On Projectively Flat Finsler Warped Product Metrics of Constant Flag Curvature

Huaifu Liu¹ · Xiaohuan Mo²

Received: 14 June 2020 / Accepted: 27 April 2021 / Published online: 24 May 2021
© The Author(s) 2021

Abstract

In this paper, we study locally projectively flat Finsler metrics of constant flag curvature. We find equations that characterize these metrics by warped product. Using the obtained equations, we manufacture new locally projectively flat Finsler warped product metrics of vanishing flag curvature. These metrics contain the metric introduced by Berwald and the spherically symmetric metric given by Mo-Zhu.

Keywords Finsler warped product metric · Locally projectively flat · Flag curvature · Spherically symmetric

Mathematics Subject Classification 22E46 · 53C30

1 Introduction

In Finsler geometry, the flag curvature is analog of sectional curvature in Riemannian geometry. Furthermore, Finsler metrics of constant flag curvature are the natural extension of Riemannian metrics of constant sectional curvature. Beltrami's theorem tells us that a Riemannian metric is of constant sectional curvature if and only if it is locally projectively flat. However, the situation is much more complicated for Finsler metrics. In fact, there are lots of projectively flat Finsler metrics which are not of constant flag curvature [11]. Conversely, there are infinitely many non-locally projectively flat Finsler metrics with constant flag curvature [1]. An interesting problem then is to

X. Mo and H. Liu: Supported by the Science and Technology Project of Beijing Municipal Education Commission (KM202010005026) and the National Natural Science Foundation of China 11771020 and 12071228. The second author (Xiaohuan Mo) is the corresponding author.

✉ Xiaohuan Mo
moxh@pku.edu.cn

¹ Faculty of Science, Beijing University of Technology, Beijing 100124, China

² Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing 100871, China

study locally projectively flat Finsler metrics of constant flag curvature. Recall that a Finsler metric F on a manifold M is said to be *locally projectively flat* if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's fourth problem [5]. In this paper, we will study locally projectively flat Finsler warped product metrics of constant flag curvature.

Finsler metrics in the form $F = \check{\alpha}\phi(r, s)$ are called *warped product metrics* where $\check{\alpha}$ is a Riemannian metric (for definition, see Sect. 2). Finsler warped product metrics are the natural extension of Riemannian warped product metrics [3]. In Riemannian geometry, these metrics have mainly been used in the efforts to construct new examples of Riemannian manifolds with prescribed conditions on the curvatures.

Very recently, Chen, Shen, and Zhao have obtained the characterization of Einstein Finsler warped product metrics $F = \check{\alpha}\phi(r, s)$ by introducing function Ψ (see the first equation of (2.4) below) [4]. In this paper, we show that ϕ and Ψ are mutually determined for a Douglas warped product metric, in particular, for a locally projectively flat Finsler warped product metric (see Lemma 2.3 below). Furthermore, the functions $\phi(r, s)$ and $\Psi(r, s)$ satisfy the same second-order PDE for a locally projectively flat Finsler warped product metric of constant flag curvature. This is indeed an amazing phenomenon. Precisely we have the following:

Theorem 1.1 *On the n -dimensional product manifold $M = I \times \check{M}$ with $n \geq 3$, a Finsler warped product metric $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ is locally projectively flat with constant flag curvature if and only if $\check{\alpha}$ has constant sectional curvature κ and*

$$\phi_r - s\phi_{rs} + [f(r)s^2 + g(r)]\phi_{ss} = 0, \quad (1.1)$$

$$\Psi_r - s\Psi_{rs} + [f(r)s^2 + g(r)]\Psi_{ss} = 0, \quad (1.2)$$

where $f(r)$ and $g(r)$ are differentiable functions which satisfy

$$\frac{dg}{dr} + fg = \kappa. \quad (1.3)$$

After noting this interesting fact, we produce infinitely many locally projectively flat Finsler warped product metrics of vanishing flag curvature in Sect. 6. We have the following:

Theorem 1.2 *Let $\phi(r, s)$ be a function defined by*

$$\phi(r, s) = a \frac{\left[c^2 s - g \sqrt{c^2 s^2 + c^2 (g^2 - c^2)} \right]^2}{\sqrt{c^2 s^2 + c^2 (g^2 - c^2)}} \quad (1.4)$$

where

$$(a, c, g) = \left(\frac{c_0 h_r^4}{(h^2 - 1)^2}, \frac{1}{h_r}, \frac{h}{h_r} \right) \quad (1.5)$$

where c_0 is a constant and h is any differentiable function satisfying $|h| > 1$, $h_r \neq 0$ and $c_0 > 0$. Then on $I \times M$ the following Finsler warped product metric $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ is locally projectively flat with zero flag curvature, where $\check{\alpha}$ has constant sectional curvature $\kappa = 1$.

We have the following two interesting special cases:

- (a) When $(a, c, g) = \left(\frac{1}{r^4(1-r^2)^2}, -r^2, -r\right)$, then on $M = I \times S^{n-1}$

$$F_{Ber} = \check{\alpha}_+ \frac{[\sqrt{s^2 + r^2(1-r^2)} + rs]^2}{(1-r^2)^2 \sqrt{s^2 + r^2(1-r^2)}}$$

is the warped product form of the Berwald's metric [2,4]. F_{Ber} is projectively flat with vanishing flag curvature.

- (b) When $(a, c, g) = \left(\frac{1}{r^4(1+4r^2)}, -2r^2\sqrt{1+4r^2}, -r(1+4r^2)\right)$, then on $M = I \times S^{n-1}$

$$F_{MZ} = \check{\alpha}_+ \frac{[\sqrt{(1+4r^2)(s^2 + r^2 + 4r^4)} + 2rs]^2}{\sqrt{(1+4r^2)(s^2 + r^2 + 4r^4)}}$$

is the warped product form of the Mo-Zhu's metric [10,13]. F_{MZ} is locally projectively flat with vanishing flag curvature, but it is not projectively flat.

In Sect. 5, we also construct a lot of locally projectively flat Finsler warped product metrics where $(K, \kappa) = (0, 0)$ (see Proposition 5.1 below). In fact we will show the following result: Any locally projectively flat Finsler warped product metric of zero flag curvature must be of Berwald type or square type (see Sect. 5 below).

For related results of locally projectively flat Finsler metrics of constant flag curvature, we refer the reader to [7,12,15].

2 Preliminaries

Let M be a manifold and let $TM = \cup_{x \in M} T_x M$ be the tangent bundle of M , where $T_x M$ is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$ where $\{0\}$ stands for $\{(x, 0) | x \in M, 0 \in T_x M\}$. A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ with the following properties

- (a) F is C^∞ on TM_o ;

- (b) At each point $x \in M$, the restriction $F_x := F|_{T_x M}$ is a Minkowski norm on $T_x M$.

