^N$ in Sect. 2.2.

2.2 Moment Maps and the Centre of Mass

We review the ingredients from complex geometry that we need in this paper. Let X be a complex smooth projective variety of complex dimension n, with a very ample line bundle L and the associated embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X, L)^\vee)$.

We fix a basis for $H^0(X, L)$ once and for all and identify $\mathbb{P}(H^0(X, L)^\vee) \cong \mathbb{P}^{N-1}$, where $N := \dim H^0(X, L)$; we also note that the basis we fixed here can be identified with an orthonormal basis for the hermitian form represented by the identity matrix on $\mathbb{C}^N \cong H^0(X, L)$ (see also Remark 4). With respect to such a reference basis, we write $[Z_1 : \cdots : Z_N]$ for the homogeneous coordinates for $\mathbb{P}^{N-1}$. Furthermore, by abuse of terminology, we also write $\{Z_i\}_{i=1}^N$ for the reference basis itself. Pick $g \in SL(N, \mathbb{C})$ and write

$$Z_i(g) := \sum_{j=1}^{N} g_{ij} Z_j, \tag{7}$$

where g_{ij} is the matrix representation of g with respect to the basis $\{Z_i\}_{i=1}^N$. Note that $\{Z_i(g)\}_{i=1}^N$ defines a new basis for $H^0(X, L)$. Throughout, we shall write

$$H_g := (g^{-1})^* g^{-1} = (gg^*)^{-1}$$

for the positive definite hermitian matrix on $H^0(X, L)$ that has $\{Z_i(g)\}_{i=1}^N$ as its orthonormal basis. The hermitian conjugate (with respect to the basis $\{Z_i\}_{i=1}^N$) will be denoted by $*$, and the special unitary group $SU(N)$ is always meant to preserve the hermitian form H_e which has $\{Z_i\}_{i=1}^N$ as its orthonormal basis.

For each positive definite hermitian form on $\mathbb{C}^N$, it is a foundational result in complex geometry that we have a Kähler metric on $\mathbb{P}^{N-1}$ called the Fubini–Study metric (see e.g. [25, Chap. 0, Sect. 2] for more details).

Definition 2 The **Fubini–Study metric** $\tilde{\omega}_{H_e}$ on $\mathbb{P}^{N-1}$ defined by H_e is an $SU(N)$-invariant Kähler metric on $\mathbb{P}^{N-1}$, whose explicit formula on $\mathbb{C}^{N-1} = \{Z_1 \neq 0\} \subset \mathbb{P}^{N-1}$ is given by

$$\tilde{\omega}_{H_e} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(1 + \sum_{i=2}^{N} |z_i|^2 \right)$$

where $z_i := Z_i / Z_1$ for $i = 2, ldots, N$. By abuse of terminology, the restriction of $\tilde{\omega}_{H_e}$ to $\iota(X) \subset \mathbb{P}^{N-1}$ is also called the Fubini–Study metric on $\iota(X)$, and written $\omega_{H_e} := \iota^* \tilde{\omega}_{H_e}$.

While the above definition is often stated for a fixed hermitian matrix, different hermitian matrices lead to different Fubini–Study metrics; for the hermitian matrix H_g, the associated Fubini–Study metric $\tilde{\omega}_{H_g}$ can be written, on $\mathbb{C}^{N-1} = \{Z_1(g) \neq 0\} \subset \mathbb{P}^{N-1}$, as

$$\tilde{\omega}_{H_g} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(1 + \sum_{i=2}^{N} |z_i(g)|^2 \right)$$

by replacing z_i with $z_i(g) := Z_i(g)/Z_1(g)$. While the isometry group of $\tilde{\omega}_{H_g}$ is isomorphic to $SU(N)$, it is not the same $SU(N)$ that we fixed above; while the $SU(N)$ as above preserves the hermitian form H_e, in general it does not preserve H_g if $g \neq e$. Recall also that $\omega_{H_g} := \iota^* \tilde{\omega}_{H_g} \in c_1(L)$ for all $g \in SL(N, \mathbb{C})$.

