Spectral Stability of the $\bar{\partial}$ -Neumann Laplacian: Domain Perturbations

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Abstract

We study spectral stability of the $\bar{\partial}$ -Neumann Laplacian on a bounded domain in $\mathbb{C}^n$ when the underlying domain is perturbed. In particular, we establish upper semi-continuity properties for the variational eigenvalues of the $\bar{\partial}$ -Neumann Laplacian on bounded pseudoconvex domains in $\mathbb{C}^n$, lower semi-continuity properties on pseudoconvex domains that satisfy property (P) , and quantitative estimates on smooth bounded pseudoconvex domains of finite D'Angelo type in $\mathbb{C}^n$.

Keywords The $\bar{\partial}$ -Neumann Laplacian · Spectrum · Stability · Pseudoconvex domain · Property (P) · Finite type condition

Mathematics Subject Classification 32W05 · 32G05 · 35J25 · 35P15

1 Introduction

The $\bar{\partial}$ -Neumann Laplacian is a prototype of an elliptic operator with non-coercive boundary conditions ([27]). Since the fundamental work of Kohn [25] and Hörmander [24], it has been known that existence and regularity of the $\bar{\partial}$ -Neumann Laplacian are closely connected to the boundary geometry of the underlying domains (see, e.g., [1, 8, 12, 16, 34] for expositions on the subject). Spectral behavior of the $\bar{\partial}$ -Neumann Laplacian has also been shown to be sensitive to the geometry of the domains. Positivity of the $\bar{\partial}$ -Neumann can be used to characterize pseudoconvexity (see [17, 22] and references therein). Spectral discreteness of the $\bar{\partial}$ -Neumann Laplacian can be used

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to determine whether the boundary of a convex domain in $\mathbb{C}^n$ contains a complex variety ([19,20]) and whether the boundary of a smooth bounded pseudoconvex Hartogs domain in $\mathbb{C}^2$ satisfies property (P), a potential theoretic property introduced by Catlin [5] (see [9,21]). Asymptotic behavior of the eigenvalues can be used to establish whether a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ is of finite type ([17]).

In physical sciences, exact values of quantities are oftentimes difficult—in some cases, impossible—to obtain and approximate values are observed and utilized instead. It is, thus, important to study how these quantities are affected when there are small perturbations of other parameters. Spectral stability of the classical Dirichlet and Neumann Laplacians on domains in $\mathbb{R}^n$ has been studied extensively in literatures (see, e.g., [3,14,23] and references therein). In this paper, we study spectral stability of the $\bar{\partial}$ -Neumann Laplacian on a bounded domain Ω in $\mathbb{C}^n$ when the underlying domain is perturbed. There are several ways to measure spectral stability. Our focus here is on the variational eigenvalues. The k th-variational eigenvalues $\lambda_k^q(\Omega)$ of the $\bar{\partial}$ -Neumann Laplacian $\square$ on $(0, q)$ -forms ($1 \leq q \leq n-1$) on Ω are defined through the min-max principle and they are bona fide eigenvalues when the spectrum is discrete (see Sect. 2 below). We first establish the following upper semi-continuity property of the variational eigenvalues of the $\bar{\partial}$ -Neumann Laplacian on a pseudoconvex domain.

Theorem 1.1 *Let Ω_1 be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^1 -smooth boundary. Let k be a positive integer. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any pseudoconvex domain Ω_2 ,*

$$\lambda_k^q(\Omega_2) \leq \lambda_k^q(\Omega_1) + \varepsilon, \quad 1 \leq q \leq n-1, \quad (1.1)$$

provided $d_H(\Omega_1, \Omega_2) < \delta$, where d_H denotes the Hausdorff distance between the domains.

Spectral theory of the $\bar{\partial}$ -Neumann Laplacian differs substantially from that of the classical Laplacians because of the non-coercive nature of the $\bar{\partial}$ -Neumann boundary conditions. Unlike the classical Dirichlet or Neumann Laplacian, spectral discreteness of the $\bar{\partial}$ -Neumann Laplacian on a bounded domain Ω in $\mathbb{C}^n$ depends not only on the smoothness of the boundary, but more importantly on geometric and potential properties of the boundary. One difficulty in studying spectral stability of the $\bar{\partial}$ -Neumann Laplacian is due to the fact that unlike the classical Neumann Laplacian, the restriction $f|_{\widehat{\Omega}}$ of a form $f \in \text{Dom}(Q_\Omega)$, the domain of definition of the quadratic form associated with the $\bar{\partial}$ -Neuman Laplacian on Ω , need not belong to $\text{Dom}(Q_{\widehat{\Omega}})$, where $\widehat{\Omega}$ is a subdomain of Ω . Additionally, unlike the Dirichlet Laplacian, the extension of f to zero outside of Ω does not make it belong to $\text{Dom}(Q_{\widetilde{\Omega}})$ for a larger domain $\widetilde{\Omega}$. To overcome these difficulties, we decompose a form in $\text{Dom}(Q_\Omega)$ into tangential and normal components and treat them separately. Roughly speaking, the tangential component is treated as in the case of the Neumann Laplacian and the normal component is treated as in the case of the Dirichlet Laplacian.

To establish the lower semi-continuity property of the variational eigenvalues, we will have to assume that the targeted domain satisfies property (P). Property (P) is a potential theoretic property introduced by Catlin [5] to study compactness in the

$\bar{\partial}$ -Neumann problem. Kohn and Nirenberg [27] showed that compactness of the $\bar{\partial}$ -Green operator, the inverse of $\bar{\partial}$ -Neumann Laplacian, implies exact global regularity of the $\bar{\partial}$ -Neumann Laplacian. (Compactness of the $\bar{\partial}$ -Green operator is equivalent to spectral discreteness of the $\bar{\partial}$ -Neumann Laplacian.) Catlin showed that for a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, property (P) implies compactness of the $\bar{\partial}$ -Green operator. Straube showed that Catlin's theorem holds without the boundary smoothness assumption ([33]). It remains an open problem whether or not the converse to Catlin's theorem is also true.

Theorem 1.2 *Let Ω_1 be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^1 -smooth boundary that satisfies Property (P_{q-1}) , $2 \leq q \leq n-1$. Let k be a positive integer. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any pseudoconvex domain Ω_2 whose $\bar{\partial}$ -Neumann Laplacian has discrete spectrum on $(0, q)$ -forms,*

$$\lambda_k^q(\Omega_2) \geq \lambda_k^q(\Omega_1) - \varepsilon \quad (1.2)$$

provided $d_H(\Omega_1, \Omega_2) < \delta$.

To establish the quantitative estimates, we further assume that the domains are of finite type. A notion of finite type was introduced by Kohn for smooth bounded pseudoconvex domains in $\mathbb{C}^2$ in connection with subellipticity of the $\bar{\partial}$ -Neumann Laplacian [26]. For domains in higher dimensions, a new finite-type notion was introduced by D'Angelo [10]: A smooth bounded domain in $\mathbb{C}^n$ is of finite type in the sense of D'Angelo if the normalized order of contact of complex analytic varieties with the boundary is finite. Catlin showed that for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, subellipticity of the $\bar{\partial}$ -Neumann Laplacian is equivalent to the finite D'Angelo type condition [4, 7]. Here we study spectral stability of the $\bar{\partial}$ -Neumann Laplacian on such domains. Our main result in this regard is as follows:

Theorem 1.3 *Let Ω be a smooth bounded pseudoconvex domain of finite D_q -type in $\mathbb{C}^n$. Let Ω_j be a family of smooth bounded pseudoconvex domains of uniform finite D_q -type in $\mathbb{C}^n$, $1 \leq q \leq n-1$. Let k be a positive integer. Then there exist constants $\delta > 0$ and $C_k > 0$ such that*

$$|\lambda_k^q(\Omega_j) - \lambda_k^q(\Omega)| \leq C_k \delta_j, \quad (1.3)$$

provided $\delta_j = d_H(\Omega, \Omega_j) < \delta$.

We refer the reader to Sect. 5 for precise definition of *uniform* finite type. Our analysis is based on Catlin's construction of bounded plurisubharmonic functions with large complex Hessians. We will also use a version of sharp Hardy inequality due to Brezis and Marcus [2] and an idea from Davies [14].

This paper is organized as follows. In Sect. 2, we recall the spectral theoretic setup of the $\bar{\partial}$ -Neumann Laplacian and relevant facts regarding the variational eigenvalues. In Sect. 3, we establish upper semi-continuity property for the variational eigenvalues of the $\bar{\partial}$ -Neumann Laplacian on bounded pseudoconvex domains in $\mathbb{C}^n$ and prove Theorem 1.1. In Sect. 4, we study lower semi-continuity of the variational eigenvalues on bounded pseudoconvex domains satisfying property (P) and establish Theorem 1.2.

In Sect. 5, we obtain quantitative estimates, including Theorem 1.3, for stability of the variational eigenvalues on pseudoconvex domains of finite type. Section 6 contains further results on convergence of the $\bar{\partial}$ -Neumann Laplacian in resolvent sense.

2 Preliminary

We first review relevant elements in general spectral theory. Let Q be a non-negative, densely defined, and closed sesquilinear form on a complex Hilbert space $\mathbb{H}$ with domain $\text{Dom}(Q)$. Then Q uniquely determines a non-negative self-adjoint operator S such that $\text{Dom}(S^{1/2}) = \text{Dom}(Q)$ and

$$Q(u, v) = \langle S^{1/2}u, S^{1/2}v \rangle$$

for all $u, v \in \text{Dom}(Q)$. Furthermore,

$$\text{Dom}(S) = \{u \in \text{Dom}(Q) \mid \exists f \in \mathbb{H}, Q(u, v) = \langle f, v \rangle, \forall v \in \text{Dom}(Q)\}.$$

(See, e.g., Theorem 4.4.2 in [13].) For any subspace $L \subset \text{Dom}(Q)$, let

$$\lambda_Q(L) = \sup\{Q(u, u) \mid u \in L, \|u\| = 1\}.$$

For any positive integer k , let

$$\lambda_k(S) = \inf\{\lambda_Q(L) \mid L \subset \text{Dom}(Q), \dim(L) = k\} \quad (2.1)$$

be the k th variational eigenvalues of S . The resolvent set $\rho(S)$ of the operator S consists of all $\lambda \in \mathbb{C}$ such that $S - \lambda I: \text{Dom}(S) \rightarrow \mathbb{H}$ is both one-to-one and onto. It follows from the closed graph theorem that this operator has a bounded inverse, the resolvent operator $R_\lambda(S) = (S - \lambda I)^{-1}: \mathbb{H} \rightarrow \text{Dom}(S)$. The spectrum $\sigma(S)$ is the complement of $\rho(S)$ in $\mathbb{C}$. It is a non-empty closed subset of $[0, \infty)$. The lowest point in the spectrum is $\lambda_1(S)$. The essential spectrum $\sigma_e(S)$ is the closed subset of $\sigma(S)$ that consists of isolated eigenvalues of infinite multiplicity and accumulation points of the spectrum. The bottom of the essential spectrum, $\inf \sigma_e(S)$, is the limit of $\lambda_k(S)$ as $k \rightarrow \infty$. The essential spectrum $\sigma_e(S)$ is empty if and only if $\lambda_k(S) \rightarrow \infty$ as $k \rightarrow \infty$. In this case, the variational eigenvalue $\lambda_k(S)$ is a *bona fide* eigenvalue of S . Indeed, it is the k th eigenvalue when the eigenvalues are arranged in increasing order and repeated according to multiplicity. One approach to measuring spectral stability of a self-adjoint operator is through study of how the variational eigenvalues vary as the operator is perturbed. The following simple lemma is well known (compare [3, Theorem 3.2]):

Lemma 2.1 *Let $S_i, i = 1, 2$, be non-negative self-adjoint operators on Hilbert spaces $\mathbb{H}_i$ with associated quadratic forms Q_i . Let $T: \text{Dom}(Q_1) \rightarrow \text{Dom}(Q_2)$ be a linear transformation from the domain of Q_1 to that of Q_2 . Let k be a positive integer.*

Suppose there exist $0 < \alpha_k < 1/(2k)$ and $\beta_k > 0$ such that for any orthonormal set $\{f_1, f_2, \dots, f_k\} \subset \text{Dom}(Q_1)$,

$$|\langle Tf_h, Tf_l \rangle_2 - \delta_{hl}| \leq \alpha_k \quad \text{and} \quad |Q_2(Tf_h, Tf_l) - Q_1(f_h, f_l)| \leq \beta_k, \quad 1 \leq h, l \leq k. \quad (2.2)$$

Then

$$\lambda_k(S_2) \leq \lambda_k(S_1) + 2k(\alpha_k \lambda_k(S_1) + \beta_k). \quad (2.3)$$

Proof Let L_k be any k -dimensional linear subspace of $\text{Dom}(Q_1)$. Let $\{f_1, f_2, \dots, f_k\}$ be an orthonormal basis for L_k . For $f = \sum_{j=1}^k a_j f_j \in L_k$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} |Q_2(Tf, Tf) - Q_1(f, f)|^2 &= \left| \sum_{h,l=1}^k (Q_2(Tf_h, Tf_l) - Q_1(f_h, f_l)) a_h \bar{a}_l \right|^2 \\ &\leq \sum_{h,l=1}^k |Q_2(Tf_h, Tf_l) - Q_1(f_h, f_l)|^2 \sum_{h,l=1}^k |a_h \bar{a}_l|^2 \\ &\leq k^2 \beta_k^2 \|f\|_1^4. \end{aligned}$$

Thus

$$Q_2(Tf, Tf) \leq Q_1(f, f) + k\beta_k \|f\|_1^2.$$

Similarly, we have

$$\|Tf\|_2^2 \geq \|f\|_1^2 - k\alpha_k \|f\|_1^2 > \frac{1}{2} \|f\|_1^2.$$

Therefore, T is one-to-one on L_k and $T(L_k)$ is a k -dimensional linear subspace of $\text{Dom}(Q_2)$. It follows from (2.1) that

$$\begin{aligned} \lambda_k(S_2) &\leq \sup \left\{ \frac{Q_2(Tf, Tf)}{\|Tf\|_2^2} \mid f \in L_k \right\} \leq \sup \left\{ \frac{Q_1(f, f) + k\beta_k \|f\|_1^2}{(1 - k\alpha_k) \|f\|_1^2} \mid f \in L_k \right\} \\ &\leq \frac{1}{1 - k\alpha_k} \lambda_{Q_1}(L_k) + \frac{k\beta_k}{1 - k\alpha_k} = \lambda_{Q_1}(L_k) + \frac{k(\alpha_k \lambda_{Q_1}(L_k) + \beta_k)}{1 - k\alpha_k} \\ &\leq \lambda_{Q_1}(L_k) + 2k(\alpha_k \lambda_{Q_1}(L_k) + \beta_k). \end{aligned} \quad (2.4)$$

Taking the infimum over all k -dimensional subspace L_k in $\text{Dom}(Q_1)$, we then obtain the desired inequality (2.3). $\square$

Remark 2.2 Condition (2.2) in Lemma 2.1 can be replaced by the following: For any k -dimensional subspace L_k of $\text{Dom}(Q_1)$ and $f \in L_k$,

$$\|Tf\|_2^2 \geq (1 - k\alpha_k)\|f\|_1^2 \quad \text{and} \quad Q_2(Tf, Tf) \leq Q_1(f, f) + k\beta_k\|f\|_1^2. \quad (2.5)$$

This is easily seen from the proof above.

