Lichnerowicz-Obata Estimate, Almost Parallel p -form and Almost Product Manifolds

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Abstract

We show a Lichnerowicz-Obata type estimate for the first eigenvalue of the Laplacian of n -dimensional closed Riemannian manifolds with an almost parallel p -form ($2 \leq p \leq n/2$) in L^2 -sense, and give a Gromov-Hausdorff approximation to a product $S^{n-p} \times X$ under some pinching conditions when $2 \leq p < n/2$.

Keywords Gromov-Hausdorff distance · Lichnerowicz-Obata estimate · Parallel p -form

Mathematics Subject Classification 53C20 · 58J50

1 Introduction

In this paper we give an estimate for the first eigenvalue of the Laplacian of closed Riemannian manifolds with positive Ricci curvature and an almost parallel form, and show the Gromov-Hausdorff closeness to a product space for the almost equality case.

One of the most famous theorem about the estimate of the first eigenvalue of the Laplacian is the Lichnerowicz-Obata theorem. Lichnerowicz showed the optimal comparison result for the first eigenvalue when the Riemannian manifold has positive Ricci curvature, and Obata showed that the equality of the Lichnerowicz estimate implies that the Riemannian manifold is isometric to the standard sphere. In the following, $\lambda_k(g)$ denotes the k -th eigenvalue of the Laplacian $\Delta := -\text{tr}_g \text{Hess}$ acting on functions.

Theorem 1.1 (*Lichnerowicz-Obata theorem*) *Take an integer $n \geq 2$. Let (M, g) be an n -dimensional closed Riemannian manifold. If $\text{Ric} \geq (n-1)g$, then $\lambda_1(g) \geq n$. The equality holds if and only if (M, g) is isometric to the standard sphere of radius 1.*

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Petersen [19], Aubry [3] and Honda [13] showed the stability result of the Lichnerowicz–Obata theorem. In the following, d_{GH} denotes the Gromov–Hausdorff distance and S^n denotes the n -dimensional standard sphere of radius 1. (see Definition 2.2 for the definition of the Gromov–Hausdorff distance).

Theorem 1.2 ([3, 13, 19]) *For given an integer $n \geq 2$ and a positive real number $\epsilon > 0$, there exists $\delta(n, \epsilon) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric} \geq (n - 1)g$ and $\lambda_n(g) \leq n + \delta$, then $d_{GH}(M, S^n) \leq \epsilon$.*

Note that Petersen considered the pinching condition on $\lambda_{n+1}(g)$, and Aubry and Honda improved it independently.

We mention some improvements of the Lichnerowicz estimate when the Riemannian manifold has a special structure. If (M, g) is a real n -dimensional Kähler manifold with $\text{Ric} \geq (n - 1)g$, then the Lichnerowicz estimate is improved as follows:

$$\lambda_1(g) \geq 2(n - 1). \quad (1)$$

See [4, Theorem 11.49] for the proof. If (M, g) is a real n -dimensional quaternionic Kähler manifold with $\text{Ric} \geq (n - 1)g$, then we have

$$\lambda_1(g) \geq \frac{2n + 8}{n + 8}(n - 1). \quad (2)$$

See [2] for the proof. In these cases, the Riemannian manifold (M, g) has a non-trivial parallel 2 and 4-form, respectively. When (M, g) is an n -dimensional product Riemannian manifold $(N_1 \times N_2, g_1 + g_2)$ with $\text{Ric} \geq (n - 1)g$, then we have

$$\lambda_1(g) \geq \min_{i \in \{1, 2\}} \left\{ \frac{\dim N_i}{\dim N_i - 1} \right\} (n - 1),$$

and M has a non-trivial parallel form if either N_1 or N_2 is orientable.

Grosjean [12] gave a unified proof of the improvements of the Lichnerowicz estimate when the Riemannian manifold has a non-trivial parallel form.

Theorem 1.3 ([12]) *Let (M, g) be an n -dimensional closed Riemannian manifold. Assume that $\text{Ric} \geq (n - p - 1)g$ and that there exists a nontrivial parallel p -form on M ($2 \leq p \leq n/2$). Then, we have*

$$\lambda_1(g) \geq n - p. \quad (3)$$

Moreover, if $p < n/2$ and if in addition M is simply connected, then the equality in (3) implies that (M, g) is isometric to a product $S^{n-p} \times (X, g')$, where (X, g') is some p -dimensional closed Riemannian manifold.

Remark 1.1 We give several remarks on this theorem.

- When $\text{Ric} \geq (n - p - 1)g$, the Lichnerowicz estimate is $\lambda_1(g) \geq n(n - p - 1)/(n - 1)$. Since $n - p > n(n - p - 1)/(n - 1)$ for $2 \leq p \leq n/2$, the estimate (3) improves the Lichnerowicz estimate.

- Grosjean also showed this type theorem when M has a convex smooth boundary.
- Though Grosjean originally assumed the manifold is orientable, the assumption can be easily removed by taking the orientable double covering.
- If (M, g) is either a Kähler manifold with $n \geq 6$ or a quaternionic Kähler manifold, then the estimate (1) or (2) (with scaling) is stronger than (3).
- There exists no non-trivial parallel 1-form on any closed Riemannian manifold with positive Ricci curvature.
- The assumption $2 \leq p \leq n/2$ (resp. $2 \leq p < n/2$) implies $n \geq 4$ (resp. $n \geq 5$). For the case $n = 4$ and $p = n/2 = 2$, the complex projective space $\mathbb{C}P^2$ also satisfies the equality in (3).
- If there exists a non-trivial parallel p -form ω ($1 \leq p \leq n-1$) on an n -dimensional Riemannian manifold (M, g) , then $\omega(x) \in \bigwedge^p T_x^* M$ ($x \in M$) is invariant under the Holonomy action, and so the Holonomy group coincides with neither $SO(n)$ nor $O(n)$.

The main aim of this paper is to show the almost version of Grosjean's result. We also give the almost version of the estimate (1) in Appendix B.

We first note that, for a closed Riemannian manifold (M, g) , there exists a non zero p -form ω with $\|\nabla\omega\|_2^2 \leq \delta\|\omega\|_2^2$ for some $\delta > 0$ if and only if $\lambda_1(\Delta_{C,p}) \leq \delta$ holds, where $\lambda_1(\Delta_{C,p})$ is defined by

$$\lambda_1(\Delta_{C,p}) := \inf \left\{ \frac{\|\nabla\omega\|_2^2}{\|\omega\|_2^2} : \omega \in \Gamma\left(\bigwedge^p T^*M\right) \text{ with } \omega \neq 0 \right\}.$$

Let us state our eigenvalue estimate.

Main Theorem 1 *For given integers $n \geq 4$ and $2 \leq p \leq n/2$, there exists a constant $C(n, p) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq (n - p - 1)g$, then we have*

$$\lambda_1(g) \geq n - p - C(n, p)\lambda_1(\Delta_{C,p})^{1/2}.$$

We immediately have the following corollary:

Corollary 1.4 *For given integers $n \geq 4$ and $2 \leq p \leq n/2$, there exists a constant $C(n, p) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq (n - p - 1)g$ and*

$$\frac{n(n - p - 1)}{n - 1} \leq \lambda_1(g) \leq n - p,$$

then we have

$$\lambda_1(\Delta_{C,p}) \geq \left(\frac{n - p - \lambda_1(g)}{C(n, p)} \right)^2.$$

Note that we always have the lower bound on the eigenvalue of the Laplacian $\lambda_1(g) \geq n(n-p-1)/(n-1)$ if $\text{Ric}_g \geq (n-p-1)g$ by the Lichnerowicz estimate. An upper bound on $C(n, p)$ is computable. However, we do not know the optimal value of it.

We next state the eigenvalue pinching result.

Main Theorem 2 *For given integers $n \geq 5$ and $2 \leq p < n/2$ and a positive real number $\epsilon > 0$, there exists $\delta = \delta(n, p, \epsilon) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq (n-p-1)g$,*

$$\lambda_{n-p+1}(g) \leq n-p+\delta$$

and

$$\lambda_1(\Delta_{C,p}) \leq \delta,$$

then M is orientable and

$$d_{GH}(M, S^{n-p} \times X) \leq \epsilon,$$

where X is some compact metric space.

Remark 1.2 In fact, we prove that there exist constants $C(n, p) > 0$ and $\alpha(n) > 0$ such that

$$d_{GH}(M, S^{n-p} \times X) \leq C(n, p)\delta^{\alpha(n)}$$

under the assumption of Main Theorem 2. One can easily find the explicit value of $\alpha(n)$ (see Notation 4.35 and Theorem 4.47). However, it might be far from the optimal value. By the Gromov's pre-compactness theorem, we can take X to be a geodesic space. However, we lose the information about the convergence rate in that case.

Based on Theorem 1.2, one might expect that we can replace the assumption " $\lambda_{n-p+1}(g) \leq n-p+\delta$ " in Main Theorem 2 to the weaker assumption " $\lambda_{n-p}(g) \leq n-p+\delta$ ". However, an example shows that we cannot do it even if $\delta = 0$ (see Proposition 3.3). Instead of that, replacing $\lambda_1(\Delta_{C,p})$ to $\lambda_1(\Delta_{C,n-p})$, we have the following theorems:

Main Theorem 3 *For given integers $n \geq 4$ and $2 \leq p \leq n/2$, there exists a constant $C(n, p) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq (n-p-1)g$, then we have*

$$\lambda_1(g) \geq n-p-C(n, p)\lambda_1(\Delta_{C,n-p})^{1/2}.$$

Main Theorem 4 *For given integers $n \geq 5$ and $2 \leq p < n/2$ and a positive real number $\epsilon > 0$, there exists $\delta = \delta(n, p, \epsilon) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq (n-p-1)g$,*

$$\lambda_{n-p}(g) \leq n-p+\delta$$

and

$$\lambda_1(\Delta_{C,n-p}) \leq \delta,$$

then we have

$$d_{GH}(M, S^{n-p} \times X) \leq \epsilon,$$

where X is some compact metric space.

Note that the assumption “ $\lambda_1(\Delta_{C,n-p}) \leq \delta$ ” is equivalent to the assumption “ $\lambda_1(\Delta_{C,p}) \leq \delta$ ” if the manifold is orientable.

We would like to point out that our work was motivated by Honda’s spectral convergence theorem [17], which asserts the continuity of the eigenvalues of the connection Laplacian $\Delta_{C,p}$ acting on p -forms with respect to the non-collapsing Gromov-Hausdorff convergence assuming the two-sided bound on the Ricci curvature. By virtue of his theorem, we can generalize our main theorems to Ricci limit spaces under such assumptions. Note that we show our main theorems without the non-collapsing assumption, i.e., without assuming the lower bound on the volume of the Riemannian manifold.

Our work was also motivated by the Cheeger-Colding almost splitting theorem (see [6, Theorem 9.25]), whose conclusion is the Gromov-Hausdorff approximation to a product $\mathbb{R} \times X$. As the almost splitting theorem, we need to show the almost Pythagorean theorem under the assumption of Main Theorem 2.

The structure of this paper is as follows.

In Sect. 2, we recall some basic definitions and facts, and give calculations of differential forms.

In Sect. 3, we estimate the error terms of the Grosjean’s formula when the Riemannian manifold has a non-trivial almost parallel p -form. As a consequence, we prove Main Theorem 1 and Main Theorem 3.

In Sect. 4, we prove Main Theorem 2 and Main Theorem 4. In Sect. 4.1, we list some useful techniques for our pinching problem. In Sect. 4.2, we show some pinching conditions on the eigenfunctions along geodesics under the assumption $\lambda_k(g) \leq n - p + \delta$ and $\lambda_1(\Delta_{C,p}) \leq \delta$. In Sect. 4.3, we show that similar results hold under the assumption $\lambda_k(g) \leq n - p + \delta$ and $\lambda_1(\Delta_{C,n-p}) \leq \delta$. In Sect. 4.4, we show that the eigenfunctions are almost cosine functions in some sense under our pinching condition. In Sect. 4.5, we construct an approximation map and show Main Theorem 2 except for the orientability. In Sect. 4.6, we give some lemmas to prove the remaining part of main theorems. In Sect. 4.7, we show the orientability of the manifold under the assumption of Main Theorem 2, and complete the proof of it. In Sect. 4.8, we show that the assumption of Main Theorem 4 implies that $\lambda_{n-p+1}(g)$ is close to $n - p$, and complete the proof of Main Theorem 4.

In Appendix A, we discuss Ricci limit spaces. Using the technique of Sect. 4.7, we show the stability of unorientability under the non-collapsing Gromov-Hausdorff convergence assuming the two-sided bound on the Ricci curvature and the upper bound on the diameter.

In Appendix B, we give the almost version of the estimate (1) assuming that there exists a 2-form ω which satisfies that $\|\nabla\omega\|_2$ and $\|J_\omega^2 + \text{Id}\|_1$ are small, where $J_\omega \in \Gamma(T^*M \otimes TM)$ is defined so that $\omega = g(J_\omega \cdot, \cdot)$.

2 Preliminaries

2.1 Basic Definitions

We first recall some basic definitions and fix our convention.

Definition 2.1 (Hausdorff distance) Let (X, d) be a metric space. For each point $x_0 \in X$, subsets $A, B \subset X$ and $r > 0$, define

$$\begin{aligned} d(x_0, A) &:= \inf\{d(x_0, a) : a \in A\}, \\ B_r(x_0) &:= \{x \in X : d(x, x_0) < r\}, \\ B_r(A) &:= \{x \in X : d(x, A) < r\}, \\ d_{H,d}(A, B) &:= \inf\{\epsilon > 0 : A \subset B_\epsilon(B) \text{ and } B \subset B_\epsilon(A)\} \end{aligned}$$

We call $d_{H,d}$ the Hausdorff distance.

The Hausdorff distance defines a metric on the collection of compact subsets of X .

Definition 2.2 (Gromov-Hausdorff distance) Let $(X, d_X), (Y, d_Y)$ be metric spaces. Define

$$\begin{aligned} d_{GH}(X, Y) &:= \inf \left\{ d_{H,d}(X, Y) : d \text{ is a metric on } X \coprod Y \text{ such that} \right. \\ &\quad \left. d|_X = d_X \text{ and } d|_Y = d_Y \right\}. \end{aligned}$$

The Gromov-Hausdorff distance defines a metric on the set of isometry classes of compact metric spaces (see [20, Proposition 11.1.3]).

Definition 2.3 (ϵ -Hausdorff approximation map) Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a map $f : X \rightarrow Y$ is an ϵ -Hausdorff approximation map for $\epsilon > 0$ if the following two conditions hold.

- (i) For all $a, b \in X$, we have $|d_X(a, b) - d_Y(f(a), f(b))| < \epsilon$,
- (ii) $f(X)$ is ϵ -dense in Y , i.e., for all $y \in Y$, there exists $x \in X$ with $d_Y(f(x), y) < \epsilon$.

If there exists an ϵ -Hausdorff approximation map $f : X \rightarrow Y$, then we can show that $d_{GH}(X, Y) \leq 3\epsilon/2$ by considering the following metric d on $X \coprod Y$:

$$d(a, b) = \begin{cases} d_X(a, b) & (a, b \in X), \\ \frac{\epsilon}{2} + \inf_{x \in X} (d_X(a, x) + d_Y(f(x), b)) & (a \in X, b \in Y), \\ d_Y(a, b) & (a, b \in Y). \end{cases}$$

If $d_{GH}(X, Y) < \epsilon$, then there exists a 2ϵ -Hausdorff approximation map from X to Y .

Let $C(u_1, \dots, u_l) > 0$ denotes a positive function depending only on the numbers $u_1, \dots, u_l$. For a set X , $\text{Card} X$ denotes a cardinal number of X .

Let (M, g) be a closed Riemannian manifold. For any $p \geq 1$, we use the normalized L^p -norm:

$$\|f\|_p^p := \frac{1}{\text{Vol}(M)} \int_M |f|^p d\mu_g,$$

and $\|f\|_\infty := \text{ess sup}_{x \in M} |f(x)|$ for a measurable function f on M . We also use these notation for tensors. We have $\|f\|_p \leq \|f\|_q$ for any $p \leq q \leq \infty$.

Let ∇ denotes the Levi-Civita connection. Throughout in this paper, $0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \dots \rightarrow \infty$ denotes the eigenvalues of the Laplacian $\Delta = -\text{tr Hess}$ acting on functions. We sometimes identify TM and T^*M using the metric g . Given points $x, y \in M$, let $\gamma_{x,y}$ denotes one of minimal geodesics with unit speed such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(d(x, y)) = y$. For given $x \in M$ and $u \in T_x M$ with $|u| = 1$, let $\gamma_u: \mathbb{R} \rightarrow M$ denotes the geodesic with unit speed such that $\gamma_u(0) = x$ and $\dot{\gamma}_u(0) = u$.

For any $x \in M$ and $u \in T_x M$ with $|u| = 1$, put

$$t(u) := \sup\{t \in \mathbb{R}_{>0} : d(x, \gamma_u(t)) = t\},$$

and define $I_x \subset M$ to be the complement of the cut locus at x (see also [21, p.104]), i.e.,

$$I_x := \{\gamma_u(t) : u \in T_x M \text{ with } |u| = 1 \text{ and } 0 \leq t < t(u)\}.$$

Then, I_x is open and $\text{Vol}(M \setminus I_x) = 0$ [21, III Lemma 4.4]. For any $y \in I_x \setminus \{x\}$, the minimal geodesic $\gamma_{x,y}$ is uniquely determined. The function $d(x, \cdot): M \rightarrow \mathbb{R}$ is differentiable in $I_x \setminus \{x\}$ and $\nabla d(x, \cdot)(y) = \dot{\gamma}_{x,y}(d(x, y))$ holds for any $y \in I_x \setminus \{x\}$ [21, III Proposition 4.8].

