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Ruelle Zeta Functions of Hyperbolic Manifolds and Reidemeister Torsion

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Abstract

This paper is concerned with the behavior of twisted Ruelle zeta functions of compact hyperbolic manifolds at the origin. Fried proved that for an orthogonal acyclic representation of the fundamental group of a compact hyperbolic manifold, the twisted Ruelle zeta function is holomorphic at $s = 0$ and its value at $s = 0$ equals the Reidemeister torsion. He also established a more general result for orthogonal representations, which are not acyclic. The purpose of the present paper is to extend Fried's result to arbitrary finite dimensional representations of the fundamental group. The Reidemeister torsion is replaced by the complex-valued combinatorial torsion introduced by Cappell and Miller.

Keywords Ruelle zeta function · Analytic torsion

Mathematics Subject Classification Primary: 37C30, Secondary: 58J20

1 Introduction

Let X be a d-dimensional closed, oriented hyperbolic manifold. Then there exists a discrete torsion free subgroup $\Gamma \subset \mathrm{SO}_0(d,1)$ such that $X = \Gamma\backslash\mathbb{H}^d$, where $\mathbb{H}^d = \mathrm{SO}_0(d,1)/\mathrm{SO}(d)$ is the d-dimensional hyperbolic space. Every $\gamma \in \Gamma \setminus \{e\}$ is loxodromic and the Γ-conjugacy class $[\gamma]$ corresponds to a unique closed geodesic τ_γ. Let $\ell(\gamma)$ denote the length of τ_γ. A conjugacy class is called prime if γ is not a non-trivial power of some other element of Γ. Let $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional complex representation of Γ and let $s \in \mathbb{C}$. Then the Ruelle zeta function

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$R(s, \chi)$ is defined by the following Euler product

$$R(s, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \chi(\gamma) e^{-s\ell(\gamma)}\right). \tag{1.1}$$

The infinite product is absolutely convergent in a certain half plane $\operatorname{Re}(s) > C$ and admits a meromorphic extension to the entire complex plane [11,31]. The Ruelle zeta function is a dynamical zeta function associated to the geodesic flow on the unit sphere bundle $S(X)$ of X. There are formal analogies with the zeta functions in number theory such as the Artin L-function associated to a Galois representation. Analogues to the role of zeta functions in number theory, one expects that special values of the Ruelle zeta function provide a connection between the length spectrum of closed geodesics and geometric and topological invariants of the manifold.

In [10], Fried has established such a connection. To explain his result we need to introduce some notation. Recall that a representation χ is called *acyclic*, if the cohomology $H^*(X, F_\chi)$ of X with coefficients in the flat bundle $F_\chi \to X$ associated to χ vanishes. Let χ be an orthogonal acyclic representation. Then F_χ is equipped with a canonical fiber metric which is compatible with the flat connection. Let $\Delta_{k,\chi}$ be the Laplacian acting in the space $\Lambda^k(X, F_\chi)$ of F_χ-valued k-forms. Regarded as operator in the space of L^2-forms, it is essentially self-adjoint with a discrete spectrum $\operatorname{Spec}(\Delta_{k,\chi})$ consisting of eigenvalues λ of finite multiplicity $m(\lambda)$. Let $\zeta_k(s; \chi) = \sum_{\lambda \in \operatorname{Spec}(\Delta_{k,\chi})} m(\lambda)\lambda^{-s}$ be the spectral zeta function of $\Delta_{k,\chi}$ [30]. The series converges absolutely in the half plane $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to the complex plane, which is holomorphic at $s = 0$. Then the Ray–Singer analytic torsion $T^{RS}(X, \chi) \in \mathbb{R}^+$ is defined by

$$\log T^{RS}(X, \chi) := \frac{1}{2} \sum_{k=1}^{d} (-1)^k k \frac{d}{ds} \zeta_k(s; \chi)\big|_{s=0}, \tag{1.2}$$

[28]. Now we can state the result of Fried [10, Theorem 1]. He proved that for an acyclic unitary representation χ the Ruelle zeta function $R(s, \chi)$ is holomorphic at $s = 0$ and

$$|R(0, \chi)^\varepsilon| = T^{RS}(X, \chi)^2, \tag{1.3}$$

where $\varepsilon = (-1)^{d-1}$ and the absolute value can be removed if $d > 2$. If χ is not acyclic, but still orthogonal, $R(s, \chi)$ may have a pole or zero at $s = 0$. Fried [10] has determined the order of $R(s, \chi)$ at $s = 0$ and the leading coefficient of the Laurent expansion around $s = 0$. Let $b_k(\chi) := \dim H^k(X, F_\chi)$. Assume that $d = 2n + 1$. Put

$$h = 2 \sum_{k=0}^{n} (n + 1 - k)(-1)^k b_k(\chi).$$

Then by [10, Theorem 3], the order of $R(s,\chi)$ at $s=0$ is h and the leading term of the Laurent expansion of $R(s,\chi)$ at $s=0$ is

$$C(\chi)\cdot T^{RS}(X,\chi)^2 s^h, \tag{1.4}$$

where $C(\chi)$ is a constant that depends on the Betti numbers $b_k(\chi)$. In [13, p. 66] Fried conjectured that (1.3) holds for all compact locally symmetric manifolds X and acyclic orthogonal bundles over $S(X)$. This conjecture was recently proved by Shen [29].

Let χ be a unitary acyclic representation of Γ. Let $\tau(X,\chi)$ be the Reidemeister torsion [25,28]. It is defined in terms of a smooth triangulation of X. However, it is independent of the particular C^∞-triangulation. Since χ is acyclic, $\tau(X,\chi)$ is a topological invariant, i.e., it does not depend on the metrics on X and in F_ρ. By [8,24] we have $T^{RS}(X,\chi)=\tau(X,\chi)$. Assume that d is odd. Then (1.3) can be restated as

$$R(0,\chi)=\tau(X,\chi)^2. \tag{1.5}$$

This provides an interesting relation between the length spectrum of closed geodesics and a secondary topological invariant.

Another class of interesting representations arises in the following way. Let $G:=\mathrm{SO}_0(d,1)$. Let ρ be a finite dimensional complex or real representation of G. Then $\rho|_\Gamma$ is a finite dimensional representation of Γ. In general, $\rho|_\Gamma$ is not an orthogonal representation. However, the flat vector bundle F_ρ associated with $\rho|_\Gamma$ can be equipped with a canonical fiber metric which allows the use of methods of harmonic analysis to study the Laplace operators $\Delta_{k,\rho}$. Put

$$R(s,\rho):=R(s,\rho|_\Gamma).$$

The behavior of $R(s,\rho)$ at $s=0$ has been studied by Wotzke [35]. Let $\theta\colon G\to G$ be the Cartan involution of G with respect to $K=\mathrm{SO}(d)$. Let $\rho_\theta:=\rho\circ\theta$. Also denote by $T^{RS}(X,\rho)$ the analytic torsion of X with respect to $\rho|_\Gamma$ and an admissible metric in F_ρ. Assume that ρ is irreducible. If $\rho\not\cong\rho_\theta$, then the cohomology $H^*(X,F_\rho)$ vanishes [2, Chapt. VII, Theorem 6.7]. Moreover, in this case Wotzke [35] has proved that $R(s,\rho)$ is holomorphic at $s=0$ and

$$|R(0,\rho)|=T^{RS}(X,\rho)^2. \tag{1.6}$$

If $\rho\cong\rho_\theta$, then $R(s,\rho)$ may have a zero or a pole at $s=0$. Wotzke [35] has also determined the order of $R(s,\rho)$ at $s=0$ and the coefficient of the leading term of the Laurent expansion of $R(s,\rho)$ at $s=0$. As in (1.4) the main contribution to the coefficient is the analytic torsion.

Let $\tau(X,\rho)$ be the Reidemeister torsion [25] of X with respect to $\rho|_\Gamma$. Assume that ρ is irreducible and $\rho\not\cong\rho_\theta$. As mentioned above, the cohomology $H^*(X,F_\rho)$ vanishes in this case and therefore, $\tau(X,\rho)$ is independent of the metrics on X and in F_ρ. By [25, Theorem 1] we have $T^{RS}(X,\rho)=\tau(X,\rho)$. Thus (1.6) can be restated as

$$|R(0,\rho)| = \tau(X,\rho)^2. \tag{1.7}$$

This equality has interesting consequences for arithmetic subgroups Γ. Assume that there exists a Γ-invariant lattice $M_\rho \subset V_\rho$. Let $\mathcal{M}_\rho \to X$ be the associated local system of free $\mathbb{Z}$-modules of finite rank. The cohomology $H^*(X,\mathcal{M}_\rho)$ is a finitely generated abelian group. If ρ_Γ is acyclic, $H^*(X,\mathcal{M}_\rho)$ is a finite abelian group. Denote by $|H^k(X,\mathcal{M}_\rho)|$ the order of $H^k(X,\mathcal{M}_\rho)$. By [8, (1.4)], [1, Sect. 2.2], $\tau(X,\rho)$ can be expressed in terms of $|H^k(X,\mathcal{M}_\rho)|$, $k=0,\dots,d$. Combined with (1.7) we get

$$|R(0,\rho)| = \prod_{k=0}^{d} |H^k(X,\mathcal{M}_\rho)|^{(-1)^{k+1}}. \tag{1.8}$$

This is another interesting relation between the length spectrum of X and topological invariants of X.

For arithmetic subgroups $\Gamma \subset G$, representations of G with Γ-invariant lattices in the corresponding representation space exist. See [1,20].