From the above definition, by writing in terms of polar coordinates $z_i(g) = r_i(g) e^{\sqrt{-1}\theta_i(g)}$ we have

$$\tilde{\omega}_{H_g}^{N-1} = \frac{1}{\left(1 + \sum_{i=2}^{N-1} r_i(g)^2\right)^{N-1}} \prod_{i=2}^{N} \frac{r_i(g) dr_i(g) \wedge \mathrm{d}\theta_i(g)}{2\pi}. \tag{8}$$

Note also that the restriction of $\tilde{\omega}_{H_g}^n$ to $\iota(X)$ defines a volume form on $\iota(X)$, which we write as

$$\mathrm{d}\nu_{H_g} := \frac{\omega_{H_g}^n}{n!}.$$

The total volume of X with respect to $\mathrm{d}\nu_{H_g}$ can be computed as

$$\int_X \mathrm{d}\nu_{H_g} = \int_X c_1(L)^n/n! =: \mathrm{Vol}(X, L), \tag{9}$$

which depends only on (X, L) and is independent of $g \in SL(N, \mathbb{C})$.

Recall that ($\sqrt{-1}$ times) the moment map $\mu_{SU} : \mathbb{P}^{N-1} \to \sqrt{-1}\mathfrak{su}(N)$ for the $SU(N)$-action on $\mathbb{P}^{N-1}$ is given by

$$\mu_{SU}([x_1 : \cdots : x_N])_{ij} = \frac{x_i \bar{x}_j}{\sum_{l=1}^N |x_l|^2} - \frac{\delta_{ij}}{N},$$

where δ_{ij} is the Kronecker delta and the subscript ij stands for the (i, j)th entry of the $N \times N$ matrix. The second term δ_{ij}/N is just to make μ_{SU} trace-free. Observing that $SU(N)$ acts transitively on $\mathbb{P}^{N-1}$, we find that μ_{SU} naturally defines a map $\mu_{SU,p} : SU(N) \to \sqrt{-1}\mathfrak{su}(N)$ by $\mu_{SU,p}(u) := \mu_{SU}(up)$ where $p \in \mathbb{P}^{N-1}$ is a fixed reference point.

We now consider the "complexified" version of the above moment map, defined for $SL(N, \mathbb{C}) = SU(N)^{\mathbb{C}}$. We fix a reference point $p \in \mathbb{P}^{N-1}$ represented by the homogeneous coordinates $[Z_1 : \cdots : Z_N]$, and observe that for each $g \in SL(N, \mathbb{C})$ the point $gp \in \mathbb{P}^{N-1}$ is represented by $[Z_1(g) : \cdots : Z_N(g)]$ in terms of the notation (7). We then define an $N \times N$ hermitian matrix $\mu_p(g) \in \sqrt{-1}\mathfrak{u}(N)$ whose (i, j)th entry is given by

$$\mu_p(g)_{ij} = \frac{Z_i(g)\overline{Z_j(g)}}{\sum_{l=1}^N |Z_l(g)|^2}. \tag{10}$$

This corresponds to the first term of μ_{SU} at the point gp; note that gp is in the $SU(N)^{\mathbb{C}}$-orbit of p. We choose not to normalise the trace of $\mu_p(g)$ to be zero, to be consistent with the notation in the literature. The centre of mass, which plays an important role in this paper, is defined for $g \in SL(N, \mathbb{C})$ and the embedded variety $\iota : X \hookrightarrow \mathbb{P}^{N-1}$ as the integral

$$\bar{\mu}_X(g) := \int_{p \in \iota(X)} \mu_p(g) \mathrm{d}\nu_{H_g}. \tag{11}$$

We summarise the above in the following formal definition.