We now recall a spectral theoretic setup for the $\bar{\partial}$ -Neumann Laplacian. (We refer the readers to [8, 16, 34] for an in-depth treatment on regularity theory of the $\bar{\partial}$ -Neumann Laplacian.) Let $L^2_{(0,q)}(\Omega)$ be the space of $(0, q)$ -forms with L^2 -coefficients on Ω with respect to the standard Euclidean metric. Let $\bar{\partial}_q: L^2_{(0,q)}(\Omega) \rightarrow L^2_{(0,q+1)}(\Omega)$ be the maximally defined Cauchy-Riemann operator. Thus, $\text{Dom}(\bar{\partial}_q)$, the domain of $\bar{\partial}_q$, consists of those $u \in L^2_{(0,q)}(\Omega)$ such that $\bar{\partial}_q u$, defined in the sense of distribution, is in $L^2_{(0,q+1)}(\Omega)$. That is, there exists $v \in L^2_{(0,q+1)}(\Omega)$ such that

$$\langle u, \vartheta \varphi \rangle = \langle v, \varphi \rangle$$

for all $\varphi \in \mathcal{D}_{(0,q+1)}(\Omega)$, where ϑ is the formal adjoint of $\bar{\partial}_q$ and $\mathcal{D}_{(0,q+1)}(\Omega)$ is the space of smooth $(0, q+1)$ -forms with compact support in Ω . Let $\bar{\partial}_q^*: L^2_{(0,q+1)}(\Omega) \rightarrow L^2_{(0,q)}(\Omega)$ be the adjoint of $\bar{\partial}_q$. Thus its domain is given by

$$\text{Dom}(\bar{\partial}_q^*) = \{u \in L^2_{(0,q+1)}(\Omega) \mid \exists C > 0, |\langle u, \bar{\partial}_q v \rangle| \leq C\|v\|, \forall v \in \text{Dom}(\bar{\partial}_q)\}. \quad (2.6)$$

The maximally defined $\bar{\partial}_q$ -operator can be regarded as the adjoint of the formal adjoint $\vartheta_q: L^2_{(0,q+1)}(\Omega) \rightarrow L^2_{(0,q)}(\Omega)$ whose domain $\text{Dom}(\vartheta_q) = \mathcal{D}_{(0,q+1)}(\Omega)$. The $\bar{\partial}_q^*$ -operator is then the closure of ϑ_q and it is sometimes referred to as the *minimal extension* of ϑ_q . Let $\Omega = \{z \in \mathbb{C}^n \mid \rho(z) < 0\}$ be a bounded domain with a C^1 -smooth defining function ρ such that $|\nabla \rho| = 1$ on $\partial\Omega$ and let

$$u = \sum'_{|J|=q} u_J d\bar{z}_J \in C^1_{(0,q)}(\bar{\Omega}).$$

Then $u \in \text{Dom}(\bar{\partial}_{q-1}^*)$ if and only if

$$(\bar{\partial}\rho)^* \lrcorner u = \sum'_{|K|=q-1} \left(\sum_{k=1}^n u_{kK} \frac{\partial \rho}{\partial z_k} \right) d\bar{z}_K = 0$$

on $\partial\Omega$, where

$$(\bar{\partial}\rho)^* = \sum_{j=1}^n \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}$$

is the dual $(0, 1)$ -vector field of $\bar{\partial}\rho$ and $\lrcorner$ denotes the contraction operator.

For $1 \leq q \leq n - 1$, let

$$Q_q(u, v) = \langle \bar{\partial}_q u, \bar{\partial}_q v \rangle_\Omega + \langle \bar{\partial}_{q-1}^* u, \bar{\partial}_{q-1}^* v \rangle_\Omega$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with domain $\text{Dom}(Q_q) = \text{Dom}(\bar{\partial}_q) \cap \text{Dom}(\bar{\partial}_{q-1}^*)$. The self-adjoint operator $\square_q$ associated with Q_q is called the $\bar{\partial}$ -Neumann Laplacian on $L^2_{(0,q)}(\Omega)$. Consequently, $\square_q$ is given by

$$\square_q = \bar{\partial}_{q-1} \bar{\partial}_{q-1}^* + \bar{\partial}_q^* \bar{\partial}_q$$

with

$$\text{Dom}(\square_q) = \{u \in L^2_{(0,q)}(\Omega) \mid u \in \text{Dom}(Q_q), \bar{\partial}_q u \in \text{Dom}(\bar{\partial}_q^*), \bar{\partial}_{q-1}^* u \in \text{Dom}(\bar{\partial}_{q-1})\}.$$

It is an elliptic operator with non-coercive boundary conditions [27]. We will use $\lambda_k^q(\Omega)$ to denote the k th-variational eigenvalue of the $\bar{\partial}$ -Neumann Laplacian $\square$ on $(0, q)$ -forms on Ω , defined as above by

$$\lambda_k^q(\Omega) = \inf_{\substack{L \subset \text{Dom}(Q_q) \\ \dim L = k}} \sup_{u \in L \setminus \{0\}} Q_q(u, u) / \|u\|^2, \quad (2.7)$$

where the infimum takes over all linear subspace of $\text{Dom}(Q_q)$ of dimension k . We will study spectral stability of the $\bar{\partial}$ -Neumann Laplacian as the underlying domain Ω is perturbed. There are several ways to study spectral stability of the $\bar{\partial}$ -Neumann Laplacian. In this paper, we will focus on stability of the variational eigenvalues and the convergence in resolvent sense. Let T_j and T be self-adjoint operators on a Hilbert space $\mathbb{H}$. We say T_j converges to T in norm (respectively strong) resolvent sense if for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the resolvent operator $R_\lambda(T_j) = (T_j - \lambda I)^{-1}$ converges to $R_\lambda(T) = (T - \lambda I)^{-1}$ in norm (strongly). It is well known that if T_j converges to T in norm resolvent sense, then for any $\lambda \notin \sigma(T)$, $\lambda \notin \sigma(T_j)$ for sufficiently large j , and if T_j converges to T in strong resolvent sense, then for any $\lambda \in \sigma(T)$, there exist $\lambda_j \in \sigma(T_j)$ so that $\lambda_j \rightarrow \lambda$. We refer the reader to [31, §VIII.7] for relevant material.

Perturbation of the domains will be measured by the Hausdorff distance. Recall that for two sets A and B in a metric space (X, d) , the Hausdorff distance between A and B is given by

$$\tilde{d}_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}.$$

In this paper, we will measure the closeness between two domains Ω_1 and Ω_2 in $\mathbb{C}^n$ by the Hausdorff distance between them and their complements using the Euclidean

metric. We set

$$d_H(\Omega_1, \Omega_2) = \max\{\tilde{d}_H(\Omega_1, \Omega_2), \tilde{d}_H(\Omega_1^c, \Omega_2^c)\}.$$

For $\delta > 0$, let

$$\Omega_\delta^- = \{z \in \Omega \mid \text{dist}(z, \Omega^c) > \delta\} \quad \text{and} \quad \Omega_\delta^+ = \{z \in \mathbb{C}^n \mid \text{dist}(z, \Omega) < \delta\}.$$

It is easy to see that $d_H(\Omega_1, \Omega_2) < \delta$ if and only if

$$\overline{(\Omega_2)_\delta^-} \subset \Omega_1 \subset (\Omega_2)_\delta^+ \quad \text{and} \quad \overline{(\Omega_1)_\delta^-} \subset \Omega_2 \subset (\Omega_1)_\delta^+.$$

3 Upper Semi-continuity

In this section, we establish several upper semi-continuity properties for the variational eigenvalues of the $\bar{\partial}$ -Neumann Laplacian when the underlying domain is perturbed. We first study spectral stability of the $\bar{\partial}$ -Neumann Laplacian when the underlying domain is exhausted by subdomains from inside.

We will use $Q_{q,\Omega}$ to denote the quadratic form associated with the $\bar{\partial}$ -Neumann Laplacian $\square_{q,\Omega}$ acting on $(0, q)$ -forms on Ω . Let $\Omega_2 \subset \Omega_1$ be bounded pseudoconvex domains in $\mathbb{C}^n$. Unlike the classical Neumann Laplacian, for a $(0, q)$ -form $f \in \text{Dom}(Q_{q,\Omega_1})$, its restriction to Ω_2 is no longer in $\text{Dom}(Q_{q,\Omega_2})$. The following regularization procedure was introduced by Straube [33] (compare also [30]) to overcome this difficulty: For $f \in \text{Dom}(Q_{q,\Omega_1})$, we define

$$Tf = \bar{\partial}_{q,\Omega_2}^* N_{q+1,\Omega_2}(\bar{\partial}_{q,\Omega_1} f)|_{\Omega_2} + \bar{\partial}_{q-1,\Omega_2} N_{q-1,\Omega_2}(\bar{\partial}_{q-1,\Omega_1}^* f)|_{\Omega_2}. \quad (3.1)$$

When $q = 1$, N_{0,Ω_2} is the inverse of the restriction of $\square_{0,\Omega_2}$ to the orthogonal complement $\ker(\bar{\partial}_{0,\Omega_2})^\perp = \mathcal{R}(\bar{\partial}_{0,\Omega_2}^*)$ of the kernel of $\bar{\partial}_{0,\Omega_2}$ such that $\square_{0,\Omega_2} N_{0,\Omega_2} = I - P_{0,\Omega_2}$, where P_{0,Ω_2} is the Bergman projection on Ω_2 (see [8, Theorem 4.4.3]). Hereafter, for economy of notations, we will suppress the subscripts involving q when doing this causes no confusion and instead use subscript 1 and 2 to indicate that the operators act on Ω_1 and Ω_2 , respectively. Evidently, T is a linear transformation from $\text{Dom}(Q_1)$ into $\text{Dom}(Q_2)$. In light of Lemma 2.1, in order to estimate the difference between variational eigenvalues on Ω_1 and Ω_2 , we need to compare f and Tf .

Lemma 3.1 *Let Ω_1 be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $f \in \text{Dom}(Q_1)$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any pseudoconvex domain $\Omega_2 \subset \Omega_1$ with $d_H(\Omega_1, \Omega_2) < \delta$,*

$$\|f - Tf\|_{\Omega_2} + \|\bar{\partial}(f - Tf)\|_{\Omega_2} + \|\vartheta(f - Tf)\|_{\Omega_2} < \varepsilon, \quad (3.2)$$

where ϑ is the formal adjoint of $\bar{\partial}$.

Proof Since $\mathcal{D}_{(0,q)}(\Omega_1)$ is dense in $\text{Dom}(\bar{\partial}_1^*)$ in the graph norm $\|f\|_{\Omega_1} + \|\bar{\partial}^* f\|_{\Omega_1}$, for any $0 < \varepsilon < 1$, there exists $\phi \in \mathcal{D}_{(0,q)}(\Omega_1)$ such that $\|f - \phi\|_{\Omega_1} + \|\bar{\partial}(f - \phi)\|_{\Omega_1} < \varepsilon$. We choose δ sufficiently small such that $\text{supp } \phi \subset \subset \Omega_2$. Thus $\phi \in \text{Dom}(\bar{\partial}_2^*)$. It follows that

$$\begin{aligned} Tf &= \bar{\partial}_2^* \bar{\partial}_2 N_2 f + \bar{\partial}_2 N_2 \vartheta f \\ &= f - \bar{\partial}_2 \bar{\partial}_2^* N_2 f + \bar{\partial}_2 N_2 \vartheta f \\ &= f - \bar{\partial}_2 \bar{\partial}_2^* N_2 (f - \phi) + \bar{\partial}_2 N_2 \vartheta (f - \phi). \end{aligned} \quad (3.3)$$

Here, we have used the orthogonal decomposition $u = \bar{\partial} \bar{\partial}^* N u + \bar{\partial}^* \bar{\partial} N u$ and commutative properties $N_2 \bar{\partial}_2 = \bar{\partial}_2 N_2$ on $\text{Dom}(\bar{\partial}_2)$ and $N_2 \bar{\partial}_2^* = \bar{\partial}_2^* N_2$ on $\text{Dom}(\bar{\partial}_2^*)$. Moreover, we have

$$\bar{\partial}_2 T f = \bar{\partial} f \quad (3.4)$$

and

$$\begin{aligned} \bar{\partial}_2^* T f &= \bar{\partial}_2^* \bar{\partial}_2 N_2 \vartheta f = \vartheta f - \bar{\partial}_2 \bar{\partial}_2^* N_2 \vartheta f \\ &= \vartheta f - \bar{\partial}_2 \bar{\partial}_2^* N_2 \vartheta (f - \phi). \end{aligned} \quad (3.5)$$

The desired inequality (3.2) then follows from Hörmander's L^2 -estimates for the $\bar{\partial}$ -operator which imply that $\bar{\partial}_2 N_2$ is a bounded operator whose norm is bounded from above by a constant depending only on the diameter of Ω_2 (see, e.g., [8, Theorem 4.4.1]). $\square$

We have the following upper semi-continuity property for the variational eigenvalues defined by (2.7).

Theorem 3.2 *Let Ω_1 be a bounded pseudoconvex domain in $\mathbb{C}^n$. Given $1 \leq q \leq n-1$ and $k \in \mathbb{N}$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any pseudoconvex domain $\Omega_2 \subset \Omega_1$,*

$$\lambda_k^q(\Omega_2) \leq \lambda_k^q(\Omega_1) + \varepsilon \quad (3.6)$$

provided $d_H(\Omega_1, \Omega_2) < \delta$.

