Compact Differences of Composition Operators on Bergman Spaces Induced by Doubling Weights

Bin Liu^{1,2} · Jouni Rättyä² · Fanglei Wu^{2,3}

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Abstract

Bounded and compact differences of two composition operators acting from the weighted Bergman space A_ω^p to the Lebesgue space L^q , where $0 < q < p < \infty$ and ω belongs to the class $\mathcal{D}$ of radial weights satisfying two-sided doubling conditions, are characterized. On the way to the proofs a new description of q -Carleson measures for A_ω^p , with $p > q$ and $\omega \in \mathcal{D}$, involving pseudohyperbolic discs is established. This last-mentioned result generalizes the well-known characterization of q -Carleson measures for the classical weighted Bergman space A_α^p with $-1 < \alpha < \infty$ to the setting of doubling weights. The case $\omega \in \widehat{\mathcal{D}}$ is also briefly discussed and an open problem concerning this case is posed.

Keywords Bergman space · Carleson measure · Doubling weight · Composition operator

Mathematics Subject Classification 47B33 · 30H20

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✉ Fanglei Wu
fangleiwu1992@gmail.com

Bin Liu
liubin_taixi@163.com; binl@uef.fi

Jouni Rättyä
jouni.rattya@uef.fi

¹ College of Mathematics and Physics, Chengdu University of Technology, Sichuan 610059, China

² University of Eastern Finland, P.O.Box 111, 80101 Joensuu, Finland

³ Department of Mathematics, Shantou University, Shantou 515063, Guangdong, China

1 Introduction and Main Results

Each analytic self-map φ of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ induces the composition operator C_φ , defined by $C_\varphi f = f \circ \varphi$, acting on the space $\mathcal{H}(\mathbb{D})$ of analytic functions in $\mathbb{D}$. These operators have been extensively studied in a variety of function spaces, see for example [4,23,24,28]. Some studies on topological properties of the space of composition operators attracted attention towards differences of composition operators. The question of when the difference $C_\phi - C_\psi$ of two composition operators is compact on the Hardy space H^p was posed by Shapiro and Sundberg [25]. Differences of composition operators have been studied ever since by many authors on several function spaces, see, for example, [1,2,6,9]. In 2004, Nieminen and Saksman [13] showed that the compactness of $C_\phi - C_\psi$ on the Hardy space H^p , with $1 \leq p < \infty$, is independent of p . By using this, Choe et al. [3] then characterized compact operators $C_\phi - C_\psi$ on Hardy spaces by using Carleson measures in 2020. Further, Moorhouse [11,12] characterized the compactness of $C_\phi - C_\psi$ on the standard weighted Bergman space A_α^2 . Saukko [21,22] generalized Moorhouse's results by characterizing the compactness from A_α^p to A_β^q if either $1 < p \leq q$, or $p > q \geq 1$. Very recently, Shi et al. [27] extended Saukko's results for the complete range $0 < p, q < \infty$ and to higher dimensions.

In this paper, we are interested in the compactness of $C_\phi - C_\psi$ on weighted Bergman spaces induced by doubling weights. We proceed with necessary definitions. For $0 < p < \infty$ and a positive Borel measure ν on $\mathbb{D}$, the Lebesgue space L_ν^p consists of complex valued ν -measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L_\nu^p}^p = \int_{\mathbb{D}} |f(z)|^p d\nu(z) < \infty.$$

If ν is continuous with respect to the Lebesgue measure, that is, $d\nu(z) = \omega(z)dA(z)$ for some non-negative function ω , then we adopt the notation $L_\nu^p = L_\omega^p$ without arising any confusion. For a non-negative function $\omega \in L^1([0, 1))$, its extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and a radial weight ω , the weighted Bergman space A_ω^p consists of analytic functions in L_ω^p . As usual, A_α^p stands for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1 - |z|^2)^\alpha$, where $-1 < \alpha < \infty$. For the theory of these spaces consult [5,8].

For a radial weight ω , write $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s) ds$ for all $z \in \mathbb{D}$. In this paper, we always assume $\widehat{\omega}(z) > 0$, for otherwise $A_\omega^p = \mathcal{H}(\mathbb{D})$ for each $0 < p < \infty$. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) \geq 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}(1 - \frac{1-r}{K})$ for all $0 \leq r < 1$, then we write $\omega \in \check{\mathcal{D}}$. In other words, $\omega \in \check{\mathcal{D}}$ if there exists $K = K(\omega) > 1$ and $C' = C'(\omega) > 0$ such that

$$\widehat{\omega}(r) \leq C' \int_r^{1-\frac{1-r}{K}} \omega(t) dt, \quad 0 \leq r < 1.$$

The intersection $\widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ is denoted by $\mathcal{D}$, and this is the class of weights that we mainly work with. For recent developments on Bergman spaces induced by weights in $\widehat{\mathcal{D}}$, see [16] and the reference therein.

In this paper, we consider compact differences of two composition operators from the weighted Bergman space A_ω^p to the Lebesgue space L_ν^q when $0 < p, q < \infty$ and $\omega \in \widehat{\mathcal{D}}$. To state the first main result, write

$$\delta(z) = \frac{\psi(z) - \varphi(z)}{1 - \overline{\psi(z)}\varphi(z)}, \quad z \in \mathbb{D}.$$

The next result generalizes [22, Theorem 1.2] to the setting of doubling weights.

Theorem 1 *Let $0 < q < p < \infty$ and $\omega \in \mathcal{D}$, and let ν be a positive Borel measure on $\mathbb{D}$. Let ϕ and ψ be analytic self-maps of $\mathbb{D}$. Then the following statements are equivalent:*

- (i) $C_\phi - C_\psi : A_\omega^p \rightarrow L_\nu^q$ is bounded;
- (ii) $C_\phi - C_\psi : A_\omega^p \rightarrow L_\nu^q$ is compact;
- (iii) δC_ϕ and δC_ψ are compact (or equivalently bounded) from A_ω^p to L_ν^q .