Let F be a Finsler metric on an n -dimensional manifold M . For a non-zero vector $y \in T_x M$, F induces an inner product $\mathbf{g}_y$ on $T_x M$ by

$$\mathbf{g}_y(u, v) = \frac{1}{2}[F^2]_{y^i y^j}(x, y)u^i v^j.$$

Here (x^i, y^i) denotes the standard local coordinate system in TM , i.e., y^i 's are determined by $y = y^i \frac{\partial}{\partial x^i} |_x$.

For a two-dimensional plane $P \subset T_x M$ and a non-zero vector $y \in T_x M$, the *flag curvature* $\mathbf{K}(y, P)$ is defined by

$$\mathbf{K}(y, P) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}$$

where $P = y \wedge u$ and $\mathbf{R}_y$ is the Riemannian curvature of F [9, 15]. A Finsler metric F on a manifold M is said to be of *scalar flag curvature* if the flag curvature $\mathbf{K}(y, P) = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle $TM \setminus \{0\}$. In particular, F is said to be of *constant flag curvature* if $\mathbf{K}(y, P) = \text{constant}$. In general, the flag curvature is a function $\mathbf{K}(y, P)$ of tangent planes $P \in T_x M$ and directions $y \in P$.

Let I be an interval of $\mathbb{R}$ and $\tilde{M}$ be an $(n-1)$ -dimensional manifold equipped with a Riemannian metric $\check{\alpha}$. Finsler metrics on the product manifold $M := I \times \tilde{M}$, given in the form

$$F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$$

where $u = (u^1, \check{u})$, $v = v^1 \frac{\partial}{\partial u^1} + \check{v}$, and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$ are called *Finsler warped product metrics* [4].

Let $\mathbb{B}^n(r)$ denote the Euclidean ball of radius r and let F be a Finsler metric on $\mathbb{B}^n(r)$. F is said to be *spherically symmetric* if it satisfies $F(Ax, Ay) = F(x, y)$ for all $x \in \mathbb{B}^n(r)$, $y \in T_x \mathbb{B}^n(r)$, and $A \in O(n)$. A Finsler metric F on $\mathbb{B}^n(r)$ is spherically symmetric if and only if there is a function $\phi : [0, r) \times \mathbb{R} \rightarrow \mathbb{R}$ such that $F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y \rangle}{|y|}\right)$ where $(x, y) \in T\mathbb{B}^n(r) \setminus \{0\}$ [6].

Lemma 2.1 [4] *A spherically symmetric metric is a Finsler warped product metric.*

Proof In fact,

$$F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y \rangle}{|y|}\right) = \check{\alpha}_+ \tilde{\phi}(r, s) \quad (2.1)$$

where $\check{\alpha}_+$ is the standard Euclidean metric on the unit sphere S^{n-1} ,

$$r := |x|, \quad s := \frac{v^1}{\check{\alpha}_+}, \quad \tilde{\phi}(r, s) = \sqrt{r^2 + s^2}\phi\left(r, \frac{rs}{\sqrt{r^2 + s^2}}\right) \quad (2.2)$$

where $v^1 = dr(y)$. □

Lemma 2.2 *Let $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ be a Finsler warped product metric on $M = I \times \tilde{M}$. Then*

$$(\phi - s\phi_s)\Psi + \phi(\Psi - s\Psi_s) = -\phi_s(2\Theta - s\Theta_s) \quad (2.3)$$

where

$$\Psi := \frac{\phi_r(\phi_s + s\phi_{ss}) - s\phi_s\phi_{rs}}{2\phi\phi_{ss}}, \quad \Theta := -\frac{\phi_r - s\phi_{rs}}{2\phi_{ss}}. \quad (2.4)$$

Proof By (2.4) we have

$$(\Psi, \Theta) = (\phi_r, \phi_{rs}) \begin{pmatrix} A & C \\ B & D \end{pmatrix} \quad (2.5)$$

where $\begin{pmatrix} A & C \\ B & D \end{pmatrix} := \frac{1}{2\phi\phi_{ss}} \begin{pmatrix} \phi_s + s\phi_{ss} & -\phi \\ -s\phi_s & s\phi \end{pmatrix}$ where we have made use of the fact $\phi > 0$, $\phi_{ss} > 0$ [4,9]. It follows that $\det \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \frac{s^2}{4\phi\phi_{ss}}$. Taking $s := \frac{v^1}{\alpha}$ with $s \neq 0$. Then $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is a non-singular matrix. Thus

$$\begin{pmatrix} \mathcal{E} & \mathcal{G} \\ \mathcal{F} & \mathcal{H} \end{pmatrix} := \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1} = \frac{2}{s^2} \begin{pmatrix} s & \phi \\ s\phi_s & \phi_s + s\phi_{ss} \end{pmatrix}. \quad (2.6)$$

From (2.5) and (2.6) one obtains

$$(\phi_r, \phi_{rs}) = (\Psi, \Theta) \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1} = (\Psi, \Theta) \begin{pmatrix} \mathcal{E} & \mathcal{G} \\ \mathcal{F} & \mathcal{H} \end{pmatrix}.$$

It implies that

$$\phi_r = \mathcal{E}\Psi + \mathcal{F}\Theta, \quad \phi_{rs} = \mathcal{G}\Psi + \mathcal{H}\Theta. \quad (2.7)$$

By using (2.7), we have

$$\mathcal{G}\Psi + \mathcal{H}\Theta = (\mathcal{E}\Psi + \mathcal{F}\Theta)_s = \mathcal{E}_s\Psi + \mathcal{E}\Psi_s + \mathcal{F}_s\Theta + \mathcal{F}\Theta_s,$$

that is,

$$(\mathcal{G} - \mathcal{E}_s)\Psi - \mathcal{E}\Psi_s = (\mathcal{F}_s - \mathcal{H})\Theta + \mathcal{F}\Theta_s. \quad (2.8)$$

Using (2.6), we get

$$\mathcal{E} = \frac{2\phi}{s}, \quad \mathcal{F} = \frac{2\phi_s}{s}, \quad \mathcal{G} = \frac{2\phi}{s^2}, \quad \mathcal{H} = 2\frac{\phi_s + s\phi_{ss}}{s^2}. \quad (2.9)$$

It follows that

$$\mathcal{G} - \mathcal{E}_s = \frac{2\phi}{s^2} - \left(\frac{2\phi}{s}\right)_s = 2\frac{2\phi - s\phi_s}{s^2}$$

$$\mathcal{F}_s - \mathcal{H} = \left(\frac{2\phi_s}{s} \right)_s - 2 \frac{\phi_s + s\phi_s}{s^2} = -4 \frac{\phi_s}{s^2}.$$