Proof For any $0 < \tilde{\varepsilon} < 1$, there exists a k -dimensional subspace $L_k \subset \text{Dom}(Q_1)$ such that $\lambda_{Q_1}(L_k) = \sup\{Q_1(f, f) \mid f \in L_k, \|f\|_{\Omega_1} = 1\} \leq \lambda_k(\Omega_1) + \tilde{\varepsilon}$. (As before, we will drop the superscript q for economy of notations when doing so causes no confusion.) Consider $Q_1(\cdot, \cdot)$ as a sesquilinear form on $L_k \times L_k$. Then there exists an orthonormal basis $\{f_1, \dots, f_k\}$ of L_k such that $Q_1(f_h, f_l) = \gamma_l \delta_{hl}$, $1 \leq h, l \leq k$, and $0 \leq \gamma_1 \leq \dots \leq \gamma_k = \lambda_{Q_1}(L_k)$. Note that

$$\|\bar{\partial} f_l\|_{\Omega_1}^2 + \|\bar{\partial}^* f_l\|_{\Omega_1}^2 \leq \lambda_k(\Omega_1) + \tilde{\varepsilon}. \quad (3.7)$$

Furthermore, by choosing δ sufficiently small, we can assume that

$$\|f_l\|_{\Omega_1 \setminus \Omega_2} + \|\bar{\partial} f_l\|_{\Omega_1 \setminus \Omega_2} + \|\bar{\partial}^* f_l\|_{\Omega_1 \setminus \Omega_2} \leq \tilde{\varepsilon} \quad (3.8)$$

for all $1 \leq l \leq k$. We have

$$\begin{aligned} |\langle Tf_h, Tfi \rangle_{\Omega_2} - \delta_{hl}| &= |\langle Tf_h, Tfi \rangle_{\Omega_2} - \langle f_h, fi \rangle_{\Omega_1}| \\ &\leq |\langle Tf_h - f_h, Tfi \rangle_{\Omega_2}| + |\langle f_h, Tfi - fi \rangle_{\Omega_2}| + |\langle f_h, fi \rangle_{\Omega_1 \setminus \Omega_2}| \\ &\leq \|Tf_h - f_h\|_{\Omega_2} \|Tfi\|_{\Omega_2} + \|f_h\|_{\Omega_2} \|Tfi - fi\|_{\Omega_2} + \|f_h\|_{\Omega_1 \setminus \Omega_2} \|fi\|_{\Omega_1 \setminus \Omega_2} \\ &\leq C\tilde{\varepsilon}. \end{aligned} \quad (3.9)$$

Since

$$\begin{aligned} Q_2(Tf_h, Tfi) - Q_1(f_h, fi) &= \langle \bar{\partial}_2 Tf_h, \bar{\partial}_2 Tfi - \bar{\partial} fi \rangle_{\Omega_2} + \langle \bar{\partial}_2 Tf_h - \bar{\partial} f_h, \bar{\partial} fi \rangle_{\Omega_2} \\ &\quad - \langle \bar{\partial} f_h, \bar{\partial} fi \rangle_{\Omega_1 \setminus \Omega_2} + \langle \bar{\partial}_2^* Tf_h, \bar{\partial}_2^* Tfi - \bar{\partial}^* fi \rangle_{\Omega_2} \\ &\quad + \langle \bar{\partial}_2^* Tf_h - \bar{\partial}^* f_h, \bar{\partial}^* fi \rangle_{\Omega_2} - \langle \bar{\partial}^* f_h, \bar{\partial}^* fi \rangle_{\Omega_1 \setminus \Omega_2}, \end{aligned} \quad (3.10)$$

it follows from (3.2), (3.7), (3.8) and the Cauchy–Schwarz inequality that

$$|Q_2(Tf_h, Tfi) - Q_1(f_h, fi)| \leq C(\lambda_k^{1/2}(\Omega_1) + \tilde{\varepsilon}^{1/2})\tilde{\varepsilon}. \quad (3.11)$$

By Lemma 2.1, we have

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1) + \varepsilon,$$

provided $\tilde{\varepsilon}$ is sufficiently small. $\square$

As a direct consequence of Theorem 3.2, we have the following:

Corollary 3.3 *Let Ω , Ω_j be bounded pseudoconvex domains in $\mathbb{C}^n$ such that $\Omega_j \subset \Omega$ and $d_H(\Omega_j, \Omega) \rightarrow 0$ as $j \rightarrow \infty$. Let $1 \leq q \leq n - 1$ and $k \in \mathbb{N}$. Then*

$$\limsup_{j \rightarrow \infty} \lambda_k^q(\Omega_j) \leq \lambda_k^q(\Omega). \quad (3.12)$$

Remark 3.4 The above result for the first eigenvalues (*i.e.*, the case when $k = 1$), is implicit in the paper of Michel and Shaw (see the proof of Theorem 1.2 on p. 123 in [29]; see also the proof of Theorem 4.3.4 on p. 76 in [8]).

We now study stability of variational eigenvalues of the $\bar{\partial}$ -Neumann Laplacian on a bounded pseudoconvex domain Ω as it is encroached—not necessarily from inside—by pseudoconvex domains. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with C^1 -smooth boundary. Let $\rho(z)$ be the signed distance function such that $\rho(z) = -\text{dist}(z, \partial\Omega)$ on Ω and $\rho(z) = \text{dist}(z, \partial\Omega)$ on $\mathbb{C}^n \setminus \Omega$. Then ρ is C^1 in a neighborhood U of $\partial\Omega$ and

$|\nabla \rho(z)| = 1$ on U (see [28]). Let $z' \in \partial\Omega$ and let $U' \subset U$ be a tubular neighborhood of z' such that $|\nabla \rho(z) - \nabla \rho(z')| < 1/2$ when $z \in U'$. Denote $\vec{n}(z) = \nabla \rho(z)$ and

$$\Omega_\delta^\pm = \{z \in \mathbb{C}^n \mid \rho(z) < \pm \delta\}.$$

Shrinking U' if necessary, then for sufficiently small $\delta > 0$, we have $z - 2\delta\vec{n}(z') \in \Omega$ for any $z \in U' \cap \Omega_\delta^+$ and $z + 2\delta\vec{n}(z') \notin \Omega$ for any $z \in U' \setminus \Omega_\delta^-$. Furthermore,

$$\text{dist}(z - 2\delta\vec{n}(z'), \partial\Omega) \geq \text{dist}(z - 2\delta\vec{n}(z), \partial\Omega) - 2\delta|\vec{n}(z) - \vec{n}(z')| > 2\delta - \delta = \delta$$

for all $z \in U' \cap \Omega_\delta^+$. We choose a finite covering $\{U^l\}_{l=0}^m$ of $\bar{\Omega}$ such that U^0 is relatively compact in Ω and each U^l , $1 \leq l \leq m$, is a tubular neighborhood about some $z^l \in \partial\Omega$ constructed as above. Write $\vec{n}^l = \vec{n}(z^l)$. We then have

$$\bigcup_{l=1}^m \left\{ z - 2\delta\vec{n}^l \mid z \in U^l \cap \Omega \right\} \bigcup U^0 \subset \Omega_\delta^-$$

and

$$\bigcup_{l=1}^m \left\{ z + 2\delta\vec{n}^l \mid z \in U^l \cap \Omega \right\} \bigcup U^0 \supset \Omega_\delta^+.$$

Let $\{\psi^l\}_{l=0}^m$ be a partition of unity subordinated the covering $\{U^l, 0 \leq l \leq m\}$ such that $\text{supp } \psi^l \subset U^l$. Let $f \in \text{Dom}(Q_{q,\Omega})$. Let $\tilde{f}$ be the form obtained by extending f to 0 outside of Ω . Then $\vartheta \tilde{f} = \widetilde{\bar{\partial}_\Omega^* f} \in L^2_{(0,q-1)}(\mathbb{C}^n)$ (see [8, p. 31]). Let

$$\widehat{f}_\delta(z) = \psi^0(z)f(z) + \sum_{l=1}^m \psi^l(z)f(z - 2\delta\vec{n}^l) \quad (3.13)$$

for $z \in \Omega_\delta^+$ and let

$$\check{f}_\delta(z) = \psi^0(z)\tilde{f}(z) + \sum_{l=1}^m \psi^l(z)\tilde{f}(z + 2\delta\vec{n}^l) \quad (3.14)$$

for $z \in \Omega$. Here we use $f(z \pm 2\delta\vec{n}^l)$ to denote the form obtained by replacing the coefficient $f_J(z)$ of the form f by $f_J(z \pm 2\delta\vec{n}^l)$:

$$f(z \pm 2\delta\vec{n}^l) = \sum'_{|J|=q} f_J(z \pm 2\delta\vec{n}^l) d\bar{z}_J.$$

Notice that $\check{f}_\delta(z)$ is supported on Ω_δ^- . Roughly speaking, the form $\widehat{f}_\delta$ and $\check{f}_\delta$ are respectively the *push-out* and *push-in* of f along the normal direction by δ unit. These

constructions are used to counter the fact that the restriction of f to a subdomain does not necessarily belong to $\text{Dom}(\bar{\partial}^*)$ on the subdomain and the extension of f to zero outside of Ω does not necessarily belong to $\text{Dom}(\bar{\partial})$ on a larger domain.

Lemma 3.5 *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 -smooth boundary. Let $f \in \text{Dom}(Q_{q,\Omega})$. Then $\widehat{f}_\delta \in \text{Dom}(\bar{\partial}_{\Omega_\delta^+})$, $\check{f}_\delta \in \text{Dom}(\bar{\partial}_{\Omega_\delta^+}^*)$, and*

$$\|\widehat{f}_\delta\|_{\Omega_\delta^+} \leq C\|f\|_\Omega, \quad \|\bar{\partial}\widehat{f}_\delta\|_{\Omega_\delta^+} \leq C(\|f\|_\Omega + \|\bar{\partial}f\|_\Omega) \quad (3.15)$$

and

$$\|\check{f}_\delta\|_\Omega \leq C\|f\|_\Omega, \quad \|\bar{\partial}^*\check{f}_\delta\|_\Omega \leq C(\|f\|_\Omega + \|\bar{\partial}^*f\|_\Omega) \quad (3.16)$$

for some constant $C > 0$ independent of δ . Furthermore,

$$\|\widehat{f}_\delta - \widetilde{f}\|_{\Omega_\delta^+} + \|\bar{\partial}\widehat{f}_\delta - \bar{\partial}\widetilde{f}\|_{\Omega_\delta^+} \rightarrow 0 \quad (3.17)$$

and

$$\|\check{f}_\delta - f\|_\Omega + \|\vartheta\check{f}_\delta - \vartheta f\|_\Omega \rightarrow 0 \quad (3.18)$$

as $\delta \rightarrow 0$, where $\bar{\partial}\widetilde{f}$, as before, is the extension of $\bar{\partial}f$ to 0 outside of Ω .

Proof The first part of the lemma follows directly from the definitions of $\widehat{f}_\delta$ and $\check{f}_\delta$. Notice that $\text{supp } \check{f}_\delta \subset \Omega_\delta^-$ and $\vartheta\check{f}_\delta \in L^2_{(0,q)}(\mathbb{C}^n)$. Hence $\check{f}_\delta \in \text{Dom}(\bar{\partial}_{\Omega_\delta^+}^*)$. Since

$$\begin{aligned} & \|\widehat{f}_\delta - \widetilde{f}\|_{\Omega_\delta^+} + \|\bar{\partial}\widehat{f}_\delta - \bar{\partial}\widetilde{f}\|_{\Omega_\delta^+} \\ & \lesssim \sum_{l=1}^m \|f(z - 2\delta\vec{n}^l) - \widetilde{f}(z)\|_{\Omega_\delta^+ \cap U^l} + \sum_{l=1}^m \|\bar{\partial}f(z - 2\delta\vec{n}^l) - \bar{\partial}\widetilde{f}(z)\|_{\Omega_\delta^+ \cap U^l} \\ & \leq \sum_{l=1}^m \|\widetilde{f}(z - 2\delta\vec{n}^l) - \widetilde{f}(z)\|_{\mathbb{C}^n} + \sum_{l=1}^m \|\bar{\partial}\widetilde{f}(z - 2\delta\vec{n}^l) - \bar{\partial}\widetilde{f}(z)\|_{\mathbb{C}^n}, \end{aligned} \quad (3.19)$$

we then obtain (3.17) from the dominated convergence theorem. The proof of (3.18) is similar and is left to the reader. $\square$

Theorem 3.6 *Let Ω_1 be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^1 -smooth boundary. Let $1 \leq q \leq n-1$ and $k \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any pseudoconvex domain Ω_2 , we have*

$$\lambda_k^q(\Omega_2) \leq \lambda_k^q(\Omega_1) + \varepsilon \quad (3.20)$$

provided $d_H(\Omega_1, \Omega_2) < \delta$.

Proof Since $d_H(\Omega_1, \Omega_2) < \delta$, we have $(\Omega_1)_\delta^- \subset \Omega_2 \subset (\Omega_1)_\delta^+$. For $f \in \text{Dom}(Q_1)$, let $\widetilde{f}$ be the form obtained by extending f to 0 outside Ω_1 and let $\widehat{f}_\delta$ be the forms constructed by (3.13) as above (with Ω replaced by Ω_1). Let

$$T_{\delta} f = \bar{\partial}_2^* N_2 \bar{\partial}_2 \widehat{f}_{\delta} + \bar{\partial}_2 N_2 \vartheta \widetilde{f}. \quad (3.21)$$

Then $T_{\delta} f \in \text{Dom}(\bar{Q}_2)$. Furthermore, for any $\phi \in \text{Dom}(\bar{\partial}_2^*)$, we have

$$\begin{aligned} T_{\delta} f &= \bar{\partial}_2^* \bar{\partial}_2 N_2 \widehat{f}_{\delta} + \bar{\partial}_2 N_2 \vartheta (\widetilde{f} - \phi) + \bar{\partial}_2 \bar{\partial}_2^* N_2 \phi \\ &= \bar{\partial}_2^* \bar{\partial}_2 N_2 \widetilde{f} + \bar{\partial}_2^* \bar{\partial}_2 N_2 (\widehat{f}_{\delta} - \widetilde{f}) + \bar{\partial}_2 N_2 \vartheta (\widetilde{f} - \phi) + \bar{\partial}_2 \bar{\partial}_2^* N_2 (\phi - \widetilde{f}) + \bar{\partial}_2 \bar{\partial}_2^* N_2 \widetilde{f} \\ &= \widetilde{f} + \bar{\partial}_2^* \bar{\partial}_2 N_2 (\widehat{f}_{\delta} - \widetilde{f}) + \bar{\partial}_2 N_2 \vartheta (\widetilde{f} - \phi) + \bar{\partial}_2 \bar{\partial}_2^* N_2 (\phi - \widetilde{f}). \end{aligned} \quad (3.22)$$

Moreover,

$$\bar{\partial}_2 T_{\delta} f = \bar{\partial}_2 \bar{\partial}_2^* N_2 \bar{\partial}_2 \widehat{f}_{\delta} = \bar{\partial}_2 \widehat{f}_{\delta} \quad (3.23)$$

and

$$\bar{\partial}_2^* T_{\delta} f = \bar{\partial}_2^* \bar{\partial}_2 N_2 \vartheta \widetilde{f} = \vartheta \widetilde{f} - \bar{\partial}_2 \bar{\partial}_2^* N_2 \vartheta \widetilde{f} = \vartheta \widetilde{f} - \bar{\partial}_2 \bar{\partial}_2^* N_2 \vartheta (\widetilde{f} - \phi). \quad (3.24)$$