The proof of Theorem 1 is organized as follows. We first show that $C_\phi - C_\psi$ is compact if δC_ϕ and δC_ψ are bounded. The proof of this implication is straightforward and relies on the fact that the norm of $f \in \mathcal{H}(\mathbb{D})$ in A_ω^p is comparable to the L_ω^p -norm of the non-tangential maximal function $N(f)(z) = \sup_{\zeta \in \Gamma(z)} |f(\zeta)|$, where

$$\Gamma(z) = \left\{ \zeta \in \mathbb{D} : |\theta - \arg \zeta| < \frac{1}{2} \left(1 - \frac{|\zeta|}{r} \right) \right\}, \quad z = re^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\},$$

is a non-tangential approach region with vertex at z . Then, as each compact operator is bounded, the proof boils down to showing that δC_ϕ and δC_ψ are compact whenever $C_\phi - C_\psi$ is bounded. This part of the proof is more laborious. We first observe that for each ρ -lattice $\{z_k\}$ the function

$$z \mapsto \sum_k b_k \left(\frac{1 - |z_k|}{1 - \overline{z_k}z} \right)^M \frac{1}{(\omega(T(z_k)))^{\frac{1}{p}}}$$

belongs to A_ω^p for all $b = \{b_k\} \in \ell^p$ and its A_ω^p -norm is dominated by a universal constant times $\|b\|_{\ell^p}$. Then we use this function for testing and apply Khinchine's inequality. After duality arguments we finally arrive to a situation where we must understand well the continuous embeddings $A_\omega^p \subset L_\mu^q$. Recall that a positive Borel measure μ on $\mathbb{D}$ is a q -Carleson measure for A_ω^p if the identity operator $I : A_\omega^p \rightarrow L_\mu^q$ is bounded. A complete characterization of such measures in the case $\omega \in \widehat{\mathcal{D}}$ can be found in [17], see also [14, 15, 19]. In particular, it is known that if $p > q$ and $\omega \in \widehat{\mathcal{D}}$, then μ is a q -Carleson measure for A_ω^p if and only if the function

$$\zeta \mapsto \int_{\Gamma(\zeta)} \frac{d\mu(z)}{\omega(T(z))} \tag{1}$$

belongs to $L_{\omega}^{\frac{p}{p-q}}$. Here and from now on $T(z) = \{\zeta \in \mathbb{D} : z \in \Gamma(\zeta)\}$ is the tent induced by $z \in \mathbb{D} \setminus \{0\}$. Further, $\omega(E) = \int_E \omega dA$ for each measurable set $E \subset \mathbb{D}$. Observe that in (1) we may replace the tent $T(z)$ by the Carleson square $S(z) = \{\zeta : 1 - |z| < |\zeta| < 1, |\arg \zeta - \arg z| < (1 - |z|)/2\}$ because $\omega(T(z)) \asymp \omega(S(z))$ for all $z \in \mathbb{D} \setminus \{0\}$ if $\omega \in \mathcal{D}$. To complete the proof of Theorem 1 we will need a variant of the above characterization of Carleson measures in the case $\omega \in \mathcal{D}$. Our statement involves pseudohyperbolic discs and an auxiliary weight associated to ω . The pseudohyperbolic distance between two points a and b in $\mathbb{D}$ is $\rho(a, b) = |(a - b)/(1 - \bar{a}b)|$. For $a \in \mathbb{D}$ and $0 < r < 1$, the pseudohyperbolic disc of centre a and of radius r is $\Delta(a, r) = \{z \in \mathbb{D} : \rho(a, z) < r\}$. It is well known that $\Delta(a, r)$ is an Euclidean disc centred at $(1 - r^2)a/(1 - r^2|a|^2)$ and of radius $(1 - |a|^2)r/(1 - r^2|a|^2)$. We denote $\tilde{\omega}(z) = \tilde{\omega}(z)/(1 - |z|)$ for all $z \in \mathbb{D}$ and note that

$$\|f\|_{A_{\tilde{\omega}}^p} \asymp \|f\|_{A_{\omega}^p}, \quad f \in \mathcal{H}(\mathbb{D}), \quad (2)$$

provided $\omega \in \mathcal{D}$, by [16]. Our embedding theorem generalizes the case $n = 0$ of [10, Theorem 1] to doubling weights and reads as follows.

Theorem 2 *Let $0 < q < p < \infty$ and $\omega \in \mathcal{D}$, and let μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

- (i) μ is a q -Carleson measure for A_{ω}^p ;
- (ii) $I : A_{\omega}^p \rightarrow L_{\mu}^q$ is compact;
- (iii) the function

$$\Theta_{\mu}^{\omega}(z) = \frac{\mu(\Delta(z, r))}{\omega(S(z))}, \quad z \in \mathbb{D} \setminus \{0\},$$

belongs to $L_{\tilde{\omega}}^{\frac{p}{p-q}}$ for some (equivalently for all) $r \in (0, 1)$.

Moreover,

$$\|I\|_{A_{\omega}^p \rightarrow L_{\mu}^q}^q \asymp \|\Theta_{\mu}^{\omega}\|_{L_{\tilde{\omega}}^{\frac{p}{p-q}}}^q. \quad (3)$$

We may not replace $L_{\tilde{\omega}}^{\frac{p}{p-q}}$ by $L_{\omega}^{\frac{p}{p-q}}$ in part (iii) of Theorem 2. A counter example can be constructed as follows. Write $D(z, r)$ for the Euclidean disc $\{\zeta : |\zeta - z| < r\}$. Let $r_n = 1 - 2^{-n}$ and $A_n = D(0, r_{n+1}) \setminus D(0, r_n)$ for all $n \in \mathbb{N}$. Pick up an $\omega \in \mathcal{D}$ such that it vanishes on A_{2n} for all $n \in \mathbb{N}$. A simple example of a such weight is $\sum_{n \in \mathbb{N}} \chi_{A_{2n+1}}$. Then choose μ such that for some $\varepsilon > 0$ its support is contained in the union of the discs $\Delta(a_n, \varepsilon)$ which have the property that for some fixed $r \in (0, 1)$ we have $\Delta(z, r) \subset A_{2n}$ for all $z \in \Delta(a_n, \varepsilon)$ and for all $n \in \mathbb{N}$. The choice $a_n = (r_{2n} + r_{2n+1})/2$ works if $0 < r < 1$ and $\varepsilon = \varepsilon(r) > 0$ are sufficiently small. Then, for such an r , the norm $\|\Theta_{\mu}^{\omega}\|_{L_{\omega}^{\frac{p}{p-q}}}$ vanishes and thus it cannot be comparable to $\|\Theta_{\mu}^{\omega}\|_{L_{\tilde{\omega}}^{\frac{p}{p-q}}}$ which is non-zero if μ is not a zero measure because $\tilde{\omega}$ is strictly positive. Moreover, by choosing μ appropriately the norm $\|\Theta_{\mu}^{\omega}\|_{L_{\tilde{\omega}}^{\frac{p}{p-q}}}$ can be made infinite.

The second main result of this study concerns the case when $p < q$ and $\omega \in \widehat{\mathcal{D}}$. It completes in part the main result in [9] which concerns the class $\mathcal{D}$ only. An analogue

of this result was proved for the Hardy spaces in [26]. Therefore, Theorem 3 takes care of the gap consisting of small Bergman spaces that exists between the Hardy and the standard weighted Bergman spaces.