Plugging these into (2.8) yields

$$\begin{aligned} \frac{2}{s^2}[(\phi - s\phi_s)\Psi + \phi(\Psi - s\Psi_s)] &= (\mathcal{G} - \mathcal{E}_s)\Psi - \mathcal{E}\Psi_s \\ &= (\mathcal{F}_s - \mathcal{H})\Theta + \mathcal{F}\Theta_s = \frac{2}{s^2}(-\phi_s)(2\Theta - s\Theta_s). \end{aligned}$$

Then (2.3) holds. $\square$

Lemma 2.3 Let $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ be a Douglas metric on $M = I \times \check{M}$. Then there exists a function $g(r)$ such that

$$\phi = e^{\int \frac{2\Psi - s\Psi_s}{s\Psi - g} ds} \quad (2.10)$$

or F is of Berwald type where Ψ is given in the first equation of (2.4).

Proof According to Lemma 3.3 in [8] and [9]

$$\Theta = \frac{1}{2}[f(r)s^2 + g(r)], \quad (2.11)$$

where Θ is given by (2.4). It follows that

$$2\Theta - s\Theta_s = g(r). \quad (2.12)$$

Plugging this into (2.3) yields

$$\begin{aligned} 0 &= (\phi - s\phi_s)\Psi + \phi(\Psi - s\Psi_s) + \phi_s(2\Theta - s\Theta_s) \\ &= 2\phi\Psi - s\phi_s\Psi - s\phi\Psi_s + g\phi_s = \phi(2\Psi - s\Psi_s) - \phi_s(s\Psi - g). \end{aligned}$$

Then when $s\Psi - g = 0$, F is of Berwald type [8]. Otherwise $(\ln \phi)_s = \frac{\phi_s}{\phi} = \frac{2\Psi - s\Psi_s}{s\Psi - g}$. Thus

$$\ln \phi = \int \frac{2\Psi - s\Psi_s}{s\Psi - g} ds.$$

which completes the proof of Lemma 2.3. $\square$

3 Locally Projectively Flat Finsler Metrics

In this section, we are going to find equations that characterize a locally projectively flat Finsler warped product metric $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$.

According to Theorem 1.1 in [10], F is of scalar flag curvature if and only if $\check{\alpha}$ has constant sectional curvature κ and

$$\varepsilon - \nu = \kappa \quad (3.1)$$

where

$$\begin{aligned} \varepsilon &= (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Psi_s, \\ \nu &= s[2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s] \\ &\quad + \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s \end{aligned}$$

where

$$\Phi := \Theta + s\Psi \quad (3.2)$$

where Ψ and Θ are given in (2.4).

A direct calculation gives the following formula:

$$\varepsilon - \nu = 2\Theta_r - s\Theta_{rs} + 2\Theta\Theta_{ss} - \Theta_s^2. \quad (3.3)$$

Combining this with (3.1), we obtain the following:

Lemma 3.1 *F is of scalar flag curvature if and only if $\check{\alpha}$ has constant sectional curvature κ and $2\Theta_r - s\Theta_{rs} + 2\Theta\Theta_{ss} - \Theta_s^2 = \kappa$ where Θ is given in (2.4).*

We mention that two characterizing equations of Finsler warped product metrics of scalar curvature were established first by Chen–Shen–Zhao [4]. Later, Liu–Mo–Zhang found that one equation is sufficient, using the Weyl curvature [10]. Very recently, Liu–Mo have constructed infinitely many non-spherically symmetric warped product Finsler metrics of scalar flag curvature by refining Chen–Shen–Zhao and Liu–Mo–Zhang equations characterizing Finsler warped product metrics of scalar flag curvature [9].

Theorem 3.2 *Let $\check{M}$ be an $(n-1)$ -dimensional Riemannian manifold with $n \geq 3$ and let $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ be a Finsler warped product metric on $M = I \times \check{M}$. Then F is locally projectively flat if and only if $\check{\alpha}$ has constant sectional curvature κ and ϕ satisfies*

$$\phi_r - s\phi_{rs} + \left[f(r)s^2 + g(r)\right]\phi_{ss} = 0 \quad (3.4)$$

where $f(r)$ and $g(r)$ are differentiable functions which satisfy (1.3).

Proof By (3.2), one obtains

$$\Theta = \Phi - s\Psi. \quad (3.5)$$

Applying Lemma 3.3 in [8], F has vanishing Douglas curvature if and only if

$$\Phi - s\Psi = \frac{1}{2} \left[f(r)s^2 + g(r) \right] \quad (3.6)$$

where $f = f(r)$ and $g = g(r)$ are differentiable functions. Plugging (3.5) into the second equation of (2.4) yields $\Phi - s\Psi = \frac{s\phi_{rs} - \phi_r}{2\phi_{ss}}$. It follows that (3.6) holds if and only if

$$\frac{s\phi_{rs} - \phi_r}{\phi_{ss}} = f(r)s^2 + g(r). \quad (3.7)$$

Combining (3.5) with (3.6) we have

$$\begin{aligned} 2\Theta_r &= f_r s^2 + g_r, & \Theta_s &= f s, \\ \Theta_{rs} &= f_r s, & \Theta_{ss} &= f. \end{aligned} \quad (3.8)$$

Hence, we get

$$2\Theta_r - s\Theta_{rs} + 2\Theta\Theta_{ss} - \Theta_s^2 = f_r s^2 + g_r - f_r s^2 + (f s^2 + g)f - f^2 s^2 = g_r + f g. \quad (3.9)$$

Furthermore, we assume that F is of scalar flag curvature. Combining (3.9) with Lemma 3.1, we conclude that (1.3) holds.

According to Douglas' result, Finsler metric $F(u, v)$ on M^n with $n \geq 3$ is locally projectively flat if and only if F has vanishing Douglas curvature and scalar flag curvature.

First suppose that F is locally projectively flat. Then (3.7) and (1.3) hold where κ is the constant sectional curvature of $(\check{M}, \check{\alpha})$. Conversely, suppose that $\check{\alpha}$ has constant sectional curvature κ such that (3.4) and (1.3) hold. Equation (3.4) implies (3.7). It follows that F has vanishing Douglas curvature. Furthermore, the constancy of flag curvature of $\check{\alpha}$, (3.9) and (1.3) tell us that F is of scalar flag curvature. Note that $n \geq 3$. Hence, F is locally projectively flat. $\square$

4 Finsler Metrics of Constant Flag Curvature

First we refine Chen–Shen–Zhao and Liu–Mo–Zhang equations that characterize Finsler warped product metrics of constant flag curvature.

