Foliation of an Asymptotically Flat End by Critical Capacitors

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Abstract

We construct a foliation of an asymptotically flat end of a Riemannian manifold by hypersurfaces which are critical points of a natural functional arising in potential theory. These hypersurfaces are perturbations of large coordinate spheres, and they admit solutions of a certain over-determined boundary value problem involving the Laplace–Beltrami operator. In a key step we must invert the Dirichlet-to-Neumann operator, highlighting the nonlocal nature of our problem.

Keywords Over-determined problem · Foliation · Asymptotically flat ends

Mathematics Subject Classification 58J05 · 58J32 · 58J37 · 35N10 · 35N25

1 Introduction

Riemannian manifolds with asymptotically flat ends play an important role in general relativity and cosmology, and so their general properties are of great interest. In particular, it is often useful to foliate an asymptotically flat end with special surfaces. Huisken and Yau [5] famously proved one can foliate a three-dimensional, asymptotically flat end with constant mean curvature spheres. Furthermore they prove these spheres share a common center, which one can take as the physical center of mass of the system. Previously, R. Ye [14] had shown one can foliate an asymptotically flat end in any dimension $n \geq 3$ provided the mass at infinity is nonzero.

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Subsequently, others have found special foliations by constant expansion surfaces [8], by Willmore surfaces [7], and by isoperimetric surfaces [2]. Here we investigate surfaces which are critical points of the Newton capacity. Recall that, if $K \subset \mathbb{R}^n$, with $n \geq 3$, is a compact set with Lipschitz boundary, one can define its Newton capacity as

$$\text{Cap}(K) = \frac{1}{n(n-2)\omega_n} \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx : u \in H^1(\mathbb{R}^n), u|_K \equiv 1 \right\}, \quad (1.1)$$

where ω_n is the Euclidean volume of an n -dimensional unit ball and $H^1(\mathbb{R}^n)$ is the Sobolev space of functions with one weak derivative in $L^2(\mathbb{R}^n)$. Standard results in potential theory imply this infimum is realized by the equilibrium potential function U_K , which solves the boundary value problem

$$\Delta_0 U_K = 0 \text{ in } \Omega = \mathbb{R}^n \setminus K, \quad U_K|_{\partial K} = 1, \quad \lim_{|y| \rightarrow \infty} U_K(y) = 0 \quad (1.2)$$

where Δ_0 is the usual, flat Laplacian. Moreover, the solution to (1.2) is unique among all functions which satisfy an appropriate decay condition. It is straight-forward to generalize both (1.1) and (1.2) to the setting of a compact set K in a complete, non-compact Riemannian manifold (M, g) with an asymptotically flat end.

The functional Cap is not scale-invariant in Euclidean space, one should not expect it to have critical points as a domain functional. Thus it is natural to seek critical, and even extremal, domains either subject to a constraint or of a modified functional which is scale invariant. One can normalize Cap using the volume of K or the surface area of ∂K ; both choices are natural and have roots in physics and potential theory [4]. Below we will seek critical sets of a volume-normalized functional, which leads us to the over-determined boundary value problem

$$\Delta_0 u = 0 \text{ on } \mathbb{R}^n \setminus K, \quad u|_{\partial K} = 1, \quad \lim_{|x| \rightarrow \infty} u(x) = 0, \quad \left. \frac{\partial u}{\partial \eta} \right|_{\partial K} = \Lambda, \quad (1.3)$$

where Λ is a constant. See Sect. A for a derivation of (1.3) as the Euler-Lagrange equation of our normalized domain functional. This computation is standard, but we include it in Appendix A for the reader's convenience.

Of course, one can also choose to normalize using the surface area of ∂K , which leads one to a slightly different over-determined boundary value problem, namely

$$\Delta_0 u = 0 \text{ on } \mathbb{R}^n \setminus K, \quad u|_{\partial K} = 1, \quad \lim_{|x| \rightarrow \infty} u(x) = 0, \quad \left. \frac{\partial u}{\partial \eta} \right|_{\partial K} = \Lambda H,$$

where H is the mean curvature of ∂K . We derive this Euler-Lagrange equation as well, even though we do not require it.

A classical theorem of Serrin [12] implies that the only solutions of the over-determined boundary value problem (1.3) in Euclidean space are round spheres, but one expects the situation to be more complicated in a general Riemannian manifold.

Our setting is that of a Riemannian manifold (M, g) of dimension $n \geq 3$ with one asymptotically flat end. In general, the Riemannian manifold (M, g) has one end that is asymptotically flat to order $\tau > 0$ if there exists a compact set $K \subset M$ and a diffeomorphism

$$\Phi : \mathbb{R}^n \setminus \mathbf{B} \rightarrow M \setminus K, \quad (1.4)$$

such that in these coordinates

$$g_{ij}(y) = \delta_{ij} + \mathcal{O}(|y|^{-\tau}), \quad \partial^k g_{ij}(y) = \mathcal{O}(|y|^{-\tau-k}),$$

where $\mathbf{B}$ is the unit ball in $\mathbb{R}^n$ and ∂^k refers to any collection of partial derivatives of order less than or equal to k . For our main result, we will require the metric to decay more rapidly to the flat metric, so that

$$g_{ij}(y) = (1 + \sigma|y|^{1-n})\delta_{ij} + h_{ij}(y), \quad |h_{ij}(y)| = \mathcal{O}(|y|^{-n}), \quad \partial^k h_{ij}(y) = \mathcal{O}(|y|^{-k-n}) \quad (1.5)$$

holds for some $\sigma \in \mathbb{R}$. Our decay condition (1.5) will in particular imply the ADM mass of the flat end (which is defined in the next paragraph below) is zero.

Despite its origins in classical potential theory, capacity appears naturally in the context of the geometry associated to scalar curvature and asymptotically flat manifolds. Schoen and Yau first discussed this connection in [11]. Thereafter Bray [1] proved the inequality: if (M, g) is a 3-dimensional Riemannian manifold with $R_g \geq 0$ and one asymptotically flat end of order $\tau > 1/2$ then

$$\text{Cap}(\Sigma) \geq m_{\text{ADM}} = \frac{1}{4|\mathbf{S}^{n-1}|} \lim_{R \rightarrow \infty} \int_{|x|=R} (\partial_i g_{ij} - \partial_j g_{ii}) \frac{x^j}{R} d\sigma.$$

Most recently, J. Jauregui [6] defined a capacity version of the mass of a three-dimensional asymptotically flat end as

$$m_{\text{CV}} = \sup \limsup_{j \rightarrow \infty} \left(\left(\frac{3|K_j|}{4\pi} \right)^{1/3} - \text{Cap}(K_j) \right),$$

where the outer supremum is over all possible compact exhaustions of M . Jauregui [6] further showed that if (M, g) is a three-dimensional manifold with one asymptotically flat end and $R_g \geq 0$ then $m_{\text{CV}} \geq m_{\text{ADM}}$. Moreover, if (M, g) is harmonically flat then $m_{\text{CV}} = m_{\text{ADM}}$.

In the same paper Jauregui conjectures that such an asymptotically flat end carries a foliation by critical points of the capacity function. Our main theorem provides some evidence towards this conjecture, at least for metrics with stronger asymptotics.

Theorem 1 *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ with one asymptotically flat end $M \setminus K$, parameterized as in (1.4) and (1.5). Then there exists $\rho_0 > 1$ and compact sets K_ρ indexed by $\rho \in (\rho_0, \infty)$ such that the domains $\Omega_\rho = M \setminus K_\rho$*

are critical capacitors. In other words, there exist functions $\bar{u}_\rho$ which solve the over-determined boundary value problem

$$\begin{cases} \Delta_g \bar{u}_\rho = 0 & \text{in } \Omega_\rho \\ \bar{u}_\rho = 1 & \text{on } \partial\Omega_\rho = \partial K_\rho \\ \lim_{|y| \rightarrow \infty} \bar{u}_\rho(\Phi(y)) = 0 \\ \frac{\partial \bar{u}_\rho}{\partial \eta} = C(\rho, \sigma, n) & \text{on } \partial\Omega_\rho = \partial K_\rho, \end{cases} \quad (1.6)$$

where η is the unit interior normal to K_ρ and

$$C(\rho, \sigma, n) = \frac{n-2}{\rho} \left(1 - \frac{(n-1)}{2\rho^{n-1}} \sigma \right).$$

The hypersurfaces $\{\partial K_\rho\}_{\rho > \rho_0}$ foliate the end $M \setminus K_{\rho_0}$.

Our result builds naturally on earlier work, particularly that of the first and second authors [3]. More precisely, they perturb small geodesic balls to produce a family of domains Ω_ρ , parameterized by $\rho \in (0, \rho_0)$, which admit solutions to the overdetermined boundary value problem

$$\Delta_g u = 1 \text{ on } \Omega_\rho, \quad u|_{\partial\Omega_\rho} = 0, \quad \frac{\partial u}{\partial \eta} \Big|_{\partial\Omega} = \text{constant}.$$

In our case, the sets K_ρ will be perturbations of large coordinate spheres, as defined by the parameterization Φ in (1.4).

We end this introduction with a brief outline of the rest of the paper. We begin by reformulating our problem in Sect. 2 to take place on a fixed set. We parameterize this reformulated problem by a radius ρ , a translation τ , and a function $w \in \mathcal{C}^{2,\alpha}(\mathbf{S})$. Sect. 3 has some preliminary computations, such as expansions of the metric and the Laplace-Beltrami operators for our reformulated problem, as well as a study of the mapping properties of the Laplace-Beltrami operator on certain weighted function spaces in Sect. 3.3. In Sect. 4 we construct an approximate solution v , given in (4.1), and perturb it by a translation to the eventual solution $\hat{u}_{\rho,\tau,w}$, given in (4.10). The function $\hat{u}_{\rho,\tau,w}$ already satisfies most of our desired properties: it is harmonic, decays appropriately, and has constant Dirichlet data. It only remains for us to choose parameters ρ , τ , and w so that $\hat{u}_{\rho,\tau,w}$ also has constant Neumann data. To correctly choose these parameters we must invert the Dirichlet-to-Neumann operator of the Laplace-Beltrami operator. We do this in two steps, first writing out an expansion of the normal derivative of $\hat{u}_{\rho,\tau,w}$ and performing a linear analysis of this expansion in Sect. 5, and then completing our nonlinear analysis using the implicit function theorem in Sect. 6. Finally, in Sect. 7 we show that we do in fact produce a foliation of the asymptotically flat end.

2 Reformulation of the Problem

In this section we reformulate our problem so that we can solve a family of PDEs on the fixed Euclidean domain $\mathbb{R}^n \setminus \mathbf{B}$. Intuitively, we accomplish three things with this reformulation. First, we rescale by $\rho > 0$, which one should take to be large. Second, we translate the center of the ball by a small parameter $\tau \in \mathbb{R}^n$. Third, we deform the unit sphere $\mathbf{S} = \partial \mathbf{B}$ by a function $w \in \mathcal{C}^{2,\alpha}(\mathbf{S})$.

We should imagine ρ to be large and both w and τ to be small. So that our parameters are all of the same scale, we require

$$\|w\|_{\mathcal{C}^{2,\alpha}(\mathbf{S})} = \mathcal{O}(\rho^{-n}), \quad |\tau| = \mathcal{O}(\rho^{-n}) \quad (2.1)$$

for the remainder of the paper.