Theorem 3 *Let $0 < p < q < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let ν be a positive Borel measure on $\mathbb{D}$. Let ϕ and ψ be analytic self-maps of $\mathbb{D}$. Then $C_\phi - C_\psi : A_\omega^p \rightarrow L_\nu^q$ is bounded (resp. compact) if and only if δC_ϕ and δC_ψ are bounded (resp. compact) from A_ω^p to L_ν^q .*

If $q = p$ then the boundedness (resp. compactness) of δC_ϕ and δC_ψ implies the same property for $C_\phi - C_\psi$ by Proposition 1. Further, Proposition 2 shows that $C_\phi - C_\psi$ is compact if δC_ϕ and δC_ψ are bounded when $p > q$. But we do not know if the boundedness of $C_\phi - C_\psi$ implies that of δC_ϕ and δC_ψ if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$ if $p \geq q$. The methods used in this paper do not seem to give this implication and hence this case remains unsettled.

The rest of the paper is organized as follows. In the next section, we consider Carleson embeddings and prove Theorem 2. The sufficiency of the conditions on δC_ϕ and δC_ψ are established in Sect. 3, while Sect. 4 is devoted to their necessity. Finally, in Sect. 5 we indicate how our main findings on $C_\phi - C_\psi$ follow from these results.

To this end, couple of words about the notation are used in this paper. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable.

2 Carleson Measures

If $\omega \in \mathcal{D}$, then there exist constants $0 < \alpha = \alpha(\omega) \leq (\omega) < \infty$ and $C = C(\omega) \geq 1$ such that

$$\frac{1}{C} \left(\frac{1-r}{1-t} \right)^\alpha \leq \frac{\widehat{\omega}(r)}{\widehat{\omega}(t)} \leq C \left(\frac{1-r}{1-t} \right)^\beta, \quad 0 \leq r \leq t < 1. \quad (4)$$

In fact, these inequalities characterize the class $\mathcal{D}$ because the right-hand inequality is satisfied if and only if $\omega \in \widehat{\mathcal{D}}$ by [14, Lemma 2.1], while the left-hand inequality describes the class $\overline{\mathcal{D}}$ in an analogous manner [16, (2.27)]. The inequalities (4) will be frequently used throughout the paper.

Proof If μ is a q -Carleson measure for A_ω^p , then $I : A_\omega^p \rightarrow L_\mu^q$ is automatically compact by [19, Theorem 3(iii)]. Therefore, it suffices to show that (i) and (iii) are equivalent and establish (3).

Assume first $\Theta_\mu^\omega \in L_{\omega}^{\frac{p}{p-q}}$ for some $r \in (0, 1)$. The subharmonicity of $|f|^q$, Fubini's theorem, Hölder's inequality and (2) imply

$$\begin{aligned} \|f\|_{L_\mu^q}^q &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(z,r)} \frac{|f(\zeta)|^q}{(1-|\zeta|^2)^2} dA(\zeta) \right) d\mu(z) \\ &= \int_{\mathbb{D}} \left(\int_{\Delta(z,r)} |f(\zeta)|^q \frac{\tilde{\omega}(\zeta)}{\widehat{\omega}(\zeta)(1-|\zeta|)} dA(\zeta) \right) d\mu(z) \\ &\leq \int_{\mathbb{D}} |f(\zeta)|^q \frac{\mu(\Delta(\zeta, r))}{\omega(S(\zeta))} \tilde{\omega}(\zeta) dA(\zeta) \leq \|f\|_{A_\omega^p}^q \|\Theta_\mu^\omega\|_{L_{\omega}^{\frac{p}{p-q}}} \\ &\asymp \|f\|_{A_\omega^p}^q \|\Theta_\mu^\omega\|_{L_{\omega}^{\frac{p}{p-q}}}, \quad f \in A_\omega^p. \end{aligned}$$

Therefore, μ is a q -Carleson measure for A_ω^p and $\|I\|_{A_\omega^p \rightarrow L_\mu^q}^q \lesssim \|\Theta_\mu^\omega\|_{L_{\omega}^{\frac{p}{p-q}}}^q$.

Conversely, assume that μ is a q -Carleson measure for A_ω^p . Then (2) shows that μ is also a q -Carleson measure for A_ω^p and the corresponding operator norms are comparable. Further, since $\omega \in \mathcal{D}$ by the hypothesis, an application of (4) shows that $\tilde{\omega} \in \mathcal{D}$. Therefore, [17, Theorem 1(a)] implies

$$\|B_\mu^\omega\|_{L_{\omega}^{\frac{p}{p-q}}}^{\frac{p}{p-q}} = \int_{\mathbb{D}} \left(\int_{\Gamma(z)} \frac{d\mu(\zeta)}{\tilde{\omega}(T(\zeta))} \right)^{\frac{p}{p-q}} \tilde{\omega}(z) dA(z) < \infty,$$

where

$$\Gamma(z) = \left\{ \zeta \in \mathbb{D} : |\arg \zeta - \arg z| < \frac{1}{2} \left(1 - \frac{|\zeta|}{|z|} \right) \right\}, \quad z \in \mathbb{D} \setminus \{0\},$$

is a non-tangential approach region with vertex at z . Further, by [19, Theorem 3(iii)] we have $\|I\|_{A_\omega^p \rightarrow L_\mu^q}^q \gtrsim \|B_\mu^\omega\|_{L_{\omega}^{\frac{p}{p-q}}}^{\frac{p}{p-q}}$.

Let now $r \in (0, 1)$ be given. For $K > 1$ and $z \in \mathbb{D} \setminus D(0, 1 - \frac{1}{K})$ write $z_K = (1 - K(1 - |z|))e^{i \arg z}$. Pick up $K = K(r) > 1$ and $R = R(r) \in (1 - \frac{1}{K}, 1)$ sufficiently large such that $\Delta(z_K, r) \subset \Gamma(z)$ for all $z \in \mathbb{D} \setminus D(0, R)$. Straightforward applications of the left-hand inequality in (4) show that $\tilde{\omega}(T(\zeta)) \lesssim \omega(S(\zeta))$, as $|\zeta| \rightarrow 1^-$, and $\tilde{\omega}(z) \lesssim \tilde{\omega}(z_K)$ for all $z \in \mathbb{D} \setminus D(0, R)$. In an analogous way, we deduce $\omega(S(\zeta)) \lesssim \omega(S(z_K))$ for all $\zeta \in \Delta(z_K, r)$ and $z \in \mathbb{D} \setminus D(0, R)$ by using the right-hand inequality. Therefore,

$$\begin{aligned} \infty &> \|I\|_{A_\omega^p \rightarrow L_\mu^q}^q \gtrsim \int_{\mathbb{D} \setminus D(0, R)} \left(\int_{\Delta(z_K, r)} \frac{d\mu(\zeta)}{\omega(T(\zeta))} \right)^{\frac{p}{p-q}} \tilde{\omega}(z) dA(z) \\ &\gtrsim \int_{\mathbb{D} \setminus D(0, R)} \left(\frac{\mu(\Delta(z_K, r))}{\omega(S(z_K))} \right)^{\frac{p}{p-q}} \tilde{\omega}(z_K) dA(z) \\ &\asymp \int_{\mathbb{D} \setminus D(0, 1-K(1-R))} |\Theta_\mu^\omega(z)|^{\frac{p}{p-q}} \tilde{\omega}(z) dA(z). \end{aligned}$$

It follows that $\Theta_\mu^\omega \in L_{\tilde{\omega}}^{\frac{p}{p-q}}$ and $\|I\|_{A_\omega^p \rightarrow L_\mu^q}^q \gtrsim \|B_\mu^\omega\|_{L_{\tilde{\omega}}^{\frac{p}{p-q}}}^q \gtrsim \|\Theta_\mu^\omega\|_{L_{\tilde{\omega}}^{\frac{p}{p-q}}}^q$. $\square$

3 Sufficient Conditions

In this section, we establish sufficient conditions for $C_\phi - C_\psi : A_\omega^p \rightarrow L_v^q$ to be bounded or compact. All these results are valid under the hypothesis $\omega \in \widehat{\mathcal{D}}$ despite the main results stated in the introduction concern only the class $\mathcal{D}$. We begin with the case $p \leq q$.