Theorem 4.1 *Let $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ be a Finsler warped product metric on $M = I \times \check{M}$. Then F is of constant flag curvature if and only if $\check{\alpha}$ has constant sectional curvature κ and*

$$2\Theta_r - s\Theta_{rs} + 2\Theta\Theta_{ss} - \Theta_s^2 = \kappa, \quad (4.1)$$

$$\Psi_r - s\Psi_{rs} + 2\Theta\Psi_{ss} = 0, \quad (4.2)$$

where Ψ and Θ are given in (2.4). In this case, the flag curvature K of F satisfies

$$K\phi^2 = \Psi^2 - s\Psi_r + 2\Theta\Psi_s + \kappa. \quad (4.3)$$

We mention that the Chen–Shen–Zhao paper was published in 2018, in which the authors obtain two PDEs depending on the constant K that characterize Finsler warped product metrics of constant flag curvature. Theorem 4.1 tells us that these PDEs can be replaced by ones independent of the constant K .

Proof of Theorem 4.1 According to Theorem 1.2 in [10], F is of constant flag curvature if and only if $\check{\alpha}$ has constant sectional curvature κ and

$$\varepsilon - \nu = \kappa, \quad 2\tau + \mu_s = 0 \quad (4.4)$$

where ε and ν are given in Sect. 3, and

$$\tau := 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \quad (4.5)$$

$$\mu := \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s \quad (4.6)$$

where Φ is given in (3.2). In this case, the flag curvature of F is given by

$$K = \frac{\kappa + \mu}{\phi^2}. \quad (4.7)$$

By (3.3), the first equation of (4.4) holds if and only if (4.1) holds. From (3.5), (4.5), and (4.6), we have

$$\begin{aligned} 2\tau + \mu_s &= 3\Psi_r - 3s\Psi_{rs} - 6s\Psi\Psi_{ss} + 6\Phi\Psi_{ss} \\ &= 3[\Psi_r - s\Psi_{rs} + 2(\Phi - s\Psi)\Psi_{ss}] = 3(\Psi_r - s\Psi_{rs} + 2\Theta\Psi_{ss}). \end{aligned}$$

It follows that the second equation of (4.4) holds if and only if (4.2) holds. By (3.5), (4.7) and (4.6), we obtain

$$\begin{aligned} K\phi^2 &= \mu + \kappa \\ &= \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s + \kappa \\ &= \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2(\Theta + s\Psi)\Psi_s + \kappa = \Psi^2 - s\Psi_r + 2\Theta\Psi_s + \kappa \end{aligned}$$

that is, (4.3) holds.

The following proposition will be used in Sect. 5.

Proposition 4.2 Let $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ be a Finsler warped product metric of constant flag curvature K on $M = I \times \check{M}$. Then

$$\Psi(\Psi - s\Psi_s) + \Psi_s(2\Theta - s\Theta_s) + \kappa = K\phi(\phi - s\phi_s) \quad (4.8)$$

where κ is the constant sectional curvature of $\check{\alpha}$.

Proof Differentiating (4.3) with respect to s and using (4.2) one obtains

$$\begin{aligned} K\phi\phi_s &= \frac{1}{2}(K\phi^2)_s \\ &= \frac{1}{2}(\Psi^2 - s\Psi_r + 2\Theta\Psi_s + \kappa)_s \\ &= \frac{1}{2}(2\Psi\Psi_s - \Psi_r - s\Psi_{rs} + 2\Theta_s\Psi_s + 2\Theta\Psi_{ss}) = \Psi\Psi_s - \Psi_r + \Theta_s\Psi_s. \end{aligned}$$

Together with (4.3), we have

$$\begin{aligned} &\Psi(\Psi - s\Psi_s) + \Psi_s(2\Theta - s\Theta_s) + \kappa \\ &= \Psi^2 - s\Psi_r + 2\Theta\Psi_s + \kappa - s(\Psi\Psi_s - \Psi_r + \Theta_s\Psi_s) \\ &= K\phi^2 - sK\phi\phi_s = K\phi(\phi - s\phi_s). \end{aligned}$$

Thus we complete the proof of the proposition. $\square$

Proof of Theorem 1.1 First suppose that F is locally projectively flat with constant flag curvature. Then $\check{\alpha}$ has constant sectional curvature, and (3.4), (1.3), and (4.2) hold. Note that (3.4) is just (1.1). By (3.4) and the second equation of (2.4), we get

$$\Theta = -\frac{\phi_r - s\phi_{rs}}{2\phi_{ss}} = \frac{1}{2}[f(r)s^2 + g(r)]. \quad (4.9)$$

Plugging this into (4.2) yields (1.2).

Conversely, we suppose that $\check{\alpha}$ has constant sectional curvature, and (1.1), (1.2), and (1.3) hold. According to Theorem 3.2, F is locally projectively flat. Equation (1.1) and the second equation of (2.4) imply (4.9). Hence we have (3.9). Combining this with (1.3) we have (4.1). Plugging (4.9) into (1.2) yields (4.2). Hence we obtain that F has constant flag curvature by Theorem 4.1.

5 Projective Finsler Metrics of Zero Flag Curvature

Let $F(u, v) = \check{\alpha}(\check{u}, \check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})}\right)$ be a locally projectively flat Finsler metric of vanishing flag curvature. Then F is of Douglas type and $K = 0$. Plugging (2.12) into (4.8) yields

$$0 = \Psi(\Psi - s\Psi_s) + \Psi_s(2\Theta - s\Theta_s) + \kappa = \Psi(\Psi - s\Psi_s) + g\Psi_s + \kappa. \quad (5.1)$$

Our classification of locally projectively flat Finsler warped product metrics with vanishing flag curvature is given in two steps. First, we are going to study the case when $\check{\alpha}$ satisfies $\kappa = 0$ by a deep analyzing. Then we shall study the case when $\kappa \neq 0$. In particular, we are going to manufacture locally projectively flat Finsler warped product metrics with $K = 0$ (see Proposition 5.1 and the proof of Theorem 1.2 below).

Case 1. $\kappa = 0$

By using (5.1) we obtain

$$(g - s\Psi)\Psi_s + \Psi^2 = 0. \quad (5.2)$$

If $\Psi = 0$ then $\Phi = \frac{1}{2}[f(r)s^2 + g(r)]$ where we have used (3.6). According to Corollary 3.2 in [8], F is of Berwald type.