Putting all these transformations together we obtain a parameterization

$$\Phi_{\rho,w,\tau} : \mathbb{R}^n \setminus \mathbf{B} \rightarrow M, \quad \Phi_{\rho,w,\tau}(x) = \Phi \left(\rho\tau + \rho x + \rho w \left(\frac{x}{|x|} \right) x \right) \quad (2.2)$$

and let $\widehat{\Omega}_{\rho,\tau,w} = \Phi_{\rho,\tau,w}(\mathbb{R}^n \setminus \mathbf{B})$. Finally we can use $\Phi_{\rho,w,\tau}$ to pull problem (1.6) back to $\mathbb{R}^n \setminus \mathbf{B}$. Under this change of coordinates, we have now reformulated our original problem (1.6) as

$$\begin{cases} \Delta_{\widehat{g}} \widehat{u}_{\rho,w,\tau} = 0 & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \widehat{u}_{\rho,w,\tau} = 1 & \text{on } \partial \mathbf{B} \\ \lim_{|x| \rightarrow \infty} \widehat{u}_{\rho,w,\tau}(x) = 0 \\ \frac{\partial \widehat{u}_{\rho,w,\tau}}{\partial \widehat{\eta}} = \text{Const.} & \text{on } \partial \mathbf{B}, \end{cases} \quad (2.3)$$

where $\widehat{g} = \Phi_{\rho,w,\tau}^*(g)$, $\widehat{\eta}$ is the inward pointing unit normal to $\mathbf{S} = \partial \mathbf{B}$ with respect to the metric $\widehat{g}$. Both our new metric $\widehat{g}$ and the function $\widehat{u}_{\rho,w,\tau}$ depend on the three parameters $\rho \in (0, \infty)$, $w \in \mathcal{C}^{2,\alpha}(\mathbf{S})$, and $\tau \in \mathbb{R}^n$.

3 Preliminary Computations

In this section we carry out some preliminary computations, in preparation for solving (2.3) We first write out a Taylor expansion of the metric $\widehat{g}$.

3.1 Notation

All our computations in this section are perturbation expansions, when ρ is large and $|\tau|$ and $\|w\|_{\mathcal{C}^{k,\alpha}(\mathbf{S})}$ are small. In the computations below we will sometimes wish to extend a function v defined on the sphere to a tubular neighborhood, and we do so by

making it constant in the radial direction, taking $w(x) = w\left(\frac{x}{|x|}\right)$. Similarly, for each $w \in \mathcal{C}^{2,\alpha}(\mathbf{S})$ we let

$$w_i\left(\frac{x}{|x|}\right) = \partial_i w\left(\frac{x}{|x|}\right), \quad w_{ij}\left(\frac{x}{|x|}\right) = \partial_{ij} w\left(\frac{x}{|x|}\right) \quad (3.1)$$

and

$$\Delta_0 w = \sum_{i=1}^n w_{ii}, \quad (3.2)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

To make the computations below tractable, we adopt the following notation throughout the rest of the paper.

For $i \in \{0, 1, 2\}$ we let L^i denote a linear partial differential operator of order i whose coefficients depend smoothly on ρ and x and that satisfies the bound

$$\|L^i(v)\|_{\mathcal{C}^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B})} \leq c \|v\|_{\mathcal{C}^{k+i,\alpha}(\mathbf{S})} \quad (3.3)$$

for each $v \in \mathcal{C}^{k,\alpha}(\mathbf{S})$, where the constant $c > 0$ is independent of ρ . Similarly we let Q^i denote a nonlinear operator of order $i \in \{0, 1, 2\}$ such that $Q^i(0, 0) = 0$ and that satisfies the bound

$$\begin{aligned} \|Q^i(v_1, \tau_1) - Q^i(v_2, \tau_2)\|_{\mathcal{C}^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B})} &\leq c \left(\|v_1\|_{\mathcal{C}^{k+i,\alpha}(\mathbf{S})} + \|v_2\|_{\mathcal{C}^{k+i,\alpha}(\mathbf{S})} + |\tau_1| + |\tau_2| \right) \\ &\quad \times \left(\|v_1 - v_2\|_{\mathcal{C}^{k+i,\alpha}(\mathbf{S})} + |\tau_1 - \tau_2| \right) \end{aligned}$$

provided $\|v_1\|_{\mathcal{C}^{k+i,\alpha}(\mathbf{S})} + \|v_2\|_{\mathcal{C}^{k+i,\alpha}(\mathbf{S})} + |\tau_1| + |\tau_2| \leq 1$.

Finally we let P_i be a function of the form

$$P_i(\rho, x, v, \tau) = \rho^{1-n} |x|^{1-n} L^i(v) + Q^i(v, \tau) + \mathcal{O}(\rho^{-n} |x|^{-n}) \quad (3.4)$$

such that for every $k, \ell, m \in \mathbb{N}$, $x \in \mathbb{R}^n \setminus \mathbf{B}$, $\tau \in \mathbb{R}^n$ and $\rho > \rho_0 > 0$

$$\|\partial_\rho^\ell \partial_v^k \partial_\tau^m P_i(\rho, \cdot, v, \tau)\| \leq c \left(\rho^{-n-\ell} + \rho^{1-n-\ell} (\|v\|_{\mathcal{C}^{i,\alpha}(\mathbf{S})} + \|v\|_{\mathcal{C}^{i,\alpha}(\mathbf{S})}^2 + |\tau|^2) \right), \quad (3.5)$$

for some positive constant depending only on $h, n, k, \ell, m, i, \alpha$ and ρ_0 . For brevity we write

$$P_i(\rho, x, v) = P_i(\rho, x, v, 0).$$

It is important to observe that the product of any two terms, each of which has the form of either L^i or Q^i , has the form of P_i .

3.2 Metric Expansions and the Laplacian

We have the following expansions.

Lemma 3.1 *We have*

$$\rho^{-2}\widehat{g}_{ij}(x) = (1 + 2w + \sigma|z|^{1-n})\delta_{ij} + x_i w_j + x_j w_i + P_1(\rho, x, w, \tau) \quad (3.6)$$

and

$$\rho^2\widehat{g}^{ij}(x) = (1 + 2w + \sigma|z|^{1-n})^{-1}\delta_{ij} - (x^i w^j + x^j w^i) + P_1(\rho, x, w, \tau), \quad (3.7)$$

where

$$|z|^{1-n} := \rho^{1-n} r^{1-n} \left(1 - \frac{n-1}{r^2} \langle x, \tau \rangle \right). \quad (3.8)$$

Proof Letting $\{e_1, \dots, e_n\}$ be the standard orthonormal basis for $\mathbb{R}^n$, we see

$$\begin{aligned} D\Phi_{\rho, w, \tau}(e_i) &= \sum_k D\Phi(e_k) \left(\left(\rho + \rho w \left(\frac{x}{|x|} \right) \right) \delta_{ik} + \rho x^k w_k \right) \\ &= \rho \left(\left(1 + w \left(\frac{x}{|x|} \right) \right) D\Phi(e_i) + \sum_k x^k w_k D\Phi(e_k) \right), \end{aligned}$$

where we evaluate derivatives of Φ at $\rho\tau + \rho x + \rho w(x/|x|)x$. Hence

$$\begin{aligned} \rho^{-2}\widehat{g}_{ij} &= g \left(\left(1 + w \left(\frac{x}{|x|} \right) \right) D\Phi(e_i) + \sum_k x^k w_k D\Phi(e_k), \right. \\ &\quad \left. \left(1 + w \left(\frac{x}{|x|} \right) \right) D\Phi(e_j) + \sum_l x^l w_l D\Phi(e_l) \right) \\ &= (1+w)^2 g_{ij} + (1+w)w_j x^l g_{il} + (1+w)w_i x^k g_{jk} + w_i w_j x^k x^l g_{kl}. \quad (3.9) \end{aligned}$$

Next, we write $|x| = r$ and

$$y = \rho \left(1 + w \left(\frac{x}{|x|} \right) \right) x + \rho\tau,$$

so that

$$\begin{aligned} |y|^2 &= \rho^2 r^2 \left(1 + 2w + \frac{2}{r^2} \langle x, \tau \rangle + w^2 + 2r^{-2} w \langle x, \tau \rangle + r^{-2} |\tau|^2 \right) \\ |y|^{1-n} &= \rho^{1-n} r^{1-n} \left(1 - (n-1)w - \frac{n-1}{r^2} \langle x, \tau \rangle + w^2 + r^{-2} w \langle x, \tau \rangle \right) \end{aligned}$$

$$\begin{aligned}
& +r^{-2}|\tau|^2 + \dots) \\
& = \rho^{1-n}r^{1-n} \left(1 - \frac{n-1}{r^2} \langle x, \tau \rangle \right) + \rho^{1-n}r^{1-n}L^0(w, \tau) + Q^0(w, \tau). \quad (3.10)
\end{aligned}$$

Observe that we absorb the term $(1-n)\rho^{1-n}r^{1-n}w$ above into $L^0(w)$, while the corresponding linear term with respect to τ is kept. Indeed when solving the nonlinear equation for small ρ in Sect. 6, we have to replace w with $\rho^{n-1}w$, which increases the power of ρ in $(1-n)\rho^{n-1}r^{1-n}w$ by $n-1$.

Using (1.5) and (3.1), it follows from (3.9) and (3.10) that both (3.6) and (3.7) hold. $\square$

In the next sections, we will work with the metric

$$g_\rho := \rho^{-2} \hat{g}. \quad (3.11)$$

Lemma 3.2 *For ρ sufficiently large,*

$$\begin{aligned}
\Delta_{g_\rho} &= (1 - 2w - \sigma|z|^{1-n})\Delta_0 - \left[x_i w_j + x_j w_i + P_1(\rho, x, w, \tau) \right] \partial_{ij}^2 \\
&\quad - \left[\left(3w_j - \frac{n-2}{2} \sigma \partial_j |z|^{1-n} \right) + x_i w_{ij} + [\Delta_0 w] x_j + P_2(\rho, x, w, \tau) \right] \partial_j,
\end{aligned}$$

where Δ_0 is the usual flat Laplacian.