The main purpose of this paper is to extend the above results about the behavior of the Ruelle zeta function at $s=0$ to every finite dimensional representation χ of Γ. To this end we use a complex version $T^{\mathbb{C}}(X,\chi)$ of the analytic torsion, which was introduced by Cappell and Miller [7] and which is closely related to the refined analytic torsion of Braverman and Kappeler [3]. It is defined in terms of the flat Laplacians $\Delta^\sharp_{k,\chi}$, $k=0,\dots,d$, which are obtained by coupling the Laplacian Δ_k on k-forms to the flat bundle F_χ (see Sect. 2 for its definition). In general, the flat Laplacian $\Delta^\sharp_{k,\chi}$ is not self-adjoint. However, its principal symbol equals the principal symbol of a Laplace type operator. Therefore, it has good spectral properties which allows to carry over most of the results from the self-adjoint case. The Cappell–Miller torsion $T^{\mathbb{C}}(X,\chi)$ is defined as an element of the determinant line

$$T^{\mathbb{C}}(X,\chi) \in \det H^*(X,F_\chi) \otimes (\det H_*(X,F_\chi))^*.$$

For an acyclic representation $T^{\mathbb{C}}(X,\chi)$ is a complex number and

$$|T^{\mathbb{C}}(X,\chi)| = T^{RS}(X,\chi)^2, \tag{1.9}$$

where $T^{\mathbb{C}}(X,\chi)$ is the Ray–Singer analytic torsion with respect to any choice of a fiber metric in F_χ. Since χ is acyclic, $T^{RS}(X,\chi)$ is independent of the choice of the metric in F_χ.

Let V_0^k be the generalized eigenspace of $\Delta^\sharp_{k,\chi}$, $k=0,\dots,d$, with generalized eigenvalue 0. Let $d_\chi^{*,\sharp}$ be the coupling of the co-differential $d_\chi^*\colon \Lambda^*(X) \to \Lambda^*(X)$ to the flat bundle F_χ. Then $(V_0^*, d_\chi, d_\chi^{*,\sharp})$ is a double complex in the sense of [7, §6]. Let

$$T_0(X,\chi) \in \det H^*(X,F_\chi) \otimes (\det H_*(X,F_\chi))^*. \tag{1.10}$$

be its torsion [7, §6]. We note that $T^{\mathbb{C}}(X,\chi)$ and $T_0(X,\chi)$ are both non-zero elements of the determinant line $\det H^*(X,F_\chi)\otimes(\det H_*(X,F_\chi))^*$. Hence there exists $\lambda\in\mathbb{C}$ with $T^{\mathbb{C}}(X,\chi)=\lambda T_0(X,\chi)$. Set

$$\frac{T^{\mathbb{C}}(X,\chi)}{T_0(X,\chi)}:=\lambda.$$

Put

$$h_k:=\dim V_0^k,\quad k=0,\dots,d. \tag{1.11}$$

Furthermore, let $d=2n+1$ and put

$$h:=2\sum_{k=0}^{n}(n+1-k)(-1)^k h_k \tag{1.12}$$

and

$$C(d,\chi):=\prod_{k=0}^{d-1}\prod_{\substack{p=k\\ p\neq n}}^{d-1}(2(n-p))^{(-1)^k h_k}. \tag{1.13}$$

Then our main result is the following theorem.

Theorem 1.1 *Let χ be a finite dimensional complex representation of Γ. Let h be defined by (1.12). Then the order of the singularity of $R(s,\chi)$ at $s=0$ is h and*

$$\lim_{s\to 0}s^{-h}R(s,\chi)=C(d,\chi)\cdot\frac{T^{\mathbb{C}}(X,\chi)}{T_0(X,\chi)}. \tag{1.14}$$

As above there is a combinatorial formula. Let $\tau_{\mathrm{comb}}(X,\chi)\in\det H^*(X,F_\chi)\otimes\det H^*(X,F_\chi^*)$ be the combinatorial torsion defined by Cappell and Miller [7, Sect. 9], which is defined in terms of a triangulation of X, but is independent of the choice of the triangulation. An equivalent definition is as follows. For a unimodular complex representation ρ of Γ, one can define a complex valued Reidemeister torsion $\tau^{\mathbb{C}}(X,\rho)\in\det H^*(X,F_\rho)$ [9, §3]. See also Sect. 6.1. Now let χ^* be the contragredient representation to χ. The representation $\chi\oplus\chi^*$ is unimodular and therefore, the complex Reidemeister torsion $\tau^{\mathbb{C}}(X,\chi\oplus\chi^*)\in\det H^*(X,F_\chi\oplus F_\chi^*)$ is well defined. Then by [5] it follows that with respect to the canonical isomorphism

$$\det H^*(X,F_\chi)\otimes\det H^*(X,F_\chi^*)\cong\det H^*(X,F_\chi\oplus F_\chi^*)$$

we have $\tau_{\mathrm{comb}}(X,\chi)=\pm\tau^{\mathbb{C}}(X,\chi\oplus\chi^*)$. By [7, Theorem 10.1] we have $T^{\mathbb{C}}(X,\chi)=\tau_{\mathrm{comb}}(X,\chi)$. Thus we can restate Theorem 1.1 as

$$\lim_{s\to 0}s^{-h}R(s,\chi)=C(d,\chi)\cdot\frac{\tau_{\mathrm{comb}}(X,\chi)}{T_0(X,\chi)}. \tag{1.15}$$

If χ is acyclic, then $T^{\mathbb{C}}(X,\chi)$, $T_0(X,\chi)$ and $\tau_{\mathrm{comb}}(X,\chi)$ are complex numbers and on the right hand side of (1.14) and (1.15) appear quotients of complex numbers.

Now we apply Theorem 1.1 to representations of Γ which are restrictions of representations of G. Denote by $\operatorname{Rep}(G)$ the space of finite dimensional complex representations of G. Then we have

Corollary 1.2 *Let $\rho \in \operatorname{Rep}(G)$ be irreducible and assume that $\rho \not\cong \rho_\theta$. Then $R(s,\rho)$ is holomorphic at $s=0$ and*

$$R(0,\rho) = C(d,\rho)\cdot \frac{T^{\mathbb{C}}(X,\rho)}{T_0(X,\rho)}. \tag{1.16}$$

Using (1.6) and (1.9), it follows that

$$|T_0(X,\rho)| = C(d,\rho). \tag{1.17}$$

Let $d=3$. Then $\mathbb{H}^3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. For $m\in\mathbb{N}$ let $\rho_m\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(S^m(\mathbb{C}^2))$ be the m-th symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. For a compact, oriented hyperbolic 3-manifold $X=\Gamma\backslash\mathbb{H}^3$ and the representations ρ_m, $m\in\mathbb{N}$, Corollary 1.2 was proved by Park [27, (5.5)]. He also determined the constant $C(3,\rho_m)$ and $|T_0(X,\rho_m)|$. By [27, Prop. 5.1] we have

$$\begin{aligned} &h_0=1 \quad \text{and} \quad |T_0(X,\rho_m)|=2, \quad \text{if } m \text{ is even}\\ &h_0=0 \quad \text{and} \quad \ \ T_0(X,\rho_m)=1, \quad \text{if } m \text{ is odd.}\end{aligned} \tag{1.18}$$

Moreover $C(3,\rho_m)=(-4)^{h_0}$. The order h of $R(s,\rho_m)$ at $s=0$ is zero. Thus by (1.12) we have $h_1=2h_0$. Note that ρ_m is acyclic. Let Δ_{k,ρ_m} be the usual Laplacian in $\Lambda^k(X,F_{\rho_m})$ with respect to the admissible metric in F_{ρ_m}. Then for $m\in\mathbb{N}$ even we have

$$\ker \Delta_{k,\rho_m}=0, \quad \ker \Delta^{\sharp}_{k,\rho_m}\neq 0, \quad k=0,\dots, \tag{1.19}$$

which shows that for acyclic representations χ, in general, the flat Laplacian $\Delta^{\sharp}_{k,\chi}$ need not be invertible.

Again we can replace $T^{\mathbb{C}}(X,\rho)$ in (1.16) by the combinatorial torsion $\tau_{\mathrm{comb}}(X,\rho)$. Using (1.16) we get

$$R(0,\rho) = C(d,\rho)\cdot \frac{\tau_{\mathrm{comb}}(X,\rho)}{T_0(X,\rho)}. \tag{1.20}$$

Since G is a connected semi-simple Lie group and ρ a representation of G, it follows from [25, Lemma 4.3] that ρ is actually a representation in $\mathrm{SL}(n,\mathbb{C})$. Thus $\rho|_\Gamma$ and $\rho^*|_\Gamma$ are unimodular. Moreover, if $\rho\not\cong\rho_\theta$, we have $H^*(X,F_\chi\oplus F^*_\chi)=0$. Thus $\tau_{\mathrm{comb}}(X,\rho)=\tau^{\mathbb{C}}(X,\rho\oplus\rho^*)\in\mathbb{C}^*$ and

$$\tau^{\mathbb{C}}(X,\rho\oplus\rho^*) = \pm\tau^{\mathbb{C}}(X,\rho)\cdot\tau^{\mathbb{C}}(X,\rho^*). \tag{1.21}$$

Another case to which Theorem 1.1 can be applied are deformations of unitary acyclic representations. Let $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ be the set of all m-dimensional complex representations of Γ equipped with the usual topology. Let $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m) \subset \mathrm{Rep}(\Gamma, \mathbb{C}^m)$ be the subset of all unitary acyclic representations (see Sect. 6.2). If $d = 3$, then by [14, Theorem 1.1] we have $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m) \neq \emptyset$. Hence there exists a neighborhood V of $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m)$ in $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ such that $\Delta_\chi^\sharp$ is invertible for all $\chi \in V$. In dimensions > 3 this is not known.