Let V be an n -dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. We define inner products on $\bigwedge^k V$ and $V \otimes \bigwedge^k V$ as follows:

$$\begin{aligned} \langle v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k \rangle &= \det\{\langle v_i, w_j \rangle\}_{i,j}, \\ \langle v_0 \otimes v_1 \wedge \dots \wedge v_k, w_0 \otimes w_1 \wedge \dots \wedge w_k \rangle &= \langle v_0, w_0 \rangle \det\{\langle v_i, w_j \rangle\}_{i,j}, \end{aligned}$$

for $v_0, \dots, v_k, w_0, \dots, w_k \in V$. For $\alpha \in V$ and $\omega \in \bigwedge^k V$, there exists a unique $\iota(\alpha)\omega \in \bigwedge^{k-1} V$ such that $\langle \iota(\alpha)\omega, \eta \rangle = \langle \omega, \alpha \wedge \eta \rangle$ holds for any $\eta \in \bigwedge^{k-1} V$. If $k = 0$, we define $\iota(\alpha)\omega = 0$ and $\bigwedge^{-1} V = \{0\}$. Then, ι defines a bi-linear map:

$$\iota: V \times \bigwedge^k V \rightarrow \bigwedge^{k-1} V.$$

By identifying V and V^* using $\langle \cdot, \cdot \rangle$, we also use the notation ι for the bi-linear map:

$$\iota: V^* \times \bigwedge^k V \rightarrow \bigwedge^{k-1} V.$$

For any Riemannian manifold (M, g) , we define operators $\nabla^*: \Gamma(T^*M \otimes \bigwedge^k T^*M) \rightarrow \Gamma(\bigwedge^k T^*M)$ and $d^*: \Gamma(\bigwedge^k T^*M) \rightarrow \Gamma(\bigwedge^{k-1} T^*M)$ by

$$\begin{aligned}\nabla^*(\alpha \otimes \beta) &:= -\text{tr}_{T^*M} \nabla(\alpha \otimes \beta) = -\sum_{i=1}^n (\nabla_{e_i} \alpha)(e_i) \cdot \beta - \sum_{i=1}^n \alpha(e_i) \cdot \nabla_{e_i} \beta. \\ d^* \omega &:= -\sum_{i=1}^n \iota(e_i) \nabla_{e_i} \omega\end{aligned}$$

for all $\alpha \otimes \beta \in \Gamma(T^*M \otimes \bigwedge^k T^*M)$ and $\omega \in \Gamma(\bigwedge^k T^*M)$, where $n = \dim M$ and $\{e_1, \dots, e_n\}$ is an orthonormal basis of TM . If M is closed, then we have

$$\begin{aligned}\int_M \langle T, \nabla \alpha \rangle d\mu_g &= \int_M \langle \nabla^* T, \alpha \rangle d\mu_g, \\ \int_M \langle \omega, d\eta \rangle d\mu_g &= \int_M \langle d^* \omega, \eta \rangle d\mu_g\end{aligned}$$

for all $T \in \Gamma(T^*M \otimes \bigwedge^k T^*M)$, $\alpha \in \Gamma(\bigwedge^k T^*M)$, $\omega \in \Gamma(\bigwedge^k T^*M)$ and $\eta \in \Gamma(\bigwedge^{k-1} T^*M)$ by the divergence theorem. The Hodge Laplacian $\Delta: \Gamma(\bigwedge^k T^*M) \rightarrow \Gamma(\bigwedge^k T^*M)$ is defined by

$$\Delta := dd^* + d^*d.$$

Notation 2.4 For an n -dimensional Riemannian manifold (M, g) , we can take orthonormal basis of TM only locally in general. However, for example, the tensor

$$\sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \in \Gamma(T^*M \otimes \bigwedge^{k-1} T^*M) \quad (f \in C^\infty(M), \omega \in \Gamma(\bigwedge^k T^*M))$$

is defined independently of the choice of the orthonormal basis $\{e_1, \dots, e_n\}$ of TM , where $\{e^1, \dots, e^n\}$ denotes its dual. Thus, we sometimes use such notation without taking a particular orthonormal basis.

Finally, we list some important notation. Let (M, g) be a closed Riemannian manifold.

- d denotes the Riemannian distance function.
- Ric denotes the Ricci curvature tensor.
- diam denotes the diameter.
- Vol or μ_g denotes the Riemannian volume measure.
- $\|\cdot\|_p$ denotes the normalized L^p -norm for each $p \geq 1$, which is defined by

$$\|f\|_p^p := \frac{1}{\text{Vol}(M)} \int_M |f|^p d\mu_g$$

for any measurable function f on M .

- $\|f\|_\infty$ denotes the essential sup of $|f|$ for any measurable function f on M .
- ∇ denotes the Levi-Civita connection.
- ∇^2 denotes the Hessian for functions.
- $\Delta: \Gamma(\bigwedge^k T^*M) \rightarrow \Gamma(\bigwedge^k T^*M)$ denotes the Hodge Laplacian defined by $\Delta := dd^* + d^*d$. We frequently use the Laplacian acting on functions. Note that $\Delta = -\text{tr}_g \nabla^2$ holds for functions under our sign convention.
- $0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \dots \rightarrow \infty$ denotes the eigenvalues of the Laplacian acting on functions.
- $\gamma_{x,y}: [0, d(x, y)] \rightarrow M$ denotes one of minimal geodesics with unit speed such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(d(x, y)) = y$ for any $x, y \in M$.
- $\gamma_u: \mathbb{R} \rightarrow M$ denotes the geodesic with unit speed such that $\gamma_u(0) = x$ and $\dot{\gamma}(0) = u$ for any $x \in M$ and $u \in T_x M$ with $|u| = 1$.
- $I_x \subset M$ denotes the complement of the cut locus at $x \in M$. We have $\text{Vol}(M \setminus I_x) = 0$. We have that $\gamma_{x,y}$ is uniquely determined and $\nabla d(x, \cdot) = \dot{\gamma}_{x,y}(d(x, y))$ holds for any $y \in I_x \setminus \{x\}$.
- $\Delta_{C,k} = \nabla^* \nabla: \Gamma(\bigwedge^k T^*M) \rightarrow \Gamma(\bigwedge^k T^*M)$ denotes the connection Laplacian acting on k -forms.
- $0 \leq \lambda_1(\Delta_{C,k}) \leq \lambda_2(\Delta_{C,k}) \leq \dots \rightarrow \infty$ denotes the eigenvalues of the connection Laplacian $\Delta_{C,k}$ acting on k -forms.
- $S^n(r)$ denotes the n -dimensional standard sphere of radius r .
- $S^n := S^n(1)$.

Note that the lowest eigenvalue of the Laplacian Δ acting on function is always equal to 0, and so we start counting the eigenvalues of it from $i = 0$. This is not the case with the connection Laplacian $\Delta_{C,k}$ acting on k -forms, and so we start counting the eigenvalues of it from $i = 1$. For any $i \in \mathbb{Z}_{>0}$, we have

$$\lambda_i(\Delta_{C,0}) = \lambda_{i-1}(g).$$

2.2 Calculus of Differential Forms

In this subsection, we recall some facts about differential forms, and do some calculations.

We first recall the decomposition:

$$T^*M \otimes \bigwedge^k T^*M = T^{k,1}M \oplus \bigwedge^{k+1} T^*M \oplus \bigwedge^{k-1} T^*M.$$

See also [23, Section 2].

Let V be an n -dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. We put

$$\begin{aligned} P_1: V \otimes \bigwedge^k V &\rightarrow \bigwedge^{k+1} V, & P_1(\alpha \otimes \omega) &:= \left(\frac{1}{k+1} \right)^{\frac{1}{2}} \alpha \wedge \omega, \\ P_2: V \otimes \bigwedge^k V &\rightarrow \bigwedge^{k-1} V, & P_2(\alpha \otimes \omega) &:= \left(\frac{1}{n-k+1} \right)^{\frac{1}{2}} \iota(\alpha)\omega, \end{aligned}$$

$$Q_1: \bigwedge^{k+1} V \rightarrow V \otimes \bigwedge^k V, \quad Q_1(\zeta) := \left(\frac{1}{k+1} \right)^{\frac{1}{2}} \sum_{i=1}^n e^i \otimes \iota(e^i)\zeta,$$

$$Q_2: \bigwedge^{k-1} V \rightarrow V \otimes \bigwedge^k V, \quad Q_2(\eta) := \left(\frac{1}{n-k+1} \right)^{\frac{1}{2}} \sum_{i=1}^n e^i \otimes e^i \wedge \eta,$$

where $\{e^1, \dots, e^n\}$ is orthonormal basis of V . Then, we have

- $\text{Im } Q_1 \perp \text{Im } Q_2$,
- $P_i \circ Q_i = \text{Id}$ for each $i = 1, 2$,
- Q_1 and Q_2 preserve the norms,
- $Q_i \circ P_i: V \otimes \bigwedge^k V \rightarrow V \otimes \bigwedge^k V$ is symmetric and $(Q_i \circ P_i)^2 = Q_i \circ P_i$ for each $i = 1, 2$.

Therefore, $Q_i \circ P_i$ is the orthogonal projection $V \otimes \bigwedge^k V \rightarrow \text{Im } Q_i$. Since $\bigwedge^{k+1} V \cong \text{Im } Q_1$ and $\bigwedge^{k-1} V \cong \text{Im } Q_2$, we can regard $\bigwedge^{k+1} V$ and $\bigwedge^{k-1} V$ as subspaces of $V \otimes \bigwedge^k V$.

Take an n -dimensional Riemannian manifold (M, g) and consider the case when $V = T_x^* M$ ($x \in M$). We can take a sub-bundle $T^{k,1} M$ of $T^* M \otimes \bigwedge^k T^* M$ such that

$$T^* M \otimes \bigwedge^k T^* M = T^{k,1} M \oplus \bigwedge^{k+1} T^* M \oplus \bigwedge^{k-1} T^* M$$

is an orthogonal decomposition. Then, for $\omega \in \Gamma(\bigwedge^k T^* M)$, we can decompose $\nabla \omega \in \Gamma(T^* M \otimes \bigwedge^k T^* M)$, the $\bigwedge^{k+1} T^* M$ -component is equal to $(1/(k+1))^{1/2} d\omega$ and the $\bigwedge^{k-1} T^* M$ -component is equal to $-(1/(n-k+1))^{1/2} d^* \omega$. Let $T(\omega)$ denotes the remaining part ($T: \Gamma(\bigwedge^k T^* M) \rightarrow \Gamma(T^{k,1} M)$). Then, we have

$$\nabla \omega = T(\omega) + \left(\frac{1}{k+1} \right)^{\frac{1}{2}} Q_1(d\omega) - \left(\frac{1}{n-k+1} \right)^{\frac{1}{2}} Q_2(d^* \omega).$$

Therefore, we get

$$|\nabla \omega|^2 = |T(\omega)|^2 + \frac{1}{k+1} |d\omega|^2 + \frac{1}{n-k+1} |d^* \omega|^2. \quad (4)$$

If $d^* \omega = 0$ and $T(\omega) = 0$, then ω is called a Killing k -form (see also [23, Definition 2.1]).

We next recall the Bochner-Weitzenböck formula.

Definition 2.5 Let (M, g) be an n -dimensional Riemannian manifold. We define a homomorphism $\mathcal{R}_k: \bigwedge^k T^* M \rightarrow \bigwedge^k T^* M$ as

$$\mathcal{R}_k \omega = - \sum_{i,j} e^i \wedge \iota(e_j) (R(e_i, e_j) \omega)$$

for any $\omega \in \bigwedge^k T^*M$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of TM , $\{e^1, \dots, e^n\}$ is its dual and $R(e_i, e_j)\omega$ is defined by

$$R(e_i, e_j)\omega = \nabla_{e_i}\nabla_{e_j}\omega - \nabla_{e_j}\nabla_{e_i}\omega - \nabla_{[e_i, e_j]}\omega \in \Gamma(\bigwedge^k T^*M).$$

Note that if $k = 1$, then we have $\mathcal{R}_1\omega = \text{Ric}(\omega, \cdot)$ for any $\omega \in \Gamma(T^*M)$.

The Bochner-Weitzenböck formula is stated as follows:

Theorem 2.6 (Bochner-Weitzenböck formula) *For any $\omega \in \Gamma(\bigwedge^k T^*M)$, we have*

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{R}_k\omega.$$

In particular, we have the following theorem when $k = 1$:

Theorem 2.7 (Bochner-Weitzenböck formula for 1-forms) *For any $\omega \in \Gamma(T^*M)$, we have*

$$\Delta\omega = \nabla^*\nabla\omega + \text{Ric}(\omega, \cdot).$$

Let us do some calculations of differential forms.

Lemma 2.8 *Let (M, g) be an n -dimensional Riemannian manifold. Take a vector field $X \in \Gamma(TM)$, a p -form $\omega \in \Gamma(\bigwedge^p T^*M)$ ($p \geq 1$) and a local orthonormal bases $\{e_1, \dots, e_n\}$ of TM .*

(i) *We have*

$$\mathcal{R}_{p-1}(\iota(X)\omega) = \iota(X)\mathcal{R}_p\omega + \iota(\text{Ric}(X))\omega + 2 \sum_{i=1}^n \iota(e_i)(R(X, e_i)\omega).$$

(ii) *We have*

$$\begin{aligned} \Delta(\iota(X)\omega) &= \iota(\Delta X)\omega + \iota(X)\Delta\omega + 2 \sum_{i=1}^n \iota(e_i)(R(X, e_i)\omega) \\ &\quad - 2 \sum_{i=1}^n \iota(\nabla_{e_i}X)(\nabla_{e_i}\omega). \end{aligned}$$

(iii) *We have*

$$\sum_{i=1}^n \iota(e_i)(R(X, e_i)\omega) = -\nabla_X d^*\omega + d^*\nabla_X\omega + \sum_{i,j=1}^n \langle \nabla_{e_j}X, e_i \rangle \iota(e_j)\nabla_{e_i}\omega.$$

Proof Let $\{e^1, \dots, e^n\}$ be the dual basis of $\{e_1, \dots, e_n\}$.

We first show (i). If $p = 1$, both sides are equal to 0. Let us assume $p \geq 2$. We have

$$\begin{aligned}
 & \iota(\text{Ric}(X))\omega \\
 &= \frac{1}{(p-1)!} \sum_{i, i_1, \dots, i_{p-1}} \omega(R(X, e_i)e_i, e_{i_1}, \dots, e_{i_{p-1}})e^{i_1} \wedge \dots \wedge e^{i_{p-1}} \\
 &= \frac{-1}{(p-1)!} \sum_{i, i_1, \dots, i_{p-1}} (R(X, e_i)\omega)(e_i, e_{i_1}, \dots, e_{i_{p-1}})e^{i_1} \wedge \dots \wedge e^{i_{p-1}} \\
 &\quad - \frac{1}{(p-1)!} \sum_{i, i_1, \dots, i_{p-1}} \sum_{l=1}^{p-1} \omega(e_i, e_{i_1}, \dots, R(X, e_i)e_{i_l}, \dots, e_{i_{p-1}})e^{i_1} \wedge \dots \wedge e^{i_{p-1}} \\
 &= - \sum_{i=1}^n \iota(e_i)(R(X, e_i)\omega) \\
 &\quad - \frac{1}{(p-1)!} \sum_{i, i_1, \dots, i_{p-1}} \sum_{l=1}^{p-1} \omega(e_i, e_{i_1}, \dots, R(X, e_i)e_{i_l}, \dots, e_{i_{p-1}})e^{i_1} \wedge \dots \wedge e^{i_{p-1}}
 \end{aligned} \tag{5}$$

We calculate the second term.

$$\begin{aligned}
 & - \frac{1}{(p-1)!} \sum_{i, i_1, \dots, i_{p-1}} \sum_{l=1}^{p-1} \omega(e_i, e_{i_1}, \dots, R(X, e_i)e_{i_l}, \dots, e_{i_{p-1}})e^{i_1} \wedge \dots \wedge e^{i_{p-1}} \\
 &= \frac{1}{(p-1)!} \sum_{l=1}^{p-1} \sum_{i, j, i_1, \dots, i_{p-1}} \langle R(e_j, e_{i_l})X, e_i \rangle \omega(e_i, e_j, e_{i_1}, \dots, \widehat{e_{i_l}}, \dots, e_{i_{p-1}}) \\
 &\quad e^{i_l} \wedge e^{i_1} \wedge \dots \wedge \widehat{e^{i_l}} \wedge \dots \wedge e^{i_{p-1}} \\
 &= \sum_{j, k} e^k \wedge \iota(e_j)\iota(R(e_j, e_k)X)\omega \\
 &= \sum_{j, k} e^k \wedge \iota(e_j)R(e_j, e_k)(\iota(X)\omega) - \sum_{j, k} e^k \wedge \iota(e_j)\iota(X)R(e_j, e_k)\omega \\
 &= \mathcal{R}_{p-1}(\iota(X)\omega) - \iota(X)\mathcal{R}_p\omega - \sum_{j=1}^n \iota(e_j)(R(X, e_j)\omega)
 \end{aligned}$$

Combining this and (5), we get (i).

Let us show (ii). We have

$$\nabla^* \nabla \iota(X)\omega = \iota(\nabla^* \nabla X)\omega - 2 \sum_i \iota(\nabla_{e_i} X) \nabla_{e_i} \omega + \iota(X) \nabla^* \nabla \omega.$$

Thus, by (i), we get

$$\begin{aligned}\Delta(\iota(X)\omega) &= \nabla^* \nabla \iota(X)\omega + \mathcal{R}_{p-1} \iota(X)\omega \\ &= \iota(\Delta X)\omega + \iota(X)\Delta\omega + 2 \sum_{i=1}^n \iota(e_i)(R(X, e_i)\omega) - 2 \sum_{i=1}^n \iota(\nabla_{e_i} X)(\nabla_{e_i} \omega).\end{aligned}$$

This gives (ii).