Definition 3 The **centre of mass** $\bar{\mu}_X(g)$, defined for $g \in SL(N, \mathbb{C})$ and $\iota : X \hookrightarrow \mathbb{P}^{N-1}$, is a hermitian matrix of size N whose (i, j)th entry is given in terms of the notation (7) by

$$\bar{\mu}_X(g)_{ij} := \int_{\iota(X)} \frac{Z_i(g)\overline{Z_j(g)}}{\sum_{l=1}^N |Z_l(g)|^2} \mathrm{d}\nu_{H_g},$$

where $d\nu_{H_g}$ is the measure on $\iota(X)$ defined by the Fubini–Study metric on $\mathbb{P}^{N-1}$ with respect to H_g, and integrates with respect to the variables $\{Z_i\}_{i=1}^N$ over the locus $\{[Z_1 : \cdots : Z_N] \in \iota(X)\} \subset \mathbb{P}^{N-1}$.

It is easy to see how $\mu_p(g)$ in (10) changes when g is pre-multiplied by a unitary matrix u, as in the following lemma.

Lemma 2 *For any $g \in SL(N, \mathbb{C})$, $u \in SU(N)$, and $p \in \mathbb{P}^{N-1}$, we have*

$$\mu_p(ug) = u \cdot \mu_p(g) \cdot u^*$$

Proof It is an obvious consequence of $\sum_{l=1}^N |Z_l(g)|^2 = \sum_{l=1}^N |Z_l(ug)|^2$ for any unitary matrix u. □

Note, on the other hand, that we do not have an analogous formula for $\mu_p(gu)$.

Remarks 3 We observe some other elementary properties of the centre of mass which immediately follow from the definition.

1. Both $\mu_p(g)$ and $\bar{\mu}_X(g)$ are positive definite as a hermitian matrix for each $g \in SL(N, \mathbb{C})$.
2. We observe that $\bar{\mu}_X(g)$ is nothing but the integral of $\mu_p(e)$ over $p \in g \cdot \iota(X)$ with respect to $d\nu_{H_g}$; $\bar{\mu}_X(g)$ can be regarded as the centre of mass of the Kodaira embedding $g \cdot \iota(X) \subset \mathbb{P}^{N-1}$.
3. $\bar{\mu}_X(g)$ is independent of the overall scaling of g, so depends only on its class in $PSL(N, \mathbb{C})$. Moreover, we observe that each entry of the integrand $\mu_p(g)$ of the centre of mass is manifestly bounded as a function of $g \in SL(N, \mathbb{C})$ for each $p \in \mathbb{P}^{N-1}$.

Computing the centre of mass is in general difficult since $\bar{\mu}_X(g)$ depends on $g \in SL(N, \mathbb{C})$ (and the embedding $\iota : X \hookrightarrow \mathbb{P}^{N-1}$) in a highly nonlinear manner and the size N of the matrices is typically large. However, there are some special cases in which we can explicitly compute it.

Example 3 Take $X := \mathbb{P}^{N-1}$ and $L := \mathcal{O}_{\mathbb{P}^{N-1}}(1)$. Then, by using (8) and the polar coordinates for $\mathbb{C}^{N-1}$, we find that $\bar{\mu}_{\mathbb{P}^{N-1}}(g)$ is a constant multiple of the identity matrix for all $g \in SL(N, \mathbb{C})$; this computation is well-known to the experts and reduces to the periodicity of the angle coordinates, but the details can be found e.g. in [26, Lemma 2.7]. In particular, $\mathbb{E}[\bar{\mu}_{\mathbb{P}^{N-1}}(g)]$ is a constant multiple of the identity matrix for any probability measure $d\sigma$ on $SL(N, \mathbb{C})$.

Example 4 The above method using the polar coordinates also work for the case when $\mathbb{P}^n$ is embedded in a higher dimensional projective space by the Veronese embedding, i.e. when $L = \mathcal{O}_{\mathbb{P}^n}(m)$ for $m > 1$, and $\{Z_i(g)\}_{i=1}^N$ is given by the monomial basis for $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$, where $N = \dim_{\mathbb{C}} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$. As in the previous example, $\bar{\mu}_{\mathbb{P}^n}(g)$ can be easily seen to be a diagonal matrix for $g \in SL(N, \mathbb{C})$ such that $\{Z_i(g)\}_{i=1}^N$ is a monomial basis. By appropriately scaling the monomial basis, we find that there exists $g \in SL(N, \mathbb{C})$ such that $\bar{\mu}_{\mathbb{P}^n}(g)$ is a constant multiple of the identity, and the explicit scaling can be written down as in [27, Example 2.4].