Let L_k be a k -dimensional subspace of $\text{Dom}(\bar{Q}_1)$ with an orthonormal basis $\{f_1, \dots, f_k\}$. For any $0 < \varepsilon < 1$, by choosing δ sufficiently small, we have that

$$\sum_{l=1}^k (\|f_l\|_{\Omega_1 \setminus \Omega_2}^2 + \|\bar{\partial} f_l\|_{\Omega_1 \setminus \Omega_2}^2 + \|\bar{\partial}^* f_l\|_{\Omega_1 \setminus \Omega_2}^2) < \varepsilon^2. \quad (3.25)$$

Since $\mathcal{D}_{(0,q)}(\Omega_1)$ is dense in $\text{Dom}(\bar{\partial}_1^*)$ in the graph norm $\|f\|_{\Omega_1} + \|\bar{\partial}^* f\|_{\Omega_1}$, there exists a $\phi_l \in \mathcal{D}_{(0,q)}(\Omega_1)$ such that

$$\sum_{l=1}^k (\|f_l - \phi_l\|_{\Omega_1}^2 + \|\bar{\partial}^* f_l - \bar{\partial}^* \phi_l\|_{\Omega_1}^2) < \varepsilon^2. \quad (3.26)$$

By choosing δ sufficiently small, we have $\text{supp } \phi_l \subset \subset \Omega_2$. Thus $\phi_l \in \text{Dom}(\bar{\partial}_2^*)$. Let

$$f = \sum_{l=1}^k c_l f_l \quad \text{and} \quad \phi = \sum_{l=1}^k c_l \phi_l.$$

It follows from the Cauchy-Schwarz inequality that

$$\|f\|_{\Omega_1 \setminus \Omega_2}^2 + \|\bar{\partial} f\|_{\Omega_1 \setminus \Omega_2}^2 + \|\bar{\partial}^* f\|_{\Omega_1 \setminus \Omega_2}^2 \leq k\varepsilon^2 \|f\|_{\Omega_1}^2. \quad (3.27)$$

From (3.22), we have

$$\|T_{\delta} f - \widetilde{f}\|_{\Omega_2} \leq \|\widehat{f}_{\delta} - \widetilde{f}\|_{\Omega_2} + \|\widetilde{f} - \phi\|_{\Omega_2} + C \|\vartheta (\widetilde{f} - \phi)\|_{\Omega_2},$$

where the constant C depending only on the diameter of Ω_2 , which can be assumed to be uniformly bounded from above. It follows from Lemma 3.5 and (3.26) that

$$\begin{aligned} |\|T_\delta f\|_{\Omega_2}^2 - \|f\|_{\Omega_1}^2| &= |\|T_\delta f\|_{\Omega_2}^2 - \|\tilde{f}\|_{\Omega_2}^2 - \|f\|_{\Omega_1 \setminus \Omega_2}^2| \\ &\leq (\|T_\delta f\|_{\Omega_2} + \|\tilde{f}\|_{\Omega_2})\|T_\delta f - \tilde{f}\|_{\Omega_2} + \|f\|_{\Omega_1 \setminus \Omega_2}^2 \quad (3.28) \\ &\leq C\varepsilon\|f\|_{\Omega_1}^2. \end{aligned}$$

From (3.23), (3.24), (3.26) and Lemma 3.5, we have

$$\|\bar{\partial} T_\delta f - \tilde{\partial} \tilde{f}\|_{\Omega_2} = \|\bar{\partial} \hat{f}_\delta - \tilde{\partial} \tilde{f}\|_{\Omega_2} \leq C\varepsilon\|f\|_{\Omega_1}$$

and

$$\|\bar{\partial}_2^* T_\delta f - \vartheta \tilde{f}\|_{\Omega_2} \leq \|\vartheta(\tilde{f} - \phi)\|_{\Omega_2} \leq C\varepsilon\|f\|_{\Omega_1}.$$

Therefore, similar to (3.28), we have

$$|Q_2(T_\delta f, T_\delta f) - Q_1(f, f)| \leq C\varepsilon\|f\|_{\Omega_1}^2.$$

By Lemma 2.1 and the subsequent remark, we then have

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1) + C\varepsilon.$$

□

As a direct consequence of Theorem 3.6, we have:

Corollary 3.7 *Let Ω, Ω_j be bounded pseudoconvex domains in $\mathbb{C}^n$ such that $\partial\Omega$ is C^1 -smooth and $d_H(\Omega_j, \Omega) \rightarrow 0$ as $j \rightarrow \infty$. Then for any $k \in \mathbb{N}$,*

$$\limsup_{j \rightarrow \infty} \lambda_k^q(\Omega_j) \leq \lambda_k^q(\Omega), \quad 1 \leq q \leq n-1. \quad (3.29)$$

The upper semi-continuity property of the variational eigenvalues also holds without the pseudoconvexity assumption when restricted to level sets.

Theorem 3.8 *Let $\Omega = \{z \in \mathbb{C}^n \mid \rho < 0\}$ be a bounded domain in $\mathbb{C}^n$ with C^2 -smooth boundary where $\rho \in C^2$ is a defining function of Ω with $|\nabla \rho| = 1$ on $\partial\Omega$. For $\delta > 0$, let $\Omega_\delta^- = \{z \in \Omega \mid \rho_\delta^- = \rho + \delta < 0\}$ and $\Omega_\delta^+ = \{z \in \mathbb{C}^n \mid \rho_\delta^+ = \rho - \delta < 0\}$. Then for any $k \in \mathbb{N}$,*

$$\limsup_{\delta \rightarrow 0^+} \lambda_k(\Omega_\delta^\pm) \leq \lambda_k(\Omega).$$

Proof Since $C_{(0,q)}^1(\overline{\Omega}) \cap \text{Dom}(\bar{\partial}^*)$ is dense in $\text{Dom}(Q_\Omega)$ in the graph norm $(\|f\|_\Omega^2 + Q_\Omega(f, f))^{1/2}$ (see, e.g., [8, Lemma 4.3.2]), it is sufficient to work on forms in

$C^1_{(0,q)}(\bar{\Omega}) \cap \text{Dom}(\bar{\partial}^*_{\Omega})$. Let $f \in C^1_{(0,q)}(\bar{\Omega}) \cap \text{Dom}(\bar{\partial}^*_{\Omega})$. For $z \in \Omega$ near the boundary, we write

$$f^{\nu} = \bar{\partial}\rho \wedge f_N / |\bar{\partial}\rho|^2, \quad f^{\tau} = f - f^{\nu},$$

where

$$f_N := (\bar{\partial}\rho)^* \lrcorner f = \frac{1}{q} \sum'_{|K|=q-1} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial z_j} f_{jK} \right) d\bar{z}_K.$$

Notice that $f_N = 0$ on $\partial\Omega$ and $(\bar{\partial}\rho)^* \lrcorner f^{\tau} = 0$ on Ω . We extend f_N to be 0 outside of Ω . Let $\{U^l\}_{l=0}^m$ be an open covering of $\bar{\Omega}$ and let $\{\psi^l\}_{l=0}^m$ be a partition of unity subordinated to the covering as in the setup preceding Lemma 3.5. Set

$$f_{\delta}^{-\nu}(z) = \psi^0(z) f_N(z) \wedge \bar{\partial}\rho + \sum_{l=1}^m \psi^l(z) f_N(z + 2\delta \bar{n}^l) \wedge \bar{\partial}\rho$$

and

$$f_{\delta}^{-\tau}(z) = f^{\tau}(z).$$

for $z \in \Omega_{\delta}^{-}$. Define $T_{\delta}^{-} f = f_{\delta}^{-\nu} + f_{\delta}^{-\tau}$. Since

$$(\bar{\partial}\rho_{\delta})^* \lrcorner T_{\delta}^{-} f = (\bar{\partial}\rho)^* \lrcorner f^{\tau} + (\bar{\partial}\rho)^* \lrcorner f_{\delta}^{-\nu} = \psi^0(z) f_N(z) + \sum_{l=1}^m \psi^l(z) f_N(z + 2\delta \bar{n}^l) = 0$$

on $\partial\Omega_{\delta}^{-}$, we have $T_{\delta}^{-} f \in \text{Dom}(Q_{\Omega_{\delta}^{-}})$. Furthermore,

$$\|T_{\delta}^{-} f - f\|_{\Omega_{\delta}^{-}} + \|\bar{\partial} T_{\delta}^{-} f - \bar{\partial} f\|_{\Omega_{\delta}^{-}} + \|\vartheta T_{\delta}^{-} f - \vartheta f\|_{\Omega_{\delta}^{-}} \rightarrow 0$$

as $\delta \rightarrow 0^{+}$. The proof for the case Ω_{δ}^{-} then follows along the same lines as in the proof of Theorem 3.6. For Ω_{δ}^{+} , we set

$$f_{\delta}^{+\nu}(z) = f^{\nu}(z)$$

and

$$f_{\delta}^{+\tau}(z) = \psi^0(z) f^{\tau}(z) + \sum_{l=1}^m \psi^l(z) f^{\tau}(z - 2\delta \bar{n}^l)$$

and define $T_{\delta}^{+} f = f_{\delta}^{+\tau} + f_{\delta}^{+\nu}$, and then proceed similarly. $\square$

4 Lower Semi-continuity and Property (P)

Property (P) was introduced by Catlin as a potential theoretic sufficient condition for compactness of the inverse of the $\bar{\partial}$ -Neumann Laplacian on bounded pseudoconvex domains in $\mathbb{C}^n$. A compact set $K \subset \mathbb{C}^n$ is said to satisfy Property (P) if for any $M > 0$, there exists a neighborhood U of K and a function $\varphi \in C^\infty(U)$ such that $0 \leq \varphi \leq 1$ and any eigenvalue of the hermitian matrix $(\partial^2 \varphi / \partial z_j \partial \bar{z}_k)_{j,k=1}^n$ is greater than or equal to M on U . It is said to satisfy Property (P_q) , $1 \leq q \leq n$, if any sum of q eigenvalues of the hermitian metric is greater than or equal to M on U . We start with the following well-known lemma.

Lemma 4.1 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $b \in C^2(\overline{\Omega})$ with $-1 \leq b \leq 0$. Then*

$$Q_\Omega(f, f) \geq \frac{1}{e} \int_\Omega H_q(b)(f) dV \quad (4.1)$$

for all $f \in \text{Dom}(Q_{q,\Omega})$, where

$$H_q(b)(f) = \sum'_{|K|=q-1} \sum_{j,k=1}^n \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} f_{jK} \bar{f}_{kK}$$

is the complex Hessian of b , acting on $(0, q)$ -forms $f = \sum'_{|J|=q} f_J d\bar{z}_J$.

Proof When $\partial\Omega$ is smooth, the above lemma is essentially due to Catlin (see (2.3) in [5]; see also (2–10) in [1]). When no boundary smoothness is assumed, the lemma was proved in [33] (see also [34, Corollary 2.13]). It can also be proved by exhausting Ω from inside by pseudoconvex domains with smooth boundaries and applying Lemma 3.1. $\square$

Lemma 4.2 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ satisfies property (P_q) . Then for any $\varepsilon > 0$, there exists a $\delta > 0$, such that for any pseudoconvex domain Ω_j with $d_H(\Omega, \Omega_j) < \delta$, we have*

$$\|f_j\|_{A_{j\delta}}^2 \leq \varepsilon^2 Q_{\Omega_j}(f_j, f_j) \quad (4.2)$$

for all $f_j \in \text{Dom}(Q_{q,\Omega_j})$, where $A_{j\sigma} = \{z \in \Omega_j \mid \text{dist}(z, \partial\Omega_j) < \sigma\}$. Furthermore, if f_j is an eigenform of $\square_{\Omega_j}$ with associated eigenvalue $\lambda(\Omega_j)$, then

$$\|\bar{\partial}_j f_j\|_{A_{j\delta}}^2 \leq \varepsilon^2 \lambda^2(\Omega_j) \|f_j\|_{\Omega_j}^2. \quad (4.3)$$

Moreover, if $\partial\Omega$ satisfies property (P_{q-1}) , then

$$\|\bar{\partial}_j^* f_j\|_{A_{j\delta}}^2 \leq \varepsilon^2 \lambda^2(\Omega_j) \|f_j\|_{\Omega_j}^2. \quad (4.4)$$

Proof For any $\varepsilon > 0$, since $\partial\Omega$ satisfies property (P_q) , there exists a neighborhood U of $\partial\Omega$ and $b \in C^\infty(U)$ with $-1 < b \leq 0$ such that

$$\sum_K' \sum_{j,k=1}^n \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} f_{jK} \bar{f}_{kK} \geq \frac{e}{\varepsilon^2} |f|^2 \quad (4.5)$$

for any $(0, q)$ -form f on U . Let $\delta = \frac{1}{2} \text{dist}(\partial\Omega, \partial U)$. If $d_H(\Omega, \Omega_j) < \delta$, then $\partial\Omega_j \subset U$ and $A_{j\delta} \subset U \cap \Omega_j$. Applying Lemma 4.1 to Ω_j and f_j , we have

$$\int_{A_{j\delta}} |f_j|^2 dV \leq \int_{\Omega_j \cap U} |f_j|^2 dV \leq \frac{\varepsilon^2}{e} \int_{\Omega_j \cap U} H_q(b)(f_j) dV \leq \varepsilon^2 Q_{\Omega_j}(f_j, f_j). \quad (4.6)$$

This concludes the proof of (4.2).