Proposition 1 *Let $0 < p \leq q < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let ν be a positive Borel measure on $\mathbb{D}$. Let ϕ and ψ be analytic self-maps of $\mathbb{D}$. If δC_ϕ and δC_ψ are bounded (resp. compact) from A_ω^p to L_ν^q , then $C_\phi - C_\psi : A_\omega^p \rightarrow L_\nu^q$ is bounded (resp. compact).*

Proof We begin with the statement on the boundedness. Let first $q > p$. Let $f \in A_\omega^p$ with $\|f\|_{A_\omega^p} \leq 1$. Fix $0 < r < R < 1$, and denote $E = \{z \in \mathbb{D} : |\delta(z)| < r\}$ and $E' = \mathbb{D} \setminus E$. Write

$$(C_\phi - C_\psi)(f) = (C_\phi - C_\psi)(f)\chi_{E'} + (C_\phi - C_\psi)(f)\chi_E,$$

and observe that it is enough to prove that the quantities

$$\|(C_\phi - C_\psi)(f)\chi_{E'}\|_{L_\nu^q} \quad \text{and} \quad \|(C_\phi - C_\psi)(f)\chi_E\|_{L_\nu^q} \quad (5)$$

are bounded.

We begin with considering the first quantity in (5). By the definition of the set E , we have the estimate

$$|(C_\phi - C_\psi)(f)\chi_{E'}| \leq \frac{1}{r}(|\delta C_\phi(f)| + |\delta C_\psi(f)|), \quad (6)$$

on $\mathbb{D}$. Since the operators δC_ϕ and δC_ψ both are bounded from A_ω^p to L_ν^q by the hypothesis, the first term in (5) is bounded by (6).

We next show that also the second term in (5) is bounded. Let μ be a finite non-negative Borel measure on $\mathbb{D}$ and h a measurable function on $\mathbb{D}$. For an analytic self-map φ of $\mathbb{D}$, the weighted pushforward measure is defined by

$$\varphi_*(h, \mu)(M) = \int_{\varphi^{-1}(M)} h d\mu \quad (7)$$

for each measurable set $M \subset \mathbb{D}$. If μ is the Lebesgue measure, we omit the measure in the notation and write $\varphi_*(h)(M)$ for the left-hand side of (7). By the measure theoretic change of variable [7, Sect. 39], we have $\|\delta C_\varphi(f)\|_{L_\nu^q} = \|f\|_{L_{\varphi_*(|\delta|^q \nu)}^q}$ for each $f \in A_\omega^p$. Therefore, the identity operator from A_ω^p to $L_{\varphi_*(|\delta|^q \nu)}^q$ is bounded by the hypothesis. Hence $\varphi_*(|\delta|^q \nu)(\Delta(\zeta, R)) \lesssim \omega(S(\zeta))^{\frac{q}{p}}$ for all $\zeta \in \mathbb{D} \setminus \{0\}$ by [17,

Theorem 1(c)]. Further, by [9, Lemma 3], with $\omega \equiv 1$ and $q = p$, there exists a constant $C = C(p, r, R) > 0$ such that

$$|f(z) - f(a)|^q \leq C \frac{\rho(z, a)^p}{(1 - |a|)^2} \int_{\Delta(a, R)} |f(\zeta)|^p dA(\zeta), \quad a \in \mathbb{D}, \quad z \in \Delta(a, r),$$

for all $f \in A_\omega^p$ with $\|f\|_{A_\omega^p} \leq 1$. This and Fubini's theorem yield

$$\begin{aligned} \|(C_\varphi - C_\psi)(f)\chi_E\|_{L_v^q}^q &= \int_E |f(\varphi(z)) - f(\psi(z))|^q v(z) dA(z) \\ &\lesssim \int_E \frac{|\delta(z)|^q}{(1 - |\varphi(z)|)^2} \int_{(\varphi(z), R)} |f(\zeta)|^q dA(\zeta) v(z) dA(z) \\ &\leq \int_{\mathbb{D}} |f(\zeta)|^q \left(\int_{\varphi^{-1}(\Delta(\zeta, R)) \cap E} \frac{|\delta(z)|^q}{(1 - |\varphi(z)|)^2} v(z) dA(z) \right) dA(\zeta) \\ &\lesssim \int_{\mathbb{D}} |f(\zeta)|^q \left(\int_{\varphi^{-1}(\Delta(\zeta, R))} \frac{|\delta(z)|^q}{(1 - |\zeta|)^2} v(z) dA(z) \right) dA(\zeta) \\ &= \int_{\mathbb{D}} |f(\zeta)|^q \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta) \\ &\lesssim \int_{\mathbb{D}} |f(\zeta)|^q \frac{\omega(S(\zeta))^{\frac{q}{p}}}{(1 - |\zeta|)^2} dA(\zeta) = \int_{\mathbb{D}} |f(\zeta)|^q d\mu(\zeta). \end{aligned} \quad (8)$$

Standard arguments show that $\mu(S(a)) \lesssim \omega(S(a))^{\frac{q}{p}}$ for all $a \in \mathbb{D} \setminus \{0\}$. Hence [17, Theorem 1(c)] yields $\|(C_\varphi - C_\psi)(f)\chi_E\|_{L_v^q}^q \lesssim \|f\|_{L_\mu^q}^q \lesssim \|f\|_{A_\omega^p}^q$. Therefore, also the second term in (5) is bounded. This finishes the proof of the case $q > p$.