Finally, we show (iii). We have

$$\begin{aligned}\sum_{i=1}^n \iota(e_i)(R(X, e_i)\omega) &= \sum_{i=1}^n \iota(e_i) \left(\nabla_X \nabla_{e_i} \omega - \nabla_{e_i} \nabla_X \omega - \nabla_{\nabla_X e_i} \omega + \nabla_{\nabla_{e_i} X} \omega \right) \\ &= -\nabla_X d^* \omega + d^* \nabla_X \omega + \sum_{i,j=1}^n \langle \nabla_{e_j} X, e_i \rangle \iota(e_j) \nabla_{e_i} \omega.\end{aligned}$$

This gives (iii). □

When ω is parallel, we have the following corollary.

Corollary 2.9 *Let (M, g) be an n -dimensional Riemannian manifold. Take a vector field $X \in \Gamma(TM)$ and a parallel p -form $\omega \in \Gamma(\bigwedge^p T^*M)$ ($p \geq 1$).*

(i) *We have*

$$\mathcal{R}_{p-1}(\iota(X)\omega) = \iota(\text{Ric}(X))\omega.$$

(ii) *We have*

$$\Delta(\iota(X)\omega) = \iota(\Delta X)\omega.$$

Finally, we give some easy equations for later use. Let (M, g) be an n -dimensional Riemannian manifold. Take a local orthonormal basis $\{e_1, \dots, e_n\}$ of TM . Let $\{e^1, \dots, e^n\}$ be its dual. For any $\omega, \eta \in \Gamma(\bigwedge^k T^*M)$, we have

$$\sum_{i=1}^n \langle e^i \wedge \omega, e^i \wedge \eta \rangle = (n-k) \langle \omega, \eta \rangle, \quad \sum_{i=1}^n \langle \iota(e_i)\omega, \iota(e_i)\eta \rangle = k \langle \omega, \eta \rangle.$$

For any $\alpha_1, \dots, \alpha_k \in \Gamma(T^*M)$, we have

$$\mathcal{Q}_1(\alpha_1 \wedge \dots \wedge \alpha_k) = \left(\frac{1}{k}\right)^{1/2} \sum_{i=1}^k (-1)^{i-1} \alpha_i \otimes \alpha_1 \wedge \dots \wedge \widehat{\alpha_i} \wedge \dots \wedge \alpha_k.$$

Since Q_1 preserves the norms, we have

$$\begin{aligned} & k |\alpha_1 \wedge \cdots \wedge \alpha_k|^2 \\ &= \left| \sum_{i=1}^k (-1)^{i-1} \alpha_i \otimes \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_k \right|^2 \end{aligned} \quad (6)$$

for any $\alpha_1, \dots, \alpha_k \in \Gamma(T^*M)$.

Suppose that M is oriented. For any k , the Hodge star operator $*$: $\bigwedge^k T^*M \rightarrow \bigwedge^{n-k} T^*M$ is defined so that

$$\langle *\omega, \eta \rangle V_g = \omega \wedge \eta$$

for all $\omega \in \Gamma(\bigwedge^k T^*M)$ and $\eta \in \Gamma(\bigwedge^{n-k} T^*M)$, where V_g denotes the volume form on (M, g) . For any $\alpha \in \Gamma(T^*M)$, $\omega \in \Gamma(\bigwedge^k T^*M)$ and $\eta \in \Gamma(\bigwedge^{k-1} T^*M)$, we have

$$\begin{aligned} \langle *(\omega \wedge \alpha), \eta \rangle V_g &= \omega \wedge \alpha \wedge \eta, \\ \langle \iota(\alpha) * \omega, \eta \rangle V_g &= \langle *\omega, \alpha \wedge \eta \rangle V_g = \omega \wedge \alpha \wedge \eta. \end{aligned}$$

Thus, we get

$$*(\omega \wedge \alpha) = \iota(\alpha) * \omega. \quad (7)$$

Therefore, for any $\alpha, \beta \in \Gamma(T^*M)$ and $\omega, \eta \in \Gamma(\bigwedge^k T^*M)$, we have

$$\begin{aligned} \langle \iota(\alpha)\omega, \iota(\beta)\eta \rangle &= \langle \omega, \alpha \wedge \iota(\beta)\eta \rangle \\ &= -\langle \beta \wedge \omega, \alpha \wedge \eta \rangle + \langle \alpha, \beta \rangle \langle \omega, \eta \rangle = -\langle \iota(\beta) * \omega, \iota(\alpha) * \eta \rangle + \langle \alpha, \beta \rangle \langle \omega, \eta \rangle, \end{aligned}$$

and so

$$\langle \iota(\alpha)\omega, \iota(\beta)\eta \rangle + \langle \iota(\beta) * \omega, \iota(\alpha) * \eta \rangle = \langle \alpha, \beta \rangle \langle \omega, \eta \rangle. \quad (8)$$

3 Almost Parallel p -form

In this section, we show Main Theorems 1 and 3.

3.1 Parallel p -form

In this subsection, we show some easy results when the Riemannian manifold has a non-trivial parallel p -form. We first give an easy proof of what Grosjean called a new Bochner-Reilly formula [12, Proposition 3.1] for closed Riemannian manifolds with a non-trivial parallel p -form ω . Similarly, we also get the formula [12, Proposition 3.1] for Riemannian manifold with boundary. In the next subsection, we estimate the error terms when ω is not parallel.

Proposition 3.1 (Bochner-Reilly-Grosjean formula [12]) *Let (M, g) be an n -dimensional closed Riemannian manifold. For any $f \in C^\infty(M)$ and any parallel p -form ω ($1 \leq p \leq n-1$) on M , we have*

$$\begin{aligned} & \int_M |T(\iota(\nabla f)\omega)|^2 d\mu_g \\ &= \frac{p-1}{p} \int_M \langle \iota(\nabla f)\omega, \iota(\nabla \Delta f)\omega \rangle d\mu_g - \int_M \langle \iota(\text{Ric}(\nabla f))\omega, \iota(\nabla f)\omega \rangle d\mu_g. \end{aligned}$$

See Sect. 2.2 for the definition of $T: \Gamma(\bigwedge^{p-1} T^*M) \rightarrow \Gamma(T^{p-1,1}M)$.

Proof Since $d^*\iota(\nabla f)\omega = -d^*d^*(f\omega) = 0$, we have

$$\begin{aligned} & \int_M \langle \iota(\text{Ric}(\nabla f))\omega, \iota(\nabla f)\omega \rangle d\mu_g \\ &= \int_M \langle \mathcal{R}_{p-1}(\iota(\nabla f)\omega), \iota(\nabla f)\omega \rangle d\mu_g \quad (9) \\ &= \int_M \langle d(\iota(\nabla f)\omega), d(\iota(\nabla f)\omega) \rangle d\mu_g - \int_M \langle \nabla(\iota(\nabla f)\omega), \nabla(\iota(\nabla f)\omega) \rangle d\mu_g \end{aligned}$$

by Corollary 2.9 (i), Bochner-Weitzenböck formula and the divergence theorem. By (4) and Corollary 2.9 (ii), we have

$$\begin{aligned} & \int_M \langle d(\iota(\nabla f)\omega), d(\iota(\nabla f)\omega) \rangle d\mu_g - \int_M \langle \nabla(\iota(\nabla f)\omega), \nabla(\iota(\nabla f)\omega) \rangle d\mu_g \quad (10) \\ &= \frac{p-1}{p} \int_M \langle \iota(\nabla \Delta f)\omega, \iota(\nabla f)\omega \rangle d\mu_g - \int_M |T(\iota(\nabla f)\omega)|^2 d\mu_g \end{aligned}$$

By (9) and (10), we get the proposition. $\square$

Based on Proposition 3.1, Grosjean showed Theorem 1.3. Assuming more strong condition on eigenvalues, we remove the assumption that the manifold is simply connected from Theorem 1.3.

Corollary 3.2 *Let (M, g) be an n -dimensional closed Riemannian manifold. Assume that $\text{Ric} \geq (n-p-1)g$ and there exists a non-trivial parallel p -form on M ($2 \leq p < n/2$). If $\lambda_{n-p+1}(g) = n-p$, then (M, g) is isometric to a product $S^{n-p} \times (X, g')$, where (X, g') is some p -dimensional closed Riemannian manifold.*

Proof Let f_k be the k -th eigenfunction of the Laplacian on S^{n-p} . Note that the functions $f_1, \dots, f_{n-p+1}$ are height functions.

By Theorem 1.3, the universal cover $(\tilde{M}, \tilde{g})$ of (M, g) is isometric to a product $S^{n-p} \times (X, g')$, where (X, g') is some p -dimensional closed Riemannian manifold. We regard the function f_i as a function on $\tilde{M}$. Since $\lambda_{n-p+1}(g) = n-p$, each $f_i \in C^\infty(\tilde{M})$ ($i = 1, \dots, n-p+1$) is a pull back of some function on M . Thus, the covering transformation preserves $f_1, \dots, f_{n-p+1}$. Therefore, the covering transformation does not act on S^{n-p} , and so we get the corollary. $\square$

The almost version of this corollary is Main Theorem 2.

Finally, we show that the assumption of Corollary 3.2 is optimal in some sense by giving an example.

Take a positive odd integer p with $p \geq 3$ and a positive integer n with $n > 2p$. Put $a := \sqrt{(p-1)/(n-p-1)}$. We define an equivalence relation $\sim$ on $S^{n-p} \times S^p(a)$ as follows:

$$\begin{aligned} &((x_0, \dots, x_{n-p}), (y_0, \dots, y_p)) \sim ((x'_0, \dots, x'_{n-p}), (y'_0, \dots, y'_p)) \\ \Leftrightarrow &\text{there exists } k \in \mathbb{Z} \text{ such that} \\ &((x'_0, \dots, x'_{n-p}), (y'_0, \dots, y'_p)) = ((-1)^k x_0, x_1, \dots, x_{n-p}), (-1)^k (y_0, \dots, y_p) \end{aligned}$$

for any $((x_0, \dots, x_{n-p}), (y_0, \dots, y_p)), ((x'_0, \dots, x'_{n-p}), (y'_0, \dots, y'_p)) \in S^{n-p} \times S^p(a)$. Then, we have the following:

Proposition 3.3 *We have the following properties:*

- $(M, g) = (S^{n-p} \times S^p(a))/\sim$ is an n -dimensional closed Riemannian manifold with a non-trivial parallel p -form.
- $\text{Ric} = (n-p-1)g$.
- $\lambda_{n-p}(g) = n-p$.
- (M, g) is not isometric to any product Riemannian manifolds.

Proof Let ω be the volume form on $S^p(a)$. Since the action on $S^{n-p} \times S^p(a)$ preserves ω , there exists a non-trivial parallel p -form on (M, g) . We also denote it by ω . Since the action on $S^{n-p} \times S^p(a)$ preserves the function

$$x_i: S^{n-p} \times S^p(a) \rightarrow \mathbb{R}, ((x_0, \dots, x_{n-p}), (y_0, \dots, y_p)) \mapsto x_i$$

for each $i = 1, \dots, n-p$, we have $\lambda_{n-p}(g) = n-p$.

Suppose that (M, g) is isometric to a product $(M_1^{n-k}, g_1) \times (M_2^k, g_2)$ ($k \leq n-k$) for some $(n-k)$ and k -dimensional closed Riemannian manifolds (M_1, g_1) and (M_2, g_2) . Since we have the irreducible decomposition $T_{[(x,y)]}M \cong T_x S^{n-p} \oplus T_y S^p(a)$ of the restricted holonomy action, we get $k = p$. Since $\lambda_1(g) = n-p$, we have that (M_1, g_1) is isometric to S^{n-p} . Thus, we get $\lambda_{n-p+1}(g) = n-p$. However the action on $S^{n-p} \times S^p(a)$ does not preserve the function

$$x_0: S^{n-p} \times S^p(a) \rightarrow \mathbb{R}, ((x_0, \dots, x_{n-p}), (y_0, \dots, y_p)) \mapsto x_0,$$

and so $\lambda_{n-p+1}(g) \neq n-p$. This is a contradiction. $\square$

3.2 Error Estimates

In this subsection, we give error estimates about Proposition 3.1. Lemma 3.8 (vii) corresponds to Proposition 3.1.

We list the assumptions of this subsection.

Assumption 3.4 In this subsection, we assume the following:

- (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq -Kg$ and $\text{diam}(M) \leq D$ for some positive real numbers $K > 0$ and $D > 0$.
- $1 \leq k \leq n - 1$.
- A k -form $\omega \in \Gamma(\wedge^k T^*M)$ satisfies $\|\omega\|_2 = 1$, $\|\omega\|_\infty \leq L_1$ and $\|\nabla\omega\|_2^2 \leq \lambda$ for some $L_1 > 0$ and $0 \leq \lambda \leq 1$.
- A function $f \in C^\infty(M)$ satisfies $\|f\|_\infty \leq L_2\|f\|_2$, $\|\nabla f\|_\infty \leq L_2\|f\|_2$ and $\|\Delta f\|_2 \leq L_2\|f\|_2$ for some $L_2 > 0$.

Note that we have

$$\|\nabla^2 f\|_2^2 = \|\Delta f\|_2^2 - \frac{1}{\text{Vol}(M)} \int_M \text{Ric}(\nabla f, \nabla f) d\mu_g \leq (1 + K)L_2^2\|f\|_2^2 \quad (11)$$

by the Bochner formula.

We first show the following:

Lemma 3.5 *There exists a positive constant $C(n, K, D) > 0$ such that $\||\omega| - 1\|_2 \leq C\lambda^{1/2}$ holds.*

Proof Put $\bar{\omega} := \int_M |\omega| d\mu_g / \text{Vol}(M)$. Since we have $|\omega| \in W^{1,2}(M)$, we get

$$\||\omega| - \bar{\omega}\|_2^2 \leq \frac{1}{\lambda_1(g)} \|\nabla|\omega|\|_2^2 \leq \frac{1}{\lambda_1(g)} \|\nabla\omega\|_2^2 \leq \frac{\lambda}{\lambda_1(g)} \leq C\lambda$$

by the Kato inequality and the Li-Yau estimate [22, p.116]. Therefore, we get

$$|1 - \bar{\omega}| = \||\omega\|_2 - \|\bar{\omega}\|_2| \leq \||\omega| - \bar{\omega}\|_2 \leq C\lambda^{1/2},$$

and so $\||\omega| - 1\|_2 \leq C\lambda^{1/2}$. $\square$

Let us give error estimates about Proposition 3.1.

Lemma 3.6 *There exists a positive constant $C = C(n, k, K, D, L_1, L_2) > 0$ such that the following properties hold:*

(i) *We have*

$$\frac{1}{\text{Vol}(M)} \int_M |d^*(\iota(\nabla f)\omega)|^2 d\mu_g \leq C\|f\|_2^2\lambda.$$

(ii) *We have*

$$\left| \frac{1}{\text{Vol}(M)} \int_M \left(\langle \iota(\text{Ric}(\nabla f))\omega, \iota(\nabla f)\omega \rangle - \langle \mathcal{R}_{k-1}(\iota(\nabla f)\omega), \iota(\nabla f)\omega \rangle \right) d\mu_g \right| \leq C\|f\|_2^2\lambda^{1/2}.$$

(iii) *We have*

$$\left| \frac{1}{\text{Vol}(M)} \int_M \left(\langle \Delta(\iota(\nabla f)\omega), \iota(\nabla f)\omega \rangle - \langle \iota(\nabla \Delta f)\omega, \iota(\nabla f)\omega \rangle \right) d\mu_g \right|$$

$$\leq C\|f\|_2^2\lambda^{1/2}.$$

(iv) We have

$$\frac{1}{\text{Vol}(M)} \int_M \left| \nabla(\iota(\nabla f)\omega) - \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f)\omega \right|^2 d\mu_g \leq C\|f\|_2^2\lambda.$$

(v) We have

$$\frac{1}{\text{Vol}(M)} \int_M \left| d(\iota(\nabla f)\omega) - \sum_{i=1}^n e^i \wedge \iota(\nabla_{e_i} \nabla f)\omega \right|^2 d\mu_g \leq C\|f\|_2^2\lambda.$$

(vi) We have

$$\frac{1}{\text{Vol}(M)} \int_M |\nabla(\iota(\nabla f)\omega)|^2 d\mu_g \leq C\|f\|_2^2.$$

(vii) We have

$$\begin{aligned} & \left| \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\text{Ric}(\nabla f))\omega, \iota(\nabla f)\omega \rangle d\mu_g \right. \\ & \quad \left. - \frac{k-1}{k} \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla \Delta f)\omega, \iota(\nabla f)\omega \rangle d\mu_g + \|T(\iota(\nabla f)\omega)\|_2^2 \right| \\ & \leq C\|f\|_2^2\lambda^{1/2}. \end{aligned}$$

(viii) If M is oriented and $1 \leq k \leq n/2$, then we have

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \text{Ric}(\nabla f, \nabla f)|\omega|^2 d\mu_g \\ & \leq \frac{n-k-1}{n-k} \frac{1}{\text{Vol}(M)} \int_M \langle \nabla \Delta f, \nabla f \rangle |\omega|^2 d\mu_g - \|T(\iota(\nabla f)\omega)\|_2^2 \\ & \quad - \|T(\iota(\nabla f) * \omega)\|_2^2 \\ & \quad - \left(\frac{n-k-1}{n-k} - \frac{k-1}{k} \right) \|d(\iota(\nabla f)\omega)\|_2^2 + C\|f\|_2^2\lambda^{1/2}. \end{aligned}$$

Although an orthonormal basis $\{e_1, \dots, e_n\}$ of TM is defined only locally, $\sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f)\omega$ and $\sum_{i=1}^n e^i \wedge \iota(\nabla_{e_i} \nabla f)\omega$ are well-defined as tensors.