Moreover,

$$\begin{aligned}
\partial_j |z|^{1-n} &= -(n-1)\rho^{1-n}r^{-1-n}x_j \left(1 - \frac{n-3}{r^2} \langle x, \tau \rangle \right) \\
&\quad - (n-1)\rho^{1-n}r^{-1-n}\tau_j.
\end{aligned}$$

Proof Recall that the Laplace-Beltrami operator has the form

$$\Delta_{g_\rho} u = \frac{1}{\sqrt{|g_\rho|}} \partial_i g_\rho^{ij} \sqrt{|g_\rho|} \partial_j u, \quad |g_\rho| = \det(g_\rho).$$

Then using (3.6) and (3.7), we have

$$\begin{aligned}
|g_\rho| &= 1 + 2nw + n\sigma|z|^{1-n} + P_1(\rho, x, w, \tau) \\
\sqrt{|g_\rho|} &= 1 + nw + \frac{n\sigma}{2}|z|^{1-n} + P_1(\rho, x, w, \tau) \\
\frac{1}{\sqrt{|g_\rho|}} &= 1 - nw - \frac{n\sigma}{2}|z|^{1-n} + P_1(\rho, x, w, \tau) \\
g_\rho^{ij} \sqrt{|g_\rho|} &= \left[\left(1 + (n-2)w + \frac{n-2}{2} \sigma |z|^{1-n} \right) \delta_{ij} \right. \\
&\quad \left. - (w^i x^j + w^j x^i) + P_1(\rho, x, w, \tau) \right]
\end{aligned}$$

$$\begin{aligned}
\partial_i(g_\rho^{ij}\sqrt{|g_\rho|}) &= \left((n-3)w_i + \frac{n-2}{2}\sigma\partial_i|z|^{1-n}\right)\delta_{ij} \\
&\quad - w_{ii}x_j - w_{ij}x_i - w_j + P_2(\rho, x, w, \tau) \\
\frac{1}{\sqrt{|g_\rho|}}\partial_i(g_\rho^{ij}\sqrt{|g_\rho|}) &= \left((n-3)w_i + \frac{n-2}{2}\sigma\partial_i|z|^{1-n}\right)\delta_{ij} \\
&\quad - w_{ii}x_j - w_{ij}x_i - w_j + P_2(\rho, x, w, \tau) \\
\Delta_{g_\rho}u &= g_\rho^{ij}\partial_i\partial_ju + \frac{1}{\sqrt{|g_\rho|}}\partial_i \\
&\quad (g_\rho^{ij}\sqrt{|g_\rho|})\partial_ju \\
&= \left[(1-2w-\sigma|z|^{1-n})\delta_{ij} - (x^iw^j + x^jw^i) \right. \\
&\quad \left. + P_1(\rho, x, w, \tau)\right]\partial_i\partial_ju \\
&\quad - \left[\left(3w_i - \frac{n-2}{2}\sigma\partial_i|z|^{1-n}\right)\partial_iu \right. \\
&\quad \left. - (x^j\Delta_0w + x_iw_{ij})\partial_ju\right] + P_2(\rho, x, w, \tau)\partial_ju. \quad (3.12)
\end{aligned}$$

In addition we have from (3.8)

$$\partial_j|z|^{1-n} = -(n-1)\rho^{1-n}r^{-1-n}x_j\left(1 - \frac{n-3}{r^2}\langle x, \tau \rangle\right) - (n-1)\rho^{1-n}r^{-1-n}\tau_j, \quad (3.13)$$

which yield the expansions in Lemma 3.2. $\square$

3.3 Weighted Spaces

The best setting in which to perform our linear analysis is that of weighted Hölder spaces. Following Pacard and Rivière [9], we use the following definition.

Definition 1 Let $v \in \mathbb{R}$, $k \in \mathbb{N}$ and $0 < \alpha < 1$. Then we say $u \in \mathcal{C}_v^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$ if $u \in \mathcal{C}_{\text{loc}}^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$ and

$$\|u\|_{k,\alpha,v} = \sup_{s>1} (s^{-v}[u]_{k,\alpha,s}) < \infty.$$

Here

$$[u]_{k,\alpha,s} := \sum_{i=0}^k s^i \sup_{A_s} |\nabla^i u| + s^{k+\alpha} \sup_{x,x' \in A_s} \frac{|\nabla^k u(x) - \nabla^k u(x')|}{|x - x'|^\alpha},$$

where $A_s = \{x \in \mathbb{R}^n : s < |x| < 2s\}$. We denote the space of functions vanishing on the boundary by

$$\mathcal{C}_{v,\mathcal{D}}^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B}) := \left\{u \in \mathcal{C}_v^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B}) : u|_{\partial B} = 0\right\}.$$

Intuitively, one can think of $\mathcal{C}_v^{0,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$ as those functions which grow at most like $|x|^v$ when $|x|$ is large.

Remark 1 Pacard and Rivire perform their analysis on weighted Hlder spaces on $\mathbf{B} \setminus \{0\}$, whereas we want to examine functions on $\mathbb{R}^n \setminus \mathbf{B}$. It is straight-forward to transfer between the two settings using the Kelvin transform $\mathbb{K}$, defined by

$$\mathbb{K} : \mathcal{C}_v^{k,\alpha}(\mathbf{B} \setminus \{0\}) \rightarrow \mathcal{C}_{2-n-v}^{k,\alpha}(\mathbb{R}^n \setminus \mathbf{B}), \quad \mathbb{K}(u)(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right). \quad (3.14)$$

It will be convenient to also note the transformation law

$$\Delta_0(\mathbb{K}(u))(x) = |x|^{-4} \mathbb{K}(\Delta_0 u)(x). \quad (3.15)$$

One can show the following theorem (see Sect. 2.2 of [9]).

Theorem 2 *The mapping*

$$\Delta_0 : \mathcal{C}_{v,\mathcal{D}}^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_{v-2}^{0,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

is injective if $v < 0$ and surjective if $v > 2 - n$.

The mapping properties of Δ_0 change when the weight v crosses over one of the indicial roots $\gamma_j^\pm$, where

$$\gamma_j^\pm = \frac{2-n}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda_j}.$$

and λ_j is the j th eigenvalue of the Laplace-Beltrami operator on the sphere. Thus one can recover the indicial roots $\gamma_j^\pm$ as growth/decay rates of solutions to the ODE

$$w'' + \frac{n-1}{r} w' - \frac{\lambda_j}{r^2} w = 0, \quad w = r^{\gamma_j^\pm}.$$

A slightly more refined analysis uncovers the following theorem.

Theorem 3 *Let $v < 0$ with $v \notin \{\gamma_j^\pm : j \in \mathbf{N}\}$, and let j_0 be the least non-negative integer such that $v > \gamma_{j_0}^-$. Then the cokernel of the mapping*

$$\Delta_0 : \mathcal{C}_{v,\mathcal{D}}^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_{v-2}^{0,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

has dimension j_0 . Alternatively let $v > 2 - n$ with $v \notin \{\gamma_j^\pm : j \in \mathbf{N}\}$, and let j_0 be the least positive integer such that $v < \gamma_{j_0}^+$ then kernel of the mapping

$$\Delta_0 : \mathcal{C}_{v,\mathcal{D}}^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_{v-2}^{0,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

has dimension j_0 .

Again, we refer the reader to Sect. 2.2 of [9] for details.

Replacing ρ by $1/\rho$ in Lemmas 3.1–3.2 and keeping the notation g_ρ for the metric $g_{1/\rho}$, we have the following result.

Lemma 3.3 *Let $2 - n < \nu < 0$. There exist $\rho_0 > 0$ and $c_0 > 0$ such that the map*

$$\mathcal{L} : (0, \rho_0) \times \mathbf{B}_{c_0}(0) \times \mathbf{B}_{c_0}(0) \times \mathcal{C}_{\nu, \mathcal{D}}^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

defined by

$$\mathcal{L}(\rho, w, \tau, u) := \Delta_{g_\rho} u$$

is well defined and smooth. Here $\mathbf{B}_r(0)$ will refer to a ball centered at the origin of radius r in the appropriate space, either $\mathbb{R}^n$ or a space of functions depending on the context.

Furthermore, for every $(\rho, w, \tau) \in (0, \rho_0) \times \mathbf{B}_{c_0}(0) \times \mathbf{B}_{c_0}(0)$, the linear map

$$\mathcal{L}_{\rho, w, \tau} : \mathcal{C}_{\nu, \mathcal{D}}^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B}), \quad u \mapsto \mathcal{L}_{\rho, w, \tau}(u) := \mathcal{L}(\rho, w, \tau, u)$$

is invertible and for all $u \in \mathcal{C}_{\nu, \mathcal{D}}^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$ we have the inequalities

$$C \|u\|_{\mathcal{C}_\nu^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B})} \leq \|\mathcal{L}_{\rho, w, \tau}(u)\|_{\mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})} \leq C^{-1} \|u\|_{\mathcal{C}_\nu^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B})}, \quad (3.16)$$

where $C > 0$ is independent of (ρ, w, τ) .

Proof By Lemma 3.2, we can find $\rho_0 > 0$ and $c_0 > 0$ such that the map

$$\mathcal{L} : (0, \rho_0) \times \mathbf{B}_{c_0}(0) \times \mathbf{B}_{c_0}(0) \times \mathcal{C}_{\nu, \mathcal{D}}^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

defined by

$$\mathcal{L}(\rho, w, \tau, u) = \Delta_{g_\rho} u$$

is well defined and smooth. We shall show that the linear map

$$u \mapsto \mathcal{L}(\rho, w, \tau, u) : \mathcal{C}_{\nu, \mathcal{D}}^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

is invertible for every $(\rho, w, \tau) \in (0, \rho_0) \times \mathbf{B}_{c_0}(0)$. To see this, we pick $f \in \mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$ and we define

$$\mathcal{F} : (-\rho_0, \rho_0) \times \mathbf{B}_{c_0}(0) \times \mathbf{B}_{c_0}(0) \times \mathcal{C}_{\nu, \mathcal{D}}^{2, \alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow \mathcal{C}_\nu^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

by

$$\mathcal{F}(\varepsilon, w, \tau, u) := \mathcal{L}(|\varepsilon|, w, \tau, u) - f.$$

It is clear from Lemma 3.2 that $\mathcal{F}$ is of class C^2 , since $n \geq 3$. Let $u_0 \in C_{v,D}^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$ be the unique solution to $\Delta_0 u_0 = f$. We have $\mathcal{F}(0, 0, 0, u_0) = 0$ and by Theorem 2, $\partial_u \mathcal{F}(0, 0, 0, u_0)$ is invertible. By the implicit function theorem, there exists a unique $u_{\varepsilon,w,\tau}$ satisfying $\mathcal{F}(\varepsilon, w, \tau, u_{\varepsilon,w,\tau}) = 0$, with $u_{0,0,0} = u_0$. We then conclude that, provided c_0 and ρ_0 small, the linear map

$$\mathcal{L}(\rho, w, \tau) : C_{v,D}^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{B}) \rightarrow C_v^{0,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

is invertible for every $\rho, w, \tau \in (0, \rho_0) \times \mathbf{B}_{c_0}(0) \times \mathbf{B}_{c_0}(0)$. The bound (3.16) also follows from the implicit function theorem. $\square$

4 Approximate and Actual Solutions

In this section we construct an approximate solution using the standard Greens function in Euclidean space and compare it to the solution of a corresponding Dirichlet problem.

For $w \in \mathbf{B}_{c_0}(0)$, we define

$$v(x) = v_{\rho,w}(x) = \left| \left(1 + w \left(\frac{x}{|x|} \right) \right) x \right|^{2-n}. \quad (4.1)$$

We have the following expansion.