To prove (4.3) and (4.4), we first note that if f_j is an eigenform for $\square_{\Omega_j}$ associated with eigenvalue $\lambda(\Omega_j)$, then

$$\bar{\partial}_j^* \bar{\partial}_j f_j = \lambda(\Omega_j) f_j - \bar{\partial}_j \bar{\partial}_j^* f_j \in \text{Dom}(\bar{\partial}_j) \quad (4.7)$$

and

$$\bar{\partial}_j \bar{\partial}_j^* f_j = \lambda(\Omega_j) f_j - \bar{\partial}_j^* \bar{\partial}_j f_j \in \text{Dom}(\bar{\partial}_j^*). \quad (4.8)$$

Moreover,

$$\square_{\Omega_j} \bar{\partial}_j f_j = \bar{\partial}_j \square_{\Omega_j} f_j = \lambda(\Omega_j) \bar{\partial}_j f_j \quad (4.9)$$

and

$$\square_{\Omega_j} \bar{\partial}_j^* f_j = \bar{\partial}_j^* \square_{\Omega_j} f_j = \lambda(\Omega_j) \bar{\partial}_j^* f_j. \quad (4.10)$$

Since $\partial\Omega$ satisfies property (P_q) , it also satisfies property (P_{q+1}) . Therefore, applying (4.2) to $\bar{\partial}_j f_j$, we have

$$\begin{aligned} \|\bar{\partial}_j f_j\|_{A_{j\delta}}^2 &\leq \varepsilon^2 Q_{\Omega_j}(\bar{\partial}_j f_j, \bar{\partial}_j f_j) = \varepsilon^2 \lambda(\Omega_j) \|\bar{\partial}_j f_j\|_{\Omega_j}^2 \\ &\leq \varepsilon^2 \lambda(\Omega_j) Q_{\Omega_j}(f_j, f_j) = \varepsilon^2 \lambda^2(\Omega_j) \|f_j\|_{\Omega_j}^2. \end{aligned}$$

Similarly, when $\partial\Omega$ satisfies property (P_{q-1}) , we have

$$\|\bar{\partial}_j^* f_j\|_{A_{j\delta}}^2 \leq \varepsilon^2 Q_{\Omega_j}(\bar{\partial}_j^* f_j, \bar{\partial}_j^* f_j) \leq \varepsilon^2 \lambda^2(\Omega_j) \|f_j\|_{\Omega_j}^2.$$

This concludes the proof of Lemma 4.2. $\square$

Lemma 4.3 Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ that satisfies property (P_q) . Let M be a positive constant. Let $\{\Omega_j\}$ be a family of pseudoconvex domains such that $d_H(\Omega, \Omega_j) \rightarrow 0$ as $j \rightarrow \infty$. Suppose $f_j \in \text{Dom}(Q_{\Omega_j})$ is a sequence of $(0, q)$ -forms such that $\|f_j\|_{\Omega_j}^2 + Q_{\Omega_j}(f_j, f_j) \leq M$. Let $\tilde{f}_j$ be the extension of f_j to 0 outside of Ω_j . Then $\{\tilde{f}_j\}$ is a pre-compact family in $L^2_{(0,q)}(\mathbb{C}^n)$.

Proof For any $\varepsilon > 0$, it follows from (4.6) that there exist a neighborhood U of $\partial\Omega$ such that

$$\int_U |\tilde{f}_j|^2 dV = \int_{\Omega_j \cap U} |f_j|^2 dV \leq \varepsilon^2 Q_{\Omega_j}(f_j, f_j) \leq \varepsilon^2 M$$

for sufficiently large j . Let $V \subset\subset U$ be a neighborhood of $\partial\Omega$. Choosing sufficiently large j such that $(\Omega \setminus \Omega_j) \cup (\Omega_j \setminus \Omega) \subset V$. Let $\eta \in C_0^\infty(\mathbb{C}^n)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $\Omega \setminus U$ and $\text{supp } \eta \subset \Omega \setminus V$. Then there exists a constant M_1 , such that $Q_{\Omega_j}(\eta f_j, \eta f_j) \leq M_1$ and hence $\|\eta \tilde{f}_j\|_{W^1(\Omega)} \leq M_1$. (Hereafter $\|f\|_{W^\alpha}$ denotes the norm of L^2 -Sobolev space of order α .) By Rellich's compactness theorem, $\{\eta \tilde{f}_j\}$ has a subsequence $\{\eta \tilde{f}_{j_l}\}$ that converges in $L^2_{(0,q)}(\Omega)$. Thus

$$\begin{aligned} \|\tilde{f}_{j_h} - \tilde{f}_{j_l}\|_{\mathbb{C}^n} &= \|\tilde{f}_{j_h} - \tilde{f}_{j_l}\|_U + \|\tilde{f}_{j_h} - \tilde{f}_{j_l}\|_{\Omega \setminus U} \\ &\leq 2M^{1/2}\varepsilon + \|\eta \tilde{f}_{j_h} - \eta \tilde{f}_{j_l}\|_{\Omega}, \end{aligned}$$

when h and l are sufficiently large. Thus $\{\tilde{f}_{j_l}\}$ is a subsequence of $\{\tilde{f}_j\}$ that converges in $L^2_{(0,q)}(\mathbb{C}^n)$. $\square$

Remark 4.4 Let f_j be an eigenform associated with k^{th} eigenvalue $\lambda_k(\Omega_j)$ of $\square_{\Omega_j}$. From the proof of Lemma 4.2, we know that $\bar{\partial}_j f_j$ is also an eigenform of $\square_{\Omega_j}$. Moreover,

$$\|\bar{\partial}_j f_j\|_{\Omega_j}^2 + Q_{\Omega_j}(\bar{\partial}_j f_j, \bar{\partial}_j f_j) \leq \lambda_k(\Omega_j)(1 + \lambda_k(\Omega_j))\|f_j\|_{\Omega_j}^2,$$

which, by Corollary 3.3, is bounded from above by a constant independent of j . Therefore, $\{\bar{\partial}_j f_j\}$ is also a pre-compact family in $L^2_{(0,q+1)}(\mathbb{C}^n)$. Similarly, when $\partial\Omega$ satisfies property (P_{q-1}) , $\{\partial \tilde{f}_j\}$ is also a pre-compact family in $L^2_{(0,q-1)}(\mathbb{C}^n)$.

Theorem 4.5 Let Ω be a bounded pseudoconvex domains in $\mathbb{C}^n$ with C^1 -smooth boundary that satisfies Property (P_{q-1}) , $2 \leq q \leq n-1$. Let Ω_j be a sequence of bounded pseudoconvex domains whose $\bar{\partial}$ -Neumann Laplacian $\square_{\Omega_j}$ has purely discrete spectrum on $(0, q)$ -forms. If $d_H(\Omega_j, \Omega) \rightarrow 0$ as $j \rightarrow \infty$, then for any $k \in \mathbb{N}$,

$$\liminf_{j \rightarrow \infty} \lambda_k^q(\Omega_j) \geq \lambda_k^q(\Omega). \quad (4.11)$$

Proof The proof is similar in some respects to Theorem 3.6. The difference here is to use Lemma 4.3 and the Kolmogorov-Riesz theorem to establish estimates that are uniform with regard to j .

We first construct the transition operator $T_{j\delta}$ from $\text{Dom}(Q_{\Omega_j})$ into $\text{Dom}(Q_\Omega)$. Let $\{U^l\}_{l=0}^m$ be an open covering of $\bar{\Omega}$ and let $\{\psi^l\}_{l=0}^m$ be a partition of unity subordinated to this covering, constructed as in the setup preceding Lemma 3.5. Let $U = \cup_{l=1}^m U^l$ and let $V \subset\subset U$ be a tubular neighborhood of ∂V , $\partial\Omega$ such that $\text{dist}(\partial V, \partial\Omega) < \text{dist}(\partial U, \partial\Omega)$. We assume that j is sufficiently large so that $(\Omega \setminus \Omega_j) \cup (\Omega_j \setminus \Omega) \subset\subset V$. Let $f_j \in \text{Dom}(Q_{\Omega_j})$. For any $\delta < \text{dist}(\partial V, \partial\Omega)$ and any sufficiently large j such that $\delta_j = d_H(\Omega_j, \Omega) < \delta$, we define

$$\widehat{f}_{j\delta}(z) = \psi^0(z)f_j(z) + \sum_{l=1}^m \psi^l(z)f_j(z - 2\delta\bar{n}^l) \quad (4.12)$$

and

$$\check{f}_{j\delta}(z) = \psi^0(z)\widetilde{f}_j(z) + \sum_{l=1}^m \psi^l(z)\widetilde{f}_j(z + 2\delta\bar{n}^l). \quad (4.13)$$

(Throughout this proof, we will use $\widetilde{f}_j$ to denote the form obtained by extending f_j to 0 outside of Ω_j .) Notice that $z - 2\delta\bar{n}^l \in \Omega_j$ and $z + 2\delta\bar{n}^l \notin \Omega_j$ for $z \in \Omega \cap U^l$ (see the proof of Lemma 3.5). It follows that $\widehat{f}_{j\delta} \in \text{Dom}(\bar{\partial}_\Omega)$ and $\check{f}_{j\delta} \in \text{Dom}(\bar{\partial}_\Omega^*)$. Define $T_{j\delta}: \text{Dom}(Q_{\Omega_j}) \rightarrow \text{Dom}(Q_\Omega)$ by

$$T_{j\delta}f_j = \bar{\partial}^* N \bar{\partial} \widehat{f}_{j\delta} + \bar{\partial} N \bar{\partial}^* \check{f}_{j\delta}. \quad (4.14)$$

Then

$$T_{j\delta}f_j = \bar{\partial}^* \bar{\partial} N \widehat{f}_{j\delta} + \bar{\partial} \bar{\partial}^* N \check{f}_{j\delta} \quad (4.15)$$

and

$$\bar{\partial} T_{j\delta}f_j = \bar{\partial} \bar{\partial}^* \bar{\partial} N \widehat{f}_{j\delta} = \bar{\partial} \widehat{f}_{j\delta} \quad \text{and} \quad \bar{\partial}^* T_{j\delta}f_j = \bar{\partial}^* \bar{\partial} \bar{\partial}^* N \check{f}_{j\delta} = \bar{\partial}^* \check{f}_{j\delta}. \quad (4.16)$$

We first fix $1 \leq l \leq k$ and let f_j be the normalized eigenform of $\square_{\Omega_j}$ associated eigenvalue $\lambda_l(\Omega_j)$. Since

$$T_{j\delta}f_j - \widetilde{f}_j = \bar{\partial}^* \bar{\partial} N(\widehat{f}_{j\delta} - \widetilde{f}_j) + \bar{\partial} \bar{\partial}^* N(\check{f}_{j\delta} - \widetilde{f}_j),$$

we have

$$\begin{aligned} \|T_{j\delta}f_j - \widetilde{f}_j\|_\Omega &\leq \|\widehat{f}_{j\delta} - \widetilde{f}_j\|_\Omega + \|\check{f}_{j\delta} - \widetilde{f}_j\|_\Omega \\ &\leq \sum_{l=1}^m \left(\|\widetilde{f}_j(z - 2\delta\bar{n}^l) - \widetilde{f}_j(z)\|_{\Omega \cap U^l} + \|\widetilde{f}_j(z + 2\delta\bar{n}^l) - \widetilde{f}_j(z)\|_{\Omega \cap U^l} \right) \\ &\leq \sum_{l=1}^m \left(\|\widetilde{f}_j(z - 2\delta\bar{n}^l) - \widetilde{f}_j(z)\|_{\mathbb{C}^n} + \|\widetilde{f}_j(z + 2\delta\bar{n}^l) - \widetilde{f}_j(z)\|_{\mathbb{C}^n} \right). \end{aligned}$$

By Lemma 4.3 and the subsequent remark, $\{\tilde{f}_j\}$ is a pre-compact family in $L^2_{(0,q)}(\mathbb{C}^n)$. For any $0 < \varepsilon < 1$, it follows from the Kolmogorov-Riesz theorem that

$$\|T_{j\delta}f_j - \tilde{f}_j\|_{\Omega} < \varepsilon$$

for all sufficiently small δ and sufficiently large j .

Furthermore, we have

$$\begin{aligned} \|\bar{\partial}T_{j\delta}f_j - \widetilde{\bar{\partial}f_j}\|_{\Omega} &= \|\bar{\partial}\tilde{f}_{j\delta} - \widetilde{\bar{\partial}f_j}\|_{\Omega} \\ &\leq C \sum_{l=1}^m (\|\tilde{f}_j(z - 2\delta\vec{n}^l) - \tilde{f}_j(z)\|_{\Omega \cap U^l} + \|\bar{\partial}\tilde{f}_j(z - 2\delta\vec{n}^l) - \widetilde{\bar{\partial}f_j}(z)\|_{\Omega \cap U^l}) \\ &\leq C \sum_{l=1}^m (\|\tilde{f}_j(z - 2\delta\vec{n}^l) - \tilde{f}_j(z)\|_{\mathbb{C}^n} + \|\widetilde{\bar{\partial}\tilde{f}_j}(z - 2\delta\vec{n}^l) - \widetilde{\bar{\partial}\tilde{f}_j}(z)\|_{\mathbb{C}^n}), \end{aligned}$$

where the constant C depends only on the partition of unity and is independent of δ or j . Note that $\widetilde{\bar{\partial}^*f_j} = \bar{\partial}^*\tilde{f}_j$. Using the pre-compactness of the families $\{\tilde{f}_j\}$ and $\{\widetilde{\bar{\partial}f_j}\}$ in L^2 -spaces, we then have

$$\|\bar{\partial}T_{j\delta}f_j - \widetilde{\bar{\partial}f_j}\|_{\Omega} < \varepsilon$$

for all sufficiently small δ and all sufficiently large j . Similarly,

$$\|\bar{\partial}^*T_{j\delta}f_j - \widetilde{\bar{\partial}^*f_j}\|_{\Omega} = \|\bar{\partial}^*\tilde{f}_{j\delta} - \widetilde{\bar{\partial}^*f_j}\|_{\Omega} < \varepsilon.$$

The rest of the proof follows the same lines of arguments as in the proof of Theorem 3.6. We sketch the proof below. Let f_{jl} be the normalized eigenform of $\square_{\Omega_j}$ associated with eigenvalue $\lambda_l(\Omega_j)$. Let L_{jk} be the k -dimensional linear subspace of $\text{Dom}(Q_{\Omega_j})$ spanned by $\{f_{jl}\}_{l=1}^k$. Let $f_j = \sum_{l=1}^k c_l f_{jl}$ be a $(0, q)$ -form in L_{jk} . Then

$$\begin{aligned} |\|T_{j\delta}f_j\|_{\Omega}^2 - \|f_j\|_{\Omega_j}^2| &= |\|T_{j\delta}f_j\|_{\Omega}^2 - \|\tilde{f}_j\|_{\Omega}^2 - \|f_j\|_{\Omega_j \setminus \Omega}^2| \\ &\leq (\|T_{j\delta}f_j\|_{\Omega} + \|\tilde{f}_j\|_{\Omega})\|T_{j\delta}f_j - \tilde{f}_j\|_{\Omega} + \|f_j\|_{\Omega_j \setminus \Omega}^2 \\ &\leq C\varepsilon\|f_j\|_{\Omega_j}^2 \end{aligned} \tag{4.17}$$

for all sufficiently small δ and sufficiently large j . Note that in the last inequality, we have used Lemma 4.2. Similarly, we have

$$|Q_{\Omega}(T_{j\delta}f_j, T_{j\delta}f_j) - Q_{\Omega_j}(f_j, f_j)| \leq C\varepsilon\|f_j\|_{\Omega_j}^2.$$

The desired inequality (4.11) then follows from Lemma 2.1 and the subsequent remark. $\square$

Theorem 1.2 is a direct consequence of Theorem 4.5 by *reductio ad absurdum*. Combining Theorems 3.2 and 1.2, we then have:

Corollary 4.6 *Let Ω, Ω_j be bounded pseudoconvex domains in $\mathbb{C}^n$. Suppose $\partial\Omega$ is C^1 -smooth and satisfies Property (P_{q-1}) , $2 \leq q \leq n-1$ and $\partial\Omega_j$ satisfies property (P_q) . If $d_H(\Omega_j, \Omega) \rightarrow 0$ as $j \rightarrow \infty$, then for any $k \in \mathbb{N}$,*

$$\lim_{j \rightarrow \infty} \lambda_k^q(\Omega_j) = \lambda_k^q(\Omega). \quad (4.18)$$

Remark 4.7 Unlike Theorem 3.8, lower semicontinuity property does not hold on level sets of a smooth bounded domain without additional assumption. For example, let Ω be the Diederich-Fornaess worm domain with winding greater than π . Since Ω does not have Stein neighborhood basis ([15]), we have $\liminf_{\delta \rightarrow 0^+} \lambda_k^1(\Omega_\delta^+) = 0$ but $\lambda_k^1(\Omega) > 0$. This follows from the fact that pseudoconvexity of a smooth bounded domain in $\mathbb{C}^2$ is characterized by positivity of one of the variational eigenvalues $\lambda_k^1(\Omega)$ (see [18]).