Using Theorem 1.1, we get

Proposition 1.3 *Assume that* $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m) \neq \emptyset$. *There exists an open neighborhood* V *of* $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m)$ *in* $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ *such that for every* $\chi \in V$, $R(s, \chi)$ *is regular at* $s = 0$ *and*

$$R(0, \chi) = T^{\mathbb{C}}(X, \chi).$$

This proposition was first proved by Spilioti [33] using the odd signature operator [3]. She also discusses the relation with the refined analytic torsion. As above we can also express $R(0, \chi)$ in terms of the combinatorial torsion and by (1.21) we have

$$R(0, \chi) = \pm\tau^{\mathbb{C}}(X, \rho) \cdot \tau^{\mathbb{C}}(X, \rho^*). \tag{1.22}$$

This agrees with (1.7).

2 Coupling Differential Operators to a Flat Bundle

We recall a construction of the flat extension of a differential operator introduced in [7]. Let X be a smooth manifold and E_1 and E_2 complex vector bundles over X. Let

$$D\colon C^\infty(X, E_1) \to C^\infty(X, E_2)$$

be a differential operator. Let $F \to X$ be a flat vector bundle. Then there is a canonically operator

$$D_F^\sharp\colon C^\infty(X, E_1 \otimes F) \to C^\infty(X, E_2 \otimes F)$$

associated to D, which is defined as follows. Let $U \subset X$ be an open subset such that $F|_U$ is trivial. Let $s_1, \dots, s_k \in C^\infty(U, F|_U)$ be a local frame field of flat sections. Every section φ of $(E_1 \otimes F)|_U$ can be written as

$$\varphi = \sum_{i=1}^{k} \psi_i \otimes s_i$$

for some sections $\psi_1, \dots, \psi_k \in C^\infty(U, E_1|_U)$. Then define

$$D_F^\sharp|_U\colon C^\infty(U, (E_1 \otimes F)|_U) \to C^\infty(U, (E_2 \otimes F)|_U)$$

by

$$(D_F^\sharp|_U)(\varphi) := \sum_{i=1}^k D(\psi_i) \otimes s_i.$$

Let $s'_1, \dots, s'_k$ be another local frame field of flat sections of $F|_U$. Then $s_i = \sum_{j=1}^k f_{ij} s'_j$, $i = 1, \dots, k$, with $f_{ij} \in C^\infty(U)$, and it follows that the transition functions f_{ij} are constant. Since D is linear, $(D_F^\sharp|_U)(\varphi)$ is independent of the choice of the local frame field of flat sections and therefore, $D_F^\sharp$ is globally well defined. Let $\sigma(D)$ be the principal symbol of D. Then the principal symbol $\sigma(D_F^\sharp)$ of $D_F^\sharp$ is given by $\sigma(D_F^\sharp) = \sigma(D) \otimes \mathrm{Id}_F$. Thus if D is elliptic, then $D_F^\sharp$ is also an elliptic differential operator.

As an example consider a Riemannian manifold X and the Laplace operator Δ_p on p-forms. Let F be a flat bundle over X. Denote by $\Lambda^p(X, F)$ the space of smooth F-valued p-forms, i.e., $\Lambda^p(X, F) = C^\infty(X, \Lambda^p T^*(X) \otimes F)$. By the construction above we obtain the flat Laplacian $\Delta_{p,F}^\sharp \colon \Lambda^p(X, F) \to \Lambda^p(X, F)$. If the flat bundle is fixed, we will denote the flat Laplacian simply by $\Delta_p^\sharp$. The flat Laplacian can be also described as the usual Laplacian. Let $d_F \colon \Lambda^{p-1}(X, F) \to \Lambda^p(X, F)$ be the exterior derivative defined as above. Let $\star \colon \Lambda^p(X) \to \Lambda^{n-p}(X)$ denote the Hodge $\star$-operator. Then the flat extension

$$d_F^{*,\sharp} \colon \Lambda^p(X, F) \to \Lambda^{p-1}(X, F)$$

of the co-differential $d^* \colon \Lambda^*(X) \to \Lambda^{p-1}(X)$ is given by

$$d_F^{*,\sharp} = (-1)^{np+n+1}(\star \otimes \mathrm{Id}_F) \circ d_F \circ (\star \otimes \mathrm{Id}_F).$$

Then $d_F^{*,\sharp}$ satisfies $d_F^{*,\sharp} \circ d_F^{*,\sharp} = 0$ and we have

$$\Delta_F^\sharp = (d_F + d_F^{*,\sharp})^2.$$

If we choose a Hermitian fiber metric on F, we can define the usual Laplace operator Δ_F in $\Lambda^p(X, F)$, which is defined by

$$\Delta_F = (d_F + d_F^*)^2 = d_F d_F^* + d_F^* d_F,$$

which is formally self-adjoint. Now note that $d_F^{*,\sharp} = d_F^* + B$, where B is a smooth homomorphism of vector bundles. Thus it follows that $\Delta_F^\sharp = \Delta_F + (B d_F + d_F B)$. Hence the principal symbol $\sigma(\Delta_F^\sharp)(x, \xi)$ of $\Delta_F^\sharp$ is given by

$$\sigma(\Delta_F^\sharp)(x, \xi) = \|\xi\|_x^2 \, \mathrm{Id}_{\Lambda^p T_x^*(X) \otimes F_x}, \quad x \in X, \ \xi \in F_x. \tag{2.1}$$

More generally, let $E \to X$ be a Hermitian vector bundle over X. Let ∇ be a covariant derivative in E which is compatible with the Hermitian metric. We denote by $C^\infty(X, E)$ the space of smooth sections of E. Let

$$\Delta_E = \nabla^* \nabla$$

be the Bochner–Laplace operator associated to the connection ∇ and the Hermitian fiber metric. Then Δ_E is a second order elliptic differential operator. Its leading symbol $\sigma(\Delta_E)\colon \pi^* E \to \pi^* E$, where π is the projection of $T^* X$, is given by

$$\sigma(\Delta_E)(x, \xi) = \|\xi\|_x^2 \cdot \mathrm{Id}_{E_x}, \quad x \in X,\ \xi \in T_x^* X. \tag{2.2}$$

Let $F \to X$ be a flat vector bundle and

$$\Delta^\sharp_{E\otimes F}\colon C^\infty(X, E \otimes F) \to C^\infty(X, E \otimes F)$$

the coupling of Δ_E to F. Then the principal symbol of $\Delta^\sharp_{E\otimes F}$ is given by

$$\sigma(\Delta^\sharp_{E\otimes F})(x, \xi) = \|\xi\|_x^2 \cdot \mathrm{Id}_{E_x \otimes F_x}\,. \tag{2.3}$$

3 Regularized Determinants and Analytic Torsion

Let Δ_E be as above. Let

$$P\colon C^\infty(X, E) \to C^\infty(X, E)$$

be an elliptic second order differential operator which is a perturbation of Δ_E by a first order differential operator, i.e.,

$$P = \Delta_E + D, \tag{3.1}$$

where $D\colon C^\infty(X, E) \to C^\infty(X, E)$ is a first order differential operator. This implies that P is an elliptic second order differential operator with leading symbol $\sigma(P)(x, \xi)$ given by

$$\sigma(P)(x, \xi) := \|\xi\|_x^2 \cdot \mathrm{Id}_{E_x}\,. \tag{3.2}$$

Though P is not self-adjoint in general, it still has nice spectral properties [30, Chapt. I, §8]. We recall the basic facts. For $I \subset [0, 2\pi]$ let

$$\Lambda_I = \{re^{i\theta} : 0 \le r < \infty,\ \theta \in I\}. \tag{3.3}$$

The following lemma describes the structure of the spectrum of P.

Lemma 3.1 *For every $0 < \varepsilon < \pi/2$ there exists $R > 0$ such that the spectrum of P is contained in the set $D_R(0) \cup \Lambda_{[-\varepsilon,\varepsilon]}$. Moreover the spectrum of P is discrete.*

Proof The first statement follows from [30, Theorem 9.3]. The discreteness of the spectrum follows from [30, Theorem 8.4]. □

For $\lambda \in \mathbb{C} \setminus \operatorname{spec}(P)$ let $R_\lambda(P) := (P - \lambda\,\mathrm{Id})^{-1}$ be the resolvent. Given $\lambda_0 \in \operatorname{spec}(P)$, let Γ_{λ_0} be a small circle around λ_0 which contains no other points of $\operatorname{spec}(P)$. Put

$$\Pi_{\lambda_0} = \frac{i}{2\pi} \int_{\Gamma_{\lambda_0}} R_\lambda(P)\, d\lambda. \tag{3.4}$$

Then Π_{λ_0} is the projection onto the *root subspace* V_{λ_0}. This is a finite-dimensional subspace of $C^\infty(X, E)$ which is invariant under P and there exists $N \in \mathbb{N}$ such that $(P - \lambda_0\,\mathrm{I})^N V_{\lambda_0} = 0$. Furthermore, there is a closed complementary subspace V'_{λ_0} to V_{λ_0} in $L^2(X, E)$ which is invariant under the closure $\bar{P}$ of P in L^2 and the restriction of $(\bar{P} - \lambda_0\,\mathrm{I})$ to V'_{λ_0} has a bounded inverse. The *algebraic multiplicity* $m(\lambda_0)$ of λ_0 is defined as

$$m(\lambda_0) := \dim V_{\lambda_0}.$$

Moreover $L^2(X, E)$ is the closure of the algebraic direct sum of finite-dimensional P-invariant subspaces V_k

$$L^2(X, E) = \overline{\bigoplus_{k\geq 1} V_k} \tag{3.5}$$

such that the restriction of P to V_k has a unique eigenvalue λ_k, for each k there exists $N_k \in \mathbb{N}$ such that $(P - \lambda_k\,\mathrm{I})^{N_k} V_k = 0$, and $|\lambda_k| \to \infty$. In general, the sum (3.5) is not a sum of mutually orthogonal subspaces. See [23, Sect. 2] for details.