Lemma 4.1 *For ρ sufficiently small, the Laplacian of v is given by*

$$\begin{aligned} \Delta_{g_\rho} v &= \frac{(n-1)(n-2)^2}{2} \sigma \rho^{n-1} r^{1-2n} \\ &\quad - \frac{(n-1)(n-4)(n-2)^2}{2} \sigma \rho^{n-1} r^{1-2n} \left\langle \frac{x}{r^2}, \tau \right\rangle + P_2(1/\rho, x, w, \tau). \end{aligned} \quad (4.2)$$

Proof By definition

$$v(x) = |x|^{2-n} \left(1 - (n-2)w + Q^0(w, \tau) \right), \quad (4.3)$$

and so

$$\begin{aligned} \frac{\partial v}{\partial x_i} &= -(n-2)r^{-n} \left(1 - (n-2)w + Q^0(w, \tau) \right) x_i \\ &\quad - (n-2)r^{2-n} w_i + Q^1(w, \tau) \\ \partial_{ij}^2 v &= -(n-2)r^{-n} \left(1 - (n-2)w \right) \delta_{ij} \\ &\quad + n(n-2)r^{-2-n} \left(1 - (n-2)w \right) x_i x_j + (n-2)^2 r^{-n} w_j x_i \\ &\quad + (n-2)^2 r^{-n} w_i x_j - (n-2)r^{2-n} w_{ij} + Q^2(w, \tau) \end{aligned}$$

$$\Delta_0(v) = -(n-2)r^{2-n}\Delta_0w + Q^2(w, \tau). \quad (4.4)$$

With this, we have

$$(1 - 2w - \sigma|z|^{1-n})\Delta_0(v) = -(n-2)r^{2-n}\Delta_0w + Q^2(w, \tau) \quad (4.5)$$

$$\begin{aligned} & \left[x_i w_j + x_j w_i + P_1(\rho, x, w, \tau) \right] \partial_{ij}^2(v) = P_2(\rho, x, w, \tau) \\ & - \left[3w_j + x_i w_{ij} + [\Delta_0 w]x_j + P_2(\rho, x, w, \tau) \right] \partial_j(v) \\ & = (n-2)r^{2-n}\Delta_0w + (n-2)r^{-n}x_i x_j w_{ij} \\ & \quad \frac{n-2}{2}\sigma \partial_j |y|^{1-n} \partial_j(v) = \frac{(n-1)(n-2)^2}{2}\sigma \rho^{1-n} r^{1-2n} \\ & \quad - \frac{(n-1)(n-4)(n-2)^2}{2}\sigma \rho^{1-n} r^{1-2n} \left\langle \frac{x}{r^2}, \tau \right\rangle. \end{aligned} \quad (4.6)$$

However, $0 = \partial_i(x^j w_j) = \delta_{ij} w_j + x^j w_{ij} \Rightarrow x^i x^j w_{ij} = -x^i \delta_{ij} j w_j = -x^i w_i = 0$. The expansion in Lemma 4.1 now follows from Lemma 3.2 after replacing ρ by $1/\rho$. $\square$

Using Lemma 3.3, we construct a unique solution $\Psi_{\rho, w}(x)$ to the equation

$$\begin{cases} \Delta_{g_\rho} \Psi_{\rho, \tau, w} = -\Delta_{g_\rho} v & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \Psi_{\rho, w} = 1 - v = (n-2)w + Q^0(w) & \text{on } \partial \mathbf{B}. \end{cases} \quad (4.7)$$

First choose $\chi \in C_c^\infty(\mathbb{R}_+)$ such that $\chi(t) = 1$ for $1/2 \leq t \leq 1$, and define

$$f(\rho, x, w) = -\Delta_{g_\rho} \left(((n-2)w + Q^0(w))\chi(|x|) \right) - \Delta_{g_\rho} v \in C_v^{0, \alpha}(\mathbb{R}^n \setminus \mathbf{B})$$

for any $v \in (2-n, 0)$. Next we use Lemma 3.3 to let $\tilde{\Psi}_{\rho, \tau, w}$ be the unique solution of

$$\Delta_{g_\rho} \tilde{\Psi}_{\rho, \tau, w}(x) = f(\rho, x, w) \text{ for } x \in \mathbb{R}^n \setminus \mathbf{B}, \quad \tilde{\Psi}_{\rho, \tau, w} = 0 \text{ on } \partial \mathbf{B}. \quad (4.8)$$

Observe that f depends smoothly on (ρ, w, τ) and that the mapping $A \mapsto (A)^{-1}$ is smooth in the open set of linear invertible operator between two Banach spaces. We deduce that the mapping

$$(\rho, \tau, w) \mapsto \tilde{\Psi}_{\rho, \tau, w}$$

is smooth. We then have

$$\Psi_{\rho, \tau, w} = \tilde{\Psi}_{\rho, \tau, w} + ((n-2)w + Q^0(w))\chi(|x|), \quad (4.9)$$

which uniquely solves (4.7).

Finally we define

$$\widehat{u}_{\rho,\tau,w}(x) = v(x) + \Psi_{\rho,\tau,w}(x), \quad (4.10)$$

which satisfies

$$\begin{cases} \Delta_{g_\rho} \widehat{u}_{\rho,\tau,w} = 0 & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \widehat{u}_{\rho,\tau,w} = 1 & \text{on } \partial \mathbf{B}. \end{cases} \quad (4.11)$$

Since $f(\rho, \cdot, w) \in C_v^{0,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$, (4.8) and (3.16) imply that $\tilde{\Psi}_{\rho,w}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. We conclude with (4.1), (4.9) and (4.10) that

$$\widehat{u}_{\rho,\tau,w}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

5 Expansion of the Normal Derivative and Linear Analysis

The aim of this section is to derive an expansion of the normal derivative of the solution $\widehat{u}_{\rho,\tau,w}$. We start by the computation of the interior unit normal vector to $\mathbf{B}$.

Lemma 5.1 *Let $\Theta : \mathbb{R}^{n-1} \rightarrow S$ be a local parameterization of S and set*

$$\Theta_k := \partial_k \Theta \quad k = 1, \dots, n-1.$$

The interior unit normal vector field to $\mathbf{B}$ with respect to the metric g_ρ is given by

$$\begin{aligned} \widehat{v}_{g_\rho} &= \frac{\widehat{\eta}}{\sqrt{\langle \widehat{\eta}, \widehat{\eta} \rangle_{g_\rho}}} = \left[1 - w - \frac{\sigma}{2} \rho^{1-n} \left(1 - (n-1) \langle \Theta, \tau \rangle \right) + P_1(\rho, \Theta, w, \tau) \right] \\ &\quad \times (-\Theta + \Upsilon), \end{aligned} \quad (5.1)$$

where

$$\Upsilon = \sum_{m=1}^{n-1} a_m \Theta_m \quad \text{and} \quad a_k = \langle \nabla_S w, \Theta_k \rangle + P_1(\rho, \Theta, w, \tau).$$

Proof For each $\Theta \in S$, the vectors

$$\Theta_\ell := \partial_\ell \Theta, \quad \ell = 1, \dots, n-1 \quad (5.2)$$

span the tangent space $T_\Theta S$ and since $T_\Theta S \simeq \mathbb{R}^{n-1}$, we may assume after orthonormalization that $\langle \Theta_i, \Theta_j \rangle = \delta_{ij}$. In addition having $\langle \Theta, \Theta \rangle = 1$, it follows that $\langle \Theta, \Theta_\ell \rangle = 0$ and $\langle \nabla_S w, \Theta \rangle_{g_S} = 0$ on S . With these identities, Lemma 3.1 yields

$$\begin{aligned}\langle \Theta_i, \Theta_j \rangle_{g_\rho} &= (1 + 2w + \sigma |z|^{1-n}) \langle \Theta_i, \Theta_j \rangle + P_1(\rho, \Theta, w, \tau) \\ &= (1 + 2w + \sigma |z|^{1-n}) \delta_{ij} + P_1(\rho, \Theta, w, \tau)\end{aligned}\quad (5.3)$$

and

$$\langle \Theta, \Theta_j \rangle_{g_\rho} = P_1(\rho, \Theta, w, \tau). \quad (5.4)$$

We look for a normal vector $\widehat{\eta}$ of $\mathbf{S}$ with respect to the metric $\widehat{g}$ in the form

$$\widehat{\eta} = -\Theta + \Upsilon, \quad \Upsilon = \sum_{k=1}^{n-1} a_k \Theta_k. \quad (5.5)$$

The condition that $\widehat{\eta}$ is normal is thus equivalent to

$$\langle \widehat{\eta}, \Theta_\ell \rangle_{g_\rho} = 0, \quad \ell = 1, \dots, n-1 \Leftrightarrow \langle \Upsilon, \Theta_\ell \rangle_{g_\rho} = \langle \Theta, \Theta_\ell \rangle_{g_\rho}, \quad \ell = 1, \dots, n-1. \quad (5.6)$$

By Lemma 3.1,

$$\begin{aligned}\langle \Theta, \Theta_\ell \rangle_{g_\rho} &= (1 + 2w + \sigma |z|^{1-n}) \langle \Theta, \Theta_\ell \rangle \\ &\quad + [\langle \nabla_{\mathbf{S}} w, \Theta_\ell \rangle + \langle \nabla_{\mathbf{S}} w, \Theta \rangle_{g_{\mathbf{S}}} + P_1(\rho, \Theta, w, \tau)] \\ &= \langle \nabla_{\mathbf{S}} w, \Theta_\ell \rangle + P_1(\rho, \Theta, w, \tau).\end{aligned}\quad (5.7)$$

We also have

$$\langle \Upsilon, \Theta_\ell \rangle_{g_\rho} = \sum_{k=1}^{n-1} a_k \langle \Theta_k, \Theta_\ell \rangle_{g_\rho}.$$

Now using (5.6) and (5.8),

$$\sum_{k=1}^{n-1} a_k \langle \Theta_k, \Theta_\ell \rangle_{g_\rho} = \langle \nabla_{\mathbf{S}} w, \Theta_\ell \rangle + P_1(\rho, \Theta, w, \tau), \quad \ell = 1, \dots, n-1. \quad (5.8)$$

Set

$$b_\ell := \langle \nabla_{\mathbf{S}} w, \Theta_\ell \rangle + P_1(\rho, \Theta, w, \tau) \quad (5.9)$$

$$\widetilde{g}_{k\ell} := \langle \Theta_k, \Theta_\ell \rangle_{g_\rho} = (1 + 2w + \sigma |z|^{1-n}) \delta_{ij} + P_1(\rho, \Theta, w, \tau), \quad (5.10)$$

where the last equality follows from (5.3). Then (5.8) reads,

$$\sum_{k=1}^{n-1} a_k \widetilde{g}_{k\ell} = b_\ell, \quad \ell = 1, \dots, n-1,$$

and we deduce with (5.9) the coefficients

$$a_k = \sum_{\ell=1}^{n-1} b_\ell \tilde{g}^{k\ell} = \langle \nabla_{\mathbf{S}} w, \Theta_k \rangle + P_1(\rho, \Theta, w, \tau), \quad k = 1, \dots, n-1. \quad (5.11)$$

Next we compute

$$\langle \widehat{\eta}, \widehat{\eta} \rangle_{g_\rho} = \langle \Theta, \Theta \rangle_{g_\rho} - 2\langle \Theta, \Upsilon \rangle_{g_\rho} + \langle \Upsilon, \Upsilon \rangle_{g_\rho}.$$

We have by Lemma 3.1,

$$\langle \Theta, \Theta \rangle_{g_\rho} = \left(1 + 2w + \sigma |z|^{1-n} + P_0(\rho, \Theta, w, \tau) \right) \quad (5.12)$$

and from (5.4),

$$\langle \Theta, \Upsilon \rangle_{g_\rho} = \sum_{k=1}^{n-1} a_k \langle \Theta, \Theta_k \rangle_{g_\rho} = P_1(\rho, \Theta, w, \tau) \quad (5.13)$$