We also have a variant of the centre of mass, introduced by Donaldson [28, Sect. 2] as follows.

Definition 4 Let $\mathrm{d}\nu$ be a fixed volume form on $\iota(X)$. We define a variant $\bar{\mu}_{X,\nu}(g)$ of (11) by the following formula

$$\bar{\mu}_{X,\nu}(g) := \int_{p \in \iota(X)} \mu_p(g) \mathrm{d}\nu,$$

in which we replaced $\mathrm{d}\nu_{H_g}$ in (11) by the fixed volume form $\mathrm{d}\nu$.

As we shall see later, it is straightforward to extend the results for $\bar{\mu}_X(g)$ to the variant $\bar{\mu}_{X,\nu}(g)$; indeed, the volume form $\mathrm{d}\nu$ not depending on g means that $\bar{\mu}_{X,\nu}(g)$ depends on g in a much less nonlinear manner than $\bar{\mu}_X(g)$, and the proof turns out to be simpler.

2.3 Proof of Theorem 2

The properties of the centre of mass presented in Sect. 2.2 are sufficient for the proof of Theorem 2, which is elementary. We compute

$$\mathbb{E}_{SU}[\bar{\mu}_X(u)] := \int_{u \in SU(N)} \bar{\mu}_X(u) \mathrm{d}\sigma_{SU} = \int_{u \in SU(N)} \mathrm{d}\sigma_{SU} \int_{p \in \iota(X)} \mu_p(u) \mathrm{d}\nu_{H_u}.$$

Note first that $\mathrm{d}\nu_{H_u} = \mathrm{d}\nu_{H_e}$ for all $u \in SU(N)$ since $H_u = (uu^*)^{-1} = H_e$. Lemma 2 further implies that the above is equal to

$$\mathbb{E}_{SU}[\bar{\mu}_X(u)] = \int_{u \in SU(N)} \mathrm{d}\sigma_{SU} \left(u \cdot \int_{p \in \iota(X)} \mu_p(e) \mathrm{d}\nu_{H_e} \cdot u^* \right).$$

We pick and fix an arbitrary $\eta \in SU(N)$, and observe that the group invariance of the Haar measure implies

$$\begin{aligned}
&\int_{u \in SU(N)} \mathrm{d}\sigma_{SU}(u) \left(u \cdot \int_{p \in \iota(X)} \mu_p(e) \mathrm{d}\nu_{H_e} \cdot u^* \right) \\
&= \int_{\eta u \in SU(N)} \mathrm{d}\sigma_{SU}(\eta u) \left(\eta u \cdot \int_{p \in \iota(X)} \mu_p(e) \mathrm{d}\nu_{H_e} \cdot u^* \eta^* \right) \\
&= \int_{u \in SU(N)} \mathrm{d}\sigma_{SU}(u) \left(\eta u \cdot \int_{p \in \iota(X)} \mu_p(e) \mathrm{d}\nu_{H_e} \cdot u^* \eta^* \right) \\
&= \eta \cdot \int_{u \in SU(N)} \mathrm{d}\sigma_{SU}(u) \left(u \cdot \int_{p \in \iota(X)} \mu_p(e) \mathrm{d}\nu_{H_e} \cdot u^* \right) \cdot \eta^*,
\end{aligned}$$

which implies that we have

$$\mathbb{E}_{SU}[\bar{\mu}_X(u)] = \eta \cdot \mathbb{E}_{SU}[\bar{\mu}_X(u)] \cdot \eta^*$$

for any $\eta \in SU(N)$. Recalling that the centre of mass $\bar{\mu}_X(u)$ is an $N \times N$ hermitian matrix, this implies that $\mathbb{E}_{SU}[\bar{\mu}_X(u)]$ must be a constant multiple of the identity matrix since it is a hermitian matrix that commutes with all elements of $SU(N)$. Noting that $\mathrm{tr}(\bar{\mu}_X(u)) = \mathrm{Vol}(X, L)$ for all $u \in SU(N)$, we find more explicitly that

$$\mathbb{E}_{SU}[\bar{\mu}_X(u)] = \frac{\mathrm{Vol}(X, L)}{N} \cdot \mathrm{id}_{N\times N},$$

which completes the proof of Theorem 2.