Assume that P is invertible. Recall that an angle $\theta \in [0, 2\pi)$ is called an *Agmon angle* for P, if there exists $\varepsilon > 0$ such that

$$\operatorname{spec}(P) \cap \Lambda_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset. \tag{3.6}$$

By Lemma 3.1 it is clear that an Agmon angle always exists for P. Choose an Agmon angle θ for P. Define the complex power P_θ^{-s}, $s \in \mathbb{C}$, as in [30, §10]. Let $d = \dim X$. For $\operatorname{Re}(s) > d/2$, the complex power P_θ^{-s} is a trace class operator and the zeta function $\zeta_\theta(s, P)$ of P is defined by

$$\zeta_\theta(s, P) := \operatorname{Tr}(P_\theta^{-s}), \quad \operatorname{Re}(s) > \frac{d}{2}. \tag{3.7}$$

The zeta function admits a meromorphic extension to the entire complex plane which is holomorphic at $s = 0$ [30, Theorem 13.1]. Let $R_\theta := \{\rho e^{i\theta} \colon \rho \in \mathbb{R}^+\}$. Denote by $\log_\theta(\lambda)$ the branch of the logarithm in $\mathbb{C} \setminus R_\theta$ with $\theta < \operatorname{Im}\log_\theta < \theta + 2\pi$. We enumerate the eigenvalues of P such that

$$\operatorname{Re}(\lambda_1) \le \operatorname{Re}(\lambda_2) \le \cdots \le \operatorname{Re}(\lambda_k) \le \cdots .$$

By Lidskii's theorem [16, Theorem 8.4] if follows that for $\operatorname{Re}(s) > d/2$ we have

$$\zeta_\theta(s, P) = \operatorname{Tr}(P_\theta^{-s}) = \sum_{k=1}^{\infty} m(\lambda_k)(\lambda_k)_\theta^{-s}, \tag{3.8}$$

where $(\lambda_k)_\theta^{-s} = e^{-s\log_\theta(\lambda_k)}$. We will need a different description of the zeta function in terms of the heat operator e^{-tP}, which can be defined using the functional calculus developed in [23, Sect. 2] by

$$e^{-tP} := \frac{i}{2\pi} \int_\Gamma e^{-t\lambda^2}(P^{1/2} - \lambda)^{-1} d\lambda, \tag{3.9}$$

where $\Gamma \subset \mathbb{C}$ is the same contour as in [23, (2.18)]. As in [23, Lemma 2.4] one can show that e^{-tP} is an integral operator with a smooth kernel. By [23, Prop. 2.5] it follows that e^{-tP} is a trace class operator. Using Lidskii's theorem as above we get

$$\operatorname{Tr}(e^{-tP}) = \sum_{k=1}^{\infty} m(\lambda_k) e^{-t\lambda_k}. \tag{3.10}$$

The absolute convergence of the right hand side follows from Weyl's law [23, Lemma 2.2]. Assume that there exists $\delta > 0$ such that $\operatorname{Re}(\lambda_k) \ge \delta$ for all $k \in \mathbb{N}$. Then by (3.10) and Weyl's law it follows that there exist $C, c > 0$ such that

$$|\operatorname{Tr}(e^{-tP})| \le C e^{-ct} \tag{3.11}$$

for $t \ge 1$. Since $\operatorname{spec}(P)$ is contained in the half plane $\operatorname{Re}(s) > 0$, we can choose the Agmon angle as $\theta = \pi$. Using the asymptotic expansion of $\operatorname{Tr}(e^{-tP})$ as $t \to 0$, it follows from (3.8) and (3.11) that

$$\zeta(s, P) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}(e^{-tP}) t^{s-1} dt$$

for $\operatorname{Re}(s) > d/2$.

Then the regularized determinant of P is defined by

$$\det_\theta(P) := \exp\left(-\frac{d}{ds}\zeta_\theta(s, P)\Big|_{s=0} \right). \tag{3.12}$$

As shown in [3, 3.10], $\det_\theta(P)$ is independent of θ. Therefore we will denote the regularized determinant simply by $\det(P)$.

Assume that the vector bundle E is $\mathbb{Z}/2\mathbb{Z}$-graded, i.e., $E = E^+ \oplus E^-$ and P preserves the grading, i.e., assume that with respect to the decomposition

$$C^\infty(Y, E) = C^\infty(Y, E^+) \oplus C^\infty(Y, E^-)$$

P takes the form

$$P = \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix}.$$

Then we define the graded determinant $\det_{\mathrm{gr}}(P)$ of P by

$$\det_{\mathrm{gr}}(P) = \frac{\det(P^+)}{\det(P^-)}. \tag{3.13}$$

Next we introduce the analytic torsion defined in terms of the non-self-adjoint operators $\Delta^\sharp_{p,\chi}$. We use the definition given in [7, Sect. 8]. Recall that the principal symbol of $\Delta^\sharp_{p,\chi}$ is given by (2.3). Therefore, $\Delta^\sharp_{p,\chi}$ satisfies the assumptions of Sect. 3.

Let $r > 0$ be such that $\mathrm{Re}(\lambda) \neq r$ for all generalized eigenvalues λ of $\Delta^\sharp_{p,\chi}$. Let $\Pi_{p,r}$ be the spectral projection on the span of the generalized eigenvectors with eigenvalues with real part less than r. Let $\Delta^\sharp_{p,\chi,r} := (1-\Pi_{p,r})\Delta^\sharp_{p,\chi}$. Let $S(p, \chi, r)$ be the set of all nonzero generalized eigenvalues with real part less than r. Furthermore, let V_0^p be the generalized eigenspace of $\Delta^\sharp_{p,\chi}$ with generalized eigenvalues 0. Then $(V_0^*, d, d^{*,\sharp})$ is double complex in the sense of [7]. Let

$$T_0(X, \chi) \in (\det H^*(X, F_\chi)) \otimes (\det H_*(X, F_\chi))^* \tag{3.14}$$

be the torsion of the double complex. Then the Cappell–Miller torsion is defined by

$$T^{\mathbb{C}}(X, \chi) := \prod_{p=1}^{d} \det(\Delta^\sharp_{p,\chi,r})^{(-1)^{p+1}p} \cdot \prod_{p=1}^{d} \Big(\prod_{\lambda \in S(p,\chi,r)} \lambda^{m(\lambda)} \Big)^{(-1)^{p+1}p} \cdot T_0(X, \chi), \tag{3.15}$$

where $m(\lambda)$ denotes the algebraic multiplicity of λ. Let $\Pi_{k,0}$ be the spectral projection on the generalized eigenspace of $\Delta^\sharp_{k,\chi}$ with generalized eigenvalue 0. Let

$$(\Delta^\sharp_{k,\chi})' := (\mathrm{Id} - \Pi_{k,0})\Delta^\sharp_{k,\chi}. \tag{3.16}$$

If we choose an Agmon angle we can also write

$$T^{\mathbb{C}}(X, \chi) = \prod_{k=1}^{d} \Big[\det(\Delta^\sharp_{k,\chi})' \Big]^{(-1)^{k+1}k} \cdot T_0(X, \chi). \tag{3.17}$$

If χ is acyclic, i.e., $H^*(X, E_\chi) = 0$, then $T_0(X, \chi)$ and $T^{\mathbb{C}}(X, \chi)$ are complex numbers.

4 Twisted Ruelle Zeta Functions

In this section we consider compact oriented hyperbolic manifolds of odd dimension $d = 2n + 1$ and we recall some basic properties of Ruelle type zeta functions.

To begin with we fix some notation. Let $G = \mathrm{SO}_0(d, 1)$ and $K = \mathrm{SO}(d)$. Then G/K equipped with the normalized invariant metric is isometric to the d-dimensional hyperbolic space $\mathbb{H}^d$. Let $G = KAN$ be the standard Iwasawa decomposition. Let M be the centralizer of A in K. Then $M \cong \mathrm{SO}(d-1)$. Let $\widehat{K}$ and $\widehat{M}$ be the set of equivalence classes of irreducible unitary representations of K and M, respectively.

Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{a}$ the Lie algebras of G, K, M, N, and A, respectively. Let $W(A) \cong \mathbb{Z}/2\mathbb{Z}$ be the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Let $\alpha \in \mathfrak{a}^*$ be the unique positive root of $(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{a}^+ := \{H \in \mathfrak{a} \colon \alpha(H) > 0\}$. Put $A^+ := \exp(\mathfrak{a}^+)$. Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ and denote by $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Let $\Gamma \subset G$ be a discrete, torsion free, co-compact subgroup. Then Γ acts fixed point free on $\mathbb{H}^d$. The quotient $X = \Gamma\backslash\mathbb{H}^d$ is a closed, oriented hyperbolic manifold and each such manifold is of this form. Given $\gamma \in \Gamma$, we denote by $[\gamma]$ the Γ-conjugacy class of γ. The set of all conjugacy classes of Γ will be denoted by $C(\Gamma)$. Let $\gamma \neq 1$. Then there exist $g \in G$, $m_\gamma \in M$, and $a_\gamma \in A^+$ such that

$$g\gamma g^{-1} = m_\gamma a_\gamma. \tag{4.1}$$

By [34, Lemma 6.6], a_γ depends only on γ and m_γ is determined up to conjugacy in M. Let $H \in \mathfrak{a}$ such that $\alpha(H) = 1$. By definition there exists $\ell(\gamma) > 0$ such that

$$a_\gamma = \exp\left(\ell(\gamma)H\right). \tag{4.2}$$

Then $\ell(\gamma)$ is the length of the unique closed geodesic in X that corresponds to the conjugacy class $[\gamma]$. An element $\gamma \in \Gamma - \{e\}$ is called primitive, if it can not be written as $\gamma = \gamma_0^k$ for some $\gamma_0 \in \Gamma$ and $k > 1$. For every $\gamma \in \Gamma - \{e\}$ there exist a unique primitive element $\gamma_0 \in \Gamma$ and $n_\Gamma(\gamma) \in \mathbb{N}$ such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$. Let $\chi \colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional complex representation of Γ and let $s \in \mathbb{C}$. Then the Ruelle zeta function $R(s, \chi)$ is defined by the following Euler product

$$R(s, \chi) := \prod_{\substack{[\gamma]\neq e \\ [\gamma]\,\mathrm{prime}}} \det\left(\mathrm{Id} - \chi(\gamma)e^{-s\ell(\gamma)}\right). \tag{4.3}$$

In order to verify that the product converges in some half plane, we first recall that there exists $C > 0$ such that for all $R > 0$ we have

$$\#\{[\gamma] \in C(\Gamma) \colon \ell(\gamma) \leq R\} \leq Ce^{(n-1)R} \tag{4.4}$$

[6, (1.31)]. We also need the following auxiliary lemma.

Lemma 4.1 *Let $\chi \colon \Gamma \to \mathrm{GL}(V)$ be a finite dimensional representation of Γ. There exist $C, c > 0$ such that*

$$|\operatorname{tr}(\chi(\gamma))| \leq Ce^{c\ell(\gamma)}, \quad \forall \gamma \in \Gamma - \{e\}. \tag{4.5}$$

For the proof see [31, Lemma 3.3]. It follows from (4.4) and Lemma 4.1 that the product on the right hand side of (4.3) converges absolutely and uniformly in some half-plane $\operatorname{Re}(s) > C$ (see [31, Prop. 3.5]). Furthermore, $R(s, \chi)$ admits a meromorphic extensions to the entire complex plane [31] and satisfies the following functional equation [32]:

$$R(s, \chi) = R(-s, \chi) \cdot \exp\left((-1)^{n+1} \frac{2\pi(d+1)\operatorname{vol}(V_\chi)\operatorname{vol}(X)}{\operatorname{vol}(S^d)} s\right). \tag{4.6}$$

For unitary representations χ, these results were proved by Bunke and Olbrich [6]. The main technical tool is the Selberg trace formula. For the extension to the non-unitary case the Selberg trace formula is replaced by a Selberg trace formula for non-unitary twists, developed in [23]. The proofs are similar except that on has to deal with non-self-adjoint operators.

There are also expressions of the zeta functions in terms of determinants of certain elliptic operators. To explain the formulas we need to recall the definition of the relevant differential operators. Given $\tau \in \widehat{K}$, let $\widetilde{E}_\tau \to \widetilde{X}$ be the homogeneous vector bundle associated to τ and let $E_\tau := \Gamma \backslash \widetilde{E}_\tau$ be the corresponding locally homogeneous vector bundle over X. Denote by $C^\infty(X, E_\tau)$ the space of smooth sections of E_τ. There is a canonical isomorphism

$$C^\infty(X, E_\tau) \cong (C^\infty(\Gamma \backslash G) \otimes V_\tau)^K \tag{4.7}$$

[21, §1]. Let $\Omega \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ be the Casimir element and denote by R_Γ the right regular representation of G in $C^\infty(\Gamma \backslash G)$. Then $R_\Gamma(\Omega)$ acts on the right hand side of (4.7) and via this isomorphism, defines an operator in $C^\infty(X, E_\tau)$. We denote the operator induced by $-R_\Gamma(\Omega)$ by A_τ.

Denote by $\widetilde{\nabla}^\tau$ the canonical connection in $\widetilde{E}_\tau$ and let ∇^τ be the induced connection in E_τ. Let $\Delta_\tau := (\nabla^\tau)^* \nabla^\tau$ be the associated Bochner–Laplace operator acting in $C^\infty(X, E_\tau)$. Let $\Omega_K \in \mathcal{Z}(\mathfrak{k}_{\mathbb{C}})$ be the Casimir element of K. Assume that τ is irreducible. Let $\lambda_\tau := \tau(\Omega_K)$ denote the Casimir eigenvalue of τ. Then we have

$$A_\tau := \Delta_\tau - \lambda_\tau \operatorname{Id}. \tag{4.8}$$

[21, §1]. Thus A_τ is a formally self-adjoint second order elliptic differential operator. Let $F_\chi \to X$ be the flat vector bundle defined by χ. Let

$$A^{\sharp}_{\tau,\chi}: C^{\infty}(X, E_{\tau} \otimes F_{\chi}) \to C^{\infty}(X, E_{\tau} \otimes F_{\chi})$$

be the coupling of A_{τ} to F_{χ}.

Denote by $R(K)$ and $R(M)$ the representation rings of K and M, respectively. Let $i: M \to K$ be the inclusion and $i^{*}: R(K) \to R(M)$ the induced map of the representation rings. The Weyl group $W(A)$ acts on $R(M)$ in the canonical way. Let $R^{\pm}(M)$ denote the ± 1-eigenspaces of the non-trivial element $w \in W(A)$. Let $\sigma \in R(M)$. It follows from the proof of Proposition 1.1 in [6] that there exist $m_{\tau}(\sigma) \in \{-1, 0, 1\}$, depending on $\tau \in \widehat{K}$, which are equal to zero except for finitely many $\tau \in \widehat{K}$, such that

$$\sigma = \sum_{\tau \in \widehat{K}} m_{\tau}(\sigma) i^{*}(\tau), \tag{4.9}$$

if $\sigma \in R^{+}(M)$, and

$$\sigma + w\sigma = \sum_{\tau \in \widehat{K}} m_{\tau}(\sigma) i^{*}(\tau), \tag{4.10}$$

if $\sigma \neq w\sigma$. Let

$$E(\sigma) := \bigoplus_{\substack{\tau \in \widehat{K} \\ m_{\tau}(\sigma) \neq 0}} E_{\tau}. \tag{4.11}$$

Then $E(\sigma)$ has a grading

$$E(\sigma) = E^{+}(\sigma) \oplus E^{-}(\sigma)$$

defined by the sign of $m_{\tau}(\sigma)$. Let $\sigma \in \widehat{M}$. Denote by ν_{σ} the highest weight of σ. Let $\mathfrak{b}$ be the standard Cartan subalgebra of $\mathfrak{m}$ [26, Sect. 2]. Let $\varrho_{\mathfrak{m}}$ be the half-sum of positive roots of $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ and $\varrho_{\mathfrak{a}}$ the half-sum of positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$. Put

$$c(\sigma) := -\|\varrho_{\mathfrak{a}}\|^{2} - \|\varrho_{\mathfrak{m}}\|^{2} + \|\nu_{\sigma} + \varrho_{\mathfrak{m}}\|^{2}. \tag{4.12}$$

We define the operator $A^{\sharp}_{\chi}(\sigma)$ acting in $C^{\infty}(X, E(\sigma) \otimes F_{\chi})$ by

$$A^{\sharp}_{\chi}(\sigma) := \bigoplus_{\substack{\tau \in \widehat{K} \\ m_{\tau}(\sigma) \neq 0}} A^{\sharp}_{\tau,\chi} + c(\sigma). \tag{4.13}$$

For $p \in \{0, \ldots, d-1\}$ let σ_{p} be the standard representation of $M = \mathrm{SO}(d-1)$ on $\Lambda^{p}\mathbb{R}^{d-1} \otimes \mathbb{C}$. We note that σ_{p} is irreducible except when $p = n$, in which case σ_{n} is the direct sum of the two spin representations $\sigma_{n}^{+}, \sigma_{n}^{-}$. Moreover, $\sigma_{n}^{\mp} = w\sigma_{n}^{\pm}$. Thus $\sigma_{n} = \sigma_{n}^{+} + w\sigma_{n}^{+}$ and

$$A^{\sharp}_{\chi}(\sigma_{n}) = A^{\sharp}_{\chi}(\sigma_{n}^{+}) \oplus A^{\sharp}_{\chi}(\sigma_{n}^{-}). \tag{4.14}$$

Recall that $A^\sharp_\chi(\sigma_p)$ acts in the space of sections of a graded vector bundle. Then by [32, Prop. 1.7] we have the following determinant formula.

Proposition 4.2 *For every $p = 0, \dots, d-1$ we have*

$$\begin{aligned} R(s,\chi) = \prod_{p=0}^{d-1} \det{}_{\mathrm{gr}} \left(A^\sharp_\chi(\sigma_p) + (s+n-p)^2 \right)^{(-1)^p} \\ \times \exp\left(- \frac{2\pi(n+1)\dim(V_\chi)\operatorname{vol}(X)}{\operatorname{vol}(S^d)} s \right), \end{aligned} \tag{4.15}$$

where $\operatorname{vol}(S^d)$ denotes the volume of the d-dimensional unit sphere.

For unitary χ this was proved in [6, Prop. 4.6].