In addition,

$$\langle \Upsilon, \Upsilon \rangle_{g_\rho} = \sum_{k,\ell=1}^{n-1} a_k a_\ell \langle \Theta_k, \Theta_\ell \rangle_{g_\rho} = \sum_{\ell=1}^{n-1} a_\ell \left(\sum_{k=1}^{n-1} a_k \tilde{g}^{k\ell} \right) = \sum_{\ell=1}^{n-1} a_\ell b_\ell = \sum_{k,\ell=1}^{n-1} b_k b_\ell \tilde{g}^{k\ell}, \quad (5.14)$$

where we have used (5.11) to get the last equality. Using (5.10) and (5.9), it follows that

$$\langle \Upsilon, \Upsilon \rangle_{g_\rho} = \sum_{\ell=1}^{n-1} b_\ell^2 + P_1(\rho, \Theta, w, \tau) = P_1(\rho, \Theta, w, \tau). \quad (5.15)$$

We gather (5.13), (5.14) and (5.15) to obtain

$$\langle \widehat{\eta}, \widehat{\eta} \rangle_{g_\rho} = \left(1 + 2w + \sigma |z|^{1-n} + P_1(\rho, \Theta, w, \tau) \right).$$

The normal interior unit vector field to $\mathbf{B}$ for the metric g_ρ is then given by

$$\widehat{v}_{g_\rho} = \frac{\widehat{\eta}}{\sqrt{\langle \widehat{\eta}, \widehat{\eta} \rangle_{g_\rho}}} = \left(1 - w - \frac{\sigma}{2} |z|^{1-n} + P_1(\rho, \Theta, w, \tau) \right) (-\Theta + \Upsilon), \quad (5.16)$$

and (5.1) follows from (3.8). $\square$

5.1 Expansion of the Normal Derivative

The following proposition yields the expansion of the normal derivative of $\widehat{u}_{\rho,\tau,w}$ with respect to the metric g_ρ .

Proposition 4 For ρ sufficiently small the normal derivative of $\widehat{u} = \widehat{u}_{\rho,\tau,w} = v + \Psi_{\rho,\tau,w}$ with respect to the metric g_ρ on $\partial\mathbf{B}$ is given by

$$\frac{\partial \widehat{u}}{\partial \widehat{v}_{g_\rho}} = (n-2)\left(1 - \frac{\sigma}{2}\rho^{n-1}\right) + H(\rho, \tau, w), \quad (5.17)$$

where

$$\begin{aligned} H(\rho, \tau, w)(x) &:= (n-1)(n-2)\left(\frac{\sigma}{2}\rho^{n-1}\langle x, \tau \rangle - w + P_1(1/\rho, x, w, \tau)\right) \\ &+ \widehat{v}_{g_\rho}^i \frac{\partial \Psi_{\rho,\tau,w}}{\partial x_i} \end{aligned} \quad (5.18)$$

and $\Psi_{\rho,\tau,w}$ is solution of (4.7).

Proof Recall our solution $\widehat{u}_{\rho,\tau,w}(x) = v(x) + \Psi_{\rho,\tau,w}(x)$, where v is given by (4.1) and $\Psi_{\rho,\tau,w}$ satisfies (4.7).

We first compute $g_\rho(\nabla_{g_\rho} v, \widehat{v}_{g_\rho})$. By Lemma 3.1 and (4.4)

$$\begin{aligned} \nabla_{g_\rho}^j v &= \sum_{i=1}^n g_\rho^{ij} \frac{\partial v}{\partial x_i} = \left[(1 - 2w - \sigma|z|^{1-n})\delta_{ij} - (x_i w_j + x_j w_i) + P_0(\rho, x, w, \tau) \right] \times \\ &\quad \left[-(n-2)\left(1 - (n-2)w + Q^0(w)\right)x_i - (n-2)w_i + Q^1(w) \right] \\ &= -(n-2)\left((1 - \sigma|z|^{1-n})x_j - nwx_j + P_1(\rho, x, w, \tau) \right) \quad \text{on } \partial\mathbf{B}, \end{aligned}$$

which yields

$$\begin{aligned} &g_\rho(-\Theta, \nabla_{g_\rho} v) \\ &= -\left[(1 + 2w + \sigma|z|^{1-n})\langle \Theta, \nabla_{g_\rho} v \rangle + \langle \nabla w, \nabla_{g_\rho} v \rangle \right. \\ &\quad \left. + \langle \nabla w, \Theta \rangle \langle \Theta, \nabla_{g_\rho} v \rangle + P_1(\rho, \Theta, w, \tau) \right] \\ &= -\left[(1 + 2w + \sigma|z|^{1-n})\langle \Theta, \nabla_{g_\rho} v \rangle + P_1(\rho, \Theta, w, \tau) \right]. \end{aligned}$$

On the other hand,

$$\langle \Theta, \nabla_{g_\rho} v \rangle = -(n-2)\left(1 - \sigma|z|^{1-n} - nw + P_1(\rho, \Theta, w, \tau)\right),$$

so we may rewrite our expression for $g_\rho(-\Theta, \nabla_{g_\rho} v)$ as

$$g_\rho(-\Theta, \nabla_{g_\rho} v) = (n-2) \left(1 + (2-n)w + P_1(\rho, \Theta, w, \tau) \right). \quad (5.19)$$

Additionally,

$$\begin{aligned} g_\rho(\Upsilon, \nabla_{g_\rho} v) &= \sum_{k=1}^n a_k g_\rho(\Theta_k, \nabla_{g_\rho} v) = \sum_{k=1}^n a_k g_\rho(\Theta_k, \Theta) \\ &+ P_1(\rho, \Theta, w, \tau) = P_1(\rho, \Theta, w, \tau), \end{aligned} \quad (5.20)$$

where we have used (5.8) in the last equality. Finally, making use of (5.16), (5.20) and (5.19), we obtain

$$\begin{aligned} g_\rho(\nabla_{g_\rho} v, \widehat{v}_{g_\rho}) &= \left(1 - w - \frac{\sigma}{2} |z|^{1-n} + P_1(\rho, \Theta, w, \tau) \right) g_\rho(-\Theta + \Upsilon, \nabla_{\widehat{g}} v) \\ &= (n-2) \left(1 - w - \frac{\sigma}{2} |z|^{1-n} + P_1(\rho, \Theta, w, \tau) \right) \\ &\quad \left(1 + (2-n)w + P_1(\rho, \Theta, w) \right) \\ &= (n-2) \left(1 - \frac{\sigma}{2} \rho^{1-n} + \frac{\sigma(n-1)}{2} \rho^{1-n} \langle x, \tau \rangle - (n-1)w \right. \\ &\quad \left. + P_1(\rho, \Theta, w, \tau) \right) \end{aligned} \quad (5.21)$$

and

$$g_\rho(\nabla_{g_\rho} \Psi_{\rho, \tau, w}, \widehat{v}_{g_\rho}) = g_\rho^{ij} (g_\rho)_{jk} \widehat{v}_{g_\rho}^k \frac{\partial \Psi_{\rho, \tau, w}}{\partial x_i} = \delta_{ik} \widehat{v}_{g_\rho}^k \frac{\partial \Psi_{\rho, \tau, w}}{\partial x_i} = \widehat{v}_{g_\rho}^i \frac{\partial \Psi_{\rho, \tau, w}}{\partial x_i}. \quad (5.22)$$

The expansion (5.17) then follows from (5.21) and (5.22) replacing ρ by $1/\rho$. $\square$

5.2 Linear Analysis of the Normal Derivative

In this section, we analyse (5.17) and (5.18) in detail and compute the operators that will allow us to solve

$$\frac{\partial \widehat{u}}{\partial \widehat{v}_{g_\rho}} = \text{Const.} \quad (5.23)$$

Scanning (5.18), (4.7) and Lemma (4.1), we observe that

$$H(0, 0, 0) \equiv 0. \quad (5.24)$$

With the condition (5.24), an obvious attempt is to consider the equation

$$H(\rho, \tau, w) = 0 \quad (5.25)$$

and apply the implicit function theorem to find τ and w as functions of a small variable ρ . As shown in Proposition 5 below, the linearised operator of H with respect to w has a non trivial kernel. To solve (5.25), we then need to project on this kernel and first express τ in terms of ρ and w . Still there are some challenges to overcome. Indeed, since τ appears with a factor of ρ^{n-1} in (5.18) and (4.2), the derivative of H with respect to τ evaluated at $(0, 0, 0)$ is zero. We compensate for this by multiplying the equation (5.25) by ρ^{1-n} after replacing w by $\rho^{n-1}w$, leading naturally to the function

$$G(\rho, \tau, w) := \rho^{1-n} H(\rho, \tau, \rho^{n-1}w). \quad (5.26)$$

From (5.33) and (5.18),

$$\begin{aligned} G(\rho, \tau, w)(x) &= (n-1)(n-2) \left(\frac{\sigma}{2} \langle x, \tau \rangle - w + \rho^{1-n} P_1(1/\rho, \Theta, \rho^{n-1}w, \tau) \right) \\ &\quad + \widehat{v}_{g_\rho}^i(\rho^{n-1}w) \frac{\partial \overline{\Psi}_{\rho, \tau, w}}{\partial x_i}, \end{aligned} \quad (5.27)$$

where

$$\overline{\Psi}_{\rho, \tau, w} := \rho^{1-n} \Psi_{\rho, \tau, \rho^{n-1}w}.$$

We emphasise that the function $\Psi_{\rho, \tau, \rho^{n-1}w}$ is the unique solution of (4.7) with w replaced by $\rho^{n-1}w$ in both the metric coefficients of g_ρ involved in the operator Δ_{g_ρ} , and the right hand side of the interior and boundary equations. We keep the notation Δ_{g_ρ} for this new Laplacian. The quantity $\widehat{v}_{g_\rho}^i(\rho^{n-1}w)$ is the i th-component of the unit vector $\widehat{v}_{g_\rho}$ in (5.17) where w is also replaced with $\rho^{n-1}w$.

By Lemma 4.1 that $\overline{\Psi}_{\rho, \tau, w}$ is solution the unique solution of

$$\begin{cases} \Delta_{g_\rho} \overline{\Psi}_{\rho, \tau, w} = -\frac{(n-1)(n-2)^2}{2} \sigma r^{1-2n} + \frac{(n-1)(n-4)(n-2)^2}{2} \sigma r^{1-2n} \langle \frac{x}{r^2}, \tau \rangle \\ \quad + \rho^{1-n} P_2(1/\rho, x, \rho^{n-1}w, \tau) \quad \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \overline{\Psi}_{\rho, \tau, w} = (n-2)w + \rho^{1-n} Q^0(\rho^{n-1}w, \tau) \quad \text{on } \partial \mathbf{B}. \end{cases} \quad (5.28)$$

In addition,

$$\overline{\Psi}_{0,0,0} = C_1(n, \sigma) \mathcal{K}, \quad (5.29)$$

where

$$C_1(n, \sigma) = -\frac{(n-1)(n-2)^2}{2} \sigma$$

and $\mathcal{K}$ is the unique solution of

$$\begin{cases} \Delta \mathcal{K} = r^{1-2n} & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \mathcal{K} = 0 & \text{on } \partial \mathbf{B}. \end{cases} \quad (5.30)$$

It is plain that the function $\mathcal{K}$ is radial, and one obtain by Gauss-Green formula $\partial_\nu \mathcal{K} = -\mathcal{K}'(1) = \frac{1}{n-1}$.