Remark 5 Note that the above proof applies word by word to prove

$$\mathbb{E}_{SU}[\bar{\mu}_{X,\nu}(u)] = \frac{\mathrm{Vol}(X, L)}{N} \cdot \mathrm{id}_{N\times N}$$

for the variant in Definition 4, by noting that dν is fixed and remains invariant under the $SU(N)$-action.

3 Proof of Theorem 1

Observe first that the definition of the centre of mass (11) implies

$$\begin{aligned}\mathbb{E}[\bar{\mu}_X(g)] &= \int_{SL(N,\mathbb{C})} d\sigma(g) \int_{x\in\iota(X)} \mu_x(g) d\nu_{H_g}\\ &= \int_{SL(N,\mathbb{C})} d\sigma(g) \int_{x\in\iota(X)} \mu_x(g) \frac{\omega^n_{H_g}}{\omega^n_{H_e}} d\nu_{H_e},\end{aligned}$$

where μ_x is as defined in (10) and we endow $SL(N, \mathbb{C}) \times \iota(X)$ with the product measure $d\sigma \times d\nu_{H_e}$. We swap the order of the above integrals by Fubini's theorem to find

$$\mathbb{E}[\bar{\mu}_X(g)] = \int_{x\in\iota(X)} d\nu_{H_e} \int_{SL(N,\mathbb{C})} \mu_x(g) \frac{\omega^n_{H_g}(x)}{\omega^n_{H_e}(x)} d\sigma(g).$$

We first fix $x \in \iota(X)$, pick a hermitian $h \in SL(N, \mathbb{C})$ such that $\pi(g) = hh^*$, and compute the second integral in the above as

$$\begin{aligned}&\int_{SL(N,\mathbb{C})} \mu_x(g) \frac{\omega^n_{H_g}(x)}{\omega^n_{H_e}(x)} d\sigma(g)\\ &= \frac{1}{\mathrm{Vol}(\mathcal{B})} \int_{\mathcal{B}} d\sigma_B(hh^*) \int_{SU(N)} \mu_x(hu) \frac{\omega^n_{H_{hu}}(x)}{\omega^n_{H_e}(x)} d\sigma_{SU}(u)\end{aligned}$$

by using Lemma 1, where we note that each entry of $\mu_x(g)$ is bounded (Remark 3) and that $\pi^{-1}(hh^*) = h \cdot SU(N)$. Observe that we may write $h = \eta\Lambda\eta^*$ for some

$\eta \in SU(N)$ and a diagonal matrix $\Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_N)$ which we can identify with a vector in $\mathbb{R}^N$. With this notation we may write

$$\pi(g) = (hu) \cdot (hu)^* = hh^* = \eta\Lambda^2\eta^*,$$

where $u \in SU(N)$. We also note

$$\omega_{H_g} = \omega_{H_{hu}} = \iota^* \left(\sqrt{-1}\partial\bar{\partial} \log \left(\sum_{i=1}^{N} \Lambda_i^2 |Z_i(\eta^{-1}u)|^2 \right) \right),$$

which implies that $\omega^n_{H_g}(x)/\omega^n_{H_e}(x)$ is bounded over $SL(N, \mathbb{C})$, since an overall scaling of Λ leaves the above metric invariant.