5 Proof of the Main Theorem

To prove Theorem 1.1 we apply Proposition 4.2. Let $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ be the standard Cartan subalgebra of $\mathfrak{g}$ and let $e_1, \dots, e_{n+1} \in \mathfrak{h}^*_{\mathbb{C}}$ be the standard basis [26, Sect. 2]. Thus $e_2, \dots, e_{n+1}$ is a basis of $\mathfrak{b}$. Then $\rho_{\mathfrak{m}} = \sum_{j=2}^{n+1}(n+1-j)e_j$ and

$$\nu_{\sigma_p} = \begin{cases} e_2 + \cdots + e_{p+1}, & \text{if } \ p \le n, \\ e_2 + \cdots + e_{2n+1-p}, & \text{if } \ p > n, \end{cases}$$

[17, Chap. IV, §7]. Moreover, $|\rho| = n$. An explicit computation shows that $c(\sigma_p) = -(n-p)^2$. Let λ_p be the p-th exterior power of the standard representation of $\mathrm{SO}(d)$. Then for $p = 0, \dots, d-1$ we have $i^*(\lambda_p) = \sigma_p + \sigma_{p-1}$. Put $\tau_p := \sum_{k=0}^{p}(-1)^k \lambda_{p-k}$. Then it follows that $i^*(\tau_p) = \sigma_p$, $p = 0, \dots, d-1$. Using (4.13) and (4.14), we obtain

$$A^\sharp_\chi(\sigma_p) + (n-p)^2 = \bigoplus_{k=0}^{p} A^\sharp_{\lambda_{p-k},\chi} = \bigoplus_{k=0}^{p} A^\sharp_{\lambda_k,\chi}. \tag{5.1}$$

Now recall that $A^\sharp_{\lambda_k,\chi}$ is the coupling of A_{λ_k} to F_χ. Furthermore, with respect to the isomorphism (4.7), A_{λ_k} corresponds to the action of $-R_\Gamma(\Omega)$ on $(C^\infty(\Gamma\backslash G) \otimes \Lambda^k \mathbb{C}^d)$. By the Lemma of Kuga, this operator corresponds to the Laplacian Δ_k on $\Lambda^k(X)$. Let $\Delta^\sharp_{k,\chi}$ be the coupling of Δ_k to F_χ. Then by (5.1) we get

$$A^\sharp_\chi(\sigma_p) + (n-p)^2 = \bigoplus_{k=0}^{p} \Delta^\sharp_{k,\chi}. \tag{5.2}$$

Inserting this equality into the alternating product of graded determinants on the right hand side of (4.15), we obtain

$$\begin{aligned}
&\prod_{p=0}^{d-1} \det\nolimits_{\mathrm{gr}}(A^{\sharp}_{\chi}(\sigma_p) + (s+n-p)^2)^{(-1)^p} \\
&\quad = \prod_{p=0}^{d-1}\prod_{k=0}^{p} \det(\Delta^{\sharp}_{p-k,\chi} + s(s+2(n-p)))^{(-1)^{p+k}} \\
&\quad = \prod_{k=0}^{d-1}\prod_{p=k}^{d-1} \det(\Delta^{\sharp}_{k,\chi} + s(s+2(n-p)))^{(-1)^{k}}.
\end{aligned} \tag{5.3}$$

Applying the determinant formula (4.15) together with (5.3), we finally get

$$\begin{aligned}
R(s;\chi) = &\prod_{k=0}^{d-1}\prod_{p=k}^{d-1} \det(\Delta^{\sharp}_{k,\chi} + s(s+2(n-p)))^{(-1)^{k}} \\
&\times \exp\left(-\frac{2\pi(n+1)\dim(V_\chi)\,\mathrm{vol}(X)}{\mathrm{vol}(S^d)} s\right).
\end{aligned} \tag{5.4}$$

Let h_k be the dimension of the generalized eigenspace of $\Delta^{\sharp}_{k,\chi}$ with eigenvalue zero.

Lemma 5.1 *We have $h_p = h_{d-p}$ for $p = 0, \ldots, d$.*

Proof Let $\star\colon \Lambda^p(X, F_\chi) \to \Lambda^{d-p}(X, F_\chi)$ be the extension of the Hodge $\star$-star operator, which acts locally as $\star(\omega \otimes f) = (\star\omega) \otimes f$, where ω is a usual p-form and f a local section of F_χ. Since $\star\Delta_p = \Delta_{d-p}\star$, it follows from the definition of the Laplacians coupled to F_χ that $\star\Delta^{\sharp}_{p,\chi} = \Delta^{\sharp}_{d-p,\chi}\star$. It follows that for every $k \in \mathbb{N}$ we have $\star(\Delta^{\sharp}_{p,\chi})^k = (\Delta^{\sharp}_{d-p,\chi})^k\star$.This proves the lemma. □

Denote by h the order of the singularity of $R(s,\chi)$ at $s = 0$. Using (5.4) and Lemma 5.1 it follows that

$$h = \sum_{k=0}^{n}(d+1-k)(-1)^k h_k + \sum_{k=n+1}^{d-1}(d-k)(-1)^k h_k = \sum_{k=0}^{n}(d+1-2k)(-1)^k h_k. \tag{5.5}$$

Let $(\Delta^{\sharp}_{k,\chi})'$ be the operator defined by (3.16). We note that for $s \in \mathbb{C}$, $|s| \ll 1$, there is a common Agmon angle for the operator $(\Delta^{\sharp}_{k,\chi})' + s(s+2(n-p))$. Therefore, in order to study the limit of $\det((\Delta^{\sharp}_{k,\chi})' + s(s+2(n-p)))$ as $s \to 0$, we can use one and the same Agmon angle.

If $p \neq n$, we get

$$\begin{aligned}\lim_{s\to 0} s^{-h_k}\det(\Delta^{\sharp}_{k,\chi}+s(s+2(n-p))) &= \lim_{s\to 0}\Big[\det((\Delta^{\sharp}_{k,\chi})'+s(s+2(n-p))\\ &\quad\times \frac{(s(s+2(n-p)))^{h_k}}{s^{h_k}}\Big]\\ &= (2(n-p))^{h_k}\cdot\det((\Delta^{\sharp}_{k,\chi})').\end{aligned}\tag{5.6}$$

For $p = n$ we get a similar formula

$$\lim_{s\to 0} s^{-2h_k}\det(\Delta^{\sharp}_{k,\chi}+s^2) = \lim_{s\to 0}\det((\Delta^{\sharp}_{k,\chi})'+s^2) = \det((\Delta^{\sharp}_{k,\chi})').\tag{5.7}$$

Let

$$C(d,\chi) := \prod_{k=0}^{d-1}\prod_{\substack{p=k\\ p\neq n}}^{d-1}(2(n-p))^{(-1)^k h_k}.\tag{5.8}$$

Using (5.5), (5.6) and (5.7) we get

$$\begin{aligned}\lim_{s\to 0} s^{-h}R(s;\chi) &= \prod_{k=0}^{d-1}\prod_{\substack{p=k\\ p\neq n}}^{d-1}\lim_{s\to 0}\left[s^{-h_k}\det(\Delta^{\sharp}_{k,\chi}+s(s+2(n-p)))\right]^{(-1)^k}\\ &\quad\times\prod_{k=0}^{n}\lim_{s\to 0}\left[s^{-2h_k}\det(\Delta^{\sharp}_{k,\chi}+s^2)\right]^{(-1)^k}\\ &= C(d,\chi)\cdot\prod_{k=0}^{d-1}\det((\Delta^{\sharp}_{k,\chi})')^{(d-k)(-1)^k} = C(d,\chi)\cdot\prod_{k=1}^{d}\det((\Delta^{\sharp}_{k,\chi})')^{k(-1)^{k+1}}.\end{aligned}\tag{5.9}$$

For the last equality we used that $\Delta^{\sharp}_{k,\chi}\cong\Delta^{\sharp}_{d-k,\chi}$. Let $T_0(X,\chi)$ be the torsion (3.14) of the double complex $(V_0^*, d, d^{*,\sharp})$ and $T^{\mathbb{C}}(X,\chi)$ the Cappell–Miller torsion defined by (3.15). We note that $T^{\mathbb{C}}(X,\chi)$ and $T_0(X,\chi)$ are both non-zero elements of the determinant line $\det H^*(X,F_\chi)\otimes(\det H_*(X,F_\chi))^*$. Hence there exists $\lambda\in\mathbb{C}$ with $T^{\mathbb{C}}(X,\chi)=\lambda T_0(X,\chi)$. Set

$$\frac{T^{\mathbb{C}}(X,\chi)}{T_0(X,\chi)} := \lambda.$$

If we combine this convention with the definition of the Cappell–Miller torsion (3.17), then (5.9) implies Theorem 1.1.