Now from (5.29) and (5.27) it follows that

$$G(0, 0, 0) = C_1(n, \sigma) \partial_\nu \mathcal{K} = \frac{C_1(n, \sigma)}{n-1} = -\frac{(n-2)^2}{2} \sigma. \quad (5.31)$$

The non linear problem under consideration is then to show that provided ρ is small, we can find τ and w such that

$$G(\rho, \tau, w) + \frac{(n-2)^2}{2} \sigma = 0. \quad (5.32)$$

We start by studying the operator G .

Proposition 5 *Let*

$$G(\rho, \tau, w) := \rho^{1-n} H(\rho, \tau, \rho^{n-1} w). \quad (5.33)$$

Then

$$[D_\tau|_{(\rho, \tau, w)=(0,0,0)} G(\rho, \tau, w)] \cdot \tau = \frac{(n-1)(n-2)}{2} \left(1 + \frac{(n-2)(n-4)}{2n-1} \right) \sigma \langle x, \tau \rangle \quad (5.34)$$

and

$$[D_w|_{(\rho, \tau, w)=(0,0,0)} G(\rho, \tau, w)] \cdot w = (n-2) \left(\partial_\nu \psi_w - (n-1)w \right), \quad (5.35)$$

where $\nu = -x$ and ψ_w is the unique solution to

$$\begin{cases} \Delta_0 \psi_w = 0 & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \psi_w = w & \text{on } \partial \mathbf{B}. \end{cases} \quad (5.36)$$

Proof We first prove (5.35). Set

$$\mathcal{U} := \{w \in C^{2,\alpha}(\partial \mathbf{B}) : w > 0\}$$

and consider the smooth map $T : \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathcal{U} \rightarrow C^{0,\alpha}(\partial \mathbf{B})$ defined by

$$T(\rho, \tau, w) := \Delta_{g_\rho} \bar{\Psi}_{\rho, \tau, w} = g_\rho^{ij} \partial_{ij} \bar{\Psi}_{\rho, \tau, w} + \frac{1}{\sqrt{|g_\rho|}} \partial_i \left(\sqrt{|g_\rho|} g_\rho^{ij} \right) \partial_j \bar{\Psi}_{\rho, \tau, w}.$$

For every $\omega \in C^{2,\alpha}(\partial\mathbf{B})$ we have

$$[D_w T(\rho, \tau, w)]\omega = \Delta_{g_\rho}[D_w \bar{\Psi}_{\rho, \tau, w}]\omega + h_w^{ij} \partial_{ij} \bar{\Psi}_{\rho, \tau, w} + \ell_w^j \partial_j \bar{\Psi}_{\rho, \tau, w} \quad (5.37)$$

with

$$h_w^{ij} := [D_w g_\rho^{ij}]\omega \quad \text{and} \quad \ell_w^j := \left[D_w \frac{1}{\sqrt{|g_\rho|}} \partial_i (\sqrt{|g_\rho|} g_\rho^{ij}) \right] \omega.$$

As already explained before (5.28), w is replaced by $\rho^{n-1}w$ in the metric coefficients of g_ρ involved in Δ_{g_ρ} . Hence evaluating (5.37) at $(\rho, \tau, w) = (0, 0, 0)$ and using (3.12) and Lemmas 3.1–3.2 we find after differentiating the right hand of (5.28) that the function

$$\psi_w := \frac{1}{n-2} [D_w]_{(\rho, \tau, w)=(0,0,0)} \bar{\Psi}_{\rho, \tau, w} \cdot w \quad (5.38)$$

satisfies

$$\begin{cases} \Delta \psi_w = 0 & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ \psi_w = w & \text{on } \partial\mathbf{B}. \end{cases} \quad (5.39)$$

We also have from (5.1),

$$[D_w]_{(\rho, \tau, w)=(0,0,0)} \widehat{v}_{g_\rho}^i(\rho^{n-1}w) \cdot w = 0 = [D_\tau]_{(\rho, \tau, w)=(0,0,0)} \widehat{v}_{g_\rho}^i(\rho^{n-1}w) \cdot \tau. \quad (5.40)$$

Indeed, $\widehat{v}_{g_\rho}^i(\rho^{n-1}w)$ is the i th-component of the unit vector $\widehat{v}_{g_\rho}$ in Lemma (5.1) with $\rho^{n-1}w$ in place of w and ρ replace by $1/\rho$. This together with the definition of the functions P_i in (3.4) allows us to recover (5.40). Now returning to (5.27) and using (5.38), (5.39) and (5.40) we get

$$[D_w]_{(\rho, \tau, w)=(0,0,0)} G(\rho, \tau, w) \cdot w = (n-2) \left(\partial_v \psi_w - (n-1)w \right),$$

which is the desired expression in (5.35).

Next we prove (5.34). We use a similar argument as above and differentiate (5.28) with respect to τ at $(\rho, \tau, w) = (0, 0, 0)$. This yields a function

$$F_\tau := [D_\tau]_{(\rho, \tau, w)=(0,0,0)} \bar{\Psi}_{\rho, \tau, w} \cdot \tau$$

satisfying

$$\begin{cases} \Delta F_\tau = C(n, \sigma) r^{1-2n} \langle \frac{x}{r^2}, \tau \rangle & \text{in } \mathbb{R}^n \setminus \mathbf{B} \\ F_\tau = 0 & \text{on } \partial\mathbf{B}, \end{cases} \quad (5.41)$$

where

$$C(n, \sigma) := \frac{(n-1)(n-4)(n-2)^2}{2} \sigma. \quad (5.42)$$

This combined with (5.27) and (5.40) yields

$$[D_\tau|_{(\rho, \tau, w)=(0,0,0)} G(\rho, \tau, w)] \cdot \tau = \frac{(n-1)(n-2)}{2} \sigma \langle x, \tau \rangle + \partial_\nu F_\tau. \quad (5.43)$$

Next, we show that

$$\partial_\nu F_\tau = \frac{C(n, \sigma)}{2n-1} \langle \cdot, \tau \rangle \quad \text{on } \mathbf{S}, \quad \text{with } \nu = -x. \quad (5.44)$$

Indeed, let

$$\Pi : L^2(\mathbf{S}) \rightarrow L^2(\mathbf{S})$$

be the orthogonal projection on $\text{span}\{x_1; \dots; x_n\}$ and consider the function

$$X^i(x) := \mathbb{K}(x^i) = |x|^{2-n} \frac{x^i}{|x|^2}.$$

We know that $\Delta_0 X^i = 0$ in $\mathbb{R}^n \setminus \{0\}$. We multiply (5.41) with X^i and integrate by parts to get

$$\begin{aligned} \int_{\mathbf{S}} \partial_\nu F_\tau(x) x^i \, d\mu_S &= C(n, \sigma) \int_{\mathbb{R}^n \setminus \mathbf{B}} |x|^{-1-n} |x|^{2-n} \left\langle \frac{x}{|x|^2}, \tau \right\rangle X^i(x) \, d\mu \\ &= C(n, \sigma) \tau^i \int_{\mathbf{S}} (x^i)^2 \, d\mu \int_1^\infty r^{1-3n} r^{n-1} \, dr. \end{aligned}$$

This implies that

$$\int_{\mathbf{S}} \partial_\nu F_\tau(x) x^i \, d\mu_S = \frac{C(n, \sigma) |\partial \mathbf{B}|}{2n-1} \tau^i = \frac{C(n, \sigma)}{2n-1} \int_{\mathbf{S}} \langle x, \tau \rangle x^i \, d\mu_S. \quad (5.45)$$

From this, we deduce that

$$\Pi \partial_\nu F_\tau(x) = \frac{C(n, \sigma) \langle \tau, x \rangle}{2n-1} \quad \text{for } x \in \mathbf{S}. \quad (5.46)$$

We **claim** that

$$\Pi^\perp \partial_\nu F_\tau = 0. \quad (5.47)$$

To see this, we let Y_i^k be the spherical harmonics for which $Y_i^1 = x^i$ for $i = 1, \dots, n$, corresponding to the eigenvalues $k(k+n-2)$ on sphere. We suppose that $k \neq 1$. We then define

$$X_i^k(x) = \mathbb{K}(Y_i^k)(x).$$

Then X_i^k are admissible test functions in (5.41). We observe that the right hand side in (5.41) is in $L^2(\mathbb{R}^n \setminus \mathbf{B})$. Therefore by simple arguments, we have that $F_\tau \in H^1(\mathbb{R}^n \setminus \mathbf{B})$. Using the decomposition in spherical harmonics of F_τ , we can see that $F_\tau(x) = f(|x|)\langle x, \tau \rangle$, for some some function f . From this we can multiply (5.41) by X_i^k and use the Gauss-Green formula to deduce that

$$\int_S \partial_\nu F_\tau(x) Y_i^k(x) d\mu_S = \tau^i \int_{\mathbb{R}^n \setminus \mathbf{B}} |x|^{-n} |x|^{-n} x^i X_i^k(x) d\mu = 0, \quad k \neq 1,$$

as claimed. Gathering (5.42), (5.43) and (5.44), we obtain (5.34). $\square$

5.3 The Spectral Properties of the Operator $\mathbb{L}$

We study the spectral properties of the operator $\mathbb{L}$ in (5.35) defined by

$$\mathbb{L}(w) := \partial_\nu \psi_w - (n-1)w,$$

where ψ_w satisfies (5.36). We consider the Kelvin transform (see (3.14) and (3.15), and observe $\mathbb{K}^{-1} = \mathbb{K}$) of ψ_w

$$\mathbb{K}(\psi_w)(x) = |x|^{2-n} \psi_w\left(\frac{x}{|x|^2}\right),$$

which satisfies

$$\Delta(\mathbb{K}(\psi_w)) = 0 \quad \text{in } \mathbf{B} \setminus \{0\}.$$

Moreover by direct computations, we have

$$\nabla(\mathbb{K}(\psi_w)) \cdot x = -\nabla \psi_w \cdot x + (2-n)w = \partial_\nu \psi_w + (2-n)w \quad \text{in } \partial \mathbf{B}. \quad (5.48)$$

Since $\psi_w \in \mathcal{C}_v^{2,\alpha}(\mathbb{R}^n \setminus \mathbf{B})$ with $v \in (2-n, 0)$, we see that

$$\lim_{|x| \rightarrow \infty} |x|^{2-n} \psi_w(x) = 0.$$

Therefore

$$\lim_{|x| \rightarrow 0} |x|^{n-2} \mathbb{K}(\psi_w)(x) = 0,$$

so the origin is a removable singularity for $\mathbb{K}(\psi_w)$ and thus $\mathbb{K}(\psi_w)$ solves

$$\begin{cases} \Delta \mathbb{K}(\psi_w) = 0 & \text{in } \mathbf{B} \\ \mathbb{K}(\psi_w) = w & \text{on } \partial \mathbf{B}. \end{cases} \quad (5.49)$$

By elliptic regularity theory, $\mathbb{K}(\psi_w) \in C^{2,\alpha}(\overline{\mathbf{B}})$. From (5.48) and the definition of $\mathbb{L}$ we get

$$w \mapsto \mathbb{L}(w) = \partial_{\tilde{\nu}}(\mathbb{K}(\psi_w)) - w,$$

where $\tilde{\nu} = \Theta$.