If $\Psi \neq 0$ then

$$\left(\frac{s\Psi - \frac{1}{2}g}{\Psi^2} \right)_s = \frac{1}{\Psi^3}[\Psi^2 + (g - s\Psi)\Psi_s] = 0$$

where we have used (5.2). It follows that

$$\frac{s\Psi - \frac{1}{2}g}{\Psi^2} = \frac{1}{2}c(r). \quad (5.3)$$

Thus,

$$\Psi_{\pm} = \frac{s \pm \omega}{c(r)} \quad (5.4)$$

are the solutions of equation (5.3) where

$$\omega = \sqrt{s^2 - c(r)g(r)}. \quad (5.5)$$

It follows that

$$\omega_s = \frac{s}{\omega}. \quad (5.6)$$

Differentiating (5.4) with respect to s and using (5.6) we get

$$(\Psi_{\pm})_s = \left[\frac{s \pm \omega}{c(r)} \right]_s = \frac{1}{c(r)} \pm \frac{s}{c(r)\omega}. \quad (5.7)$$

According to Lemma 2.3, we denote the corresponding functions with respect to $\Psi_{\pm}$ by $\phi_{\pm}$. By using (2.10), (5.4), and (5.7), we have

$$\begin{aligned} \ln \phi_{\pm} &= \int \frac{2\Psi_{\pm} - s(\Psi_{\pm})_s}{s\Psi_{\pm} - g} ds \\ &= \int \frac{1}{\omega} \frac{(s \pm \omega)(2\omega \mp s)}{s(s \pm \omega) - cg} ds \\ &= - \int \frac{1}{\omega} \frac{\pm 2\omega^2 + 2s\omega \mp s^2 - s\omega}{cg - s^2 \mp s\omega} ds = \int \frac{\pm 2cg \mp s^2 - s\omega}{\omega(cg - s^2 \mp s\omega)} ds = (I) \end{aligned} \quad (5.8)$$

where

$$(I) := \int \frac{\pm cg \mp \omega^2 - s\omega}{-\omega^2(\omega \pm s)} ds = 2 \ln(s \pm \omega) - \ln \omega + \ln a(r) \quad (5.9)$$

where we have used the following formula

$$[2 \ln(s \pm \omega) - \ln \omega]_s = 2 \frac{1 \pm \omega_s}{s \pm \omega} - \frac{\omega_s}{\omega} = \frac{\pm cg \mp \omega^2 - s\omega}{-\omega^2(\omega \pm s)}$$

where we have used (5.6). Plugging (5.9) into (5.8) and using (5.5) we obtain

$$\phi_{\pm} = a(r) \frac{[s \pm \sqrt{s^2 - c(r)g(r)}]^2}{\sqrt{s^2 - c(r)g(r)}} = a \frac{(s \pm \omega)^2}{\omega}. \quad (5.10)$$

It follows that

$$F_{\pm} = \check{\alpha} \phi(r, s) = \check{\alpha} \frac{a(r)(s \pm \omega)^2}{\omega} = \frac{(\alpha \pm \beta)^2}{\alpha}$$

where

$$\begin{cases} \alpha = a(r) \sqrt{(v^1)^2 - c(r)g(r) \check{\alpha}^2}, \\ \beta = a(r) v^1. \end{cases} \quad (5.11)$$

We obtain that F is of square type. By investigating ϕ_+ , we have the following:

Proposition 5.1 *Let $\phi(r, s)$ be a function defined by*

$$\phi(r, s) = -\frac{\lambda}{2} h_r \frac{\left(s + \sqrt{s^2 + \frac{8h^3}{h_r^2}}\right)^2}{\sqrt{s^2 + \frac{8h^3}{h_r^2}}} \quad (5.12)$$

where λ is a constant and h is any differentiable function satisfying $\lambda h_r < 0$ and $h > 0$. Then on $M = I \times \check{M}$ the following Finsler metric

$$F(u, v) = \check{\alpha}(\check{u}, \check{v}) \phi \left(u^1, \frac{v^1}{\check{\alpha}(\check{u}, \check{v})} \right)$$

is locally projectively flat with zero flag curvature, where $\check{\alpha}$ has zero sectional curvature.

Proof Using (5.6), we obtain

$$\left[\frac{(s + \omega)^2}{\omega} \right]_s = \frac{s + \omega}{\omega^2} \left(2\omega + s - \frac{s^2}{\omega} \right). \quad (5.13)$$

In (5.10), we denote ϕ_+ by ϕ . Then we have

$$\phi - s\phi_s = a \frac{(s + \omega)^2}{\omega} - as \left[\frac{(s + \omega)^2}{\omega} \right]_s = \frac{aA^2}{\omega^3} \quad (5.14)$$

where $A(r) = c(r)g(r)$ and we have used (5.5). By simple calculations, we have

$$(\omega^2)_r = -A_r, \quad (\omega^3)_r = -\frac{3}{2}\omega A_r. \quad (5.15)$$

Thus we have

$$\phi_r - s\phi_{rs} = \left(\frac{aA^2}{\omega^3} \right)_r = \frac{2(aA^2)_r \omega^2 + 3aA^2 A_r}{2\omega^5} \quad (5.16)$$

and

$$\phi_{ss} = -\frac{1}{s} \left(\frac{aA^2}{\omega^3} \right)_s = 3aA^2 \frac{1}{\omega^5}. \quad (5.17)$$

By (5.16), (5.17), and (1.1), we get

$$f(r)s^2 + g(r) = -\frac{\phi_r - s\phi_{rs}}{\phi_{ss}} = -\frac{2(aA^2)_r(s^2 - A) + 3aA^2 A_r}{6aA^2}.$$

It follows that

$$f(r) = -\frac{1}{3acg}(a_r c g + 2ac_r g + 2acg_r) \quad (5.18)$$

and

$$g(r) = \frac{1}{3a}(2cga_r + acg_r + agc_r). \quad (5.19)$$

Denote Ψ_+ by Ψ . By (5.4) and (5.7), one obtains

$$\Psi - s\Psi_s = \frac{\omega^2 - s^2}{c\omega} = -\frac{A}{c\omega}.$$

From the first equation of (5.15), one has

$$\Psi_r - s\Psi_{rs} = -\left(\frac{A}{c\omega} \right)_r = \frac{1}{2c^2\omega^2}(2Ac_r\omega^2 - cAA_r - 2A_r c\omega^2) \quad (5.20)$$

and

$$\Psi_{ss} = -\frac{1}{s}(\Psi - s\Psi_s)_s = -\frac{1}{s} \left(\frac{-A}{c\omega} \right)_s = -\frac{A}{c\omega^3}. \quad (5.21)$$