Thus, by writing $\tilde{\Lambda} := \Lambda^2$, the above integral may be written as

$$\frac{1}{\mathrm{Vol}(\mathcal{B})} \int_{\tilde{\Lambda} \in \mathbb{R}^{N-1}_{>0}} \Delta^2(\tilde{\Lambda})\gamma(\tilde{\Lambda})\rho(\tilde{\Lambda})d\tilde{\Lambda} \int_{SU(N)} d\sigma_{SU}(\eta) \int_{SU(N)} \mu_x(\eta\Lambda\eta^{-1}u)\Psi_x(\Lambda, \eta^{-1}u)d\sigma_{SU}(u),$$

by (3) and (4), where we set

$$\Psi_x(\Lambda, \eta^{-1}u) := \frac{\omega^n_{H_{hu}}(x)}{\omega^n_{H_e}(x)} = \frac{\iota^* \left(\sqrt{-1}\partial\bar{\partial} \log \left(\sum_{i=1}^{N} \Lambda_i^2 |Z_i(\eta^{-1}u)|^2 \right) \right)^n (x)}{\omega^n_{H_e}(x)},$$

$\rho(\tilde{\Lambda})$ is the Radon–Nikodym density of $d\sigma_B$, and Δ, γ are as in (3). By the group invariance of the Haar measure, we have

$$\begin{aligned}
&\int_{u \in SU(N)} \mu_x(\eta\Lambda\eta^{-1}u)\Psi_x(\Lambda, \eta^{-1}u)d\sigma_{SU}(u) \\
&= \int_{\eta u \in SU(N)} \mu_x(\eta\Lambda u)\Psi_x(\Lambda, u)d\sigma_{SU}(\eta u) \\
&= \int_{u \in SU(N)} \mu_x(\eta\Lambda u)\Psi_x(\Lambda, u)d\sigma_{SU}(u) \\
&= \eta \left(\int_{u \in SU(N)} \mu_x(\Lambda u)\Psi_x(\Lambda, u)d\sigma_{SU}(u) \right) \eta^*
\end{aligned}$$

by recalling Lemma 2 and noting that

$$\Psi_x(\Lambda, u) = \frac{\iota^* \left(\sqrt{-1}\partial\bar{\partial} \log \left(\sum_{i=1}^{N} \Lambda_i^2 |Z_i(u)|^2 \right) \right)^n (x)}{\omega^n_{H_e}(x)} \tag{12}$$

does not depend on η, where the homogeneous coordinates $[Z_1 : \cdots : Z_N]$ are evaluated at $x \in \iota(X)$.

We are thus reduced to first computing

$$\int_{SU(N)} \mu_x(\Lambda u)\Psi_x(\Lambda, u)d\sigma_{SU}(u). \tag{13}$$

We claim that the off-diagonal entries of the above integral are zero. Since any $x \in \iota(X) \subset \mathbb{P}^{N-1}$ can be moved to $p_0 = [1 : 0 : \cdots : 0]$ by the $SU(N)$-action, for the moment we assume without loss of generality that $x = p_0$, by using the $SU(N)$-invariance of the Haar measure. Since p_0 is fixed by the subgroup $S(U(1)\times U(N-1))$ of $SU(N)$, the integral (13) is in fact an integral over $\mathbb{P}^{N-1} = SU(N)/S(U(1) \times U(N-1))$. We now recall that a group invariant measure on a homogeneous space (if exists) is unique up to an overall positive multiplicative constant by [30, Chapter III, §4, Theorem 1], which is a result credited to Weil in [30]. Thus, the measure on $\mathbb{P}^{N-1}$ induced by the Haar measure $d\sigma_{SU}$ agrees, up to an overall constant multiple, with the $SU(N)$-invariant Fubini–Study measure $\tilde{\omega}_{H_e}^{N-1}$. Thus, by using the homogeneous coordinate system $[Z_1 : \cdots : Z_N]$ given by the reference basis, we find that the (i, j)-th entry of (13) is equal to

$$\int_{\mathbb{P}^{N-1}} \frac{\Lambda_i \Lambda_j Z_i \bar{Z}_j}{\sum_{l=1} \Lambda_l^2 |Z_l|^2} \Psi_x(\Lambda, [Z_1 : \cdots : Z_N])\tilde{\omega}_{H_e}^{N-1} \tag{14}$$