Let now $q = p$. By following the proof above, it suffices to show that

$$\int_S \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta) \lesssim \omega(S) \quad (9)$$

for every Carleson square $S \subset \mathbb{D}$. By the hypothesis, the identity operator from A_ω^p to $L_{\varphi_*(|\delta|^q v)}^p$ is bounded, and hence $\varphi_*(|\delta|^q v)(S) \lesssim \omega(S)$ for all S by [17, Theorem 1(b)]. But for each positive Borel measure μ on $\mathbb{D}$, Fubini's theorem yields

$$\begin{aligned} \int_{S(a)} \frac{\mu(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta) &= \int_{\{z \in \mathbb{D} : S(a) \cap \Delta(z, R) \neq \emptyset\}} \left(\int_{S(a) \cap \Delta(z, R)} \frac{dA(\zeta)}{(1 - |\zeta|)^2} \right) d\mu(z) \\ &\leq \int_{S(b)} \left(\int_{\Delta(z, R)} \frac{dA(\zeta)}{(1 - |\zeta|)^2} \right) d\mu(z) \asymp \mu(S(b)), \quad |a| > R, \end{aligned} \quad (10)$$

where $b = b(a, R) \in \mathbb{D}$ satisfies $\arg b = \arg a$ and $1 - |b| \asymp 1 - |a|$ for all $a \in \mathbb{D} \setminus \overline{D(0, R)}$. By applying this to $\mu = \varphi_*(|\delta|^q v)$ and using the hypothesis $\omega \in \widehat{\mathcal{D}}$, we deduce

$$\int_{S(a)} \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta) \lesssim \varphi_*(|\delta|^q v)(S(b)) \lesssim \omega(S(b)) \lesssim \omega(S(a)), \quad |a| > R.$$

This estimate implies (9), and thus the case $q = p$ is proved.

To obtain the compactness statement, it suffices to show that the quantities

$$\|(C_\varphi - C_\psi)(f_n)\chi_{E'}\|_{L_v^q}, \quad \text{and} \quad \|(C_\varphi - C_\psi)(f_n)\chi_E\|_{L_v^q} \quad (11)$$

tend to zero as $n \rightarrow \infty$ for each sequence $\{f_n\}_{n \in \mathbb{N}}$ in A_ω^p which tends to zero uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$ and satisfies $\|f_n\|_{A_\omega^p} \leq 1$ for all $n \in \mathbb{N}$. Since δC_φ and δC_ψ are compact from A_ω^p to L_v^q by the hypothesis, an application of (6) to $f = f_n$ shows that the first quantity in (11) tends to zero as $n \rightarrow \infty$. As for the second quantity, observe that (8) implies

$$\|(C_\varphi - C_\psi)(f_n)\chi_E\|_{L_v^q}^q \lesssim \int_{\mathbb{D}} |f_n(\zeta)|^q \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta), \quad n \in \mathbb{N}. \quad (12)$$

Let first $q > p$. Since the identity operator from A_ω^p to $L_{\varphi_*(|\delta|^q v)}^q$ is compact by the hypothesis, we have $\varphi_*(|\delta|^q v)(S(\zeta))/\omega(S(\zeta))^{\frac{q}{p}} \rightarrow 0$ as $|\zeta| \rightarrow 1^-$ by [19, Theorem 3(ii)]. Now, for each $\zeta \in \mathbb{D} \setminus \{0\}$ pick up $\zeta' = \zeta'(\zeta, R) \in \mathbb{D}$ such that $\arg \zeta' = \arg \zeta$, $\Delta(\zeta', R) \subset S(\zeta)$ and $1 - |\zeta'| \asymp 1 - |\zeta|$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. Then

$$\frac{\varphi_*(|\delta|^q v)(S(\zeta))}{\omega(S(\zeta))^{\frac{q}{p}}} \geq \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta', R))}{\omega(S(\zeta))^{\frac{q}{p}}} \gtrsim \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta', R))}{\omega(S(\zeta'))^{\frac{q}{p}}}, \quad \zeta \in \mathbb{D} \setminus \{0\},$$

and hence $\sup_{\zeta \in \mathbb{D} \setminus \{0\}} \varphi_*(|\delta|^q v)(\Delta(\zeta, R))/\omega(S(\zeta))^{\frac{q}{p}} < \infty$ and, for a given $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) \in (0, 1)$ such that $\varphi_*(|\delta|^q v)(\Delta(\zeta, R))/\omega(S(\zeta))^{\frac{q}{p}} < \varepsilon$ for all $\zeta \in \mathbb{D} \setminus D(0, \eta)$. Further, by the uniform convergence, there exists $N = N(\varepsilon, \eta, q) \in \mathbb{N}$ such that $|f_n|^q \leq \varepsilon$ on $D(0, \eta)$ for all $n \geq N$. These observations together with the proof of the boundedness case and (12) yield

$$\begin{aligned} \|(C_\varphi - C_\psi)(f_n)\chi_E\|_{L_v^q}^q &\lesssim \sup_{\zeta \in D(0, \eta) \setminus \{0\}} \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta, R))}{\omega(S(\zeta))^{\frac{q}{p}}} \\ &\quad \times \int_{D(0, \eta)} |f_n(\zeta)|^q \frac{\omega(S(\zeta))^{\frac{q}{p}}}{(1 - |\zeta|)^2} dA(\zeta) \\ &\quad + \sup_{\zeta \in \mathbb{D} \setminus D(0, \eta)} \frac{\varphi_*(|\delta|^q v)(\Delta(\zeta, R))}{\omega(S(\zeta))^{\frac{q}{p}}} \\ &\quad \times \int_{\mathbb{D} \setminus D(0, \eta)} |f_n(\zeta)|^q \frac{\omega(S(\zeta))^{\frac{q}{p}}}{(1 - |\zeta|)^2} dA(\zeta) \\ &\lesssim \varepsilon \int_{\mathbb{D}} \frac{\omega(S(\zeta))^{\frac{q}{p}}}{(1 - |\zeta|)^2} dA(\zeta) + \varepsilon \|f_n\|_{A_\omega^p}^q \lesssim \varepsilon, \quad n \geq N. \end{aligned}$$

Thus also the second quantity in (11) tends to zero as $n \rightarrow \infty$ in the case $q > p$.

Finally, let $q = p$. The compactness of the identity operator from A_ω^p to $L_{\varphi_*(|\delta|^p\nu)}^p$ implies $\varphi_*(|\delta|^p\nu)(S(\zeta))/\omega(S(\zeta)) \rightarrow 0$ as $|\zeta| \rightarrow 1^-$ by [19, Theorem 3(ii)]. By following the proof above the only different step consists of making the quantity

$$J(\eta) = \int_{\mathbb{D} \setminus D(0, \eta)} |f_n(\zeta)|^p \frac{\varphi_*(|\delta|^p\nu)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta)$$

small, uniformly in n , by choosing $\eta \in (0, 1)$ sufficiently large. But an application of [17, Theorem 1(b)] together with (10) yields

$$J(\eta) \lesssim \|f_n\|_{A_\omega^p}^p \sup_{b \in \mathbb{D} \setminus \{0\}} \frac{\varphi_*(|\delta|^p\nu)(S(b) \setminus D(0, \eta))}{\omega(S(b))}.$$

Standard arguments can now be used to make the right-hand side smaller than a pregiven $\varepsilon > 0$ for η sufficiently large by using $\varphi_*(|\delta|^p\nu)(S(\zeta))/\omega(S(\zeta)) \rightarrow 0$ as $|\zeta| \rightarrow 1^-$, see, for example, [15, pp. 26–27] for details. This completes the proof of the proposition.