Proof We first prove (i). Since $d^*(f\omega) = -\iota(\nabla f)\omega + f d^*\omega$ and $d^* \circ d^* = 0$, we have $d^*(\iota(\nabla f)\omega) = -\iota(\nabla f)d^*\omega$. Thus, we get

$$\frac{1}{\text{Vol}(M)} \int_M |d^*(\iota(\nabla f)\omega)|^2 d\mu_g \leq C\|\nabla f\|_\infty^2 \|\nabla \omega\|_2^2 \leq C\|f\|_2^2\lambda.$$

To prove (ii) and (iii), we estimate following terms:

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla f) \Delta \omega, \iota(\nabla f) \omega \rangle d\mu_g, \\ & \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla f) \nabla^* \nabla \omega, \iota(\nabla f) \omega \rangle d\mu_g, \\ & \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla f) \mathcal{R}_k \omega, \iota(\nabla f) \omega \rangle d\mu_g, \\ & \frac{1}{\text{Vol}(M)} \int_M \left\langle \sum_{i=1}^n \iota(\nabla_{e_i} \nabla f) (\nabla_{e_i} \omega), \iota(\nabla f) \omega \right\rangle d\mu_g, \\ & \frac{1}{\text{Vol}(M)} \int_M \left\langle \sum_{i=1}^n \iota(e_i) (R(\nabla f, e_i) \omega), \iota(\nabla f) \omega \right\rangle d\mu_g. \end{aligned}$$

We have

$$\begin{aligned} & \int_M \langle \iota(\nabla f) \Delta \omega, \iota(\nabla f) \omega \rangle d\mu_g \\ &= \int_M \langle d\omega, d(df \wedge \iota(\nabla f) \omega) \rangle d\mu_g + \int_M \langle d^* \omega, d^*(df \wedge \iota(\nabla f) \omega) \rangle d\mu_g \end{aligned}$$

and

$$\begin{aligned} & |\langle d\omega, d(df \wedge \iota(\nabla f) \omega) \rangle| \\ &= |\langle d\omega, \sum_{i=1}^n df \wedge e^i \wedge (\iota(\nabla_{e_i} \nabla f) \omega + \iota(\nabla f) \nabla_{e_i} \omega) \rangle| \\ &\leq C |\nabla \omega| |\nabla f| (|\nabla^2 f| |\omega| + |\nabla f| |\nabla \omega|), \\ & |\langle d^* \omega, d^*(df \wedge \iota(\nabla f) \omega) \rangle| \\ &= |\langle d^* \omega, \sum_{i=1}^n \iota(e_i) (\nabla_{e_i} df \wedge \iota(\nabla f) \omega + df \wedge \iota(\nabla_{e_i} \nabla f) \omega + df \wedge \iota(\nabla f) \nabla_{e_i} \omega) \rangle| \\ &\leq C |\nabla \omega| |\nabla f| (|\nabla^2 f| |\omega| + |\nabla f| |\nabla \omega|). \end{aligned}$$

Thus, we get

$$\left| \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla f) \Delta \omega, \iota(\nabla f) \omega \rangle d\mu_g \right| \leq C \|f\|_2^2 \lambda^{1/2}. \quad (12)$$

We have

$$\int_M \langle \iota(\nabla f) \nabla^* \nabla \omega, \iota(\nabla f) \omega \rangle d\mu_g = \int_M \langle \nabla \omega, \nabla(df \wedge \iota(\nabla f) \omega) \rangle d\mu_g$$

and $|\langle \nabla \omega, \nabla(df \wedge \iota(\nabla f)\omega) \rangle| \leq C|\nabla \omega||\nabla f|(|\nabla^2 f||\omega| + |\nabla f||\nabla \omega|)$. Thus, we get

$$\left| \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla f) \nabla^* \nabla \omega, \iota(\nabla f)\omega \rangle d\mu_g \right| \leq C\|f\|_2^2 \lambda^{1/2}. \quad (13)$$

By Theorem 2.6, (12) and (13), we have

$$\begin{aligned} & \left| \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla f) \mathcal{R}_k \omega, \iota(\nabla f)\omega \rangle d\mu_g \right| \\ & \leq \frac{1}{\text{Vol}(M)} \left(\left| \int_M \langle \iota(\nabla f) \Delta \omega, \iota(\nabla f)\omega \rangle d\mu_g \right| \right. \\ & \quad \left. + \left| \int_M \langle \iota(\nabla f) \nabla^* \nabla \omega, \iota(\nabla f)\omega \rangle d\mu_g \right| \right) \\ & \leq C\|f\|_2^2 \lambda^{1/2}. \end{aligned} \quad (14)$$

Since $|\langle \sum_{i=1}^n \iota(\nabla_{e_i} \nabla f)(\nabla_{e_i} \omega), \iota(\nabla f)\omega \rangle| \leq C|\omega||\nabla f||\nabla \omega||\nabla^2 f|$, we have

$$\left| \frac{1}{\text{Vol}(M)} \int_M \left\langle \sum_{i=1}^n \iota(\nabla_{e_i} \nabla f)(\nabla_{e_i} \omega), \iota(\nabla f)\omega \right\rangle d\mu_g \right| \leq C\|f\|_2^2 \lambda^{1/2}. \quad (15)$$

Let us estimate

$$\frac{1}{\text{Vol}(M)} \int_M \left\langle \sum_{i=1}^n \iota(e_i)(R(\nabla f, e_i)\omega), \iota(\nabla f)\omega \right\rangle d\mu_g.$$

We have

$$\begin{aligned} & \left| \frac{1}{\text{Vol}(M)} \int_M \langle \nabla_{\nabla f} d^* \omega, \iota(\nabla f)\omega \rangle d\mu_g \right| \\ & = \left| \frac{1}{\text{Vol}(M)} \int_M \langle d^* \omega, \nabla^*(df \otimes \iota(\nabla f)\omega) \rangle d\mu_g \right| \\ & \leq C\|f\|_2^2 \lambda^{1/2}, \\ & \left| \frac{1}{\text{Vol}(M)} \int_M \langle d^* \nabla_{\nabla f} \omega, \iota(\nabla f)\omega \rangle d\mu_g \right| \\ & = \left| \frac{1}{\text{Vol}(M)} \int_M \langle \nabla \omega, df \otimes d(\iota(\nabla f)\omega) \rangle d\mu_g \right| \\ & \leq C\|f\|_2^2 \lambda^{1/2} \end{aligned}$$

and

$$\left| \frac{1}{\text{Vol}(M)} \int_M \left\langle \sum_{i,j=1}^n \langle \nabla_{e_j} \nabla f, e_i \rangle \iota(e_j) \nabla_{e_i} \omega, \iota(\nabla f)\omega \right\rangle d\mu_g \right|$$

$$\leq C \|f\|_2^2 \lambda^{1/2}.$$

Thus, by Lemma 2.8 (iii), we get

$$\left| \frac{1}{\text{Vol}(M)} \int_M \left\langle \sum_{i=1}^n \iota(e_i)(R(\nabla f, e_i)\omega), \iota(\nabla f)\omega \right\rangle d\mu_g \right| \leq C \|f\|_2^2 \lambda^{1/2}. \quad (16)$$

By (12), (14), (15), (16) and Lemma 2.8, we get (ii) and (iii).

Since $\nabla(\iota(\nabla f)\omega) - \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f)\omega = \sum_{i=1}^n e^i \otimes \iota(\nabla f)\nabla_{e_i}\omega$, we get (iv) and (vi).

Since $d(\iota(\nabla f)\omega) - \sum_{i=1}^n e^i \wedge \iota(\nabla_{e_i} \nabla f)\omega = \sum_{i=1}^n e^i \wedge \iota(\nabla f)\nabla_{e_i}\omega$, we get (v).

By Theorem 2.6 and (4), we have

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \langle \mathcal{R}_{k-1}(\iota(\nabla f)\omega), \iota(\nabla f)\omega \rangle d\mu_g \\ &= \frac{1}{\text{Vol}(M)} \int_M \langle (\Delta - \nabla^* \nabla)(\iota(\nabla f)\omega), \iota(\nabla f)\omega \rangle d\mu_g \\ &= \frac{k-1}{k} \frac{1}{\text{Vol}(M)} \int_M \langle \Delta(\iota(\nabla f)\omega), \iota(\nabla f)\omega \rangle d\mu_g \\ & \quad + \left(\frac{n-k+1}{n-k+2} - \frac{k-1}{k} \right) \|d^*(\iota(\nabla f)\omega)\|_2^2 - \|T(\iota(\nabla f)\omega)\|_2^2. \end{aligned}$$

Thus, by (i), (ii) and (iii), we get (vii)

Finally, we prove (viii). Suppose that M is oriented and $1 \leq k \leq n/2$. Since $\nabla(*\omega) = *\nabla\omega$, we have

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\text{Ric}(\nabla f)) * \omega, \iota(\nabla f) * \omega \rangle d\mu_g \\ & \leq \frac{n-k-1}{n-k} \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla \Delta f) * \omega, \iota(\nabla f) * \omega \rangle d\mu_g - \|T(\iota(\nabla f) * \omega)\|_2^2 \\ & \quad + C \|f\|_2^2 \lambda^{1/2} \end{aligned}$$

by (vii). Thus, by (8), (i), (iii) and (vii), we get

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \text{Ric}(\nabla f, \nabla f) |\omega|^2 d\mu_g \\ & \leq \frac{n-k-1}{n-k} \frac{1}{\text{Vol}(M)} \int_M \langle \nabla \Delta f, \nabla f \rangle |\omega|^2 d\mu_g - \|T(\iota(\nabla f)\omega)\|_2^2 \\ & \quad - \|T(\iota(\nabla f) * \omega)\|_2^2 \\ & \quad - \left(\frac{n-k-1}{n-k} - \frac{k-1}{k} \right) \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla \Delta f)\omega, \iota(\nabla f)\omega \rangle d\mu_g + C \|f\|_2^2 \lambda^{1/2} \\ & \leq \frac{n-k-1}{n-k} \frac{1}{\text{Vol}(M)} \int_M \langle \nabla \Delta f, \nabla f \rangle |\omega|^2 d\mu_g - \|T(\iota(\nabla f)\omega)\|_2^2 \end{aligned}$$

$$\begin{aligned}
& - \|T(\iota(\nabla f) * \omega)\|_2^2 \\
& - \left(\frac{n-k-1}{n-k} - \frac{k-1}{k} \right) \|d(\iota(\nabla f)\omega)\|_2^2 + C\|f\|_2^2 \lambda^{1/2}.
\end{aligned}$$

This gives (viii). $\square$

3.3 Eigenvalue Estimate

In this subsection, we complete the proofs of Main Theorems 1 and 3. Recall that $\lambda_1(\Delta_{C,p})$ denotes the first eigenvalue of the connection Laplacian $\Delta_{C,p}$ acting on p -forms:

$$\Delta_{C,p} := \nabla^* \nabla : \Gamma\left(\bigwedge^p T^*M\right) \rightarrow \Gamma\left(\bigwedge^p T^*M\right).$$

It is enough to show Main Theorem 1 when $\lambda_1(\Delta_{C,p}) \leq 1$. Note that we always have $\lambda_1(\Delta_{C,1}) \geq 1$ if $\text{Ric}_g \geq (n-1)g$.

We need the following L^∞ estimates.

Lemma 3.7 *Take an integer $n \geq 2$ and positive real numbers $K > 0$, $D > 0$, $\Lambda > 0$. Let (M, g) be an n -dimensional closed Riemannian manifold with $\text{Ric} \geq -Kg$ and $\text{diam}(M) \leq D$. Then, we have the following:*

- (i) *For any function $f \in C^\infty(M)$ and any $\lambda \geq 0$ with $\Delta f = \lambda f$ and $\lambda \leq \Lambda$, then we have $\|\nabla f\|_\infty \leq C(n, K, D, \Lambda)\|f\|_2$ and $\|f\|_\infty \leq C(n, K, D, \Lambda)\|f\|_2$.*
- (ii) *For any p -form $\omega \in \Gamma\left(\bigwedge^p T^*M\right)$ and any $\lambda \geq 0$ with $\Delta_{C,p}\omega = \lambda\omega$ and $\lambda \leq \Lambda$, then we have $\|\omega\|_\infty \leq C(n, K, D, \Lambda)\|\omega\|_2$.*

Proof By the gradient estimate for eigenfunctions [19, Theorem 7.3], we get (i).

Let us show (ii). Since we have

$$\Delta|\omega|^2 = 2\langle \Delta_{C,p}\omega, \omega \rangle - 2|\nabla\omega|^2 \leq 2\Lambda|\omega|^2,$$

we get $\|\omega\|_\infty \leq C$ by [20, Proposition 9.2.7] (see also Propositions 7.1.13 and 7.1.17 in [20]). Note that our sign convention of the Laplacian is different from [20]. $\square$

We use the following proposition not only for the proofs of Main Theorems 1 and 3 but also for other main theorems.

Proposition 3.8 *For given integers $n \geq 4$ and $2 \leq p \leq n/2$, there exists a constant $C(n, p) > 0$ such that the following property holds. Let (M, g) be an n -dimensional closed oriented Riemannian manifold with $\text{Ric}_g \geq (n-p-1)g$. Suppose that an integer $i \in \mathbb{Z}_{>0}$ satisfies $\lambda_i(g) \leq n-p+1$, and there exists an eigenform ω of the connection Laplacian $\Delta_{C,p}$ acting on p -forms with $\|\omega\|_2 = 1$ corresponding to the eigenvalue λ with $0 \leq \lambda \leq 1$. Then, we have*

$$\frac{n-p-1}{n-p} \lambda_i(g) (\lambda_i(g) - (n-p)) \|f_i\|^2$$

$$\begin{aligned} &\geq \|T(\iota(\nabla f_i)\omega)\|_2^2 + \|T(\iota(\nabla f_i) * \omega)\|_2^2 \\ &\quad + \left(\frac{n-p-1}{n-p} - \frac{p-1}{p} \right) \|d(\iota(\nabla f_i)\omega)\|_2^2 - C\lambda^{1/2} \|f_i\|_2^2, \end{aligned}$$

where f_i denotes the i -th eigenfunction of the Laplacian acting on functions.

Proof By Lemma 3.6 (viii), we have

$$\begin{aligned} &\frac{n-p-1}{\text{Vol}(M)} \int_M \langle \nabla f_i, \nabla f_i \rangle |\omega|^2 d\mu_g \\ &\leq \frac{1}{\text{Vol}(M)} \int_M \text{Ric}(\nabla f_i, \nabla f_i) |\omega|^2 d\mu_g \\ &\leq \frac{n-p-1}{n-p} \frac{\lambda_i(g)}{\text{Vol}(M)} \int_M \langle \nabla f_i, \nabla f_i \rangle |\omega|^2 d\mu_g - \|T(\iota(\nabla f_i)\omega)\|_2^2 \\ &\quad - \|T(\iota(\nabla f_i) * \omega)\|_2^2 \\ &\quad - \left(\frac{n-p-1}{n-p} - \frac{p-1}{p} \right) \|d(\iota(\nabla f_i)\omega)\|_2^2 + C\lambda^{1/2} \|f_i\|_2^2. \end{aligned}$$

Thus, we get the proposition by Lemma 3.5. $\square$

Proof of Main Theorem 1 If M is orientable, we get the theorem immediately by Proposition 3.8. If M is not orientable, we get the theorem by considering the two-sheeted orientable Riemannian covering $\pi: (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ because we have $\lambda_1(g) \geq \lambda_1(\tilde{g})$ and $\lambda_1(\Delta_{C,p}, g) \geq \lambda_1(\Delta_{C,p}, \tilde{g})$. $\square$

Similarly, we get Main Theorem 3 because $\lambda_1(\Delta_{C,p}, g) = \lambda_1(\Delta_{C,n-p}, g)$ holds if the manifold is orientable.

4 Pinching

In this section, we show the remaining main theorems. Main Theorem 2 is proved in Sect. 4.5 except for the orientability, and the orientability is proved in Sect. 4.7. Main Theorem 4 is proved in Sect. 4.8.

We list assumptions of this section.

Assumption 4.1 Throughout in this section, we assume the following:

- $n \geq 5$, $2 \leq p < n/2$ and $1 \leq k \leq n-p+1$.
- (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric}_g \geq (n-p-1)g$.
- $C = C(n, p) > 0$ denotes a positive constant depending only on n and p .
- $\delta > 0$ satisfies $\delta \leq \delta_0$ for sufficiently small $\delta_0 = \delta_0(n, p) > 0$.
- $f_i \in C^\infty(M)$ ($i \in \{1, \dots, k\}$) is an eigenfunction of the Laplacian acting on functions with $\|f_i\|_2^2 = 1/(n-p+1)$ corresponding to the eigenvalue λ_i with $0 < \lambda_i \leq n-p+\delta$ such that

$$\int_M f_i f_j d\mu_g = 0$$

holds for any $i \neq j$.

Note that, for given real numbers a, b with $0 < b < a$ and a positive constant $C > 0$, we can assume that $C\delta^a \leq \delta^b$. At the beginning of each subsections, we add either one of the following assumptions if necessary.

Assumption 4.2 There exists an eigenform $\omega \in \Gamma(\bigwedge^p T^*M)$ of the connection Laplacian $\Delta_{C,p}$ with $\|\omega\|_2 = 1$ corresponding to the eigenvalue λ with $0 \leq \lambda \leq \delta$.

Assumption 4.3 There exists an eigenform $\xi \in \Gamma(\bigwedge^{n-p} T^*M)$ of the connection Laplacian $\Delta_{C,n-p}$ with $\|\xi\|_2 = 1$ corresponding to the eigenvalue λ with $0 \leq \lambda \leq \delta$.