Thanks to [10], the spectrum of the operator $\mathbb{L}$ is given by

$$\lambda_k = k - 1, \quad k \in \mathbb{N} \quad (5.50)$$

meaning that the kernel of the operator $\mathbb{L}$ is given by the space V_1 spanned by linear coordinates on the sphere $\mathbf{S}$

$$V_1 := \{\Theta^i, i = 1, \dots, n\}.$$

Moreover there exists a constant $C > 0$ such that

$$\|w\|_{C^{2,\alpha}(\mathbf{S})} \leq C \|\mathbb{L}(w)\|_{C^{1,\alpha}(\mathbf{S})}, \quad (5.51)$$

provided $w \in \Pi^\perp C^{\alpha}(\mathbf{S})$.

6 Solving the Nonlinear Problem

We want to show that provided ρ is small, we can find τ and w such that (5.32) holds. That is

$$G(\rho, \tau, w) + \frac{(n-2)^2}{2} \sigma = 0. \quad (6.1)$$

Define

$$\tilde{G}(\rho, \tau, w) := G(\rho, \tau, w) + \frac{(n-2)^2}{2} \sigma.$$

Then from (5.31),

$$\tilde{G}(0, 0, 0) = 0. \quad (6.2)$$

We denote by Π the L^2 -orthogonal projection from $\mathcal{C}^{1,\alpha}(\mathbf{S})$ onto V_1 and $T : V_1 \rightarrow \mathbb{R}^n$ the isomorphism sending $x_i|_{\partial\mathbf{B}}$ to e_i . We also define $\tilde{\Pi} := T \circ \Pi$,

$$\tilde{K} := \tilde{\Pi} \circ \tilde{G} : (0, \delta_0) \times \mathbf{B}_{c_0}(0) \times \mathbf{B}_{c_0}(0) \longrightarrow \mathbb{R}^n$$

and consider the equation

$$\tilde{K}(\rho, \tau, w) = 0. \quad (6.3)$$

The mapping $\tilde{K}$ has the following properties:

- $\tilde{K}(0, 0, 0) = 0$. This is from (6.2),
- $D_\tau|_{(\rho, \tau, w)=(0,0,0)} \tilde{K}$ is a the identity in $\mathbb{R}^n$ times a constant, which follows from (5.34).

Applying the implicit function theorem, we find a unique smooth mapping

$$(-r_0, r_0) \times \mathbf{B}_{k_1}(0) \longrightarrow \mathbf{B}_{k_2}(0) \subset \mathbb{R}^n, \quad (\rho, w) \mapsto \tau(\rho, w)$$

defined for some constants r_0, k_1 and k_2 in $\mathbb{R}_+^*$ such that

$$\tilde{K}(\rho, \tau(\rho, w), w) = 0 \quad \text{for all } (\rho, w) \in (-r_0, r_0) \times \mathbf{B}_{k_0}(0). \quad (6.4)$$

Net we provide estimates for the function $\tau(\rho, w)$ in (6.4). Observe that since $\tilde{\Pi} \circ \mathbb{L} = 0$, (6.4) is equivalent to

$$C'(n, \sigma)\tau + \tilde{\Pi}[\rho^{1-n}P_1(1/\rho, x, \rho^{n-1}w, \tau)] = 0 \quad \text{on } \mathbf{S}, \quad (6.5)$$

where $C'(n, \sigma)$ is the constant appearing in (5.34). This can be seen by writing the Taylor expansion of G using (5.34) and (5.35). We then deduce from (3.3) and (3.4) the estimates

$$|\tau(\rho, w)| \leq C\rho \quad \text{and} \quad |D_w \tau(\rho, w)| \leq C\rho^{n-1}. \quad (6.6)$$

Now replace τ by $\tau(\rho, w)$ in (6.1) and consider the equation

$$\Pi^\perp(\tilde{G}(\rho, \tau, w)) = 0. \quad (6.7)$$

From (5.35) and the estimates in (6.6), the mapping

$$D_w|_{(\rho, \tau, w)=(0,0,0)} \Pi^\perp \circ \tilde{G} = \mathbb{L} : V_1^\perp \longrightarrow \mathbb{L}(V_1^\perp),$$

is an isomorphism from Sect. 5.3. Hence, there exists a unique solution $w(\rho)$ to (6.7) for small $\rho \in (0, R_0)$. Using (5.34) and (5.35),

$$\mathbb{L}(w(\rho)) + \Pi^\perp[\rho^{1-n}P_1(1/\rho, x, \rho^{n-1}w(\rho), \tau(\rho, w))] = 0 \quad \text{on } \mathbf{S},$$

and from (5.51) and (3.4),

$$\|w(\rho)\|_{C^{2,\alpha}(\mathbf{S})} \leq C\rho. \quad (6.8)$$

Decreasing R_0 if necessary, the analysis of the previous section establishes the first statement of Theorem 1 with $\rho_0 = \frac{1}{R_0}$ and

$$\Omega_\rho = \widetilde{\Omega}_{\frac{1}{\rho}}, \quad \rho \in (\rho_0, +\infty),$$

where

$$\widetilde{\Omega}_\rho := \Phi_{\frac{1}{\rho}, \tau(\rho, w(\rho)), \rho^{n-1}w(\rho)}(\mathbb{R}^n \setminus \mathbf{B}), \quad \rho \in (0, R_0). \quad (6.9)$$

In addition combining (5.17), (5.18), (5.26) (5.27) and (6.1), we have for every $\rho \in (0, R_0)$

$$H(\rho, \tau(\rho, w(\rho)), \rho^{n-1}w(\rho)) = -\frac{(n-2)^2}{2}\sigma\rho^{n-1}$$

and

$$\frac{\partial \widehat{u}}{\partial \widehat{v}_{g_\rho}} = (n-2) \left(1 - \frac{(n-1)}{2}\sigma\rho^{n-1} \right).$$

Recalling also (3.11), $\widehat{v}_{\widehat{g}} = \rho\widehat{v}_{g_\rho}$ and $\widehat{g}(\nabla_{\widehat{g}}\widehat{u}, \widehat{v}_{\widehat{g}}) = \rho g_\rho(\nabla_{g_\rho}\widehat{u}, \widehat{v}_{g_\rho})$ we for every $\rho \in (0, R_0)$, Therefore, the constant $C(\rho, \sigma, n)$ in Theorem 1 is given by

$$C(\rho, \sigma, n) = \frac{n-2}{\rho} \left(1 - \frac{(n-1)}{2\rho^{n-1}}\sigma \right), \quad \rho > \rho_0. \quad (6.10)$$

It remains to show that the family $(\partial\Omega_\rho)_{\rho \in (\rho_0, +\infty)}$ constitutes a smooth foliation.

7 Foliation by Boundaries of Critical Capacitors

Proposition 6 *There exists a constant $\rho_0 > 1$ such that the family $(\partial K_\rho)_{\rho > \rho_0}$ constitutes a smooth foliation.*

Proof We are proving that the family $(\partial\Omega_\rho)_{\rho \in (\rho_0, +\infty)}$ constitutes a foliation of $M \setminus \Omega_{\rho_0}$. The proof is inspired by the argument in [3, Sect. 5] and [14, Pages 9-10].

Notice that $\partial\widetilde{\Omega}_\rho$ is given by

$$\partial\widetilde{\Omega}_\rho = \Phi(\widetilde{S}_{\frac{1}{\rho}, \tau(\rho, w(\rho)), \rho^{n-1}w(\rho)}),$$

where

$$\tilde{S}_{\frac{1}{\rho}, \tau, w} = \left\{ y = \frac{1}{\rho} \left(x + \tau + w(x)x \right), \quad x \in \mathbf{S} \right\}. \quad (7.1)$$

We define the functions

$$h(\rho, x) := \frac{1}{\rho} \left(x + \tau(\rho, w(\rho)) + \rho^{n-1} w_\rho(x)x \right) \quad \text{and} \quad v(\rho, x) := \frac{h(\rho, x)}{|h(\rho, x)|_{g_\rho}}, \quad x \in \mathbf{S}.$$

By the estimates in (6.6) and (6.8), the function $v(\rho, \cdot)$ extends continuously at $\rho = 0$ with $v(0, \cdot) = I_{\mathbf{S}}$ and for all ρ small $v(\rho, \cdot)$ is a diffeomorphism from $\mathbf{S}$ into itself.

Thus for all $y \in \mathbf{S}$,

$$h(\rho, v^{-1}(\rho, y)) = |h(\rho, v^{-1}(\rho, y))|_{g_\rho} y.$$

We put

$$\tilde{\varphi}(\rho, y) := |h(\rho, v^{-1}(\rho, y))|_{g_\rho},$$

where $v^{-1}(\rho, \cdot)$ denotes the inverse of $v(\rho, \cdot)$. Then

$$\tilde{S}_{\frac{1}{\rho}, \tau(\rho, w(\rho)), \rho^{n-1} w(\rho)} = \tilde{S}_\rho := \left\{ \tilde{\varphi}(\rho, y)y, \quad y \in \mathbf{S} \right\}. \quad (7.2)$$

Using the estimates in (6.6) and (6.8) once again, we find

$$\begin{aligned} D_x h &= \frac{1}{\rho} \left(I_{\mathbf{S}} + \rho^{n-1} L^1(w_\rho)x + \rho^{n-1} w_\rho(x) I_{\mathbf{S}} \right) = \frac{1}{\rho} (I_{\mathbf{S}} + O(\rho)) \\ \frac{\partial h}{\partial \rho} &= -\frac{1}{\rho^2} \left(x + \tau - \rho \frac{\partial \tau}{\partial \rho} \right. \\ &\quad \left. - \rho D_w \tau(\rho, w(\rho)) w'_\rho + \rho^n w'_\rho(x)x - (n-2) \rho^{n-1} w_\rho(x)x \right) \\ &= -\frac{1}{\rho^2} (x + O(\rho)). \end{aligned}$$

Also, since

$$|v^{-1}(\rho, y)|_{g_\rho} = 1 \quad \text{for all } \rho \in (0, R_0) \quad \text{and } y \in \mathbf{S},$$

we have

$$\langle v^{-1}(\rho, y), \partial_\rho v^{-1}(\rho, y) \rangle_{g_\rho} = 0 \quad \text{for all } \rho \in (0, R_0) \quad \text{and } y \in \mathbf{S}.$$

Using this, we then obtain

$$\begin{aligned}
 \frac{\partial \tilde{\varphi}}{\partial \rho} &= \frac{1}{|h(\rho, v^{-1}(\rho, y))|_{g_\rho}} \langle h(\rho, v^{-1}(\rho, y)), \frac{\partial h}{\partial \rho}(\rho, v^{-1}(\rho, y)) + D_x h(\partial_\rho v^{-1}) \rangle_{g_\rho} \\
 &= \frac{1}{|v^{-1}(\rho, y) + O(\rho)|_{g_\rho}} \langle v^{-1}(\rho, y) + O(\rho), -\frac{1}{\rho} v^{-1}(\rho, y) + \partial_\rho v^{-1} + O(\rho) \rangle_{g_\rho} \\
 &= \frac{1}{\rho |v^{-1}(\rho, y) + O(\rho)|_{g_\rho}} \langle v^{-1}(\rho, y) + O(\rho), -v^{-1}(\rho, y) + \rho \partial_\rho v^{-1} + O(\rho^2) \rangle_{g_\rho} \\
 &= -\frac{1}{\rho} (1 + O(\rho)).
 \end{aligned}$$

We conclude the the function $\tilde{\varphi}(\rho, x)$ is strictly decreasing with respect to ρ for ρ small or equivalently $\tilde{\varphi}(\rho^{-1}, x)$ is strictly increasing for ρ large. Thanks to (7.2), the family $(\tilde{S}_\rho)_\rho$ constitutes a foliation of $\mathbb{R}^n \setminus \mathbf{B}_{\rho_0}$. Since Φ is a diffeomorphism and $\partial\Omega_\rho = \Phi(\tilde{S}_\rho)$, we deduce that the family $(\partial\Omega_\rho)_{\rho \in (\rho_0, +\infty)}$ foliates $M \setminus \Omega_{\rho_0}$ and the proof of Theorem 1 is complete. $\square$

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Appendix A: Variational Setting

In this section we define two scale-invariant energies associated with capacity, and compute their first variations. We perform the computations below in Euclidean space, but again it is an easy exercise to carry them out in a Riemannian manifold with one asymptotically flat end.