The next result is a counter part of Proposition 1 in the case $p > q$.

Proposition 2 *Let $0 < q < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let ν be a positive Borel measure on $\mathbb{D}$. Let ϕ and ψ be analytic self-maps of $\mathbb{D}$. If δC_ϕ and δC_ψ are bounded from A_ω^p to L_ν^q , then $C_\phi - C_\psi : A_\omega^p \rightarrow L_\nu^q$ is compact.*

Proof Let $\{f_n\}$ be a bounded sequence in A_ω^p such that $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. Since δC_ϕ and δC_ψ are bounded from A_ω^p to L_ν^q by the hypothesis, they are also compact by [19, Theorem 3(iii)], and therefore

$$\lim_{n \rightarrow \infty} (\|\delta f_n(\phi)\|_{L_\nu^q} + \|\delta f_n(\psi)\|_{L_\nu^q}) = 0. \quad (13)$$

Let $0 < r < R < 1$, and denote $E = \{z \in \mathbb{D} : |\delta(z)| < r\}$ and $E' = \mathbb{D} \setminus E$. To prove the compactness of $C_\phi - C_\psi : A_\omega^p \rightarrow L_\nu^q$, it suffices to show that

$$\lim_{n \rightarrow \infty} (\|(C_\phi - C_\psi)(f_n)\chi_E\|_{L_\nu^q} + \|(C_\phi - C_\psi)(f_n)\chi_{E'}\|_{L_\nu^q}) = 0$$

since

$$\|(C_\phi - C_\psi)(f_n)\|_{L_\nu^q}^q = \|(C_\phi - C_\psi)(f_n)\chi_E\|_{L_\nu^q}^q + \|(C_\phi - C_\psi)(f_n)\chi_{E'}\|_{L_\nu^q}^q.$$

By using (6) and (13), it is easy to show that

$$\lim_{n \rightarrow \infty} \|(C_\phi - C_\psi)(f_n)\chi_{E'}\|_{L_\nu^q} = 0.$$

Further, by (8), we have

$$\|(C_\phi - C_\psi)(f_n)\chi_E\|_{L_\nu^q}^q \lesssim \int_{\mathbb{D}} |f_n(\zeta)|^q \frac{\varphi_*(|\delta|^q\nu)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta).$$

Let $\varepsilon > 0$. Since the identity operator from A_ω^p to $L_{\phi_*(|\delta|^q \nu)}^q$ is bounded by the hypothesis, [19, Theorem 3(iii)] and the dominated convergence theorem implies the existence of an $R_0 = R_0(\varepsilon) \in (0, 1)$ such that

$$\int_{\mathbb{D}} \left(\int_{\Gamma(z) \setminus \overline{D(0, R_0)}} \frac{\phi_*(|\delta|^q \nu)(\Delta(\zeta, R))}{\omega(T(\zeta))(1 - |\zeta|)^2} dA(\zeta) \right)^{\frac{p}{p-q}} \omega(z) dA(z) < \varepsilon^{\frac{p}{p-q}}. \quad (14)$$

Further, by the uniform convergence, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|f_n(z)| < \varepsilon^{\frac{1}{q}}$ for all $n \geq N$ and $z \in \overline{D(0, R_0)}$. Therefore, for all $n \geq N$, by Fubini's theorem, Hölder's inequality, [15, Lemma 4.4] and (14), we have

$$\begin{aligned} & \| (C_\phi - C_\psi)(f_n) \chi_E \|_{L_v^q}^q \\ & \lesssim \left(\int_{\overline{D(0, R_0)}} + \int_{\mathbb{D} \setminus \overline{D(0, R_0)}} \right) |f_n(\zeta)|^q \frac{\phi_*(|\delta|^q \nu)(\Delta(\zeta, R))}{(1 - |\zeta|)^2} dA(\zeta) \\ & \lesssim \varepsilon + \int_{\mathbb{D}} \left(\int_{\Gamma(z) \setminus \overline{D(0, R_0)}} |f_n(\zeta)|^q \frac{\phi_*(|\delta|^q \nu)(\Delta(\zeta, R))}{\omega(T(\zeta))(1 - |\zeta|)^2} dA(\zeta) \right) \omega(z) dA(z) \\ & \leq \varepsilon + \int_{\mathbb{D}} N(f_n)^q(z) \left(\int_{\Gamma(z) \setminus \overline{D(0, R_0)}} \frac{\phi_*(|\delta|^q \nu)(\Delta(\zeta, R))}{\omega(T(\zeta))(1 - |\zeta|)^2} dA(\zeta) \right) \omega(z) dA(z) \\ & \leq \varepsilon \left(1 + \|N(f_n)\|_{L_\omega^p}^q \right) \asymp \varepsilon. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \|(C_\phi - C_\psi)(f_n) \chi_E\|_{L_v^q} = 0$, and thus $C_\phi - C_\psi$ is compact from A_ω^p to L_v^q .

4 Necessary Conditions

In this section, we establish necessary conditions for $C_\phi - C_\psi : A_\omega^p \rightarrow L_v^q$ to be bounded or compact. In the case $p < q$ we work with the whole class $\widehat{\mathcal{D}}$, but the arguments employed in the case $p \geq q$ rely strongly on the hypothesis $\omega \in \mathcal{D}$. We begin with the case $p \leq q$.

Proposition 3 *Let either $0 < p < q < \infty$ and $\omega \in \widehat{\mathcal{D}}$ or $p = q$ and $\omega \in \mathcal{D}$, and let ν be a positive Borel measure on $\mathbb{D}$. Let ϕ and ψ analytic self-maps of $\mathbb{D}$. If $C_\phi - C_\psi : A_\omega^p \rightarrow L_v^q$ is bounded (resp. compact), then δC_ϕ and δC_ψ are bounded (resp. compact) from A_ω^p to L_v^q .*

Proof Let first $p < q$ and $\omega \in \widehat{\mathcal{D}}$. We begin with the boundedness and show in detail that δC_ϕ is bounded from A_ω^p to L_v^q . For each $a \in \mathbb{D}$, consider the function

$$f_a(z) = \left(\frac{1 - |a|}{1 - \bar{a}z} \right)^\gamma \omega(S(a))^{-\frac{1}{p}}, \quad z \in \mathbb{D},$$

induced by ω and $0 < \gamma, p < \infty$. Then [14, Lemma 2.1] implies that for each $\gamma = \gamma(\omega, p) > 0$ sufficiently large we have $\|f_a\|_{A_\omega^p} \asymp 1$ for all $a \in \mathbb{D}$. Fix such a γ .