Under our assumptions, we have $\|\omega\|_\infty \leq C$, $\|\xi\|_\infty \leq C$, $\|f_i\|_\infty \leq C$ and $\|\nabla f_i\|_\infty \leq C$ for all i by Lemma 3.7. By Main Theorems 1 and 3, we have $\lambda_i \geq n - p - C(n, p)\delta^{1/2}$ for all i . Note that we do not assume that $\lambda_i = \lambda_i(g)$.

4.1 Useful Techniques

In this subsection, we list some useful techniques for our pinching problems. Although we suppose that Assumption 4.1 holds, most assertions hold under weaker assumptions.

The following lemma is a variation of the Cheng-Yau estimate. See [1, Lemma 2.10] for the proof (see also [6, Theorem 7.1]).

Lemma 4.4 Take a positive real number $0 < \epsilon_1 \leq 1$. For any function $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$ and any point $x \in M$, we have

$$|\nabla f|^2(x) \leq \frac{C}{\epsilon_1} (f(p) - f(x) + \epsilon_1 \|f\|_2)^2,$$

where $p \in M$ denotes a maximum point of f .

The following theorem is an easy consequence of the Bishop-Gromov inequality.

Theorem 4.5 For any $p \in M$ and $0 < r \leq \text{diam}(M) + 1$, we have $r^n \text{Vol}(M) \leq C \text{Vol}(B_r(p))$.

The following theorem is due to Cheeger-Colding [7] (see also [20, Theorem 7.1.10]). By this theorem, we get integral pinching conditions along the geodesics under the integral pinching condition for a function on M .

Theorem 4.6 (segment inequality) For any non-negative measurable function $h: M \rightarrow \mathbb{R}_{\geq 0}$, we have

$$\frac{1}{\text{Vol}(M)^2} \int_{M \times M} \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h \circ \gamma_{y_1, y_2}(s) ds dy_1 dy_2 \leq \frac{C}{\text{Vol}(M)} \int_M h d\mu_g.$$

Remark 4.1 The book [20] deals with the segment $c_{y_1, y_2} : [0, 1] \rightarrow M$ for each $y_1, y_2 \in M$, defined to be $c_{y_1, y_2}(0) = y_1$, $c_{x, y}(1) = y_2$ and $\nabla_{\partial/\partial t} \dot{c} = 0$. We have $c_{x, y}(t) = \gamma_{x, y}(td(x, y))$ for all $t \in [0, 1]$ and

$$d(y_1, y_2) \int_0^1 h \circ c_{y_1, y_2}(t) dt = \int_0^{d(y_1, y_2)} h \circ \gamma_{y_1, y_2}(s) ds.$$

After getting integral pinching conditions along the geodesics, we use the following lemma to get L^∞ error estimate along them. The proof is standard (c.f. [7, Lemma 2.41]).

Lemma 4.7 Take positive real numbers $l, \epsilon > 0$ and a non-negative real number $r \geq 0$. Suppose that a smooth function $u : [0, l] \rightarrow \mathbb{R}$ satisfies

$$\int_0^l |u''(t) + r^2 u(t)| dt \leq \epsilon.$$

Then, we have

$$\left| u(t) - u(0) \cos rt - \frac{u'(0)}{r} \sin rt \right| \leq \epsilon \frac{\sinh rt}{r},$$

$$|u'(t) + ru(0) \sin rt - u'(0) \cos rt| \leq \epsilon + \int_0^t \left| u(s) - u(0) \cos rs - \frac{u'(0)}{r} \sin rs \right| ds,$$

for all $t \in [0, l]$, where we defined $\frac{1}{r} \sin rt := t$, $\frac{1}{r} \sinh rt := t$ if $r = 0$.

The following lemma is standard.

Lemma 4.8 For all $t \in \mathbb{R}$, we have

$$1 - \frac{1}{2}t^2 \leq \cos t \leq 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4.$$

For any $t \in [-\pi, \pi]$, we have $\cos t \leq 1 - \frac{1}{9}t^2$, and so $|t| \leq 3(1 - \cos t)^{1/2}$. For any $t_1, t_2 \in [0, \pi]$, we have $|t_1 - t_2| \leq 3|\cos t_1 - \cos t_2|^{1/2}$.

Finally, we recall some facts about the geodesic flow. Let UM denotes the sphere bundle defined by

$$UM := \{u \in TM : |u| = 1\}.$$

There exists a natural Riemannian metric G on UM , which is the restriction of the Sasaki metric on TM (see [21, p.55]). The Riemannian volume measure μ_G satisfies

$$\int_{UM} F d\mu_G = \int_M \int_{U_p M} F(u) d\mu_0(u) d\mu_g(p)$$

for any $F \in C^\infty(UM)$, where μ_0 denotes the standard measure on $U_p M \cong S^{n-1}$. The geodesic flow $\phi_t: UM \rightarrow UM$ ($t \in \mathbb{R}$) is defined by

$$\phi_t(u) := \left. \frac{\partial}{\partial s} \right|_{s=t} \gamma_u(s) \in U_{\gamma_u(t)} M$$

for any $u \in UM$. Though ϕ_t does not preserve the metric G in general, it preserves the measure μ_G . This is an easy consequence of [21, Lemma 4.4], which asserts that the geodesic flow on TM preserve the natural symplectic structure on TM . We can easily show the following lemma.

Lemma 4.9 *For any $f \in C^\infty(M)$ and $l > 0$, we have*

$$\frac{1}{\text{Vol}(M)} \int_M f d\mu_g = \frac{1}{l \text{Vol}(UM)} \int_{UM} \int_0^l f \circ \gamma_u(t) dt d\mu_G(u).$$

This kind of lemma was used by Colding [10] to prove that the almost equality of the Bishop comparison theorem implies the Gromov-Hausdorff closeness to the standard sphere.

4.2 Estimates for the Segments

In this subsection, we suppose that Assumption 4.2 holds. The goal is to give error estimates along the geodesics. We first list some basic consequences of our pinching condition.

Lemma 4.10 *For any $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$, we have*

- (i) $\|\iota(\nabla f)\omega\|_2^2 \leq C\delta^{1/2}\|f\|_2^2$,
- (ii) $\|\nabla(\iota(\nabla f)\omega)\|_2^2 \leq C\delta^{1/2}\|f\|_2^2$,
- (iii) $\|(|\nabla^2 f|^2 - \frac{1}{n-p}|\Delta f|^2)|\omega|^2\|_1 \leq C\delta^{1/4}\|f\|_2^2$.

Proof It is enough to consider the case when M is orientable.

We first assume that $f = f_i$ for some $i = 1, \dots, k$. Then, we have

$$\begin{aligned} \|d(\iota(\nabla f)\omega)\|_2^2 &\leq C\delta^{1/2}\|f\|_2^2, \\ \|d^*(\iota(\nabla f)\omega)\|_2^2 &\leq C\delta^{1/2}\|f\|_2^2, \quad \|T(\iota(\nabla f)\omega)\|_2^2 \leq C\delta^{1/2}\|f\|_2^2, \\ \|d^*(\iota(\nabla f) * \omega)\|_2^2 &\leq C\delta^{1/2}\|f\|_2^2, \quad \|T(\iota(\nabla f) * \omega)\|_2^2 \leq C\delta^{1/2}\|f\|_2^2 \end{aligned} \quad (17)$$

by Lemma 3.6 (i) and Proposition 3.8. Thus, by (4), we get

$$\|\nabla(\iota(\nabla f)\omega)\|_2^2 \leq C\delta^{1/2}\|f\|_2^2 \quad (18)$$

and

$$\|\nabla(\iota(\nabla f) * \omega)\|_2^2 \leq \frac{1}{n-p} \|d(\iota(\nabla f) * \omega)\|_2^2 + C\delta^{1/2}\|f\|_2^2. \quad (19)$$

Moreover, by Lemma 3.6 (iii), we have

$$\begin{aligned}\|\iota(\nabla f)\omega\|_2^2 &= \frac{1}{\lambda_i} \frac{1}{\text{Vol}(M)} \int_M \langle \iota(\nabla \Delta f)\omega, \iota(\nabla f)\omega \rangle d\mu_g \\ &\leq C \|d(\iota(\nabla f)\omega)\|_2^2 + C \|d^*(\iota(\nabla f)\omega)\|_2^2 + C \delta^{1/2} \|f\|_2^2 \\ &\leq C \delta^{1/2} \|f\|_2^2.\end{aligned}\quad (20)$$

For any $f = a_1 f_1 + \dots + a_k f_k \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$, we have (17), (18), (19), (20). For example, we have

$$\|\nabla(\iota(\nabla f)\omega)\|_2 \leq \sum_{i=1}^k |a_k| \|\nabla(\iota(\nabla f_i)\omega)\|_2 \leq C \delta^{1/4} \sum_{i=1}^k |a_k| \|f_i\|_2 \leq C \delta^{1/4} \|f\|_2.$$

Thus, we get (i) and (ii) by (18) and (20).

Finally, we prove (iii). Take arbitrary $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$. We have

$$\begin{aligned}&\left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) * \omega \right|^2 \\ &= \sum_{i=1}^n \langle \iota(\nabla_{e_i} \nabla f) * \omega, \iota(\nabla_{e_i} \nabla f) * \omega \rangle = |\nabla^2 f|^2 |\omega|^2 - \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2.\end{aligned}\quad (21)$$

Thus, we have

$$\begin{aligned}&\frac{1}{\text{Vol}(M)} \int_M \left| |\nabla(\iota(\nabla f) * \omega)|^2 - |\nabla^2 f|^2 |\omega|^2 \right| d\mu_g \\ &\leq \frac{1}{\text{Vol}(M)} \int_M \left| |\nabla(\iota(\nabla f) * \omega)|^2 - \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) * \omega \right|^2 \right| d\mu_g \\ &\quad + \frac{1}{\text{Vol}(M)} \int_M \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 d\mu_g,\end{aligned}$$

and so we get

$$\frac{1}{\text{Vol}(M)} \int_M \left| |\nabla(\iota(\nabla f) * \omega)|^2 - |\nabla^2 f|^2 |\omega|^2 \right| d\mu_g \leq C \delta^{1/2} \|f\|_2^2 \quad (22)$$

by (ii) and Lemma 3.6 (iv) and (vi). We have

$$\begin{aligned}
 & \left| \sum_{i=1}^n e^i \wedge \iota(\nabla_{e_i} \nabla f) * \omega \right|^2 \\
 &= \sum_{i=1}^n |\iota(\nabla_{e_i} \nabla f) * \omega|^2 - \sum_{i,j=1}^n \langle \iota(e_i) \iota(\nabla_{e_j} \nabla f) * \omega, \iota(e_j) \iota(\nabla_{e_i} \nabla f) * \omega \rangle \\
 &= |\nabla^2 f|^2 |\omega|^2 - \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 \\
 &\quad - \sum_{i,j,k,l=1}^n \nabla^2 f(e_i, e_k) \nabla^2 f(e_j, e_l) \langle e^i \wedge e^l \wedge \omega, e^j \wedge e^k \wedge \omega \rangle
 \end{aligned} \tag{23}$$

by (21) and (7). Since

$$\begin{aligned}
 \langle e^i \wedge e^l \wedge \omega, e^j \wedge e^k \wedge \omega \rangle &= (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) |\omega|^2 - \delta_{ij} \langle \iota(e_k) \omega, \iota(e_l) \omega \rangle \\
 &\quad + \delta_{ik} \langle \iota(e_j) \omega, \iota(e_l) \omega \rangle + \langle e^l \wedge \omega, e^j \wedge e^k \wedge \iota(e_i) \omega \rangle,
 \end{aligned}$$

we have

$$\begin{aligned}
 & \sum_{i,j,k,l=1}^n \nabla^2 f(e_i, e_k) \nabla^2 f(e_j, e_l) \langle e^i \wedge e^l \wedge \omega, e^j \wedge e^k \wedge \omega \rangle \\
 &= |\nabla^2 f|^2 |\omega|^2 - (\Delta f)^2 |\omega|^2 - \sum_{i=1}^n |\iota(\nabla_{e_i} \nabla f) \omega|^2 \\
 &\quad - \sum_{i=1}^n \Delta f \langle \iota(\nabla_{e_i} \nabla f) \omega, \iota(e_i) \omega \rangle \\
 &\quad + \sum_{j,k,l=1}^n \nabla^2 f(e_j, e_l) \langle e^l \wedge \omega, e^j \wedge e^k \wedge \iota(\nabla_{e_k} \nabla f) \omega \rangle.
 \end{aligned} \tag{24}$$

By (23) and (24), we get

$$\begin{aligned}
 \left| \sum_{i=1}^n e^i \wedge \iota(\nabla_{e_i} \nabla f) * \omega \right|^2 &= (\Delta f)^2 |\omega|^2 + \sum_{i=1}^n \Delta f \langle \iota(\nabla_{e_i} \nabla f) \omega, \iota(e_i) \omega \rangle \\
 &\quad - \sum_{j,k,l=1}^n \nabla^2 f(e_j, e_l) \langle e^l \wedge \omega, e^j \wedge e^k \wedge \iota(\nabla_{e_k} \nabla f) \omega \rangle,
 \end{aligned}$$

and so

$$\left| \left| \sum_{i=1}^n e^i \wedge \iota(\nabla_{e_i} \nabla f) * \omega \right|^2 - (\Delta f)^2 |\omega|^2 \right| \leq C |\nabla^2 f| |\omega| \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right| \quad (25)$$

By (25), (ii) and Lemma 3.6, we get

$$\frac{1}{\text{Vol}(M)} \int_M \left| |d(\iota(\nabla f) * \omega)|^2 - (\Delta f)^2 |\omega|^2 \right| d\mu_g \leq C \delta^{1/4} \|f\|_2^2. \quad (26)$$

Since we have $|\nabla(\iota(\nabla f) * \omega)|^2 \geq |d(\iota(\nabla f) * \omega)|^2 / (n - p)$ at each point by (4), we get (iii) by (19), (22) and (26). $\square$

We use the following notation.

Notation 4.11 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$ with $\|f\|_2^2 = 1/(n - p + 1)$ and put

$$\begin{aligned} h_0 &:= |\nabla^2 f|^2, \quad h_1 := ||\omega|^2 - 1|, \quad h_2 := |\nabla \omega|^2, \\ h_3 &:= |\iota(\nabla f) \omega|^2, \quad h_4 := |\nabla(\iota(\nabla f) \omega)|^2, \quad h_5 := \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 \\ h_6 &:= \left| |\nabla^2 f|^2 - \frac{1}{n - p} (\Delta f)^2 \right| |\omega|^2. \end{aligned}$$

For each $y_1 \in M$, we define

$$\begin{aligned} D_f(y_1) &:= \left\{ y_2 \in I_{y_1} \setminus \{y_1\} : \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h_0 \circ \gamma_{y_1, y_2}(s) ds \leq \delta^{-1/50} \text{ and} \right. \\ &\quad \left. \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h_i \circ \gamma_{y_1, y_2}(s) ds \leq \delta^{1/5} \text{ for all } i = 1, \dots, 6 \right\}, \\ Q_f &:= \{y_1 \in M : \text{Vol}(M \setminus D_f(y_1)) \leq \delta^{1/100} \text{Vol}(M)\}, \\ E_f(y_1) &:= \left\{ u \in U_{y_1} M : \frac{1}{\pi} \int_0^\pi h_0 \circ \gamma_u(s) ds \leq \delta^{-1/50} \text{ and} \right. \\ &\quad \left. \frac{1}{\pi} \int_0^\pi h_i \circ \gamma_u(s) ds \leq \delta^{1/5} \right. \\ &\quad \left. \text{for all } i = 1, \dots, 6 \right\}, \\ R_f &:= \{y_1 \in M : \text{Vol}(U_{y_1} M \setminus E_f(y_1)) \leq \delta^{1/100} \text{Vol}(U_{y_1} M)\}. \end{aligned}$$

Now, we use the segment inequality and Lemma 4.9. We show that we have the integral pinching condition along most geodesics.

Lemma 4.12 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$ with $\|f\|_2^2 = 1/(n - p + 1)$. Then, we have the following properties:

- (i) $\text{Vol}(M \setminus Q_f) \leq C\delta^{1/100}\text{Vol}(M)$.
(ii) $\text{Vol}(M \setminus R_f) \leq C\delta^{1/100}\text{Vol}(M)$.

Proof We have $\|h_i\|_1 \leq C\delta^{1/4}$ for all $i = 1, \dots, 6$ by the assumption, Lemmas 3.5, 3.6 (iv) and 4.10, and we have $\|h_0\|_1 \leq C$ by (11).

For any $y_1 \in M \setminus Q_f$, we have $\text{Vol}(M \setminus D_f(y_1)) > \delta^{1/100}\text{Vol}(M)$, and so we have either

$$\frac{1}{\text{Vol}(M)} \int_M \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h_0 \circ \gamma_{y_1, y_2}(s) ds dy_2 \geq \frac{1}{7}\delta^{-1/100}$$

or

$$\frac{1}{\text{Vol}(M)} \int_M \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h_i \circ \gamma_{y_1, y_2}(s) ds dy_2 \geq \frac{1}{7}\delta^{21/100}$$

for some $i = 1, \dots, 6$. Thus, we get either

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \int_M \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h_0 \circ \gamma_{y_1, y_2}(s) ds dy_1 dy_2 \\ & \geq \frac{1}{49}\delta^{-1/100}\text{Vol}(M \setminus Q_f) \end{aligned}$$

or

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \int_M \int_M \frac{1}{d(y_1, y_2)} \int_0^{d(y_1, y_2)} h_i \circ \gamma_{y_1, y_2}(s) ds dy_1 dy_2 \\ & \geq \frac{1}{49}\delta^{21/100}\text{Vol}(M \setminus Q_f) \end{aligned}$$

for some $i = 1, \dots, 6$. Therefore, we get (i) by the segment inequality (Theorem 4.6).