Now, we assume that $A = c(r)g(r) \neq 0$. Hence that $\Psi_{ss} \neq 0$. Using (1.2), (5.20), and (5.21), we have

$$f(r)s^2 + g(r) = -\frac{\Psi_r - s\Psi_{rs}}{\Psi_{ss}} = \frac{2Ac_r(s^2 - A) - cAA_r - 2cA_r(s^2 - A)}{2cA}.$$

It follows that

$$f(r) = \frac{1}{2cA}(2Ac_r - 2A_r c) = -(\ln |g|)_r \quad (5.22)$$

and

$$g(r) = \frac{1}{2cA}(-cAA_r + 2cAA_r - 2A^2c_r) = -\frac{1}{2}(cg_r - gc_r). \quad (5.23)$$

By using (5.10) we have $a(r) > 0$. Combining this with (5.18) yields

$$f(r) = -\frac{1}{3}[(\ln a)_r + 2(\ln |g|)_r + 2(\ln |c|)_r]. \quad (5.24)$$

By (5.23), we obtain

$$\frac{2}{c} = (\ln |g|)_r - (\ln |c|)_r. \quad (5.25)$$

Combining this with (5.5), we get

$$\left[\ln \left(-\frac{g}{c} \right) \right]_r = \frac{2}{c}. \quad (5.26)$$

From (5.25) and (5.22), one obtains

$$-(\ln |c|)_r = \frac{2}{c} - (\ln |g|)_r = \frac{2}{c} + f. \quad (5.27)$$

Together with (5.24) and (5.25), we have

$$f(r) = -\frac{1}{3} \left[(\ln a)_r + \frac{4}{c} + 4(\ln |c|)_r \right] = \frac{4}{3}f + \frac{4}{3c} - \frac{1}{3}(\ln a)_r.$$

It follows that

$$(\ln a)_r = f + \frac{4}{c}. \quad (5.28)$$

By simple calculations, we see that (5.22), (5.27), and (5.28) imply (5.19). It follows that (5.18), (5.19), (5.22), and (5.23) hold if and only if (5.22), (5.26), and (5.28) hold.

Let

$$h := -\frac{g}{c} > 0 \quad (5.29)$$

where we have used (5.11). Combining (5.26) we can see that

$$\frac{h_r}{h} = (\ln h)_r = \left[\ln \left(-\frac{g}{c} \right) \right]_r = \frac{2}{c}. \quad (5.30)$$

It follows that

$$c = \frac{2h}{h_r}. \quad (5.31)$$

From which together with (5.29) we obtain

$$g = -ch = -\frac{2h^2}{h_r}. \quad (5.32)$$

By using (5.28), (5.22), and (5.30), we obtain

$$(\ln |ag|)_r = (\ln a)_r + (\ln |g|)_r = f + \frac{4}{c} - f = 2(\ln h)_r = (\ln h^2)_r.$$

Hence we have $\left(\ln \frac{|ag|}{h^2} \right)_r = 0$. It follows that $\frac{ag}{h^2} = -\lambda = \text{constant}$ where $\text{sgn } \lambda = -\text{sgn } g$. Thus we get

$$a = -\frac{\lambda h^2}{g} = -\frac{\lambda}{2} h_r \quad (5.33)$$

where we have made use of (5.32). From (5.31) and (5.32), we have

$$c(r)g(r) = -\frac{4h^3}{h_r^2}. \quad (5.34)$$

Substituting (5.33) and (5.34) into (5.10), we get (5.12). By using (5.14) and (5.17)

we get $\phi - s\phi_s = \frac{a\left(\frac{8h^3}{h_r^2}\right)^3}{\left(\sqrt{s^2 + \frac{8h^3}{h_r^2}}\right)^3}$, $\phi_{ss} = \frac{3aA^2}{\omega^5}$. Combining these with Proposition 5.1

in [8] we obtain that $F = \check{\alpha}\phi\left(u^1, \frac{v^1}{\check{\alpha}}\right)$ is strongly convex if and only if $\lambda h_r < 0$ and $h > 0$. $\square$

Case 2. $\kappa \neq 0$

We assume that $\Psi^2 + \kappa \neq 0$. Otherwise we obtain that κ is negative. We will discuss it below. By a straightforward computation one obtains

$$\begin{aligned} \left(\frac{g\Psi + \kappa s}{\sqrt{\Psi^2 + \kappa}} \right)_s &= \frac{1}{\Psi^2 + \kappa} \left[(g\Psi_s + \kappa)\sqrt{\Psi^2 + \kappa} - (g\Psi + \kappa s) \frac{\Psi\Psi_s}{\sqrt{\Psi^2 + \kappa}} \right] \\ &= \frac{(I)}{(\Psi^2 + \kappa)^{\frac{3}{2}}} \end{aligned} \quad (5.35)$$

where

$$(I) := (g\Psi_s + \kappa)(\Psi^2 + \kappa) - \Psi\Psi_s(g\Psi + \kappa s) = \kappa(\Psi^2 - s\Psi\Psi_s + g\Psi_s + \kappa) = 0 \quad (5.36)$$

where we have used (5.1). Plugging (5.36) into (5.35) yields $\left(\frac{g\Psi + \kappa s}{\sqrt{\Psi^2 + \kappa}} \right)_s = 0$. It follows that $\frac{g\Psi + \kappa s}{\sqrt{\Psi^2 + \kappa}} = c(r)$ for a differentiable function $c(r)$. We obtain the following quadratic equation

$$(g^2 - c^2)\Psi^2 + \kappa g s \Psi + \kappa^2 s^2 - \kappa c^2 = 0. \quad (5.37)$$

If $g^2 - c^2 = 0$, from (5.37), we get $\kappa g s \Psi + \kappa^2 s^2 - \kappa c^2 = 0$. Note that $s = \frac{v^1}{\check{\alpha}(u, v)} \in \mathbb{R}$. By taking $s = 0$, we have $\kappa c(r) = 0$. It follows that $c(r) = 0$ from $\kappa \neq 0$. Thus $g(r) \equiv 0$. We obtain that $s \equiv 0$, which is a contradiction. Solving (5.37) for Ψ , we get

$$\Psi_{\pm} = \frac{-\kappa g s \pm \sqrt{\kappa^2 c^2 s^2 + \kappa c^2 (g^2 - c^2)}}{g^2 - c^2}. \quad (5.38)$$

Proof of Theorem 1.2 By taking $\kappa = 1$ in (5.38) and denoting the corresponding functions with respect to $\Psi_{\pm}$ by $\phi_{\pm}$. Using Lemma 2.3, we obtain

$$\phi_{\pm} = a \frac{\left[c^2 s \mp g \sqrt{c^2 s^2 + c^2 (g^2 - c^2)} \right]^2}{\sqrt{c^2 s^2 + c^2 (g^2 - c^2)}}. \quad (5.39)$$

Hence $\phi := \phi_+$ satisfies (1.4). From (5.39) we obtain that

$$F_{\pm} = \check{\alpha} \phi_{\pm}(r, s) = \frac{(\alpha \pm \beta)^2}{\alpha}$$

where

$$\begin{cases} \alpha = a g^2 \sqrt{c^2 (v^1)^2 - c^2 (g^2 - c^2) \check{\alpha}^2}, \\ \beta = -a g c^2 v^1. \end{cases}$$

Hence $F_{\pm}$ are of square type.