Since $C_\varphi - C_\psi$ is bounded, we have

$$\begin{aligned} 1 &\asymp \|f_a\|_{A_\omega^p}^q \gtrsim \|(C_\varphi - C_\psi)(f_a)\|_{L_v^q}^q \\ &= \int_{\mathbb{D}} \left| \left(\frac{1-|a|}{1-\bar{a}\varphi(z)} \right)^\gamma - \left(\frac{1-|a|}{1-\bar{a}\psi(z)} \right)^\gamma \right|^q \frac{v(z)}{\omega(S(a))^{\frac{q}{p}}} dA(z) \\ &= \int_{\mathbb{D}} \left| \frac{1-|a|}{1-\bar{a}\varphi(z)} \right|^{\gamma q} \left| 1 - \left(\frac{1-\bar{a}\varphi(z)}{1-\bar{a}\psi(z)} \right)^\gamma \right|^q \frac{v(z)}{\omega(S(a))^{\frac{q}{p}}} dA(z). \end{aligned}$$

According to [21, p. 795], for each $0 < \gamma < \infty$ and $0 < r < 1$ there exists a constant $C = C(\gamma, r) > 0$ such that

$$\left| 1 - \left(\frac{1-\bar{a}z}{1-\bar{a}w} \right)^\gamma \right| \geq C|a|\rho(z, w), \quad z \in \Delta(a, r), \quad a, w \in \mathbb{D}. \quad (15)$$

An application of this inequality gives

$$1 \gtrsim \int_{\varphi^{-1}(\Delta(a, r))} \frac{|a|^q |\delta(z)|^q}{(\omega(S(a)))^{\frac{q}{p}}} v(z) dA(z) = |a|^q \frac{\varphi_*(|\delta|^q v)(\Delta(a, r))}{(\omega(S(a)))^{\frac{q}{p}}}.$$

It follows that $\varphi_*(|\delta|^q v)$ is a bounded q -Carleson measure for A_ω^p by [17, Theorem 1(c)], and hence $\delta C_\phi : A_\omega^p \rightarrow L_v^q$ is bounded. The same argument shows that also δC_ψ is bounded.

For the compactness statement, first observe that f_a tends to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \rightarrow 1^-$. Then, if $C_\varphi - C_\psi$ is compact, we have $\lim_{|a| \rightarrow 1^-} \|(C_\varphi - C_\psi)(f_a)\|_{L_v^q} = 0$. By arguing as above we deduce

$$\lim_{|a| \rightarrow 1^-} \frac{\varphi_*(|\delta|^q v)(\Delta(a, r))}{(\omega(S(a)))^{\frac{q}{p}}} = 0.$$

Therefore, $\delta C_\varphi : A_\omega^p \rightarrow L_v^q$ is compact by [19, Theorem 3]. The same argument shows that also δC_ψ is compact.

Let now $p = q$ and $\omega \in \mathcal{D}$. The statement follows from the proof above with the modification that [9, Theorem 2], valid for $\omega \in \mathcal{D}$, is used instead of [17, Theorem 1(c)] and [19, Theorem 3]. The only extra step is to observe that for each $\omega \in \mathcal{D}$ there exists $r = r(\omega) \in (0, 1)$ such that $\omega(S(a)) \asymp \omega(\Delta(a, r))$ for all $a \in \mathbb{D} \setminus \{0\}$. This follows from (4). With this guidance we consider the proposition proved.

The next result establishes a counter part of Proposition 3 when $p > q$. Recall that for $0 < r < 1$ a sequence $\{z_k\}$ of distinct points in the unit disc is called r -lattice if $\inf_{i \neq j} \rho(z_i, z_j) > r$ and $\mathbb{D} = \bigcup_{i=1}^\infty \Delta(z_i, r)$.

Proposition 4 *Let $0 < q < p < \infty$ and $\omega \in \mathcal{D}$, and let v be a positive Borel measure on $\mathbb{D}$. Let ϕ and ψ be analytic self-maps of $\mathbb{D}$. If $C_\phi - C_\psi : A_\omega^p \rightarrow L_v^q$ is bounded, then δC_ϕ and δC_ψ are both bounded from A_ω^p to L_v^q .*

Proof Let $\{z_k\}_{k \in \mathbb{N}}$ be a ρ -lattice such that it is ordered by increasing moduli and $z_k \neq 0$ for all k . Then by [20, Theorem 1] there exist constants $M = M(p, \omega) > 1$ and $C = C(p, \omega) > 0$ such that the function

$$F(z) = \sum_k b_k \left(\frac{1 - |z_k|}{1 - \overline{z_k} z} \right)^M \frac{1}{(\omega(T(z_k)))^{\frac{1}{p}}}, \quad z \in \mathbb{D},$$

belongs to A_ω^p and satisfies $\|F\|_{A_\omega^p} \leq C \|b\|_{\ell^p}$ for all $b = \{b_k\} \in \ell^p$. Since $C_\phi - C_\psi : A_\omega^p \rightarrow L_v^q$ is bounded by the hypothesis, we deduce

$$\begin{aligned} \|b\|_{\ell^p}^q &\gtrsim \|F\|_{A_\omega^p}^q \gtrsim \int_{\mathbb{D}} |(C_\phi - C_\psi) \circ F(z)|^q \, dv(z) \\ &= \int_{\mathbb{D}} \left| \sum_k b_k \left(\left(\frac{1 - |z_k|}{1 - \overline{z_k} \phi(z)} \right)^M - \left(\frac{1 - |z_k|}{1 - \overline{z_k} \psi(z)} \right)^M \right) \frac{1}{(\omega(T(z_k)))^{\frac{1}{p}}} \right|^q \, dv(z), \quad b \in \ell^p. \end{aligned}$$

We now replace b_k by $b_k r_k(t)$, integrate with respect to t from 0 to 1, and then apply Fubini's theorem and Khinchine's inequality to get

$$\begin{aligned} \|b\|_{\ell^p}^q &\gtrsim \int_{\mathbb{D}} \left(\sum_k |b_k|^2 \left| \left(\frac{1 - |z_k|}{1 - \overline{z_k} \phi(z)} \right)^M - \left(\frac{1 - |z_k|}{1 - \overline{z_k} \psi(z)} \right)^M \right|^2 \right. \\ &\quad \left. \frac{1}{(\omega(T(z_k)))^{\frac{2}{p}}} \right)^{\frac{q}{2}} \, dv(z), \quad b \in \ell^p. \end{aligned}$$