Similarly, we get (ii) by Lemma 4.9. $\square$

Under the pinching condition along the geodesic, we get the following:

Lemma 4.13 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$ with $\|f\|_2^2 = 1/(n - p + 1)$. Suppose that a geodesic $\gamma: [0, l] \rightarrow M$ satisfies one of the following:

- There exist $x \in M$ and $y \in D_f(x)$ such that $l = d(x, y)$ and $\gamma = \gamma_{x, y}$,
- There exist $x \in M$ and $u \in E_f(x)$ such that $l = \pi$ and $\gamma = \gamma_u$.

Then, we have

$$|\omega|^2(s) - 1| \leq C\delta^{1/10}, \quad |\iota(\nabla f)\omega|(s) \leq C\delta^{1/10}$$

for all $s \in [0, l]$, and at least one of the following:

- (i) $\frac{1}{l} \int_0^l |\nabla^2 f| \circ \gamma(s) ds \leq C\delta^{1/250}$,

(ii) There exists a parallel orthonormal basis $\{E^1(s), \dots, E^n(s)\}$ of $T_{\gamma(s)}^*M$ along γ such that

$$|\omega - E^{n-p+1} \wedge \dots \wedge E^n|(s) \leq C\delta^{1/25}$$

for all $s \in [0, l]$, and

$$\frac{1}{l} \int_0^l |\nabla^2 f + f \sum_{i=1}^{n-p} E^i \otimes E^i|(s) ds \leq C\delta^{1/200},$$

where we write $|\cdot|(s)$ instead of $|\cdot| \circ \gamma(s)$.

In particular, for both cases, there exists a parallel orthonormal basis $\{E^1(s), \dots, E^n(s)\}$ of $T_{\gamma(s)}^*M$ along γ such that

$$\frac{1}{l} \int_0^l |\nabla^2 f + f \sum_{i=1}^{n-p} E^i \otimes E^i|(s) ds \leq C\delta^{1/250}.$$

Moreover, if we put $\dot{\gamma}^E := \sum_{i=1}^{n-p} \langle \dot{\gamma}, E_i \rangle E_i$, where $\{E_1, \dots, E_n\}$ denotes the dual basis of $\{E^1, \dots, E^n\}$, then $|\dot{\gamma}^E|$ is constant along γ , and

$$\begin{aligned} & \left| f \circ \gamma(s) - f(\gamma(s_0)) \cos(|\dot{\gamma}^E|(s - s_0)) - \frac{1}{|\dot{\gamma}^E|} \langle \nabla f, \dot{\gamma}(s_0) \rangle \sin(|\dot{\gamma}^E|(s - s_0)) \right| \\ & \leq C\delta^{1/250}, \\ & \left| \langle \nabla f, \dot{\gamma}(s) \rangle + f(\gamma(s_0)) |\dot{\gamma}^E| \sin(|\dot{\gamma}^E|(s - s_0)) - \langle \nabla f, \dot{\gamma}(s_0) \rangle \cos(|\dot{\gamma}^E|(s - s_0)) \right| \\ & \leq C\delta^{1/250} \end{aligned}$$

for all $s, s_0 \in [0, l]$.

Proof Let us show the first assertion. Since $\frac{d}{ds} |\omega|^2(s) = 2\langle \nabla_{\dot{\gamma}} \omega, \omega \rangle$, we have

$$\begin{aligned} \left| |\omega|^2(s) - |\omega|^2(0) \right| &= \left| \int_0^s \frac{d}{ds} |\omega|^2(t) dt \right| \\ &\leq 2 \left(\int_0^s |\nabla \omega|^2(t) dt \right)^{1/2} \left(\int_0^s |\omega|^2(t) dt \right)^{1/2} \leq C\delta^{1/10} \end{aligned}$$

for all $s \in [0, l]$. Since we have $\int_0^l ||\omega|^2 - 1| dt \leq \delta^{1/5}$, we get $||\omega|^2(s) - 1| \leq C\delta^{1/10}$. In particular, $|\omega|(s) \geq 1/2$, and so

$$\frac{1}{l} \int_0^l \left| |\nabla^2 f|^2 - \frac{1}{n-p} (\Delta f)^2 \right|(s) ds \leq 2\delta^{1/5}. \quad (27)$$

Similarly, we have $|\iota(\nabla f)\omega|(s) \leq C\delta^{1/10}$ for all $s \in [0, l]$.

We show the remaining assertions. Put

$$\begin{aligned} A_1 &:= \left\{ s \in [0, l] : \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 (s) > \delta^{1/10} \right\}, \\ A_2 &:= \left\{ s \in [0, l] : \left| |\nabla^2 f|^2 - \frac{1}{n-p} (\Delta f)^2 \right| (s) > \delta^{1/10} \right\}, \\ A_3 &:= \left\{ s \in [0, l] : |\nabla^2 f|(s) < \delta^{1/250} \right\}. \end{aligned}$$

Then, we have $H^1(A_1) \leq \delta^{1/10}l$ and $H^1(A_2) \leq 2\delta^{1/10}l$, where H^1 denotes the one dimensional Hausdorff measure. We consider the following two cases:

- (a) $[0, l] = A_1 \cup A_2 \cup A_3$,
 (b) $[0, l] \neq A_1 \cup A_2 \cup A_3$.

We first consider the case (a). Since $H^1([0, l] \setminus A_3) \leq 3\delta^{1/10}l$, we have

$$\begin{aligned} \int_{[0, l] \setminus A_3} |\nabla^2 f|(s) ds &\leq \left(\int_{[0, l] \setminus A_3} |\nabla^2 f|^2(s) ds \right)^{1/2} H^1([0, l] \setminus A_3)^{1/2} \\ &\leq C\delta^{-1/100}\delta^{1/20}l = C\delta^{1/25}l. \end{aligned}$$

On the other hand, we have $\int_{A_3} |\nabla^2 f|(s) ds \leq \delta^{1/250}l$. Therefore, we get (i). Moreover, since $|\Delta f| \leq \sqrt{n}|\nabla^2 f|$ and $\|\Delta f - (n-p)f\|_\infty \leq C\delta^{1/2}$, we get

$$\frac{1}{l} \int_0^l |\nabla^2 f + f \sum_{i=1}^{n-p} E^i \otimes E^i|(s) ds \leq C\delta^{1/250},$$

where $\{E^1(s), \dots, E^n(s)\}$ is any parallel orthonormal basis of $T_{\gamma(s)}^*M$ along γ .

We next consider the case (b). There exists $t \in [0, l]$ such that

$$\begin{aligned} \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 (t) &\leq \delta^{1/10}, \\ \left| |\nabla^2 f|^2 - \frac{1}{n-p} (\Delta f)^2 \right| (t) &\leq \delta^{1/10}, \quad |\nabla^2 f|(t) \geq \delta^{1/250}. \end{aligned}$$

Take an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_{\gamma(t)}M$ such that $\nabla^2 f(e_i, e_j) = \mu_i \delta_{ij}$ ($\mu_i \in \mathbb{R}$) for all $i, j = 1, \dots, n$. Let $\{e^1, \dots, e^n\}$ be the dual basis of $T_{\gamma(t)}^*M$. Then, we have

$$\delta^{1/10} \geq \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 (t) = \sum_{i=1}^n \mu_i^2 |\iota(e_i) \omega|^2 (t).$$

Thus, for each $i = 1, \dots, n$, we have at least one of the following:

- (1) $|\mu_i| \leq \delta^{1/100}$,
 (2) $|\iota(e_i)\omega|(t) \leq \delta^{1/25}$.

Since $|\omega|(t) \geq 1/2$, we have $\text{Card}\{i : |\iota(e_i)\omega|(t) \leq \delta^{1/25}\} \leq n - p$, and so $\text{Card}\{i : |\mu_i| \leq \delta^{1/100}\} \geq p$. Therefore, we can assume $|\mu_i| \leq \delta^{1/100}$ for all $i = n - p + 1, \dots, n$. Then, we get

$$\begin{aligned} & \left| \nabla^2 f + \frac{\Delta f}{n-p} \sum_{i=1}^{n-p} e^i \otimes e^i \right|^2(t) \\ &= |\nabla^2 f|^2(t) + \frac{2}{n-p} (\Delta f)(t) \sum_{i=1}^{n-p} \mu_i + \frac{(\Delta f)^2(t)}{n-p} \\ &= |\nabla^2 f|^2(t) - \frac{(\Delta f)^2(t)}{n-p} - \frac{2}{n-p} (\Delta f)(t) \sum_{i=n-p+1}^n \mu_i \\ &\leq C\delta^{1/100}. \end{aligned}$$

Putting $e_i \otimes e_i$ into the inside of the left hand side, we get $|\mu_i + \Delta f(t)/(n-p)|^2 \leq C\delta^{1/100}$ for all $i = 1, \dots, n-p$, and so

$$\begin{aligned} |\mu_i| &\geq \frac{|\Delta f(t)|}{n-p} - C\delta^{1/200} \geq \left(\frac{|\nabla^2 f|^2(t) - \delta^{1/10}}{n-p} \right)^{1/2} - C\delta^{1/200} \\ &\geq \left(\frac{\delta^{1/125} - \delta^{1/10}}{n-p} \right)^{1/2} - C\delta^{1/200} > \delta^{1/100}. \end{aligned}$$

Thus, we have $|\iota(e_i)\omega|(t) \leq \delta^{1/25}$ for all $i = 1, \dots, n-p$. Therefore, we get either $|\omega(t) - e^{n-p+1} \wedge \dots \wedge e^n| \leq C\delta^{1/25}$ or $|\omega(t) + e^{n-p+1} \wedge \dots \wedge e^n| \leq C\delta^{1/25}$ by $||\omega|^2(t) - 1| \leq C\delta^{1/10}$. We can assume that $|\omega(t) - e^{n-p+1} \wedge \dots \wedge e^n| \leq C\delta^{1/25}$.

Let $\{E_1, \dots, E_n\}$ be the parallel orthonormal basis of TM along γ such that $E_i(t) = e_i$, and let $\{E^1, \dots, E^n\}$ be its dual. Because

$$\int_0^l \left| \frac{d}{ds} |\omega - E^{n-p+1} \wedge \dots \wedge E^n|^2(s) \right| ds \leq C\delta^{1/10},$$

we get $|\omega - E^{n-p+1} \wedge \dots \wedge E^n|(s) \leq C\delta^{1/25}$ for all $s \in [0, l]$. Thus, we get $|\langle \iota(E_i)\omega, \iota(E_j)\omega \rangle| \leq C\delta^{1/25}$ for all $i = 1, \dots, n$ and $j = 1, \dots, n-p$, and $|\langle \iota(E_i)\omega, \iota(E_j)\omega \rangle - \delta_{ij}| \leq C\delta^{1/25}$ for all $i, j = n-p+1, \dots, n$. Therefore, we get

$$\begin{aligned} & \left| \sum_{i=1}^n E^i \otimes \iota(\nabla_{E_i} \nabla f) \omega - \sum_{i=1}^n \sum_{j=n-p+1}^n (\nabla^2 f(E_i, E_j))^2 \right| \\ &= \left| \sum_{i,j,k=1}^n \nabla^2 f(E_i, E_j) \nabla^2 f(E_i, E_k) \langle \iota(E_j)\omega, \iota(E_k)\omega \rangle \right| \end{aligned}$$

$$\left| -\sum_{i=1}^n \sum_{j=n-p+1}^n (\nabla^2 f(E_i, E_j))^2 \right| \leq C |\nabla^2 f|^2 \delta^{1/25}.$$

Thus, for all $i = 1, \dots, n$ and $j = 1, \dots, n - p$, we get

$$|\nabla^2 f(E_i, E_j)|^2 \leq \left| \sum_{k=1}^n E^k \otimes \iota(\nabla_{E_k} \nabla f) \omega \right|^2 + C |\nabla^2 f|^2 \delta^{1/25},$$

and so

$$\frac{1}{l} \int_0^l |\nabla^2 f(E_i, E_j)|^2(s) ds \leq \delta^{1/5} + C \delta^{-1/50} \delta^{1/25} \leq C \delta^{1/50}.$$

This gives

$$\frac{1}{l} \int_0^l |\nabla^2 f(E_i, E_j)|(s) ds \leq C \delta^{1/100}$$

for all $i = 1, \dots, n$ and $j = 1, \dots, n - p$. Because

$$\begin{aligned} \left| \nabla^2 f + \frac{\Delta f}{n-p} \sum_{i=1}^{n-p} E^i \otimes E^i \right|^2 &= |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n-p} \\ &\quad - 2 \frac{\Delta f}{n-p} \sum_{i=n-p+1}^n \nabla^2 f(E_i, E_i), \end{aligned}$$

we have

$$\frac{1}{l} \int_0^l \left| \nabla^2 f + \frac{\Delta f}{n-p} \sum_{i=1}^{n-p} E^i \otimes E^i \right|^2 ds \leq 2\delta^{1/5} + C \delta^{1/100} \leq C \delta^{1/100}$$

by (27). Since $\|f - \Delta f/(n-p)\|_\infty \leq C \delta^{1/2}$, we get (ii).

Let us show the final assertion. It is trivial that $|\dot{\gamma}^E|$ is constant along γ . Since we have

$$\left(\nabla^2 f + f \sum_{i=1}^{n-p} E^i \otimes E^i \right) (\dot{\gamma}, \dot{\gamma}) = \frac{d^2}{ds^2} f \circ \gamma + |\dot{\gamma}^E|^2 f \circ \gamma,$$

we get

$$\int_0^l \left| \frac{d^2}{ds^2} f \circ \gamma(s) + |\dot{\gamma}^E|^2 f \circ \gamma(s) \right| ds \leq C \delta^{1/250}.$$

Thus, we get the lemma by Lemma 4.7. $\square$

4.3 Almost Parallel $(n - p)$ -form I

In this subsection, we suppose that Assumption 4.3 holds instead of 4.2. If M is orientable, then Assumption 4.3 implies 4.2, and so we assume that M is not orientable. We use the following notation.

Notation 4.14 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$ with $\|f\|_2^2 = 1/(n - p + 1)$. Let $\pi: (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ be the two-sheeted oriented Riemannian covering. Put $\tilde{f} := f \circ \pi \in C^\infty(\tilde{M})$, $\tilde{\xi} := \pi^*\xi \in \Gamma(\bigwedge^{n-p} T^*\tilde{M})$ and $\omega := *\tilde{\xi} \in \Gamma(\bigwedge^p T^*\tilde{M})$. Define $h_0, \dots, h_6$, $Q_{\tilde{f}}$, $D_{\tilde{f}}(\tilde{y}_1)$, $R_{\tilde{f}}$ and $E_{\tilde{f}}(\tilde{y}_1)$ as Notation 4.11 for $\tilde{f}$, ω and $\tilde{y}_1 \in \tilde{M}$. Put

$$Q_f := M \setminus \pi \left(\tilde{M} \setminus Q_{\tilde{f}} \right), \quad D_f(y_1) := M \setminus \pi \left(\tilde{M} \setminus \bigcap_{\tilde{y} \in \pi^{-1}(y_1)} D_{\tilde{f}}(\tilde{y}) \right),$$

$$R_f := M \setminus \pi \left(\tilde{M} \setminus R_{\tilde{f}} \right), \quad E_f(y_1) := U_{y_1}M \setminus \bigcup_{\tilde{y} \in \pi^{-1}(y_1)} \pi_* \left(U_{\tilde{y}}\tilde{M} \setminus E_{\tilde{f}}(\tilde{y}) \right)$$

for each $y_1 \in M$.

We immediately have the following lemmas by Lemmas 4.12 and 4.13.

Lemma 4.15 *We have the following:*

- (i) $\text{Vol}(M \setminus Q_f) \leq C\delta^{1/100}\text{Vol}(M)$, and $\text{Vol}(M \setminus D_f(y_1)) \leq 2\delta^{1/100}\text{Vol}(\tilde{M}) = 4\delta^{1/100}\text{Vol}(M)$ for each $y_1 \in Q_f$.
- (ii) $\text{Vol}(M \setminus R_f) \leq C\delta^{1/100}\text{Vol}(M)$, and $\text{Vol}(U_{y_1}M \setminus E_f(y_1)) \leq 2\delta^{1/100}\text{Vol}(U_{y_1}M)$ for each $y_1 \in R_f$.
- (iii) Take $y_1 \in M$ and $y_2 \in D_f(y_1)$ and one of the lift of γ_{y_1, y_2} :

$$\tilde{\gamma}_{y_1, y_2}: [0, d(y_1, y_2)] \rightarrow \tilde{M}.$$

Put $\tilde{y}_1 := \tilde{\gamma}_{y_1, y_2}(0) \in \tilde{M}$ and $\tilde{y}_2 := \tilde{\gamma}_{y_1, y_2}(d(y_1, y_2)) \in \tilde{M}$. Then, we have $\tilde{y}_2 \in D_{\tilde{f}}(\tilde{y}_1)$.

- (iv) Take $y_1 \in M$ and $u \in E_f(y_1)$ and one of the lift of γ_u :

$$\tilde{\gamma}_u: [0, \pi] \rightarrow \tilde{M}.$$

Put $\tilde{y}_1 := \tilde{\gamma}_u(0) \in \tilde{M}$ and $\tilde{u} := \tilde{\gamma}_u(\pi) \in U_{\tilde{y}_1}\tilde{M}$. Then, we have $\tilde{u} \in E_{\tilde{f}}(\tilde{y}_1)$.

Lemma 4.16 *Suppose that a geodesic $\gamma: [0, l] \rightarrow M$ satisfies one of the following:*

- There exist $x \in M$ and $y \in D_f(x)$ such that $l = d(x, y)$ and $\gamma = \gamma_{x, y}$,
- There exist $x \in M$ and $u \in E_f(x)$ such that $l = \pi$ and $\gamma = \gamma_u$.