Now we investigate the function $\phi = \phi_+$. We can rewrite ϕ as follows

$$\phi = a(r) \frac{(c^2 s - g\omega)^2}{\omega} \quad (5.40)$$

where

$$\omega := \sqrt{c^2 s^2 + \eta(r)}, \quad \eta(r) := c^2(g^2 - c^2). \quad (5.41)$$

By simple calculations, we have

$$\omega_s := \frac{c^2 s}{\omega}, \quad \omega_r := \frac{2cc_r s^2 + \eta_r}{2\omega}. \quad (5.42)$$

It follows that $\left[\frac{(c^2 s - g\omega)^2}{\omega} \right]_s = \frac{c^2 s - g\omega}{\omega^2} (2c^2 \omega - c^2 g s - \frac{c^4 s^2}{\omega})$. Together with (5.40) yields

$$\phi - s\phi_s = a \frac{c^2 s - g\omega}{\omega} \times (I) \quad (5.43)$$

where

$$(I) := -c^2 s - g\omega + \frac{c^2 s^2}{\omega} \left(g + \frac{c^2 s}{\omega} \right) = -\eta \frac{c^2 s + g\omega}{\omega^2}.$$

Plugging it into (5.43) yields

$$\begin{aligned} \phi - s\phi_s &= -a\eta \frac{(c^2 s - g\omega)(c^2 s + g\omega)}{\omega^3} \\ &= -a\eta \frac{s^2 c^2 (c^2 - g^2) - g^2 \eta}{\omega^3} = (s^2 + g^2) \frac{a\eta^2}{\omega^3} \end{aligned} \quad (5.44)$$

where we have used (5.41). Combining this with (5.42), we obtain

$$\phi_{ss} = -\frac{1}{s} (\phi - s\phi_s)_s = -\frac{a\eta^2}{s} \left(\frac{s^2 + g^2}{\omega^3} \right)_s = \frac{a\eta^2 c^2}{\omega^5} (s^2 + g^2 + 2c^2). \quad (5.45)$$

(5.44) can be expressed by the following

$$\phi - s\phi_s = a\eta^2 \times (II) \quad (5.46)$$

where

$$(II) := \frac{s^2 + g^2}{\omega^3}. \quad (5.47)$$

Together with (5.42) and (5.41), we have

$$\begin{aligned}(II)_r &= \left(\frac{s^2 + g^2}{\omega^3} \right)_r \\ &= \frac{1}{\omega^5} \left[2gg_r(c^2s^2 + \eta) - 3(s^2 + g^2) \left(cc_rs^2 + \frac{1}{2}\eta_r \right) \right].\end{aligned}\quad (5.48)$$

By (5.46), (5.47), and (5.48), one obtains

$$\begin{aligned}(\phi - s\phi_s)_r &= (a\eta^2)_r(II) + a\eta^2(II)_r \\ &= \frac{(a\eta^2)_r(s^2 + g^2)(c^2s^2 + \eta)}{\omega^5}\end{aligned}\quad (5.49)$$

$$+ \frac{a\eta^2[2gg_r(c^2s^2 + \eta) - 3(s^2 + g^2)(cc_rs^2 + \frac{1}{2}\eta_r)]}{\omega^5}.\quad (5.50)$$

From (5.45), we get

$$\phi_{ss}(fs^2 + g)\omega^5 \equiv ac^2\eta^2fs^4 \pmod{s^0, s^2}.\quad (5.51)$$

Using (5.50), we have

$$-(\phi - s\phi_s)_r\omega^5 \equiv (III) \times s^4 \pmod{s^0, s^2}\quad (5.52)$$

where

$$(III) := 3a\eta^2cc_r - c^2(a\eta^2)_r = -ac^3\eta \left[c(g^2 - c^2)b_r - 5c^2c_r + 4cgg_r + g^2c_r \right]\quad (5.53)$$

where

$$b := \ln a.\quad (5.54)$$

By using (1.1), we get $\phi_{ss}(fs^2 + g)\omega^5 = -(\phi - s\phi_s)_r\omega^5$. Together with (5.51), (5.52), and (5.53), we have

$$\eta f = -c \left[c(g^2 - c^2)b_r - 5c^2c_r + 4cgg_r + g^2c_r \right].\quad (5.55)$$

Combining this with the second equation of (5.41), we obtain

$$c(g^2 - c^2)b_r - 5c^2c_r + 4cgg_r + g^2c_r = -c(g^2 - c^2)f.$$

From (5.38) and (5.41), we have

$$\Psi := \Psi_+ = \frac{\omega - gs}{\xi}, \quad \xi := g^2 - c^2.\quad (5.56)$$

Combining this with (5.41), we get

$$\omega = \sqrt{c^2 s^2 + c^2 \xi}. \quad (5.57)$$

By (5.56) and (5.42), we obtain

$$\Psi_s = \frac{1}{\xi} \left(\frac{c^2 s}{\omega} - g \right). \quad (5.58)$$

Combining this with (5.56), we can see that

$$\Psi - s\Psi_s = \frac{c^2}{\omega}. \quad (5.59)$$

Together with (5.42), we obtain

$$\Psi_{ss} = \frac{c^4}{\omega^3}. \quad (5.60)$$

By (5.41), (5.42) and the second equation of (5.56), we obtain $\omega_r = \frac{2cc_r s^2 + (c^2 \xi)_r}{2\omega}$. Together with (5.59), we have

$$(\Psi - s\Psi_s)_r = \frac{1}{\omega^3} \left(c^3 c_r s^2 + c^3 c_r \xi - \frac{1}{2} c^4 \xi_r \right). \quad (5.61)$$

If $c = 0$ or $g^2 - c^2 = 0$, then F is not strongly convex by (5.41), (5.45), and (5.46). Using (1.2), (5.60), and (5.61), we get $f s^2 + g = -(\ln |c|)_r s^2 + g g_r - (\ln |c|)_r g^2$. It follows that

$$f = -(\ln |c|)_r, \quad g = g g_r - g^2 (\ln |c|)_r. \quad (5.62)$$

Combining the first equation (5.62) with (5.55), we have

$$(c^2 - g^2) b_r = -2(c^2 - g^2)_r$$

Hence we have $b_r = -[\ln(c^2 - g^2)]_r$. Together with (5.54), we obtain

$$a = \frac{c_0}{(c^2 - g^2)^2} \quad (5.63)$$

where $c_0 = \text{constant}$. From the second equation of (5.62), we have

$$\frac{1}{g} = \left(\ln \left| \frac{g}{c} \right| \right)_r = \frac{h_r}{h}$$

where $h := \frac{g}{c}$. Thus we obtain $g = \frac{h}{h_r}$ and $c = \frac{1}{h_r}$. Plugging these into (5.63) yields $a = \frac{c_0 h_r^4}{(h^2 - 1)^2}$. Hence (1.5) holds.