5 Quantitative Estimates on Finite Type Domains

We continue our study of spectral stability on smooth bounded pseudoconvex domains of finite type. Our aim is to establish quantitative estimates for the stability on such domains. Notions of finite type were introduced by Kohn [26], D'Angelo [10, 11], and Catlin [4, 6, 7] in connection with subelliptic theory of the $\bar{\partial}$ -Neumann Laplacian. A smooth bounded domain Ω in $\mathbb{C}^n$ is said to be of finite D_q -type if the order of contact of $\partial\Omega$ with any q -dimensional complex analytic variety is finite. (We refer the reader to [10, 11] for precise definitions.)

A fundamental theorem of Catlin states that a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$ is of finite D_q -type if and only if the $\bar{\partial}$ -Neumann Laplacian satisfies the following subelliptic estimate

$$\|f\|_{W^\alpha}^2 \leq C Q_\Omega(f, f), \quad \forall f \in \text{Dom}(Q_{q, \Omega}) \quad (5.1)$$

for some constants $0 < \alpha \leq 1/2$ and $C > 0$. The constant α is referred to as the order of subellipticity. A key step in Catlin's theory is the construction of plurisubharmonic functions with large complex Hessians. More precisely, if Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ of finite type, then there exist constants $\alpha > 0$, $\delta_0 > 0$, and $C > 0$ such that for any $0 < \delta < \delta_0$, there exists a smooth plurisubharmonic function λ_δ on $\bar{\Omega}$ with $|\lambda_\delta| \leq 1$ and

$$H_q(\lambda_\delta)(f) \geq C|f|^2/\delta^{2\alpha} \quad (5.2)$$

on $A_\delta = \{z \in \Omega \mid d(z) = \text{dist}(z, \partial\Omega) < \delta\}$ ([7, Theorem 9.2]). Subelliptic estimate (5.1) is then a consequence of the existence of such plurisubharmonic functions. Straube [33] showed that this last step also holds on bounded pseudoconvex domains with Lipschitz boundaries: Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with

Lipschitz boundary. Suppose there exist a continuous plurisubharmonic function λ on Ω and constants $\alpha > 0$, $C > 0$ such that

$$H_q(\lambda)(f) \geq C|f|^2/(d(z))^{2\alpha} \quad (5.3)$$

on Ω as currents, then subelliptic estimate (5.1) holds. For abbreviation, a bounded pseudoconvex domain Ω is said to satisfy *property* (P_q^α) if condition (5.2) is satisfied. We have the following simple analogs of Lemma 4.2.

Lemma 5.1 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose Ω satisfies property (P_q^α) . Then there exists a constant C such that for all sufficiently small $\delta > 0$,*

$$\|f\|_{A_\delta}^2 \leq C\delta^{2\alpha} Q_\Omega(f, f), \quad \forall f \in \text{Dom}(Q_{q,\Omega}). \quad (5.4)$$

Furthermore, if f is an eigenform for $\square_\Omega$ associated with eigenvalue $\lambda(\Omega)$, then

$$\|f\|_{A_\delta}^2 \leq C\delta^{2\alpha} \lambda(\Omega) \|f\|_\Omega^2 \quad \text{and} \quad \|\bar{\partial} f\|_{A_\delta}^2 \leq C\delta^{2\alpha} \lambda^2(\Omega) \|f\|_\Omega^2. \quad (5.5)$$

Moreover, if Ω satisfies property (P_{q-1}^α) , then

$$\|\bar{\partial}^* f\|_{A_\delta}^2 \leq C\delta^{2\alpha} \lambda^2(\Omega) \|f\|_\Omega^2. \quad (5.6)$$

Lemma 5.2 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose Ω satisfies property (P_q^α) . Then there exists a constant $\delta > 0$ such that for any pseudoconvex domain Ω_j with $\delta_j = d_H(\Omega, \Omega_j) \leq \delta$, we have*

$$\|f_j\|_{A_{j\delta_j}}^2 \leq C\delta_j^{2\alpha} Q_{\Omega_j}(f_j, f_j), \quad \forall f_j \in \text{Dom}(Q_{\Omega_j}), \quad (5.7)$$

where $A_{j\sigma} := \{z \in \Omega_j \mid \text{dist}(z, \partial\Omega_j) < \sigma\}$. Furthermore, if $f_j \in \text{Dom}(\square_{\Omega_j})$ is an eigenform satisfies $\square_{\Omega_j} f_j = \lambda(\Omega_j) f_j$, then

$$\|\bar{\partial}_j f_j\|_{A_{j\delta_j}}^2 \leq C\delta_j^{2\alpha} \lambda^2(\Omega_j) \|f_j\|_{\Omega_j}^2. \quad (5.8)$$

Moreover, if Ω satisfies property (P_{q-1}^α) , then

$$\|\bar{\partial}_j^* f_j\|_{A_{j\delta_j}}^2 \leq C\delta_j^{2\alpha} \lambda^2(\Omega_j) \|f_j\|_{\Omega_j}^2. \quad (5.9)$$

These two lemmas are simple consequence of Lemma 4.1, following the same line of arguments as in Lemma 4.2. We omit the proofs. The following lemma is a direct consequence of the interior ellipticity of the $\bar{\partial} \oplus \bar{\partial}^*$.

Lemma 5.3 *Let Ω be a bounded domain in $\mathbb{C}^n$. Let $\Omega_\delta = \{z \in \Omega \mid \text{dist}(z, \partial\Omega) > \delta\}$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{W^1(\Omega_\delta)}^2 \leq C(Q_\Omega(f, f) + \frac{1}{\delta^2} \|f\|_\Omega^2) \quad (5.10)$$

for all $f \in \text{Dom}(Q_\Omega)$.

Proof The lemma is also well known. We include a proof for the reader's convenience. Let $\chi(t) = 0$ for $t < 1/2$, $\chi(t) = 2(t - 1/2)$ for $t \in [1/2, 1]$, and $\chi(t) = 1$ for $t > 1$. Let $d(z) = \text{dist}(z, \partial\Omega)$ and $\eta(z) = \chi(d(z)/\delta)$. Note that since the distance function is uniformly Lipschitz with Lipschitz constant 1, we have $|\nabla\eta(z)| \leq 2/\delta$ almost everywhere on Ω . Therefore

$$\sum_{l=1}^n \sum'_{|J|=q} \|\partial(\eta f_J)/\partial \bar{z}_l\|^2 \leq Q(\eta f, \eta f) \leq C(Q_\Omega(f, f) + \frac{1}{\delta^2} \|f\|_\Omega^2).$$

(See [34, Corollary 2.13] for a proof of the first inequality.) The desired inequality then follows from integration by part on the left-hand side. $\square$

We remark that the constant in (5.10) can be chosen to be independent of Ω . We will use this fact in the proof of the next theorem.

Theorem 5.4 *Let Ω be a bounded pseudoconvex domain with C^1 -smooth boundary. Assume that Ω satisfies property (P_{q-1}^α) . Let Ω_j be a bounded pseudoconvex domain whose $\bar{\partial}$ -Neumann Laplacian has discrete spectrum on $(0, q)$ -forms. Let $k \in \mathbb{N}$. Then there exist constants $\delta > 0$ and $C > 0$ such that*

$$|\lambda_k(\Omega_j) - \lambda_k(\Omega)| \leq Ck\delta_j^{\alpha/(\alpha+1)}(\lambda_k(\Omega) + 1)^2, \quad (5.11)$$

provided $\delta_j = d_H(\Omega, \Omega_j) < \delta$.

Proof The proof follows the same line of arguments as those for Theorem 4.5. The difference here is that we use Lemmas 5.1 and 5.2 to estimate terms near the boundary and use Lemma 5.3 to estimate terms inside the domain.

We provide the proof of the inequality

$$\lambda_k(\Omega) - \lambda_k(\Omega_j) \leq Ck\delta_j^{\alpha/(\alpha+1)}.$$

Following the same setup as in the proof of Theorem 4.5, for $f_j \in \text{Dom}(Q_{\Omega_j})$, we set

$$\widehat{f}_j(z) = \psi^0(z)f_j(z) + \sum_{l=1}^m \psi^l(z)f_j(z - 2\delta_j \vec{n}^l) \quad (5.12)$$

and

$$\check{f}_j(z) = \psi^0(z)\widetilde{f}_j(z) + \sum_{l=1}^m \psi^l(z)\widetilde{f}_j(z + 2\delta_j \vec{n}^l). \quad (5.13)$$

Define $T_j : \text{Dom}(Q_{\Omega_j}) \rightarrow \text{Dom}(Q_\Omega)$ by

$$T_j f_j = \bar{\partial}^* N \bar{\partial} \widehat{f_j} + \bar{\partial} N \bar{\partial}^* \check{f_j}. \quad (5.14)$$

We now assume that $f_j(z)$ is the normalized eigenform of $\square_{\Omega_j}$ associated with the eigenvalue $\lambda(\Omega_j)$. As in the proofs of Theorems 3.6 and 4.5, it suffices to estimate the terms

$$\|f_j\|_{\Omega_j \setminus \Omega}, \quad \|\bar{\partial} f_j\|_{\Omega_j \setminus \Omega}, \quad \|\bar{\partial}^* f_j\|_{\Omega_j \setminus \Omega}, \quad \|\widehat{f_j} - \widetilde{f_j}\|_\Omega, \quad \|\check{f_j} - \widetilde{f_j}\|_\Omega, \quad (5.15)$$

and

$$\|\bar{\partial} \widehat{f_j} - \widetilde{\bar{\partial} f_j}\|_\Omega, \quad \|\bar{\partial}^* \check{f_j} - \widetilde{\bar{\partial}^* f_j}\|_\Omega. \quad (5.16)$$

From (5.7) in Lemma 5.2, we have

$$\|f_j\|_{\Omega_j \setminus \Omega} \leq \|f_j\|_{A_{j\delta_j}} \leq C \delta_j^\alpha (\lambda(\Omega_j))^{1/2}.$$

Similarly,

$$\|\bar{\partial} f_j\|_{\Omega_j \setminus \Omega} + \|\bar{\partial}^* f_j\|_{\Omega_j \setminus \Omega} \leq \|\bar{\partial} f_j\|_{A_{j\delta_j}} + \|\bar{\partial}^* f_j\|_{A_{j\delta_j}} \leq C \delta_j^\alpha \lambda(\Omega_j).$$

Noticing that from Lemma 5.3, we have

$$\|\nabla f_j\|_{\Omega_{j,\delta_j^\beta}} \leq C \delta_j^{-\beta} (\lambda(\Omega_j) + 1)^{1/2},$$

where $\Omega_{j,\delta_j^\beta} = \{z \in \Omega_j \mid \text{dist}(z, \partial\Omega_j) > \delta_j^\beta\}$. Taking $\beta = 1/(1 + \alpha)$, we then have

$$\begin{aligned} \|\widehat{f_j} - \widetilde{f_j}\|_\Omega &\leq C \sum_{l=1}^m \left\| f_j(z - 2\delta_j \vec{n}^l) - \widetilde{f_j}(z) \right\|_{\Omega \cap U^l} \\ &\leq C \sum_{l=1}^m \left\| f_j(z - 2\delta_j \vec{n}^l) - f_j(z) \right\|_{(\Omega \cap U^l) \setminus A_{j,\delta_j^\beta}} + C \|f_j(z)\|_{A_{j,2\delta_j^\beta}} \\ &\leq C \delta_j \|\nabla f_j\|_{\Omega_{j,\frac{1}{2}\delta_j^\beta}} + C \delta_j^{\alpha/(\alpha+1)} (\lambda(\Omega_j))^{1/2} \\ &\leq C \delta_j^{\alpha/(\alpha+1)} (\lambda(\Omega_j) + 1)^{1/2}. \end{aligned} \quad (5.17)$$

Noticing that in obtaining the last inequality, we have used the facts that

$$\Omega \setminus A_{j,\delta_j^\beta} \subset \Omega_{j,\frac{1}{2}\delta_j^\beta}$$

and $\lambda(\Omega_j)$ is controlled from above by the corresponding eigenvalue $\lambda(\Omega)$ of $\square_\Omega$ as shown in Corollary 3.7. Similar estimates also hold for the other three terms in (5.15) and (5.16). Notice that plugging $\bar{\partial} f_j$ and $\bar{\partial}^* f_j$ into (5.10), we have

$$\|\nabla \bar{\partial} f_j\|_{\Omega_{j,\delta}} \leq C\delta^{-1}(\lambda(\Omega_j) + 1) \quad \text{and} \quad \|\nabla \bar{\partial}^* f_j\|_{\Omega_{j,\delta}} \leq C\delta^{-1}(\lambda(\Omega_j) + 1)$$

and the constants in the above estimates are independent of j . Using Lemma 2.1, we then obtain inequality (5.12). The proof of the other inequality in Theorem 5.4 is similar and is left to the interested reader. $\square$

Quantitative estimate (5.11) can be sharpened when more restriction is placed on the boundaries of Ω and Ω_j . A family of smoothly bounded pseudoconvex domains Ω_j in $\mathbb{C}^n$ with defining functions ρ_j is said to be of *uniform finite D_q -type* if there exist positive constants α and C such that inequality (5.2) holds for all Ω_j and the C^∞ -norm of ρ_j is uniformly bounded. The following lemma is a direct consequence of Catlin's subelliptic estimates ([7]).

Lemma 5.5 *Let Ω be a smooth bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. Let l be a non-negative integer. Let f be an eigenform of the $\bar{\partial}$ -Neumann Laplacian $\square_{q,\Omega}$ with associated eigenvalue $\lambda(\Omega)$. Then there exist positive constants α and C_l such that*

$$\|f\|_{C^l(\bar{\Omega})} \leq C_l(\lambda(\Omega))^{[\frac{n+l}{2\alpha}] + 1} \|f\|, \quad (5.18)$$

where $[(n+l)/2\alpha]$ denotes the integer part of $(n+l)/2\alpha$.