Let $\tilde{\gamma}: [0, l] \rightarrow \tilde{M}$ be one of the lift of γ . Then, we have

$$|\omega|^2(\tilde{\gamma}(s)) - 1| \leq C\delta^{1/10}, \quad |\iota(\nabla \tilde{f})(\omega)| \circ \tilde{\gamma}(s) \leq C\delta^{1/10}$$

for all $s \in [0, l]$, and at least one of the following:

- (i) $\frac{1}{l} \int_0^l |\nabla^2 f| \circ \gamma(s) ds \leq C\delta^{1/250}$,
- (ii) There exists a parallel orthonormal basis $\{E^1(s), \dots, E^n(s)\}$ of $T_{\gamma(s)}^*M$ along γ such that

$$|\xi - E^1 \wedge \dots \wedge E^{n-p}|(s) \leq C\delta^{1/25}$$

for all $s \in [0, s]$, and

$$\frac{1}{l} \int_0^l |\nabla^2 f + f \sum_{i=1}^{n-p} E^i \otimes E^i|(s) ds \leq C\delta^{1/200}.$$

4.4 Eigenfunction and Distance

In this subsection, we suppose that either Assumption 4.2 or 4.3 holds. In the following, Lemma 4.12 (resp. 4.13) shall be replaced by Lemma 4.15 (resp. 4.16) under Assumption 4.3. The following proposition, which asserts that our function is an almost cosine function in some sense, is the goal of this subsection. See Notation 4.11 (under Assumption 4.2) and Notation 4.14 (under Assumption 4.3) for the definitions of D_f , Q_f , E_f and R_f .

Proposition 4.17 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$ with $\|f\|_2^2 = 1/(n-p+1)$. There exists a point $p_f \in Q_f$ such that the following properties hold:

- (i) $\sup_M f \leq f(p_f) + C\delta^{1/100n}$ and $|f(p_f) - 1| \leq C\delta^{1/800n}$,
- (ii) For any $x \in D_f(p_f)$ with $|\nabla f|(x) \leq \delta^{1/800n}$, we have $||f(x)| - 1| \leq C\delta^{1/800n}$.
- (iii) For any $x \in D_f(p_f) \cap Q_f \cap R_f$, we have $|f(x)|^2 + |\nabla f|^2(x) - 1 \leq C\delta^{1/800n}$.
- (iv) Put $A_f := \{x \in M : |f(x) - 1| \leq \delta^{1/900n}\}$. Then, we have

$$|f(x) - \cos d(x, A_f)| \leq C\delta^{1/2000n}$$

for all $x \in M$, and $\sup_{x \in M} d(x, A_f) \leq \pi + C\delta^{1/100n}$.

Proof Take a maximum point $\tilde{p} \in M$ of f . Then, by the Bishop-Gromov theorem and Lemma 4.12, there exists a point $p_f \in Q_f$ with $d(\tilde{p}, p_f) \leq C\delta^{1/100n}$. By Lemmas 4.4 and 3.7, we have

$$|\nabla f|(p_f) \leq C\delta^{1/200n}. \quad (28)$$

Claim 4.18 For any $x \in D_f(p_f)$ with $|\nabla f|(x) \leq C\delta^{1/800n}$, we have

$$||f(x)| - |f(p_f)|| \leq C\delta^{1/800n}.$$

Proof of Claim 4.18 Since $|\nabla f|(p_f) \leq C\delta^{1/200n}$ and $|\nabla f|(x) \leq C\delta^{1/800n}$, we get

$$|f \circ \gamma_{p_f, x}(s) - f(p_f) \cos(|\dot{\gamma}_{p_f, x}^E|s)| \leq C\delta^{1/200n},$$

$$|f \circ \gamma_{p_f, x}(d(p_f, x) - s) - f(x) \cos(|\dot{\gamma}_{p_f, x}^E|s)| \leq C\delta^{1/800n}$$

for all $s \in [0, d(p_f, x)]$ by Lemma 4.13. Thus, we have

$$\begin{aligned} |f(x) - f(p_f) \cos(|\dot{\gamma}_{p_f, x}^E|d(p_f, x))| &\leq C\delta^{1/200n}, \\ |f(p_f) - f(x) \cos(|\dot{\gamma}_{p_f, x}^E|d(p_f, x))| &\leq C\delta^{1/800n}, \end{aligned}$$

and so we get $||f(x)| - |f(p_f)|| \leq C\delta^{1/800n}$. $\square$

Similarly to p_f , we take a point $q_f \in Q_f(x)$ with $d(\tilde{q}, q_f) \leq C\delta^{1/100n}$, where $\tilde{q} \in M$ is minimum point of f . By $\|f\|_\infty \geq \|f\|_2 = 1/\sqrt{n-p+1}$, we have $\max\{|f(p_f)|, |f(q_f)|\} \geq 1/\sqrt{n-p+1} - C\delta^{1/100n}$. Since $|\nabla f|(q_f) \leq C\delta^{1/200n}$, we have $|f(p_f)| \geq |f(q_f)| - C\delta^{1/800n}$ by Claim 4.18. Therefore, we get

$$f(p_f) \geq \frac{1}{\sqrt{n-p+1}} - C\delta^{1/800n} \geq \frac{1}{2\sqrt{n-p+1}}. \quad (29)$$

Claim 4.19 Take $x \in M$ and $y \in D_f(x)$. Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis along $\gamma_{x,y}$ in Lemma 4.13. If (i) holds in the lemma, we can assume that $E_1 = \dot{\gamma}_{x,y}$. Then, we have

$$|\langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle - \langle \nabla f, \dot{\gamma}_{x,y}^E(s) \rangle| \leq C\delta^{1/25}, \quad (30)$$

$$|\langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle| \leq |\nabla f(\gamma_{x,y}(s))| |\dot{\gamma}_{x,y}^E| + C\delta^{1/25} \quad (31)$$

and

$$\begin{aligned} \left| f \circ \gamma_{x,y}(s) - f(x) \cos(|\dot{\gamma}_{x,y}^E|s) - \frac{1}{|\dot{\gamma}_{x,y}^E|} \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \sin(|\dot{\gamma}_{x,y}^E|s) \right| &\leq C\delta^{1/250}, \\ \left| \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle + f(x) |\dot{\gamma}_{x,y}^E| \sin(|\dot{\gamma}_{x,y}^E|s) - \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \cos(|\dot{\gamma}_{x,y}^E|s) \right| &\leq C\delta^{1/250} \end{aligned}$$

for all $s \in [0, d(x, y)]$.

Proof of Claim 4.19 If (i) holds in the lemma, $\dot{\gamma}_{x,y} = \dot{\gamma}_{x,y}^E$, and so (30) and (31) are trivial. If (ii) in the lemma holds, we have $|\iota(\nabla f)(E^{n-p+1} \wedge \dots \wedge E^n)| \leq C\delta^{1/25}$, and so $|\langle \nabla f(x), E_i \rangle| \leq C\delta^{1/25}$ for all $i = n-p+1, \dots, n$. This gives (30) and (31). We get the remaining part of the claim by Lemma 4.13 putting $s_0 = 0$. $\square$

Claim 4.20 For any $x \in Q_f \cap R_f$ with $|\nabla f|(x) \geq \delta^{1/800n}$, we have

$$|f(x)^2 + |\nabla f|^2(x) - f(p_f)^2| \leq C\delta^{1/800n}.$$

Moreover, there exists a point $y \in D_f(p_f) \cap D_f(x)$ such that the following properties hold.

- (a) $d(x, y) < \pi$,
 (b) $|f(p_f) - f(y)| \leq C\delta^{1/800n}$,
 (c) $|f(x) - f(p_f) \cos d(x, y)| \leq C\delta^{1/800n}$,
 (d) For any $z \in M$ with $d(x, z) \leq d(x, y) - \delta^{1/2000n}$, we have $f(p_f) - f(z) \geq \frac{1}{C}\delta^{1/1000n}$.

Proof of Claim 4.20 Take $x \in Q_f \cap R_f$ with $|\nabla f|(x) \geq \delta^{1/800n}$. By the definition of R_f , there exists a vector $u \in E_f(x)$ with

$$\left| \frac{\nabla f}{|\nabla f|}(x) - u \right| \leq C\delta^{1/100n}.$$

Thus, we have

$$\left| \langle \nabla f(x), \dot{\gamma}_u(0) \rangle - |\nabla f|(x) \right| = |\nabla f|(x) - \langle \nabla f(x), u \rangle \leq C\delta^{1/100n}. \quad (32)$$

Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis along γ_u in Lemma 4.13. We first suppose that (ii) holds in the lemma. Then, for all $i = n - p + 1, \dots, n$, we have $|\langle \nabla f, E_i \rangle| \leq C\delta^{1/25}$, and so

$$\begin{aligned} |\langle u, E_i \rangle| &\leq \left| u - \frac{\nabla f}{|\nabla f|}(x) \right| + \left| \left\langle \frac{\nabla f}{|\nabla f|}(x), E_i \right\rangle \right| \\ &\leq C\delta^{1/100n} + C\delta^{1/25}\delta^{-1/800n} \leq C\delta^{1/100n}. \end{aligned}$$

Thus, we get $|\dot{\gamma}_u^E|^2 = |u^E|^2 = 1 - \sum_{i=n-p+1}^n \langle u, E_i \rangle^2 \geq 1 - C\delta^{1/100n}$. If (i) holds in the lemma, we can assume $u = E_1$, and so $|\dot{\gamma}_u^E| = |u^E| = 1$. For both cases, we get

$$\begin{aligned} |f \circ \gamma_u(s) - f(x) \cos s - |\nabla f|(x) \sin s| &\leq C\delta^{1/100n} \\ |\langle \nabla f, \dot{\gamma}_u(s) \rangle + f(x) \sin s - |\nabla f|(x) \cos s| &\leq C\delta^{1/100n} \end{aligned} \quad (33)$$

for all $s \in [0, \pi]$ by (32). Take $s_0 \in [0, \pi]$ such that

$$\begin{aligned} \frac{f(x)}{(f(x)^2 + |\nabla f|^2(x))^{1/2}} &= \cos s_0, \\ \frac{|\nabla f|(x)}{(f(x)^2 + |\nabla f|^2(x))^{1/2}} &= \sin s_0. \end{aligned}$$

Since $\sin s_0 \geq \frac{1}{C}\delta^{1/800n}$ by the assumption, we have

$$\frac{1}{C}\delta^{1/800n} \leq s_0 \leq \pi - \frac{1}{C}\delta^{1/800n}. \quad (34)$$

By the definition of s_0 and the formulas for $\cos(s - s_0)$ and $\sin(s - s_0)$, we have

$$\begin{aligned}(f(x)^2 + |\nabla f|^2(x))^{1/2} \cos(s - s_0) &= f(x) \cos s + |\nabla f|(x) \sin s, \\ (f(x)^2 + |\nabla f|^2(x))^{1/2} \sin(s - s_0) &= f(x) \sin s - |\nabla f|(x) \cos s,\end{aligned}$$

and so we get

$$\begin{aligned}|f \circ \gamma_u(s_0) - (f(x)^2 + |\nabla f|^2(x))^{1/2}| &\leq C\delta^{1/100n}, \\ |\langle \nabla f, \dot{\gamma}_u(s_0) \rangle| &\leq C\delta^{1/100n}\end{aligned}\quad (35)$$

by (33). Take $y \in D_f(p_f) \cap D_f(x)$ with $d(\gamma_u(s_0), y) \leq C\delta^{1/100n}$. We have

$$d(x, y) \leq d(x, \gamma_u(s_0)) + d(\gamma_u(s_0), y) \leq s_0 + C\delta^{1/100n}. \quad (36)$$

By (35), we get

$$|f(y) - (f(x)^2 + |\nabla f|^2(x))^{1/2}| \leq C\delta^{1/100n} \quad (37)$$

Take a parallel orthonormal basis $\{\tilde{E}^1, \dots, \tilde{E}^n\}$ of T^*M along $\gamma_{x,y}$ in Lemma 4.13. By (34) and (36), we get (a) and

$$\frac{1}{C}\delta^{1/800n} \leq |\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) + s_0 \leq 2\pi - \frac{1}{C}\delta^{1/800n},$$

and so

$$\cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) + s_0) \leq 1 - \frac{1}{C}\delta^{1/400n}. \quad (38)$$

If $|\dot{\gamma}_{x,y}^{\tilde{E}}| \leq \delta^{1/100}$, we have $|f(y) - f(x)| \leq C\delta^{1/250}$ by Claim 4.19, and so $(f(x)^2 + |\nabla f|^2(x))^{1/2} - f(x) \leq C\delta^{1/100n}$ by (37). This contradicts to $|\nabla f|(x) \geq \delta^{1/800n}$. Thus, we get $|\dot{\gamma}_{x,y}^{\tilde{E}}| \geq \delta^{1/100}$. Then, we have

$$\frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|} |\langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle| \leq |\nabla f|(x) + C\delta^{3/100} \quad (39)$$

and

$$\begin{aligned}&(f(x)^2 + |\nabla f|^2(x))^{1/2} \\ &\leq f(y) + C\delta^{1/100n} \\ &\leq f(x) \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y)) \\ &\quad + \frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|} \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \sin(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y)) + C\delta^{1/100n}\end{aligned}$$

$$\leq \left(f(x)^2 + \frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|^2} \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle^2 \right)^{1/2} + C\delta^{1/100n}$$

by Claim 4.19 and (37). Thus,

$$|\nabla f|^2(x) \leq \frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|^2} \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle^2 + C\delta^{1/100n}. \quad (40)$$

By (39) and (40), we get

$$\left| \frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|^2} \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle^2 - |\nabla f|^2(x) \right| \leq C\delta^{1/100n}. \quad (41)$$

This gives

$$\begin{aligned} & \left| \frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|} |\langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle| - |\nabla f|(x) \right| \\ & \leq \left| \frac{1}{|\dot{\gamma}_{x,y}^{\tilde{E}}|^2} \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle^2 - |\nabla f|^2(x) \right| \delta^{-1/800n} \leq C\delta^{7/800n}. \end{aligned} \quad (42)$$

We show that $\langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle > 0$. If $\langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \leq 0$, we get

$$\left| f(y) - f(x) \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y)) + |\nabla f| \sin(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y)) \right| \leq C\delta^{7/800n}$$

by (42) and Claim 4.19, and so

$$\left| f(y) - (f(x)^2 + |\nabla f|^2(x))^{1/2} \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) + s_0) \right| \leq C\delta^{7/800n}.$$

Thus, we get

$$\begin{aligned} & (f(x)^2 + |\nabla f|^2(x))^{1/2} \\ & \leq f(y) + C\delta^{1/100n} \\ & \leq (f(x)^2 + |\nabla f|^2(x))^{1/2} \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) + s_0) + C\delta^{7/800n} \\ & \leq (f(x)^2 + |\nabla f|^2(x))^{1/2} - \frac{1}{C}\delta^{3/800n} \end{aligned}$$

by (37), (38) and $|\nabla f|(x) \geq \delta^{1/800n}$. This is a contradiction. Therefore, we get $\langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle > 0$. Thus,

$$\begin{aligned} & \left| f(y) - (f(x)^2 + |\nabla f|^2(x))^{1/2} \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) - s_0) \right| \leq C\delta^{7/800n}, \\ & \left| \langle \nabla f(y), \dot{\gamma}_{x,y} \rangle + |\dot{\gamma}_{x,y}^{\tilde{E}}|(f(x)^2 + |\nabla f|^2(x))^{1/2} \sin(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) - s_0) \right| \leq C\delta^{7/800n} \end{aligned} \quad (43)$$

by (42) and Claim 4.19. Then, we have

$$(f(x)^2 + |\nabla f|^2(x))^{1/2}(1 - \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) - s_0)) \leq C\delta^{7/800n}$$

by (37), and so

$$1 - \cos(|\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) - s_0) \leq C\delta^{3/400n}.$$

by $|\nabla f|(x) \geq \delta^{1/800n}$. Since $-\pi < |\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) - s_0 < \pi$, we get

$$\left| |\dot{\gamma}_{x,y}^{\tilde{E}}|d(x, y) - s_0 \right| \leq C\delta^{3/800n}. \quad (44)$$

Thus, we have $s_0 \leq |\dot{\gamma}_{x,y}^{\tilde{E}}|s_0 + C\delta^{3/800n}$ by (36), and so

$$1 - |\dot{\gamma}_{x,y}^{\tilde{E}}| \leq C\delta^{1/400n} \quad (45)$$

by (34). Thus, we get

$$|d(x, y) - s_0| \leq C\delta^{1/400n}. \quad (46)$$

By (43) and (44), we have

$$|\langle \nabla f(y), \dot{\gamma}_{x,y}(d(x, y)) \rangle| \leq C\delta^{3/800n}. \quad (47)$$

We have

$$\begin{aligned} & \frac{d}{ds} \left(|\nabla f|^2(s) - \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle^2 \right) \\ &= 2 \left(\langle \nabla_{\dot{\gamma}_{x,y}} \nabla f, \nabla f \rangle(s) - \langle \nabla_{\dot{\gamma}_{x,y}} \nabla f, \dot{\gamma}_{x,y}(s) \rangle \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle \right) \\ &= 2 \langle \nabla^2 f + f \sum_{i=1}^{n-p} \tilde{E}^i \otimes \tilde{E}^i, \dot{\gamma}_{x,y} \otimes \nabla f \rangle(s) - 2f \langle \nabla f, \dot{\gamma}_{x,y}^{\tilde{E}} \rangle \\ &\quad - 2 \langle \nabla^2 f + f \sum_{i=1}^{n-p} \tilde{E}^i \otimes \tilde{E}^i, \dot{\gamma}_{x,y} \otimes \dot{\gamma}_{x,y} \rangle(s) \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle \\ &\quad + 2f |\dot{\gamma}_{x,y}^{\tilde{E}}|^2 \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle. \end{aligned} \quad (48)$$

Thus, we get

$$\begin{aligned} & \left| \frac{d}{ds} \left(|\nabla f|^2(s) - \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle^2 \right) \right| \\ & \leq C \left| \nabla^2 f + f \sum_{i=1}^{n-p} \tilde{E}^i \otimes \tilde{E}^i \right| + C \left| \langle \nabla f, \dot{\gamma}_{x,y}^{\tilde{E}} \rangle - |\dot{\gamma}_{x,y}^{\tilde{E}}|^2 \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle \right|. \quad (49) \\ & \leq C \left| \nabla^2 f + f \sum_{i=1}^{n-p} \tilde{E}^i \otimes \tilde{E}^i \right| + C\delta^{1/400n} \end{aligned}$$

by (30) and (45). By integration, we get

$$\int_0^{d(x,y)} \left| \frac{d}{ds} \left(|\nabla f|^2(s) - \langle \nabla f, \dot{\gamma}_{x,y}(s) \rangle^2 \right) \right| ds \leq C\delta^{1/400n},$$

and so

$$\left| |\nabla f|^2(y) - \langle \nabla f(y), \dot{\gamma}_{x,y}(d(x,y)) \rangle^2 - |\nabla f|^2(x) + \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle^2 \right| \leq C\delta^{1/400n}.$$

Thus, we get

$$|\nabla f|(y) \leq C\delta^{1/800n}.$$

by (41), (45) and (47). By Claim 4.18 and (29), we get

$$|f(y) - f(p_f)| \leq C\delta^{1/800n}.$$

Since

$$f(y) \geq (f(x)^2 + |\nabla f|^2(x))^{1/2} - C\delta^{1/100n} \geq \delta^{1/800n} - C\delta^{1/100n} > 0$$

by (37), we get (b). We get

$$|(f(x)^2 + |\nabla f|^2(x))^{1/2} - f(p_f)| \leq C\delta^{1/800n} \quad (50)$$

by (43), (44) and (b), and so we get (c) by the definition of s_0 and (46). (50) implies the first assertion.