up to an overall constant multiple, where $\Psi_x(\Lambda, [Z_1 : \cdots : Z_N])$ stands for $\Psi_x(\Lambda, u)$ with the identification given by $\mathbb{P}^{N-1} = SU(N)/S(U(1) \times U(N-1))$ as above. By recalling the formula (8) for the Fubini–Study volume form on $\mathbb{C}^{N-1} \subset \mathbb{P}^{N-1}$ and writing the above integral in terms of polar coordinates, we find that (14) is zero if $i \neq j$ because of the periodicity of the angle coordinates, by performing the computation as in [26, Lemma 2.7] (and as pointed out in Examples 3 and 4), since $\Psi_x(\Lambda, [Z_1 : \cdots : Z_N])$ does not depend on the angle coordinates as we can see from the formula (12).

Thus we find

$$\int_{SU(N)} \mu_x(\Lambda u)\Psi_x(\Lambda, u)d\sigma_{SU}(u) = \mathrm{diag}(\alpha_1(\Lambda), \ldots, \alpha_N(\Lambda))$$

for some maps $\alpha_i : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ $(i = 1, \ldots, N)$; observe that each α_i depends smoothly on Λ and is bounded over $\mathbb{R}^N$, since the (i, i)-th entry of the integrand is

$$\frac{\Lambda_i^2 |(u\tilde{x})_i|^2}{\sum_{j=1}^N \Lambda_j^2 |(u\tilde{x})_j|^2} \Psi_{u\tilde{x}}(\Lambda, e),$$

where $\tilde{x} \in \mathbb{C}^N$ is any nonzero lift (i.e. the homogeneous coordinates) of $x \in \iota(X) \subset \mathbb{P}^{N-1}$. We further observe that each α_i does not depend on $x \in \iota(X)$, since for any $x' \in \iota(X)$ there exists $u' \in SU(N)$ such that $x' = u'x$ (as $SU(N)$ acts transitively on the ambient $\mathbb{P}^{N-1}$) and hence the dependence on x is integrated out by the group invariance of the Haar measure. Moreover, the above formula and (12) imply that each

α_i can be naturally regarded as a function of $\tilde{\Lambda} = \Lambda^2$, and hence by abuse of notation we shall write $\alpha_i(\tilde{\Lambda})$ for $\alpha_i(\Lambda)$, which can be considered as a smooth bounded function on $\mathbb{R}^N_{>0}$.

Let $C_N := \{\eta_1, \dots, \eta_N\}$ be the group of cyclic permutations of N letters, which is naturally a subgroup of $U(N)$. We then find

$$\sum_{i=1}^{N} \eta_i \left(\int_{SU(N)} \mu_x(\Lambda u) \Psi_x(\Lambda, u) d\sigma_{SU}(u) \right) \eta_i^* = \alpha(\tilde{\Lambda}) \cdot \mathrm{id}_{N \times N},$$

with $\alpha(\tilde{\Lambda}) := \sum_{i=1}^{N} \alpha_i(\tilde{\Lambda})$; we also note that in the above we may assume $\eta_i \in SU(N)$ for $i = 1, \dots, N$ by dividing them by an N-th root of $\det(\eta_i) \in U(1)$ which leaves the above integral invariant. Thus we get, again by the group invariance of the Haar measure,

$$\begin{aligned}
&\int_{\eta \in SU(N)} d\sigma_{SU}(\eta) \int_{u \in SU(N)} \eta \mu_x(\Lambda u) \eta^* \Psi_x(\Lambda, u) d\sigma_{SU}(u) \\
&= \frac{1}{N} \sum_{i=1}^{N} \int_{\eta\eta_i^{-1} \in SU(N)} (\eta\eta_i^{-1}) \eta_i \\
&\qquad \left(\int_{u \in SU(N)} \mu_x(\Lambda u) \Psi_x(\Lambda, u) d\sigma_{SU}(u) \right) \eta_i^* (\eta\eta_i^{-1})^* d\sigma_{SU}(\eta\eta_i^{-1}) \\
&= \int_{\eta \in SU(N)} \eta \left(\frac{1}{N} \sum_{i=1}^{N} \eta_i \left(\int_{u \in SU(N)} \mu_x(\Lambda u) \Psi_x(\Lambda, u) d\sigma_{SU}(u) \right) \eta_i^* \right) \eta^* d\sigma_{SU}(\eta) \\
&= \frac{\alpha(\tilde{\Lambda})}{N} \cdot \mathrm{id}_{N \times N},
\end{aligned}$$