6 Acyclic Representations

In this section we assume that χ is acyclic. Then $T^{\mathbb{C}}(X,\chi)$, $T_0(X,\chi)$ and $\tau_{\text{comb}}(X,\chi)$ are complex numbers and the right hand side of (1.14) is the quotient of the two complex numbers. Besides the Cappell–Miller torsion we need another version of a complex analytic torsion for arbitrary flat vector bundles F_χ. This is the *refined analytic torsion* $T^{\text{ran}}(X,\chi)\in\det(H^*(X,F_\chi))$ introduced by Braverman and Kappeler [4]. The definition is based on the consideration of the odd signature operator B_χ [3, 2.1]. It is defined as follows. Let

$$\alpha\colon \Lambda^*(X,F_\chi)\to\Lambda^*(X,F_\chi)$$

be the chirality operator defined by

$$\alpha(\omega):=i^{n+1}(-1)^{k(k+1)/2}\star\omega,\quad \omega\in\Lambda^k(M,F_\chi).$$

Let ∇_χ be the flat connection in F_χ. Then the odd signature operator is defined as

$$B_\chi:=\alpha\nabla_\chi+\nabla_\chi\alpha. \tag{6.1}$$

It leaves the even subspace $\Lambda^{\text{ev}}(X,F_\chi)$ invariant. Let $B_{\text{ev},\chi}$ be the restriction of B_χ to $\Lambda^{\text{ev}}(X,F_\chi)$. Then $T^{\text{ran}}(X,\chi)\in\det(H^*(X,F_\chi))$ is defined in terms of $B_{\text{ev},\chi}$. If χ is acyclic, then $T^{\text{ran}}(X,\chi)$ is a complex number. In [5], Braverman and Kappeler determined the relation between the Cappell–Miller torsion and the refined analytic torsion. Let $\eta(B)$ be the eta-invariant of $B_{\text{ev},\chi}$. In general, B_χ is not self-adjoint and therefore, $\eta(B)$ is in general not real. Furthermore, let η_0 be the eta-invariant of the trivial line bundle. Then by Proposition 4.2 and Theorem 5.1 of [5] it follows that

$$T^{\mathbb{C}}(X,\chi)=\pm T^{\text{ran}}(X,\chi)^2\cdot e^{2\pi i(\eta(B)-\dim(\chi)\eta_0)}. \tag{6.2}$$

On the other hand, it follows from [4, Theorem 1.9] that

$$|T^{\text{ran}}(X,\chi)|=T^{RS}(X,\chi)\cdot e^{\pi\,\text{Im}(\eta(B))}. \tag{6.3}$$

Combining (6.2) and (6.3), we obtain (1.9).

6.1 Restriction of Representations of the Underlying Lie Group

The first case that we consider are representations which are restrictions to Γ of representations of G.

Let $\rho\colon G\to\text{GL}(V_\rho)$ be a finite dimensional real (resp. complex) representation of G. Denote by $F_\rho\to X$ the flat vector bundle associated to $\rho|_\Gamma$. Let $\widetilde{E}_\rho\to G/K$ be the homogeneous vector bundle associated to $\rho|_K$. By [18, Part I, Prop. 3.3] there is a canonical isomorphism

$$F_\rho \cong \Gamma \backslash \widetilde{E}_\rho. \tag{6.4}$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. By [18, Part I, Lemma 3.1], there exists an inner product $\langle \cdot, \cdot \rangle$ in V_ρ such that

(1) $\langle \tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
(2) $\langle \tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\rho|_K$ is unitary with respect to this inner product, it induces a metric in $\Gamma\backslash\widetilde{E}_\tau$ and by (6.4) also in F_ρ. Denote by $T^{RS}(X, \rho)$ the Ray–Singer analytic torsion of (X, F_ρ) with respect to the metric on X and the metric in F_ρ. Denote by $\theta\colon G \to G$ the Cartan involution. Let $\rho_\theta := \rho \circ \theta$. Assume that ρ is irreducible and $\rho \not\cong \rho_\theta$. Then $H^*(X, F_\rho) = 0$ [2, Chap. VII, Theorem 6.7]. Thus $\rho|_\Gamma$ is acyclic. In this case $T^{RS}(X, \rho)$ is independent of the metrics on X and in F_ρ [25, Corollary 2.7]. Let $R^\rho(s) := R(s, \rho)R(s, \rho_\theta)$. Then by [35, Theorem 8.13] $R^\rho(s)$ is holomorphic at $s = 0$ and

$$R^\rho(0) = T^{RS}(X, \rho)^4. \tag{6.5}$$

Furthermore, from the discussions in [35, Sect. 9.1] follows that both $R(s, \rho)$ and $R(s, \rho_\theta)$ are holomorphic at $s = 0$ and $R(0, \rho_\theta) = \overline{R(0, \rho)}$. Thus it follows that

$$|R(0, \rho)| = T^{RS}(X, \rho)^2. \tag{6.6}$$

Hence $R(s, \rho)$ is regular at $s = 0$ and $R(0, \rho) \neq 0$. Applying Theorem 1.1 we obtain Corollary 1.2.

Next we briefly recall the definition of the complex Reidemeister torsion [9]. Let V be $\mathbb{C}$-vector space of dimension m. Let $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_m)$ be two basis of V. Let $T = (t_{ij})$ be the matrix of the change of basis from v to w, i.e., $w_i = \sum_j t_{ij} v_j$. Put $[w/v] := \det(T)$. Let

$$C^*\colon 0 \to C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-2}} C^{n-1} \xrightarrow{\delta_{n-1}} C^n \to 0$$

be a co-chain complex of finite dimensional complex vector spaces. We assume that C^* is acyclic. Let $Z_q = \ker(\delta_q)$ and $B_q := \operatorname{Im}(\delta_{q-1}) \subset C^q$. Let c_q be a preferred base of C_q. Choose a basis b_q for $B_q, q = 0, \dots, n$, and let $\tilde{b}_{q+1}$ be an independent set in C_q such that $\delta_q(\tilde{b}_{q+1}) = b_{q+1}$. Then $(b_q, \tilde{b}_{q+1})$ is a basis of C^q and $[b_q, \tilde{b}_{q+1}/c_q]$ depends only on b_q and b_{q+1}. Therefore, we denote it by $[b_q, b_{q+1}/c_q]$. Then the complex Reidemeister torsion $\tau^{\mathbb{C}}(C^*) \in \mathbb{C}$ of the co-chain complex C^* is defined by

$$\tau^{\mathbb{C}}(C^*) := \prod_{q=0}^{n} [b_q, b_{q+1}/c_q]^{(-1)^q}. \tag{6.7}$$

Let K be a C^∞-triangulation of X and $\widetilde{K}$ the lift of K to a triangulation of the universal covering $\mathbb{H}^d$ of X. Then $C^q(\widetilde{K}, \mathbb{C})$ is a module over the complex group algebra $\mathbb{C}[\Gamma]$. Now let ρ be an acyclic representation of Γ in $\mathrm{SL}(N, \mathbb{C})$. Let

$$C^q(K, \rho) := C^q(\widetilde{K}, \mathbb{C}) \otimes_{C[\Gamma]} \mathbb{C}^N$$

be the twisted co-chain group and

$$C^*(K, \rho)\colon 0 \to C^0(K, \rho) \xrightarrow{\partial_\rho} C^1(K, \rho) \xrightarrow{\partial_\rho} \cdots \xrightarrow{\partial_\rho} C^d(K, \rho) \to 0$$

the corresponding co-chain complex. Then $C^*(K, \rho)$ is acyclic. Let $e_1, \dots, e_{r_q}$ be a preferred basis of $C^q(\tilde{K}, \mathbb{C})$ as a $\mathbb{C}[\Gamma]$-module consisting of the duals of lifts of q-simplexes and let $v_1, \dots, v_N$ be a basis of $\mathbb{C}^N$. Then $\{e_i \otimes v_j : i = 1, \dots, r_q, \quad j = 1, \dots, N\}$ is a preferred basis of $C^q(K, \rho)$. Now consider the complex-valued Reidemeister torsion $\tau^{\mathbb{C}}(C^*(K, \rho))$. Since ρ is a representation in $\mathrm{SL}(N, \mathbb{C})$, a different choice of the preferred basis $\{e_i\}$ leads at most to a sign change of $\tau^{\mathbb{C}}(C^*(K, \rho))$. If v' is a different basis of $\mathbb{C}^N$, then $\tau^{\mathbb{C}}(C^*(K, \rho))$ changes by $[v'/v]^{\chi(X)}$. Hence, if $\chi(X) = 0$, $\tau^{\mathbb{C}}(C^*(K, \rho))$ is well defined as an element of $\mathbb{C}^*/\{\pm 1\}$. Since every two smooth triangulations of X admit a common subdivision, it follows from [22] that $\tau^{\mathbb{C}}(C^*(K, \rho))$ is independent of the smooth triangulation K. Put

$$\tau^{\mathbb{C}}(X, \rho) := \tau^{\mathbb{C}}(C^*(K, \rho)). \tag{6.8}$$

This is the complex valued Reidemeister torsion of X and ρ. If ρ is not acyclic, then the same construction gives an element $\tau^{\mathbb{C}}(X, \rho) \in \det H^*(X, F_\rho)$ [9].