Proof It follows from above-mentioned work of Catlin that Ω satisfies property (P_q^α) for some $\alpha \in (0, 1/2]$ and there exists a constant $C_s > 0$ such that

$$\|N_\Omega f\|_{W^{s+2\alpha}(\Omega)} \leq C_s \|f\|_{W^s(\Omega)}. \quad (5.19)$$

Starting with $s = 0$ and repeatedly applying (5.19) to $\square_\Omega f = \lambda(\Omega)f$, we then have

$$\|f\|_{W^{2m\alpha}(\Omega)} \leq C(\lambda(\Omega))^m \|f\|, \quad m \in \mathbb{N}. \quad (5.20)$$

The desired estimates (5.18) is then an immediate consequence of Sobolev embedding theorem. $\square$

We remark that the constant in (5.18) depends only on the constant in (5.2) and the C^∞ -norm of the defining function of Ω . We will use this fact in proving the following theorem:

Theorem 5.6 *Let Ω be a smooth bounded pseudoconvex domain of finite D_q -type in $\mathbb{C}^n$. Let Ω_j be a family of bounded pseudoconvex domains. Let $1 \leq q \leq n-1$ and let $k \in \mathbb{N}$. Then there exist constants $C_k > 0$ and $\delta > 0$ such that*

$$\lambda_k^q(\Omega_j) - \lambda_k^q(\Omega) \leq C_k \delta_k^{1/2}, \quad (5.21)$$

provided $\delta_j = d_H(\Omega, \Omega_j) < \delta$. Furthermore, if Ω_j is a family of smooth bounded pseudoconvex domains of uniform finite D_q -type, then

$$-C_k \delta_j \leq \lambda_k^q(\Omega_j) - \lambda_k^q(\Omega) \leq C_k \delta_j. \quad (5.22)$$

We will prove this theorem using the following sharp Hardy's inequality due to Brezis and Marcus (for functions) [2] and an idea from Davies [14].

Lemma 5.7 *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 -smooth boundary. Then there exists a constant $A > 0$ such that*

$$\int_{\Omega} \frac{|f_N|^2}{(d(z))^2} dV \leq 16 \left(Q_{\Omega}(f_N, f_N) + A \|f_N\|^2 \right) \quad (5.23)$$

for any $f \in \text{Dom}(Q_{\Omega})$, where $f_N = (\bar{\partial}d(z))^* \lrcorner f$ is the normal component of f and $d(z) = d(z, \partial\Omega)$ is the Euclidean distance from z to the boundary $\partial\Omega$. Furthermore, if Ω is pseudoconvex, then for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon} > 0$ such that

$$Q_{\Omega}(f_N, f_N) \leq (4q + \varepsilon) Q_{\Omega}(f, f) + C_{\varepsilon} \|f\|^2. \quad (5.24)$$

Proof For functions in $W_0^1(\Omega)$, inequality (5.23) was proved in [2]. We first provide a proof for functions, using only the divergence theorem. We can assume that $f \in C_0^{\infty}(\Omega)$. Replacing $d(z)$ by a function that is identical to $d(z)$ in a neighborhood of $\partial\Omega$ and C^2 inside Ω , we may assume that $d \in C^2(\overline{\Omega})$. Then

$$\begin{aligned} \int_{\Omega} |\nabla f|^2 dV &= \int_{\Omega} \left| d^{1/2} \nabla(d^{-1/2} f) + \frac{\nabla d}{2d} f \right|^2 dV \\ &= \int_{\Omega} d |\nabla(d^{-1/2} f)|^2 dV + \frac{1}{2} \langle \nabla |d^{-1/2} f|^2, \nabla d \rangle_{\Omega} + \frac{1}{4} \int_{\Omega} \frac{|\nabla d|^2}{d^2} |f|^2 dV. \end{aligned} \quad (5.25)$$

Let $g = d^{-1/2} f$. Note that $g \in C_0^{\infty}(\Omega)$. By the divergence theorem, we have

$$\begin{aligned} 0 &= \int_{\Omega} \nabla \cdot (|g|^2 d \nabla d) dV \\ &= \int_{\Omega} (\nabla |g|^2 \cdot (d \nabla d) + |g|^2 |\nabla d|^2 + |g|^2 d \nabla^2 d) dV. \end{aligned} \quad (5.26)$$

Thus,

$$\begin{aligned} \int_{\Omega} |g|^2 |\nabla d|^2 dV &= -2 \operatorname{Re} \int_{\Omega} \bar{g} \nabla g \cdot d(\nabla d) dV - \int_{\Omega} |g|^2 d(\nabla^2 d) dV \\ &\leq \varepsilon \int_{\Omega} d |\nabla g|^2 dV \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\varepsilon} \int_{\Omega} d|\nabla d|^2 |g|^2 dV + \int_{\Omega} d|\nabla^2 d| |g|^2 dV \\
& \leq \varepsilon \int_{\Omega} d|\nabla g|^2 dV + C_{\varepsilon} \int_{\Omega} d|g|^2 dV.
\end{aligned} \tag{5.27}$$

Note that $|\nabla d| = 1$ near $\partial\Omega$. The middle term in the last expression of (5.25) is under controlled as above. Thus by choosing $A > 0$ sufficiently large, we obtain the following version of Hardy's inequality:

$$\int_{\Omega} \frac{|f|^2}{d^2} dV \leq 4 \int_{\Omega} |\nabla f|^2 dV + A \|f\|^2. \tag{5.28}$$

The above inequality holds for all $f \in W_0^1(\Omega)$ as $C_0^\infty(\Omega)$ is dense in $W_0^1(\Omega)$.

Since $C_{(0,q)}^1(\bar{\Omega}) \cap \text{Dom}(Q_{\Omega})$ is dense in $\text{Dom}(Q_{\Omega})$ in graph norm, it suffices to prove (5.23) for $f \in C_{(0,q)}^1(\bar{\Omega}) \cap \text{Dom}(Q_{\Omega})$. Notice that

$$f_N = (\bar{\partial}d(z))^* \lrcorner f = \sum'_{|K|=q-1} f_N^K d\bar{z}_K \quad \text{where} \quad f_N^K = \frac{1}{q} \sum_{j=1}^n \frac{\partial d(z)}{\partial z_j} f_{jK}.$$

Since $f \in \text{Dom}(\bar{\partial}_{\Omega}^*)$, it follows that $f_N^K = 0$ on $\partial\Omega$ and hence $f_N^K \in W_0^1(\Omega)$ for any strictly increasing $(q-1)$ -tuple K . Moreover,

$$Q_{\Omega}(f_N, f_N) = \sum'_{|K|=q-1} \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial f_N^K}{\partial \bar{z}_j} \right|^2 dV = \frac{1}{4} \|\nabla f_N\|_{\Omega}^2. \tag{5.29}$$

We then obtain (5.23) by combining (5.28) and (5.29).

To establish (5.24), we note that

$$Q_{\Omega}(f_N, f_N) = \frac{1}{q^2} \sum'_{|K|=q-1} \sum_{j=1}^n \int_{\Omega} \left| \sum_{l=1}^n \frac{\partial d(z)}{\partial z_l} \frac{\partial f_{lK}}{\partial \bar{z}_j} + \sum_{l=1}^n \frac{\partial^2 d(z)}{\partial z_l \partial \bar{z}_j} f_{lK} \right|^2 dV. \tag{5.30}$$

The desirable inequality then follows from a simple use of the Cauchy-Schwarz inequality and the facts that $\sum_{l=1}^n |\partial d(z)/\partial z_l|^2 = 1/4$ near $\partial\Omega$ and

$$Q_{\Omega}(f, f) \geq \sum'_{|J|=q} \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial f_J}{\partial \bar{z}_j} \right|^2 dV = \frac{1}{q} \sum'_{|K|=q-1} \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial f_{jK}}{\partial \bar{z}_j} \right|^2 dV.$$

□

The following lemma is a direct consequence of Lemma 5.7 and a theorem of Davies [14, Theorem 4]. We sketch the proof for the reader's convenience.

Lemma 5.8 *Let Ω be a smooth bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. If f is an eigenform for the $\bar{\partial}$ -Neumann Laplacian with associated eigenvalue $\lambda(\Omega)$, then there exist constants $0 < \alpha < 1$ and $C > 0$ such that*

$$\|f_N\|_{A_\delta} \leq C\delta^{3/2}(1 + \lambda(\Omega))^{\frac{1}{2}[\frac{1}{2\alpha}] + \frac{3}{4}}\|f\|, \quad (5.31)$$

where $A_\delta = \{z \in \Omega \mid \text{dist}(z, \partial\Omega) < \delta\}$.

Proof It follows from [14, Theorem 4] that

$$\|f_N\|_{A_\delta}^2 \leq C\delta^3\|(\Delta^D + A)f_N\|_\Omega\|(\Delta^D + A)^{1/2}f_N\|_\Omega, \quad (5.32)$$

where Δ^D is the Dirichlet Laplacian, acting componentwise on f_N . Note that

$$\|(\Delta^D + A)^{1/2}f_N\|^2 = \|\nabla f_N\|^2 + A\|f_N\|^2 = 4Q_\Omega(f_N, f_N) + A\|f_N\|^2.$$

Thus by Lemma 5.7, this term is dominated by a constant multiple of

$$Q_\Omega(f, f) + \|f\|^2 = (1 + \lambda(\Omega))\|f\|^2.$$

To estimate the term $\|(\Delta^D + A)f_N\|_\Omega$, we observe that

$$\begin{aligned} \|\Delta^D f_N\|_\Omega^2 &= \|\nabla^2 f_N\|_\Omega^2 = \left\| \nabla^2 \sum_{|K|=q-1}' \left(\frac{1}{q} \sum_{j=1}^n \frac{\partial d(z)}{\partial z_j} f_{jK} \right) d\bar{z}_K \right\|_\Omega^2 \\ &\leq C \left(\|f\|^2 + \|\nabla f\|^2 + \|\lambda(\Omega)f_N\|^2 \right) \\ &\leq C(1 + \lambda(\Omega))^{2[\frac{1}{2\alpha}] + 2} \|f\|^2, \end{aligned} \quad (5.33)$$

for some constant C depending on the C^3 -norm of the defining function. Here in the last inequality above we have used (5.20). Combining the above estimates, we then obtain the desired estimate (5.31). $\square$

We are now in a position to prove Theorem 5.6.

Proof of Theorem 5.6 Let f be a normalized $(0, q)$ -eigenform of $\square_\Omega$ associated with eigenvalue $\lambda(\Omega)$. Since Ω is of finite type, $f \in C_{(0,q)}^\infty(\bar{\Omega})$. Let $d(z) = \text{dist}(z, \partial\Omega)$ and let $\eta_{\delta_j}(z) = \chi(d(z)/\delta_j)$ where χ is a smooth function such that $\chi(t) = 0$ if $t < 1$, $\chi(t) = 1$ if $t > 2$, and $0 \leq \chi'(t) \leq 1$. Then $|\nabla \eta_{\delta_j}| \leq 1/\delta_j$, and $\text{supp } \eta_{\delta_j} \subset \Omega$ provided δ_j is sufficiently small. Set $f_{\delta_j}(z) = \eta_{\delta_j}(z)f(z)$. Then $\text{supp } f_{\delta_j} \subset\subset \Omega_j$ and hence $f_{\delta_j} \in \text{Dom}(\bar{\partial}_{\Omega_j}^*)$.

Let $\mathcal{E}: W^s(\Omega) \rightarrow W^s(\mathbb{C}^n)$ be a continuous extension operator. Recall that the norm of this operator depends only on n, s , and the Lipschitz constant of Ω ([32, Ch VI.3, Theorem 5]). We have

$$\|f_{\delta_j} - \mathcal{E}f\|_{\Omega_j}^2 \leq \int_{A_{j,3\delta_j}} |\mathcal{E}f|^2 dV \leq C\delta_j, \quad (5.34)$$

where, as before, $A_{j,3\delta_j} = \{z \in \Omega_j \mid \text{dist}(z, \partial\Omega_j) \leq 3\delta_j\}$. We have

$$\vartheta f_{\delta_j} = \eta_{\delta_j} \vartheta f + \delta_j^{-1} \chi'(d(z)/\delta_j) f_N,$$

where

$$f_N = (\bar{\partial}d(z))^* \lrcorner f = \sum'_{|K|=q-1} f_N^K(z) d\bar{z}_K \quad \text{and} \quad f_N^K(z) = \frac{1}{q} \sum_{j=1}^n \frac{\partial d(z)}{\partial z_j} f_{jK}(z).$$

Note that since $f \in \text{Dom}(\bar{\partial}_\Omega^*)$, $f_N^K(z) = 0$ on $\partial\Omega$. It follows from Lemmas 5.5 and 5.8 that

$$\|\vartheta f_{\delta_j}\|_{A_{2\delta_j}} \leq \|\vartheta f\|_{A_{2\delta_j}} + \delta_j^{-1} \|f_N\|_{A_{2\delta_j}} \leq C\delta_j^{1/2}$$

and

$$\begin{aligned} \|\vartheta f_{\delta_j} - \vartheta f\|_{\Omega_j} &\leq \|(\eta_{\delta_j} - 1)\vartheta f\|_{\Omega_j} + \delta_j^{-1} \|\chi'(d/\delta_j) f_N\|_{\Omega_j} \\ &\leq \|\vartheta f\|_{A_{2\delta_j}} + \delta_j^{-1} \|f_N\|_{A_{2\delta_j}} \leq C\delta_j^{1/2}. \end{aligned} \quad (5.35)$$

Define $T_j: \text{Dom}(\square_\Omega) \rightarrow \text{Dom}(Q_j)$ by

$$T_j f = \bar{\partial}_j^* N_j \bar{\partial}_j \mathcal{E} f + \bar{\partial}_j N_j \bar{\partial}_j^* f_{\delta_j} = \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f + \bar{\partial}_j \bar{\partial}_j^* N_j f_{\delta_j}. \quad (5.36)$$

Then

$$\begin{aligned} \|T_j f\|_{\Omega_j}^2 &= \|\bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f\|_{\Omega_j}^2 + \|\bar{\partial}_j \bar{\partial}_j^* N_j f_{\delta_j}\|_{\Omega_j}^2 \\ &= \|f_{\delta_j}\|_{\Omega_j}^2 + \|\bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f\|_{\Omega_j}^2 - \|\bar{\partial}_j^* \bar{\partial}_j N_j f_{\delta_j}\|_{\Omega_j}^2. \end{aligned} \quad (5.37)$$