and hence, by recalling that $\alpha(\tilde{\Lambda})$ does not depend on $x \in X$ as pointed out in the above, we find

$$\begin{aligned}
\mathbb{E}[\bar{\mu}_X(g)] &= \frac{1}{\mathrm{Vol}(\mathcal{B})} \int_{x \in \iota(X)} d\nu_{H_e} \int_{\tilde{\Lambda} \in \mathbb{R}^{N-1}_{>0}} \Delta^2(\tilde{\Lambda}) \gamma(\tilde{\Lambda}) \rho(\tilde{\Lambda}) d\tilde{\Lambda} \\
&\qquad \int_{\eta \in SU(N)} d\sigma_{SU}(\eta) \int_{u \in SU(N)} \mu_x(\eta \Lambda u) \Psi_x(\Lambda, u) d\sigma_{SU}(u) \\
&= \left(\frac{\mathrm{Vol}(X, L)}{N \mathrm{Vol}(\mathcal{B})} \int_{\tilde{\Lambda} \in \mathbb{R}^{N-1}_{>0}} \alpha(\tilde{\Lambda}) \Delta^2(\tilde{\Lambda}) \gamma(\tilde{\Lambda}) \rho(\tilde{\Lambda}) d\tilde{\Lambda} \right) \cdot \mathrm{id}_{N \times N}.
\end{aligned}$$

Since $\alpha(\tilde{\Lambda})$ is bounded over $\mathbb{R}^N_{>0}$, the integral

$$\int_{\tilde{\Lambda} \in \mathbb{R}^{N-1}_{>0}} \alpha(\tilde{\Lambda}) \Delta^2(\tilde{\Lambda}) \gamma(\tilde{\Lambda}) \rho(\tilde{\Lambda}) d\tilde{\Lambda}$$

is a well-defined real number by (6) because $d\sigma_B$ is of finite volume. In fact, the above integral is equal to Vol($\mathcal{B}$), by observing

$$\mathrm{tr}(\mathbb{E}[\bar{\mu}_X(g)]) = \mathrm{Vol}(X, L)$$

since $\mathrm{tr}(\bar{\mu}_X(g)) = \mathrm{Vol}(X, L)$ for all $g \in SL(N, \mathbb{C})$ and $d\sigma$ is assumed to have unit volume as in (2). Thus we finally get

$$\mathbb{E}[\bar{\mu}_X(g)] = \frac{\mathrm{Vol}(X, L)}{N} \cdot \mathrm{id}_{N\times N}$$

as claimed, with respect to the fixed reference basis $\{Z_i\}_{i=1}^N$ (see Remark 4). This completes the proof of Theorem 1.

Remark 6 It is straightforward to extend the above proof to the variant $\bar{\mu}_{X,\nu}$ in Definition 4. First note that we have

$$\mathbb{E}[\bar{\mu}_{X,\nu}(g)] = \int_{SL(N,\mathbb{C})} d\sigma(g) \int_{x\in\iota(X)} \mu_x(g) d\nu,$$

by definition. Noting that $d\nu$ is fixed and does not depend on $g \in SL(N, \mathbb{C})$, we again apply Fubini's theorem to $SL(N, \mathbb{C}) \times \iota(X)$ with the product measure $d\sigma \times d\nu$, to find

$$\mathbb{E}[\bar{\mu}_{X,\nu}(g)] = \int_{x\in\iota(X)} d\nu \int_{SL(N,\mathbb{C})} \mu_x(g) d\sigma(g) \tag{15}$$

and repeat the argument presented above.

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