We earlier defined the capacity function Cap on compact sets $K \subset \mathbb{R}^n$ in (1.1). A change of variables shows us the scaling law

$$\text{Cap}(RK) = R^{n-2} \text{Cap}(K) \quad (\text{A.1})$$

for any $R > 0$, and a quick computation gives $\text{Cap}(\bar{\mathbf{B}}_R) = R^{n-2}$, with the equilibrium potential function $U(x) = R^{n-2}|x|^{2-n}$. As we discussed in the introduction, one can either normalize using volume or surface area, leading to the following two scale-invariant functionals:

$$\mathcal{E}_0(K) = \frac{\text{Cap}(K)}{|K|^{\frac{n-2}{n}}}, \quad \mathcal{E}_1(K) = \frac{\text{Cap}(K)}{|\Sigma|^{\frac{n-2}{n-1}}}. \quad (\text{A.2})$$

By (A.1) both $\mathcal{E}_0$ and $\mathcal{E}_1$ are scale-invariant.

To compute the first variation of both $\mathcal{E}_0$ and $\mathcal{E}_1$ we let $X : (-\varepsilon, \varepsilon) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field, and let ξ be its flow, defined by

$$\xi : (-\varepsilon, \varepsilon) \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \frac{\partial \xi}{\partial t}(t, x) = X(t, x), \quad \xi(0, x) = x.$$

Let $K_t = \xi(t, \cdot)(K)$ and let U_t be the solution to (1.2) in $\Omega_t = \mathbb{R}^n \setminus K_t$. It will also be convenient to denote $\Sigma = \partial K = \partial \Omega$, and $\Sigma_t = \partial K_t = \partial \Omega_t$.

Lemma A.1 *We have*

$$\begin{aligned} \left. \frac{d}{dt} \mathcal{E}_0(K_t) \right|_{t=0} &= \frac{1}{n(n-2)\omega_n |K|^{\frac{n-2}{n}}} \left[\frac{(n-2)^2 \omega_n \text{Cap}(K)}{|K|} \int_{\Sigma} \langle X, \eta \rangle d\sigma \right. \\ &\quad \left. - \int_{\Sigma} \langle X, \eta \rangle \left(\frac{\partial U}{\partial \eta} \right)^2 d\sigma \right], \end{aligned} \quad (\text{A.3})$$

where η is the unit normal vector pointing into K .

Proof Differentiating the boundary condition $U_t|_{\Sigma_t} = 0$, we see

$$\frac{\partial U}{\partial t} = -\langle X, \eta \rangle \frac{\partial U}{\partial \eta} \text{ on } \Sigma_t.$$

Thus

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}_0(K_t) &= \frac{1}{n(n-2)\omega_n} \left. \frac{d}{dt} \right|_{t=0} |K|^{\frac{2-n}{n}} \int_{\Omega_t} |\nabla U_t|^2 dx \\ &= \frac{(n-2)}{n} \frac{\text{Cap}(K)}{n(n-2)\omega_n |K|^{\frac{n-2}{n}-1}} \int_{\Sigma} \langle X, \eta \rangle d\sigma \\ &\quad + \frac{1}{n(n-2)\omega_n |K|^{\frac{n-2}{n}}} \left[2 \int_{\Omega} \left\langle \nabla U, \nabla \frac{\partial U}{\partial t} \right\rangle dx + \int_{\Sigma} \langle X, \eta \rangle |\nabla U|^2 d\sigma \right] \\ &= \frac{(n-2)}{n} \frac{\text{Cap}(K)}{n(n-2)\omega_n |K|^{\frac{n-2}{n}-1}} \int_{\Sigma} \langle X, \eta \rangle d\sigma \\ &\quad + \frac{1}{n(n-2)\omega_n |K|^{\frac{n-2}{n}}} \left[\int_{\Sigma} \langle X, \eta \rangle |\nabla U|^2 d\sigma - 2 \int_{\Omega} \frac{\partial U}{\partial t} \Delta U dx \right] \end{aligned}$$

$$\begin{aligned}
& +2 \int_{\Sigma} \frac{\partial U}{\partial t} \frac{\partial U}{\partial \eta} \Big] \\
& = \frac{(n-2)}{n} \frac{\text{Cap}(K)}{n(n-2)\omega_n |K|^{\frac{n-2}{n}-1}} \int_{\Sigma} \langle X, \eta \rangle d\sigma \\
& \quad - \frac{1}{n(n-2)\omega_n |K|^{\frac{n-2}{n}}} \left[\int_{\Sigma} \langle X, \eta \rangle \left(\frac{\partial U}{\partial \eta} \right)^2 d\sigma \right] \\
& = \frac{1}{n(n-2)\omega_n |K|^{\frac{n-2}{n}}} \left[\frac{(n-2)^2 \omega_n \text{Cap}(K)}{|K|} \int_{\Sigma} \langle X, \eta \rangle d\sigma \right. \\
& \quad \left. - \int_{\Sigma} \langle X, \eta \rangle \left(\frac{\partial U}{\partial \eta} \right)^2 d\sigma \right].
\end{aligned}$$

Here we have used the fact that U is constant on Σ , so $|\nabla U| = \frac{\partial U}{\partial \eta}$ there. $\square$

Corollary 7 *A compact set K with nonempty interior $\mathcal{O}$ and smooth boundary Σ is a critical point of $\mathcal{E}_0$ if and only if $\Omega = \mathbb{R}^n \setminus K$ supports a solution to the over-determined boundary value problem (1.3).*

Proof Setting

$$\Lambda^2 = \frac{(n-2)^2 \omega_n \text{Cap}(K)}{|K|},$$

we use (A.3) to see that K is a critical point of $\mathcal{E}_0$ if and only if

$$\int_{\Sigma} \langle X, \eta \rangle \left[\Lambda^2 - \left(\frac{\partial U}{\partial \eta} \right)^2 \right] d\sigma = 0$$

for all possible variation fields X , which in turn implies

$$\frac{\partial U}{\partial \eta} = \Lambda = (n-2) \sqrt{\frac{\omega_n \text{Cap}(K)}{|K|}} \quad (\text{A.4})$$

along Σ . We notice that if $K = \overline{\mathbf{B}}_{\rho}$ this constant reduces to $\Lambda = \frac{n-2}{\rho}$.

Conversely, let Ω admit a solution to (1.3). By the uniqueness of solutions to (1.2), this function must be the equilibrium potential of K , so by (A.3)

$$\frac{d}{dt} \mathcal{E}_0(K_t) \Big|_{t=0} = 0$$

for all possible variation fields X . $\square$

Though we will not use it, we include the following derivation to satisfy the reader's curiosity.

Lemma A.2 *We have*

$$-\frac{d}{dt}\mathcal{E}_1\Big|_{t=0} = \frac{1}{|\Sigma|^{\frac{n-2}{n-1}}} \left[\left(\frac{n-2}{n-1} \right) \left(\frac{\text{Cap}(K)}{|\Sigma|} \right) \int_{\Sigma} \langle X, \eta \rangle H_{\Sigma} d\sigma \right. \\ \left. + \frac{1}{n(n-2)\omega_n} \int_{\Sigma} \langle X, \eta \rangle \left(\frac{\partial U}{\partial \eta} \right)^2 d\sigma \right].$$

Proof We begin by differentiating the boundary condition $U|_{\Sigma} = 1$ to see

$$\frac{\partial U}{\partial t} = -\langle X, \eta \rangle \frac{\partial U}{\partial \eta}$$

on Σ . Thus

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_1\Big|_{t=0} &= \frac{d}{dt}\Big|_{t=0} \left(\frac{1}{n(n-2)\omega_n} |\Sigma|^{\frac{2-n}{n-1}} \int_{\Omega_t} \langle \nabla U_t, \nabla U_t \rangle dx \right) \\ &= \frac{1}{|\Sigma|^{\frac{n-2}{n-1}}} \left(\frac{2-n}{n-1} \right) |\Sigma|^{-1} \frac{1}{n(n-2)\omega_n} \int_{\Omega} |\nabla U|^2 dx \int_{\Sigma} \langle X, \eta \rangle H_{\Sigma} d\sigma \\ &\quad + \frac{1}{n(n-2)\omega_n |\Sigma|^{\frac{n-2}{n-1}}} \left[2 \int_{\Omega} \left\langle \nabla U, \nabla \frac{\partial U}{\partial t} \right\rangle dx + \int_{\Sigma} \langle X, \eta \rangle |\nabla U|^2 d\sigma \right] \\ &= - \left(\frac{n-2}{n-1} \right) \frac{\text{Cap}(K)}{|\Sigma|^{\frac{n-2}{n-1}+1}} \int_{\Sigma} \langle X, \eta \rangle H_{\Sigma} d\sigma \\ &\quad + \frac{1}{n(n-2)\omega_n |\Sigma|^{\frac{n-2}{n-1}}} \left[\int_{\Sigma} \langle X, \eta \rangle |\nabla U|^2 d\sigma - 2 \int_{\Omega} \frac{\partial U}{\partial t} \Delta_0 U dx \right. \\ &\quad \left. + 2 \int_{\Sigma} \frac{\partial U}{\partial t} \frac{\partial U}{\partial \eta} d\sigma \right] \\ &= - \frac{1}{|\Sigma|^{\frac{n-2}{n-1}}} \left[\left(\frac{n-2}{n-1} \right) \frac{\text{Cap}(K)}{|\Sigma|} \int_{\Sigma} \langle X, \eta \rangle H_{\Sigma} d\sigma \right. \\ &\quad \left. + \frac{1}{n(n-2)\omega_n} \int_{\Sigma} \langle X, \eta \rangle \left(\frac{\partial U}{\partial \eta} \right)^2 d\sigma \right]. \end{aligned}$$

Here we've used the fact that H_{Σ} is the first variation of $|\Sigma|$ and that $|\nabla U| = -\frac{\partial U}{\partial \eta}$ on Σ . $\square$

One can mimic the proof of Corollary 7 to show that critical points of $\mathcal{E}_1$ are precisely those sets K which admit a solution to the over-determined boundary value problem

$$\Delta U = 0, \quad U|_{\partial K} = 1, \quad \lim_{|x| \rightarrow \infty} U(x) = 0, \quad \frac{\partial U}{\partial \eta} = \Lambda H,$$

where H is the mean curvature of $\Sigma = \partial K$.

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