By applying (15) and the estimate $|1 - \overline{z_k} z| \asymp 1 - |z_k|$, valid for all $z \in \Delta(z_k, \rho)$ and $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \left| \left(\frac{1 - |z_k|}{1 - \overline{z_k} \phi(z)} \right)^M - \left(\frac{1 - |z_k|}{1 - \overline{z_k} \psi(z)} \right)^M \right| &= \left| \frac{1 - |z_k|}{1 - \overline{z_k} \phi(z)} \right|^M \left| 1 - \left(\frac{1 - \overline{z_k} \phi(z)}{1 - \overline{z_k} \psi(z)} \right)^M \right| \\ &\gtrsim |z_k| |\delta(z)| \left| \frac{1 - |z_k|}{1 - \overline{z_k} \phi(z)} \right|^M \chi_{\phi^{-1}(\Delta(z_k, \rho))}(z) \\ &\asymp |z_k| |\delta(z)| \chi_{\phi^{-1}(\Delta(z_k, \rho))}(z), \quad z \in \mathbb{D}, \quad k \in \mathbb{N}, \end{aligned}$$

and hence

$$\|b\|_{\ell^p}^q \gtrsim |z_1|^q \int_{\mathbb{D}} \left(\sum_k |b_k|^2 |\delta(z)|^2 \frac{1}{(\omega(T(z_k)))^{\frac{2}{p}}} \chi_{\phi^{-1}(\Delta(z_k, \rho))}(z) \right)^{\frac{q}{2}} \, dv(z). \quad (16)$$

If $q \geq 2$ then the inequality $\sum_j c_j^x \leq \left(\sum_j c_j\right)^x$, valid for all $c_j \geq 0$ and $x \geq 1$, imply

$$\|b\|_{\ell^p}^q \gtrsim \int_{\mathbb{D}} \left(\sum_k |b_k|^q |\delta(z)|^q \frac{1}{(\omega(T(z_k)))^{\frac{q}{p}}} \chi_{\varphi^{-1}(\Delta(z_k, \rho))}(z) \right) d\nu(z). \quad (17)$$

To get the same estimate for $0 < q < 2$ we apply Hölder's inequality. It together with the fact that the number of discs $\Delta(z_k, r)$ to which each $\varphi(z)$ may belong is uniformly bounded yields

$$\begin{aligned} & \int_{\mathbb{D}} \left(\sum_k |b_k|^q |\delta(z)|^q \frac{1}{(\omega(T(z_k)))^{\frac{q}{p}}} \chi_{\varphi^{-1}(\Delta(z_k, \rho))}(z) \right) d\nu(z) \\ & \leq \int_{\mathbb{D}} \left(\sum_k |b_k|^2 |\delta(z)|^2 \frac{1}{(\omega(T(z_k)))^{\frac{2}{p}}} \chi_{\varphi^{-1}(\Delta(z_k, r))}(z) \right)^{\frac{q}{2}} \\ & \quad \cdot \left(\sum_k \chi_{\varphi^{-1}(\Delta(z_k, \rho))}(z) \right)^{1-\frac{q}{2}} d\nu(z) \\ & \lesssim \int_{\mathbb{D}} \left(\sum_k |b_k|^2 |\delta(z)|^2 \frac{1}{(\omega(T(z_k)))^{\frac{2}{p}}} \chi_{\varphi^{-1}(\Delta(z_k, \rho))}(z) \right)^{\frac{q}{2}} d\nu(z). \end{aligned}$$

Thus (17) holds for each $0 < q < \infty$. By using Fubini's theorem we now deduce

$$\begin{aligned} \|b\|_{\ell^p}^q & \gtrsim \sum_k |b_k|^q \frac{1}{(\omega(T(z_k)))^{\frac{q}{p}}} \int_{\varphi^{-1}(\Delta(z_k, \rho))} |\delta(z)|^q d\nu(z) \\ & = \sum_k |b_k|^q \frac{\varphi_*(|\delta|^q \nu)(\Delta(z_k, \rho))}{(\omega(T(z_k)))^{\frac{q}{p}}}, \quad b \in \ell^p. \end{aligned}$$

Therefore, the sequence

$$\left\{ \frac{\varphi_*(|\delta|^q \nu)(\Delta(z_k, \rho))}{(\omega(T(z_k)))^{\frac{q}{p}}} \right\}_{k \in \mathbb{N}}$$

belongs to $(\ell^{\frac{p}{q}})^* \simeq \ell^{\frac{p}{p-q}}$, and consequently

$$\sum_k \left(\frac{\varphi_*(|\delta|^q \nu)(\Delta(z_k, \rho))}{(\omega(T(z_k)))^{\frac{q}{p}}} \right)^{\frac{p}{p-q}} < \infty.$$

Pick up an $r = r(\rho) \in (0, 1)$ such that $\Delta(z, \rho) \subset \Delta(z_k, r)$ for all $z \in \Delta(z_k, \rho)$ and $k \in \mathbb{N}$. The right-hand inequality in (4) shows that $\widehat{\omega}(z) \asymp \widehat{\omega}(z_k)$ and $\omega(S(z)) \asymp$

$\omega(S(z_k))$ for all $z \in \Delta(z_k, \rho)$ and $k \in \mathbb{N}$. Then, as $\{z_k\}_{k \in \mathbb{N}}$ is a ρ -lattice, we deduce

$$\begin{aligned} & \int_{\mathbb{D}} \left(\frac{\varphi_*(|\delta|^q \nu)(\Delta(z, \rho))}{\omega(S(z))} \right)^{\frac{p}{p-q}} \tilde{\omega}(z) dA(z) \\ & \asymp \sum_k \int_{\Delta(z_k, \rho)} \left(\frac{\varphi_*(|\delta|^q \nu)(\Delta(z, \rho))}{\omega(S(z))} \right)^{\frac{p}{p-q}} \tilde{\omega}(z) dA(z) \\ & \lesssim \sum_k \left(\frac{\varphi_*(|\delta|^q \nu)(\Delta(z_k, \rho))}{\omega(S(z_k))} \right)^{\frac{p}{p-q}} \tilde{\omega}(z_k) (1 - |z_k|)^2 \\ & \lesssim \sum_k \left(\frac{\varphi_*(|\delta|^q \nu)(\Delta(z_k, \rho))}{(\omega(S(z_k)))^{\frac{q}{p}}} \right)^{\frac{p}{p-q}} \lesssim \sum_k \left(\frac{\varphi_*(|\delta|^q \nu)(\Delta(z_k, \rho))}{(\omega(T(z_k)))^{\frac{q}{p}}} \right)^{\frac{p}{p-q}} < \infty. \end{aligned}$$

Therefore, $\varphi_*(|\delta|^q \nu)$ is a q -Carleson measure for A_ω^p by Theorem 2. For the same reason, $\psi_*(|\delta|^q \nu)$ is a q -Carleson measure for A_ω^p . The proof is complete.

5 Proofs of Main Theorems

Here we shortly indicate how the main results stated in the introduction easily follow from the propositions proved in the previous two sections.

Proof of Theorem The theorem follows by Propositions 2 and 4. Namely, if δC_φ and δC_ψ are bounded from A_ω^p to L_v^q , then $C_\varphi - C_\psi : A_\omega^p \rightarrow L_v^q$ is compact, and thus bounded as well, by Proposition 2. Conversely, if $C_\varphi - C_\psi : A_\omega^p \rightarrow L_v^q$ is bounded, then δC_φ and δC_ψ are bounded by Proposition 4. $\square$

Proof of Theorem The theorem is an immediate consequence of Propositions 1 and 3. $\square$

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