Notice that

$$\begin{aligned} \|\bar{\partial}_j^* \bar{\partial}_j N_j f_{\delta_j}\|_{\Omega_j}^2 &= \|\bar{\partial}_j^* \bar{\partial}_j N_j (f_{\delta_j} - \mathcal{E} f) + \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f\|_{\Omega_j}^2 \\ &= \|\bar{\partial}_j^* \bar{\partial}_j N_j (f_{\delta_j} - \mathcal{E} f)\|_{\Omega_j}^2 + \|\bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f\|_{\Omega_j}^2 \\ &\quad + 2 \text{Re} \langle \bar{\partial}_j^* \bar{\partial}_j N_j (f_{\delta_j} - \mathcal{E} f), \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f \rangle_{\Omega_j}. \end{aligned} \quad (5.38)$$

The first term on the right hand side above is estimated by

$$\|\bar{\partial}_j^* \bar{\partial}_j N_j (f_{\delta_j} - \mathcal{E} f)\|_{\Omega_j}^2 \leq \|f_{\delta_j} - \mathcal{E} f\|_{\Omega_j}^2 \leq C\delta_j. \quad (5.39)$$

We also have

$$|\langle \bar{\partial}_j^* \bar{\partial}_j N_j (f_{\delta_j} - \mathcal{E} f), \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f \rangle_{\Omega_j}| = |\langle f_{\delta_j} - \mathcal{E} f, \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E} f \rangle_{\Omega_j}|,$$

which is estimated from above by

$$|\langle f_{\delta_j} - \mathcal{E}f, \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E}f \rangle_{\Omega_j}| \leq C \|f_{\delta_j} - \mathcal{E}f\|_{A_{j,3\delta_j}} \|\bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E}f\|_{A_{j,3\delta_j}}. \quad (5.40)$$

When there is no finite type assumption on Ω_j , since

$$\|\bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E}f\|_{A_{j,3\delta_j}} \leq \|\bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E}f\|_{\Omega_j} \leq \|\mathcal{E}f\|_{\Omega_j} \leq C,$$

it follows from (5.34) and (5.40) that

$$|\langle f_{\delta_j} - \mathcal{E}f, \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E}f \rangle_{\Omega_j}| \leq C \delta_j^{1/2}. \quad (5.41)$$

It then follows from (5.37), (5.38), (5.39), and (5.41) that in this case, we have

$$\left| \|T_j f\|_{\Omega_j}^2 - \|f\|_{\Omega}^2 \right| \leq C \delta_j^{1/2}. \quad (5.42)$$

Under the uniform finite type assumption on Ω_j , by Catlin's subelliptic estimate, $N_j \mathcal{E}f$ is smooth and its C^2 -norm is bounded from above by a constant independent of j (see Lemma 5.5 and the subsequent remark above). It follows that

$$|\langle f_{\delta_j} - \mathcal{E}f, \bar{\partial}_j^* \bar{\partial}_j N_j \mathcal{E}f \rangle_{\Omega_j}| \leq C \delta_j. \quad (5.43)$$

Thus, in this case, we have

$$\left| \|T_j f\|_{\Omega_j}^2 - \|f\|_{\Omega}^2 \right| \leq C \delta_j. \quad (5.44)$$

On the one hand, since

$$\|\bar{\partial}_j^* T_j f\|_{\Omega_j}^2 = \|\bar{\partial}_j^* \bar{\partial}_j N_j \bar{\partial}_j^* f_{\delta_j}\|_{\Omega_j}^2 = \|\bar{\partial}_j^* f_{\delta_j}\|_{\Omega_j}^2,$$

we have

$$\begin{aligned} & \left| \|\bar{\partial}_j^* T_j f\|_{\Omega_j}^2 - \|\bar{\partial}^* f\|_{\Omega}^2 \right| \\ & \leq \int_{\Omega} (1 - \eta_{\delta_j}^2) |\vartheta f|^2 + 2|\eta_{\delta_j} \vartheta f| \cdot |\delta_j^{-1} \chi'(d(z)/\delta_j) f_N| + \delta_j^{-2} |f_N|^2 \, dV. \end{aligned}$$

By Lemma 5.8 and the fact that the C^1 -norm of f is bounded from above (see Lemma 5.5), we have

$$\left| \|\bar{\partial}_j^* T_j f\|_{\Omega_j}^2 - \|\bar{\partial}^* f\|_{\Omega}^2 \right| \leq C \delta_j. \quad (5.45)$$

On the other hand, since $\bar{\partial}_j T_j f = \bar{\partial} \mathcal{E}f$, we have

$$|\|\bar{\partial}_j T_j f\|_{\Omega_j}^2 - \|\bar{\partial} f\|_{\Omega}^2| \leq \int_{\Omega_j \setminus \Omega} |\bar{\partial} \mathcal{E}f|^2 \, dV + \int_{\Omega \setminus \Omega_j} |\bar{\partial} f|^2 \, dV \leq C \delta_j. \quad (5.46)$$

Thus

$$|Q_{\Omega_j}(T_j f, T_j f) - Q_{\Omega}(f, f)| \leq C\delta_j. \quad (5.47)$$

When (5.47) is coupled with (5.42), we then obtain from Lemma 2.1 the inequality (5.21). When it is coupled with (5.44), we obtain the second inequality in (5.22). The first inequality of (5.22) is proved similarly and is left to the interested reader. $\square$

Remark 5.9 The power of δ_j in (5.22) is sharp. For example, let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. Then $\lambda_k^q(r_j \mathbb{B}) = \lambda_k^q(\mathbb{B})/r_j^2$. Thus

$$|\lambda_k^q(r_j \mathbb{B}) - \lambda_k^q(\mathbb{B})| = (1 - r_j)(1 + r_j)\lambda_k^q(\mathbb{B})/r_j^2 \approx \delta_j$$

as $\delta_j = 1 - r_j \rightarrow 0$.

6 Resolvent Convergence

Let Ω be a bounded domain in $\mathbb{C}^n$. We consider $L_{(0,q)}^2(\Omega)$ be a subspace of $L_{(0,q)}^2(\mathbb{C}^n)$ consisting of forms vanishing outside Ω . For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we extend the resolvent operator $R_{\lambda}(\square_{\Omega}) = (\lambda I - \square_{\Omega})^{-1}$ to act on $L_{(0,q)}^2(\mathbb{C}^n)$ by setting $R_{\lambda}(\square_{\Omega}) = 0$ on $L_{(0,q)}^2(\mathbb{C}^n) \ominus L_{(0,q)}^2(\Omega)$. Let Ω_j and Ω be bounded domains in $\mathbb{C}^n$, we say that $\square_{\Omega_j}$ converges to $\square_{\Omega}$ in strong (respectively in norm) resolvent sense if for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $R_{\lambda}(\square_{\Omega_j})$ converges strongly (respectively in norm) to $R_{\lambda}(\square_{\Omega})$ as operators acting on $L_{(0,q)}^2(\mathbb{C}^n)$. When Ω and Ω_j are pseudoconvex, we will extend $N_{\Omega} = \square_{\Omega}^{-1}$ and $N_{\Omega_j} = \square_{\Omega_j}^{-1}$ to act on $L_{(0,q)}^2(\mathbb{C}^n)$ in a likewise manner. Since the spectra of $\square_{\Omega_j}$ and $\square_{\Omega}$ are uniformly bounded away from 0, it is easy to see that $\square_{\Omega_j}$ converges to $\square_{\Omega}$ in strong resolvent sense if N_j converges to N strongly on $L_{(0,q)}^2(\mathbb{C}^n)$ (see, e.g., [31, Theorem VIII.9]). Here, as before, to economize the notation, we write N_{Ω_j} and N_{Ω} simply as N_j and N respectively.

Theorem 6.1 Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^1 -smooth boundary. Let $\{\Omega_j\}_{j \in \mathbb{N}}$ be a sequence of bounded pseudoconvex domain such that $\delta_j = d_H(\Omega, \Omega_j) \rightarrow 0$ as $j \rightarrow \infty$. Then $\square_{\Omega_j}$ converges to $\square_{\Omega}$ in strong resolvent sense.

Proof The proof follows the same lines of arguments as in Theorem 3.6. Let $f \in L_{(0,q)}^2(\mathbb{C}^n)$ and let $g = Nf$. Then $f|_{\Omega} := f|_{\Omega} = \square_{\Omega} g = \bar{\partial} \bar{\partial}^* g + \bar{\partial}^* \bar{\partial} g$. Since $g, \bar{\partial} g \in \text{Dom}(\bar{\partial}_{\Omega}^*)$, it follows from the minimality of $\bar{\partial}^*$ that for any $0 < \varepsilon < 1$, there exist $\phi \in \mathcal{D}_{(0,q)}(\Omega)$ and $\varphi \in \mathcal{D}_{(0,q+1)}(\Omega)$ such that

$$\|\phi - g\|_{\Omega} + \|\bar{\partial}^*(\phi - g)\|_{\Omega} + \|\varphi - \bar{\partial} g\|_{\Omega} + \|\bar{\partial}^*(\varphi - \bar{\partial} g)\|_{\Omega} < \varepsilon. \quad (6.1)$$

By choosing j sufficiently large, we have $\phi \in \mathcal{D}_{(0,q)}(\Omega_j)$ and $\varphi \in \mathcal{D}_{(0,q+1)}(\Omega_j)$.

As in the proof of Theorem 3.6, we will use $\tilde{F}$ to denote the extension of F to $\mathbb{C}^n$ by letting $\tilde{F} = 0$ outside of Ω and $\hat{F}$ to denote the form constructed by (3.13) with $\delta = \delta_j$. Thus $\widehat{\bar{\partial}^* g}, \widehat{g} \in \text{Dom}(\bar{\partial}_j)$. Since

$$\begin{aligned}
\|N_j f - Nf\|_{\mathbb{C}^n} &= \|N_j f - Nf\|_{\Omega_j} + \|Nf\|_{\Omega \setminus \Omega_j} \\
&\leq \|N_j \widetilde{f_\Omega} - \widetilde{g}\|_{\Omega_j} + \|N_j(f - \widetilde{f_\Omega})\|_{\Omega_j} + \|Nf\|_{\Omega \setminus \Omega_j} \\
&\leq \|N_j \widetilde{f_\Omega} - \widetilde{g}\|_{\Omega_j} + C\|f - \widetilde{f_\Omega}\|_{\Omega_j} + \|Nf\|_{\Omega \setminus \Omega_j}
\end{aligned}$$

and the last two terms above goes to 0 as $j \rightarrow \infty$, it suffices to prove that $\|N_j \widetilde{f_\Omega} - \widetilde{g}\|_{\Omega_j} \rightarrow 0$. We have

$$\|N_j \widetilde{f_\Omega} - \widetilde{g}\|_{\Omega_j} \leq \|N_j \widehat{\bar{\partial} \bar{\partial}^* g} - \bar{\partial}_j \bar{\partial}_j^* N_j \widetilde{g}\|_{\Omega_j} + \|N_j \bar{\partial}^* \widetilde{g} - \bar{\partial}_j^* \bar{\partial}_j N_j \widetilde{g}\|_{\Omega_j} =: I + II.$$

Note that

$$\begin{aligned}
I &= \|\bar{\partial}_j N_j \widehat{\bar{\partial}^* g} + N_j(\widehat{\bar{\partial} \bar{\partial}^* g} - \bar{\partial} \widehat{\bar{\partial}^* g}) - \bar{\partial}_j \bar{\partial}_j^* N_j \widetilde{g}\|_{\Omega_j} \\
&= \|\bar{\partial}_j N_j \bar{\partial}^* \widetilde{g} + \bar{\partial}_j N_j(\widehat{\bar{\partial}^* g} - \bar{\partial}^* \widetilde{g}) + N_j(\widehat{\bar{\partial} \bar{\partial}^* g} - \bar{\partial} \widehat{\bar{\partial}^* g}) - \bar{\partial}_j \bar{\partial}_j^* N_j \widetilde{g}\|_{\Omega_j} \\
&\leq \|\bar{\partial}_j \bar{\partial}_j^* N_j(\phi - \widetilde{g})\|_{\Omega_j} + \|\bar{\partial}_j N_j \bar{\partial}^*(\widetilde{g} - \phi)\|_{\Omega_j} \\
&\quad + \|\bar{\partial}_j N_j(\widehat{\bar{\partial}^* g} - \bar{\partial}^* \widetilde{g})\|_{\Omega_j} + \|N_j(\widehat{\bar{\partial} \bar{\partial}^* g} - \bar{\partial} \widehat{\bar{\partial}^* g})\|_{\Omega_j} \\
&\leq \|\phi - g\|_{\Omega} + C\|\bar{\partial}^*(g - \phi)\|_{\Omega} + C\|\widehat{\bar{\partial}^* g} - \bar{\partial}^* \widetilde{g}\|_{\Omega_j} + C\|\widehat{\bar{\partial} \bar{\partial}^* g} - \bar{\partial} \widehat{\bar{\partial}^* g}\|_{\Omega_j}.
\end{aligned}$$

It follows from (6.1) and Lemma 3.5 that $I \rightarrow 0$ as $j \rightarrow \infty$. Similarly, we have

$$\begin{aligned}
II &\leq \|\bar{\partial}_j^* \bar{\partial}_j N_j(\widehat{g} - \widetilde{g})\|_{\Omega_j} + \|\bar{\partial}_j^* N_j(\widehat{\bar{\partial} g} - \bar{\partial} \widehat{g})\|_{\Omega_j} \\
&\quad + \|\bar{\partial}_j^* N_j(\varphi - \bar{\partial} \widetilde{g})\|_{\Omega_j} + \|N_j \bar{\partial}^*(\bar{\partial} \widetilde{g} - \varphi)\|_{\Omega_j} \\
&\leq \|\widehat{g} - \widetilde{g}\|_{\Omega_j} + C\|\bar{\partial} \widetilde{g} - \bar{\partial} \widehat{g}\|_{\Omega_j} + C\|\varphi - \bar{\partial} g\|_{\Omega} + C\|\bar{\partial}^*(\bar{\partial} g - \varphi)\|_{\Omega},
\end{aligned}$$

which again goes to 0 as $j \rightarrow \infty$. Thus $\|N_j \widetilde{f_\Omega} - \widetilde{g}\|_{\Omega_j} \rightarrow 0$. $\square$

Remark 6.2 (1) As for Theorem 3.2, Theorem 6.1 holds without the C^1 assumption on $\partial\Omega$ if the Ω_j 's are contained in Ω . (2) One cannot expect that $\square_{\Omega_j}$ converges to $\square_{\Omega}$ in norm resolvent sense. For example, let Ω_j and Ω be bounded convex domains in $\mathbb{C}^n$ such that Ω is exhausted by Ω_j from inside. Suppose $\partial\Omega$ contains a complex analytic variety but $\partial\Omega_j$ does not. Then N_j is compact but N is not (see [19]). Thus in this case, N_j does not converge to N in norm.

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