6.2 Deformations of Acyclic Unitary Representations

Let $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ be the set of all m-dimensional complex representations of Γ. It is well known that $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ has a natural structure of a complex algebraic variety [3, 13.6]. Recall that $\chi \in \mathrm{Rep}(\Gamma, \mathbb{C}^m)$ is called acyclic, if $H^*(X, F_\chi) = 0$, where $F_\chi \to X$ is the flat vector bundle associated to χ. Denote by $\mathrm{Rep}_0(\Gamma, \mathbb{C}^m) \subset \mathrm{Rep}(\Gamma, \mathbb{C}^m)$ the subset of all acyclic representations. A representation $\chi \in \mathrm{Rep}(\Gamma, \mathbb{C}^m)$ is called unitary, if there exists a Hermitian scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^m$ which is preserved by all maps $\chi(\gamma)$, $\gamma \in \Gamma$. Let $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m) \subset \mathrm{Rep}_0(\Gamma, \mathbb{C}^m)$ be the subset of all unitary acyclic representations. By [14, Theorem 1.1] we get

Proposition 6.1 *For every compact hyperbolic 3-manifold $\Gamma \backslash \mathbb{H}^3$, we have $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m) \neq \emptyset$.*

For $d > 3$ it is not known if a d-dimensional compact oriented hyperbolic manifold $\Gamma \backslash \mathbb{H}^d$ admits a unitary flat bundle. Assume that $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m) \neq \emptyset$. Let $\chi \in \mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m)$. For such a representation the flat Laplacian $\Delta^\sharp_{k,\chi}$ equals the usual Laplace operator $\Delta_{k,\chi}$ and $T^{\mathbb{C}}(X, \chi) = T^{RS}(X, \chi)^2$. Moreover, $h_k = 0$, $k = 0, \dots, d$, which implies $h = 0$ and $T_0(X, \chi) = 1$. Thus $R(s, \chi)$ is regular at $s = 0$ and from Theorem 1.1 we recover Fried's result [10]

$$R(0, \chi) = T^{RS}(X, \chi)^2. \tag{6.9}$$

We equip $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ with the topology obtained from its structure as complex algebraic variety. The complement of the singular set is a complex manifold. Let $W \subset \mathrm{Rep}(\Gamma, \mathbb{C}^m)$ be the connected component of $\mathrm{Rep}(\Gamma, \mathbb{C}^m)$ which contains $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m)$. Let $\chi_0 \in W$ be a unitary acyclic representation and let E_0 be the associated flat vector bundle. By [15, Prop. 4.5] every vector bundles E_χ, $\chi \in W$, is isomorphic to E_0. Thus the flat connection on E_χ, which is induced by the trivial connection on $\widetilde{X} \times \mathbb{C}^m$, corresponds to a flat connection ∇_χ on E_0. Now recall that

$$\Delta_\chi^\sharp = (d_\chi + d_\chi^{*,\sharp})^2,$$

where

$$d_\chi^{*,\sharp}\big|_{\Lambda^p(X, E_\chi)} = (-1)^p (\star \otimes \mathrm{Id}) d_\chi (\star \otimes \mathrm{Id}).$$

Via the isomorphism $E_\chi \cong E_0$, the operator $d_\chi + d_\chi^{*,\sharp}$ corresponds to the operator

$$D_\chi^\sharp := \nabla_\chi + \nabla_\chi^{*,\sharp} \colon \Lambda^*(X, E_0) \to \Lambda^*(X, E_0),$$

where

$$\nabla_\chi^{*,\sharp} = (-1)^p (\star \otimes \mathrm{Id}) \nabla_\chi (\star \otimes \mathrm{Id}).$$

Let ∇_0 be the unitary flat connection on E_0. Let $\mathcal{C}(E_0)$ denote the space of connections on E_0. Recall that $\mathcal{C}(E_0)$ can be identified with $\Lambda^1(X, \mathrm{End}(E_0))$ by associating to a connection $\nabla \in \mathcal{C}(E_0)$ the 1-form $\nabla - \nabla_0 \in \Lambda^1(X, \mathrm{End}(E_0))$. We equip $\mathcal{C}(E_0)$ with the C^0-topology defined by the sup-norm $\|\omega\|_{\sup} := \max_{x \in X} |\omega(x)|$, $\omega \in \Lambda^1(X, \mathrm{End}(E_0))$, where $|\cdot|$ denotes the natural norm on $\Lambda^1(T^*X) \otimes E_0$. Since E_0 is acyclic, $D_0 := \nabla_0 + \nabla_0^*$ is invertible. If $\|\nabla_\chi - \nabla_0\| \ll 1$ it follows as in [3, Prop 6.8] that $D_\chi^\sharp$ is invertible and hence $\Delta_\chi^\sharp = (D_\chi^\sharp)^2$ is invertible too. Thus we get

Lemma 6.2 *There exists an open neighborhood $V \subset W$ of $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^m)$ such that $\Delta_\chi^\sharp$ is invertible for all $\chi \in V$.*

Let $\chi \in V$. Then we have $h_k = \dim(\Delta_{k,\chi}^\sharp) = 0$, $k = 0, \dots, d$, and therefore the order h of $R(s, \chi)$ at $s = 0$ vanishes. Also $C(d, \chi) = 1$ and $T_0(X, \chi) = 1$. Thus by Theorem 1.1 we obtain Proposition 1.3.

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References

1. Bergeron, N., Venkatesh, A.: The asymptotic growth of torsion homology for arithmetic groups. J. Inst. Math. Jussieu **12**(2), 391–447 (2013)
2. Borel, A., Wallach, N.: Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, Annals of Mathematical Studies, vol. 94. Princeton University Press, Princeton (1980)
3. Braverman, M., Kappeler, T.: Refined analytic torsion. J. Differ. Geom. **78**(2), 193–267 (2008)
4. Braverman, M., Kappeler, T.: Refined analytic torsion as an element of the determinant line. Geom. Topol. **11**, 139–213 (2007)
5. Braverman, M., Kappeler, T.: A canonical quadratic form on the determinant line of a flat vector bundle. Int. Math. Res. Not. IMRN 2008, no. 11, Art. ID rnn030
6. Bunke, U., Olbrich, M.: Selberg Zeta and Theta Functions. A Differential Operator Approach. Mathematical Research, 83. Akademie-Verlag, Berlin (1995)
7. Cappell, S.E., Miller, E.Y.: Complex-valued analytic torsion for flat bundles and for holomorphic bundles with (1,1) connections. Commun. Pure Appl. Math. **63**(2), 133–202 (2010)
8. Cheeger, J.: Analytic torsion and the heat equation. Ann. Math. **109**(2), 259–322 (1979)
9. Farber, M., Turaev, V.: Poincaré-Reidemeister metric, Euler structures, and torsion. J. Reine Angew. Math. **520**, 195–225 (2000)
10. Fried, D.: Analytic torsion and closed geodesics on hyperbolic manifolds. Invent. Math. **84**(3), 523–540 (1986)
11. Fried, D.: Meromorphic zeta functions of analytic flows. Commun. Math. Phys. **174**, 161–190 (1995)
12. Fried, D.: Homological identities for closed orbits. Invent. math. **71**, 419–442 (1983)
13. Fried, D.: Lefschetz Formulas for Flows, Contemporary Mathematics, Part III, vol. 58, pp. 19–69 (1987)
14. Friedl, S., Nagel, M.: 3-manifolds that can be made acyclic. Int. Math. Res. Not. IMRN **24**, 13360–13378 (2015)
15. Goldman, W., Millson, J.: The deformation theory of representations of the fundamental groups of compact Kähler manifolds. Inst. Hautes Études Sci. Publ. Math. No. **67**, 43–96 (1988)
16. Gohberg, I.C., Krein, M.G.: Introduction to the Theory of Linear Nonselfadjoint Operators. Translations of Mathematical Monographs, vol. 18. American Mathematical Society, Providence (1969)
17. Knapp, A.W.: Representation Theory of Semi-simple Groups. Princeton University Press, Princeton and Oxford (2001)
18. Matsushima, Y., Murakami, S.: On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds. Ann. Math. **78**, 365–416 (1963)
19. Markus, A.S.: Introduction to the Spectral Theory of Polynomial Operator Pencils, Translation of Mathematical Monographs, vol. 71. Providence (1988)
20. Marshall, S., Müller, W.: On the torsion in the cohomology of arithmetic hyperbolic 3-manifolds. Duke Math. J. **162**(5), 863–888 (2013)
21. Miatello, R.: The Minakshisundaram–Pleijel coefficients for the vector-valued heat kernel on compact locally symmetric spaces of negative curvature. Trans. Am. Math. Soc. **260**(1), 1–33 (1980)
22. Milnor, J.: Whitehead torsion. Bull. Am. Math. Soc. **72**, 358–426 (1966)
23. Müller, W.: A Selberg trace formula for non-unitary twists. IMRN 9, 2068–2109 (2011)
24. Müller, W.: Analytic torsion and R-torsion of Riemannian manifolds. Adv. Math. **28**(3), 233–305 (1978)
25. Müller, W.: Analytic torsion and R-torsion for unimodular representations. J. Am. Math. Soc. **6**(3), 721–753 (1993)

26. Müller, W., Paff, J.: On the asymptotics of the Ray-Singer analytic torsion for compact hyperbolic manifolds. Int. Math. Res. Not. IMRN **13**, 2945–2983 (2013)
27. Park, J.: Reidemeister torsion, complex volume and the Zograf infinite product for hyperbolic 3-manifolds. Geom. Topol. **23**(7), 3687–3734 (2019)
28. Ray, D.B., Singer, I.M.: R-torsion and the Laplacian on Riemannian manifolds. Adv. Math. **7**, 145–210 (1971)
29. Shen, S.: Analytic torsion, dynamical zeta functions, and the Fried conjecture. Anal. PDE **11**(1), 1–74 (2018)
30. Shubin, M.A.: Pseudodifferential Operators and Spectral Theory, 2nd edn. Springer, Berlin (2001)
31. Spilioti, P.: Selberg and Ruelle zeta functions for non-unitary twists. Ann. Global Anal. Geom. **53**(2), 151–203 (2018)
32. Spilioti, P.: The functional equations of the Selberg and Ruelle zeta functions for non-unitary twists. arXiv:1507.05947
33. Spilioti, P.: Twisted Ruelle zeta function and complex-valued analytic torsion, preprint (2020). arXiv:2004.13474v1
34. Wallach, N.: On the Selberg trace formula in the case of compact quotient. Bull. Am. Math. Soc. **82**, 171–195 (1976)
35. Wotzke, A.: Die Ruellesche Zetafunktion und die analytische Torsion hyperbolischer Mannigfaltigkeiten, PhD thesis, Universität Bonn (2008). https://bib.math.uni-bonn.de/downloads/bms/BMS-389.pdf