By (1.4), (5.46), (5.47), and (5.45), $F = \check{\alpha}\phi\left(u^1, \frac{v^1}{\check{\alpha}}\right)$ is strongly convex if and only if

$$\omega > 0, \quad a > 0, \quad c\eta \neq 0.$$

Notice that $s \in \mathbb{R}$ and $g^2 - c^2 = \frac{h^2 - 1}{h_r^2}$. Thus we obtain that F is strongly convex if and only if $|h| > 1$, $h_r \neq 0$, $c_0 > 0$. This completes the proof of Theorem 1.2.

Finally we discuss the case that $\kappa = -1$. By (5.38) and Lemma 2.3, we have

$$\phi = a(r) \frac{\left[c^2 s + g \sqrt{c^2 s^2 - c^2 (g^2 - c^2)} \right]^2}{\sqrt{c^2 s^2 - c^2 (g^2 - c^2)}}.$$

A simple calculation gives the following formula $\phi - s\phi_s = \frac{ac^4(g^2 - c^2)^2}{\omega}(g^2 - s^2)$. Notice that $a > 0$ and $s \in \mathbb{R}$. Hence we obtain that $F = \check{\alpha}\phi$ is not strongly convex.

6 Examples

Below are two interesting examples. These warped product Finsler metrics are locally projectively flat with zero flag curvature. By considering in Theorem 1.2, $c_0 = 1$ and $h = \frac{1}{r}$ where $r < 1$, we have

$$(a, c, g) = \left(\frac{1}{r^4(1 - r^2)^2}, -r^2, -r \right)$$

and

$$h > 1, \quad h_r \neq 0, \quad c_0 > 0. \quad (6.1)$$

Thus we obtain the following:

Example 6.1. Let $\phi(r, s)$ be a function defined by

$$\phi(r, s) = \frac{\left[\sqrt{s^2 + r^2(1 - r^2)} + rs \right]^2}{(1 - r^2)^2 \sqrt{s^2 + r^2(1 - r^2)}}.$$

Then the warped product Finsler metric given by $F = \check{\alpha}_+\phi(r, s)$ is a locally projectively flat Finsler metric with zero flag curvature, where

$$r := |x|, \quad s := \frac{v^1}{\check{\alpha}_+}, \quad (6.2)$$

and $\check{\alpha}_+$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$. In fact, F is the Finsler warped product form of the metric introduced by Berwald in 1929 [2,4].

Let us take a look at another special case of Theorem 1.2: when $c_0 = 1$, $h = \frac{\sqrt{1+4r^2}}{2r}$,

$$(a, c, g) = \left(\frac{1}{r^4(1+4r^2)}, -2r^2\sqrt{1+4r^2}, -r(1+4r^2) \right).$$

Furthermore, c_0, h, h_r satisfy (6.1). Hence we have the following:

Example 6.2. Let $\phi(r, s)$ be a function defined by

$$\phi(r, s) = \frac{\left[\sqrt{(1+4r^2)(s^2+r^2+4r^4)} + 2rs \right]^2}{\sqrt{(1+4r^2)(s^2+r^2+4r^4)}}.$$

Then the warped product Finsler metric given by

$$F = \check{\alpha}_+ \phi(r, s) \quad (6.3)$$

is a locally projectively flat Finsler metric with zero flag curvature, where r and s satisfy (6.2). Finsler metric (6.3) is the Finsler warped product form of the metric found by Mo and Zhu recently [13]. Mo and Zhu showed that F is not projectively flat.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Bao, D., Robles, C., Shen, Z.: Zermelo navigation on Riemannian manifolds. *J. Differ. Geom.* **66**, 377–435 (2004)
2. Berwald, L.: Über die n -dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind. *Math. Z.* **30**, 449–469 (1929)
3. Bishop, R.L., O'Neill, B.: Manifolds of negative curvature. *Trans. Am. Math. Soc.* **145**, 1–49 (1969)
4. Chen, B., Shen, Z., Zhao, L.: Constructions of Einstein Finsler metrics by warped product. *Int. J. Math.* **29**, 1850081 (2018)
5. Hilbert, D.: Mathematical problems. *Bull. Am. Math. Soc.* **37**, 407–436 (2001)
6. Huang, L., Mo, X.: Projectively flat Finsler metrics with orthogonal invariance. *Ann. Pol. Math.* **107**, 259–270 (2013)
7. Li, B., Shen, Z.: On a class of projectively flat Finsler metrics of constant flag curvature. *Int. J. Math.* **18**, 749–760 (2007)
8. Liu, H., Mo, X.: Finsler warped product metrics of Douglas type. *Can. Math. Bull.* **62**, 119–130 (2019)
9. Liu, H., Mo, X.: The explicit construction on Finsler warped product metrics of scalar flag curvature. *Houston J. Math.* **46**, 307–321 (2020)

10. Liu, H., Mo, X., Zhang, H.: Finsler warped product metrics with special Riemannian curvature properties. *Sci. China. Math.* **63**, 1391–1408 (2020)
11. Mo, X., Yu, C.: On some explicit constructions of Finsler metrics with scalar flag curvature. *Can. J. Math.* **62**, 1325–1339 (2010)
12. Mo, X., Zhu, H.: On a class of projectively flat Finsler metrics of negative constant flag curvature. *Int. J. Math.* **23**, 1250084 (2012)
13. Mo, X., Zhu, H.: On a class of locally projectively flat Finsler metrics. *Bull. Iran. Math. Soc.* **43**, 735–746 (2017)
14. Mo, X., Zhou, L., Zhu, H.: On a class of Finsler metrics of constant flag curvature. *Houston J. Math.* **43**, 829–846 (2017)
15. Shen, Z.: Projectively flat Randers metrics with constant flag curvature. *Math. Ann.* **325**, 19–30 (2003)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.