Finally, we show (d). Suppose that a point $z \in M$ satisfies $d(x, z) \leq d(x, y) - \delta^{1/2000n}$. Then, $d(x, y) \geq \delta^{1/2000n}$, and so

$$f(x) \leq f(p_f) \cos d(x, y) + C\delta^{1/800n} \leq f(p_f) - \frac{1}{C}\delta^{1/1000n}$$

by (29). There exists $w \in D_f(x)$ with $d(z, w) \leq C\delta^{1/100n}$. Let $\{\bar{E}^1, \dots, \bar{E}^n\}$ be a parallel orthonormal basis along $\gamma_{x,w}$ in Lemma 4.13. If (i) holds in the lemma, we

assume that $\bar{E}_1 = \dot{\gamma}_{x,w}$. If $|\dot{\gamma}_{x,w}^{\bar{E}}| \leq \delta^{1/100}$, we have

$$f(z) \leq f(w) + C\delta^{1/100n} \leq f(x) + C\delta^{1/100n} \leq f(p_f) - \frac{1}{C}\delta^{1/1000n}$$

by Claim 4.19. If $|\dot{\gamma}_{x,w}^{\bar{E}}| \geq \delta^{1/100}$, we have

$$\begin{aligned} f(z) &\leq f(w) + C\delta^{1/100n} \\ &\leq f(x) \cos(|\dot{\gamma}_{x,w}^{\bar{E}}|d(x, z)) + |\nabla f|(x) \sin(|\dot{\gamma}_{x,w}^{\bar{E}}|d(x, z)) + C\delta^{1/100n} \\ &\leq f(p_f) \cos(|\dot{\gamma}_{x,w}^{\bar{E}}|d(x, z) - d(x, y)) + \delta^{1/800n} \leq f(p_f) - \frac{1}{C}\delta^{1/1000n} \end{aligned}$$

by Claim 4.19, (46), (50) and $-\pi \leq |\dot{\gamma}_{x,w}^{\bar{E}}|d(x, z) - d(x, y) \leq -\delta^{1/2000n}$. For both cases, we get (d). $\square$

By Claims 4.18 and 4.20, we get

$$|f(x)^2 + |\nabla f|^2(x) - f(p_f)^2| \leq C\delta^{1/800n} \quad (51)$$

for all $x \in D_f(p_f) \cap Q_f \cap R_f$.

Claim 4.21 We have $|f(p_f) - 1| \leq C\delta^{1/800n}$.

Proof of Claim 4.21 Since $\|f^2 + |\nabla f|^2 - f(p_f)^2\|_\infty \leq C$ and $\text{Vol}(M \setminus (D_f(p_f) \cap Q_f \cap R_f)) \leq C\delta^{1/100}$, we get

$$\frac{1}{\text{Vol}(M)} \int_M |f(x)^2 + |\nabla f|^2(x) - f(p_f)^2| d\mu_g \leq C\delta^{1/800n}$$

by (51). By the assumption, we have

$$\frac{1}{\text{Vol}(M)} \left| \int_M (f(x)^2 + |\nabla f|^2(x) - 1) d\mu_g \right| \leq C\delta^{1/2}$$

Thus, we get $|f(p_f)^2 - 1| \leq C\delta^{1/800n}$. Since $f(p_f) > 0$, we get the claim. $\square$

By Claims 4.18, 4.21 and (51), we get (i), (ii) and (iii).

Finally, we prove (iv). Put $A_f := \{x \in M : |f(x) - 1| \leq \delta^{1/900n}\}$. Since we have $|f(w) - \cos d(w, A_f)| \leq \delta^{1/900n}$ for all $w \in A_f$, we get (iv) on A_f .

Let us show (iv) on $M \setminus A_f$. Take $w \notin A_f$ and $x \in D_f(p_f) \cap Q_f \cap R_f$ with $d(w, x) \leq C\delta^{1/100n}$.

We first suppose that $|\nabla f|(x) \geq \delta^{1/800n}$. Take $y \in D_f(p_f) \cap D_f(x)$ of Claim 4.20. Then, $|f(y) - 1| \leq C\delta^{1/800n}$, and so $y \in A_f$. Thus,

$$d(x, A_f) \leq d(x, y) < \pi. \quad (52)$$

For all $z \in A_f$, we have $|f(p_f) - f(z)| \leq C\delta^{1/900n}$, and so $d(x, z) > d(x, y) - \delta^{1/2000n}$ by Claim 4.20 (d). Thus,

$$d(x, A_f) \geq d(x, y) - \delta^{1/2000n}. \quad (53)$$

By (52) and (53), we get $|d(x, A_f) - d(x, y)| \leq \delta^{1/2000n}$. Therefore, we have $|f(x) - \cos d(x, A_f)| \leq C\delta^{1/2000n}$ by Claim 4.20 (c), and so $|f(w) - \cos d(w, A_f)| \leq C\delta^{1/2000n}$. By (52), we have $d(w, A_f) \leq \pi + C\delta^{1/100n}$.

We next suppose that $|\nabla f|(x) \leq \delta^{1/800n}$. Then, $||f|(x) - 1| \leq C\delta^{1/800n}$ by Claim 4.18. If $f(x) \geq 0$, then $w \in A_f$. This contradicts to $w \notin A_f$. Thus, we have $|f(x) + 1| \leq C\delta^{1/800n}$. We see that (i) in Lemma 4.13 cannot occur for $\gamma_{p_f, x}$ because we have

$$|\nabla^2 f| \geq \frac{1}{\sqrt{n}} |\Delta f| \geq \frac{n-p}{\sqrt{n}} |f| - C\delta^{1/2}.$$

Thus, there exists an orthonormal basis $\{e^1, \dots, e^n\}$ of T_x^*M such that $|\omega(x) - e^{n-p+1} \wedge \dots \wedge e^n| \leq C\delta^{1/25}$ if Assumption 4.2 holds, and $|\xi(x) - e^1 \wedge \dots \wedge e^{n-p}| \leq C\delta^{1/25}$ if Assumption 4.3 holds. Take $u \in E_f(x)$ with $|u - e_1| \leq C\delta^{1/100n}$. Then, we get $|f \circ \gamma_u(s) + \cos s| \leq C\delta^{1/800n}$ for all $s \in [0, \pi]$ by Lemma 4.13. Thus, we get $\gamma_u(\pi) \in A_f$, and so

$$d(w, A_f) \leq \pi + C\delta^{1/100n}. \quad (54)$$

For any $y \in A_f$, there exists $z \in D_f(x)$ with $d(y, z) \leq C\delta^{1/100n}$. Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis of T^*M along $\gamma_{x, z}$ of Claim 4.19. Then,

$$|1 + \cos(|\dot{\gamma}_{x, z}^E| d(x, z))| \leq C\delta^{1/900n}$$

by Claim 4.19. Thus, we get $d(x, z) \geq \pi - C\delta^{1/1800n}$, and so

$$d(w, A_f) \geq \pi - C\delta^{1/1800n}. \quad (55)$$

By (54) and (55), we get $|d(w, A_f) - \pi| \leq C\delta^{1/1800n}$, and so $|f(w) - \cos d(w, A_f)| \leq C\delta^{1/1800n}$.

For both cases, we get (iv). $\square$

4.5 Gromov-Hausdorff Approximation

In this subsection, we suppose that Assumption 4.1 for $k = n - p + 1$ and either Assumption 4.2 or 4.3 hold. We construct a Gromov-Hausdorff approximation map, and show that the Riemannian manifold is close to the product metric space $S^{n-p} \times X$ in the Gromov-Hausdorff topology. The following proposition is based on [19, Lemma 5.2].

Lemma 4.22 Define $\tilde{\Psi} := (f_1, \dots, f_{n-p+1}): M \rightarrow \mathbb{R}^{n-p+1}$. Then, we have

$$\| |\tilde{\Psi}|^2 - 1 \|_{\infty} \leq C\delta^{1/1000n^2}.$$

Proof We first prove the following claim:

Claim 4.23 For any $x \in M$, we have $|\tilde{\Psi}|(x) \leq 1 + C\delta^{1/800n}$

Proof of Claim 4.23 If $|\tilde{\Psi}|(x) = 0$, the claim is trivial. Thus, we assume that $|\tilde{\Psi}|(x) \neq 0$. Put

$$f_x := \frac{1}{|\tilde{\Psi}|(x)} \sum_{i=1}^{n-p+1} f_i(x) f_i.$$

Then, we have $\|f_x\|_2^2 = 1/(n-p+1)$. Thus, we get $|\tilde{\Psi}|(x) = f_x(x) \leq 1 + C\delta^{1/800n}$ by Proposition 4.17 (i). $\square$

For $x \in M$ with $|\tilde{\Psi}(x)|^2 - 1 < 0$, we have $||\tilde{\Psi}(x)|^2 - 1| = 1 - |\tilde{\Psi}(x)|^2$. For $x \in M$ with $|\tilde{\Psi}(x)|^2 - 1 \geq 0$, we have $||\tilde{\Psi}(x)|^2 - 1| = |\tilde{\Psi}(x)|^2 - 1 \leq 1 - |\tilde{\Psi}(x)|^2 + C\delta^{1/800n}$ by Claim 4.23. For both cases, we have $||\tilde{\Psi}(x)|^2 - 1| \leq 1 - |\tilde{\Psi}(x)|^2 + C\delta^{1/800n}$. Combining this and $\|\tilde{\Psi}\|_2 = 1$, we get $\| |\tilde{\Psi}|^2 - 1 \|_1 \leq C\delta^{1/800n}$. Therefore, we have

$$\text{Vol}(\{x \in M : ||\tilde{\Psi}(x)|^2 - 1| \geq \delta^{1/1000n^2}\}) \leq C\delta^{1/800n} \delta^{-1/1000n^2} \leq C\delta^{1/1000n}$$

(note that we assumed $n \geq 5$). This and the Bishop-Gromov inequality imply that, for any $x \in M$, there exists $y \in \{x \in M : ||\tilde{\Psi}(x)|^2 - 1| < \delta^{1/1000n^2}\}$ with $d(x, y) \leq C\delta^{1/1000n^2}$, and so $||\tilde{\Psi}(x)|^2 - 1| \leq C\delta^{1/1000n^2}$ by $\|\nabla |\tilde{\Psi}|^2\|_{\infty} \leq C$. Thus, we get the lemma. $\square$

Notation 4.24 In the remaining part of this subsection, we use the following notation.

- Let d_S denotes the intrinsic distance function on $S^{n-p}(1)$. Note that we have $\cos d_S(x, y) = x \cdot y$ and

$$d_{\mathbb{R}^{n-p+1}}(x, y) \leq d_S(x, y) \leq 3d_{\mathbb{R}^{n-p+1}}(x, y)$$

for all $x, y \in S^{n-p} \subset \mathbb{R}^{n-p+1}$.

- For each $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, we use the notation p_f and A_f of Proposition 4.17. Recall that we defined $A_f := \{x \in M : |f(x) - 1| \leq \delta^{1/900n}\}$.
- Define $\tilde{\Psi} := (f_1, \dots, f_{n-p+1}): M \rightarrow \mathbb{R}^{n-p+1}$ and

$$\Psi := \frac{\tilde{\Psi}}{|\tilde{\Psi}|}: M \rightarrow S^{n-p}.$$

- For each $x \in M$, put

$$f_x := \frac{1}{|\tilde{\Psi}|(x)} \sum_{i=1}^{n-p+1} f_i(x) f_i = \sum_{i=1}^{n-p+1} \Psi_i(x) f_i,$$

$$p_x := p_{f_x} \text{ and } A_x := A_{f_x}.$$

- For each $x \in M$ and $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, choose $a_f(x) \in A_f$ such that $d(x, A_f) = d(x, a_f(x))$.

The goal of this subsection is to show that

$$\Phi_f: M \rightarrow S^{n-p} \times A_f, x \mapsto (\Psi(x), a_f(x))$$

is a Gromov-Hausdorff approximation map.

Lemma 4.25 *For all $x, y \in M$, we have $|\Psi(x) - \Psi(y)| \leq Cd(x, y)$.*

Proof Since we have $\|\nabla f_i\|_{\infty} \leq C$ for all $i \in \{1, \dots, n-p+1\}$, we get $|\tilde{\Psi}(x) - \tilde{\Psi}(y)| \leq Cd(x, y)$ for all $x, y \in M$. Thus, we get the lemma by Lemma 4.22 ($|\tilde{\Psi}| \geq 1/2$). $\square$

Lemma 4.26 *Take $u \in S^{n-p}$ and put $f = \sum_{i=1}^{n-p+1} u_i f_i$. Then, we have*

$$|d_S(\Psi(y), u) - d(y, A_f)| \leq C\delta^{1/2000n^2}$$

for all $y \in M$.

Proof Since $f(y) = u \cdot \tilde{\Psi}(y)$, we have $|u \cdot \tilde{\Psi}(y) - \cos d(y, A_f)| \leq C\delta^{1/2000n}$ by Proposition 4.17, and so

$$|u \cdot \Psi(y) - \cos d(y, A_f)| \leq C\delta^{1/1000n^2}$$

by Lemma 4.22. Since $\cos d_S(\Psi(y), u) = u \cdot \Psi(y)$, this and $d(y, A_f) \leq \pi + C\delta^{1/100n}$ imply the lemma. $\square$

By the definition of A_y , we immediately get the following corollaries:

Corollary 4.27 *Take $u \in S^{n-p}$ and put $f = \sum_{i=1}^{n-p+1} u_i f_i$. Then, we have $d_S(\Psi(p_f), u) \leq C\delta^{1/2000n^2}$.*

Corollary 4.28 *For each $y_1, y_2 \in M$, we have*

$$|d_S(\Psi(y_1), \Psi(y_2)) - d(y_2, A_{y_1})| \leq C\delta^{1/2000n^2}.$$

Corollary 4.29 *For each $y \in M$, we have $d(y, A_y) \leq C\delta^{1/2000n^2}$.*

We need to show the almost Pythagorean theorem for our purpose. To do this, we regard $|\dot{\gamma}^E|_s$ in Lemma 4.13 as a moving distance in S^{n-p} . We first approximate their cosine.

Lemma 4.30 Take $y_1 \in M$, $\tilde{y}_1 \in D_{f_{y_1}}(p_{y_1}) \cap R_{f_{y_1}} \cap Q_{f_{y_1}}$ with $d(y_1, \tilde{y}_1) \leq C\delta^{1/100n}$ and $y_2 \in D_{f_{y_1}}(\tilde{y}_1)$ (note that we can take such $\tilde{y}_1$ for any y_1 by the Bishop-Gromov theorem). Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis of T^*M along $\gamma_{\tilde{y}_1, y_2}$ in Lemma 4.13 for f_{y_1} . Then, (ii) holds in the lemma, and

$$|\cos(|\dot{\gamma}_{\tilde{y}_1, y_2}^E|_s) - \cos d_S(\Psi(y_1), \Psi(\gamma_{\tilde{y}_1, y_2}(s)))| \leq C\delta^{1/2000n^2}$$

for all $s \in [0, d(\tilde{y}_1, y_2)]$. In particular, we have

$$|\cos(|\dot{\gamma}_{\tilde{y}_1, y_2}^E|_{d(\tilde{y}_1, y_2)}) - \cos d_S(\Psi(y_1), \Psi(y_2))| \leq C\delta^{1/2000n^2}.$$

Proof By Corollary 4.29, we have $d(\tilde{y}_1, A_{y_1}) \leq C\delta^{1/2000n^2}$, and so we get

$$\begin{aligned} f_{y_1} \circ \gamma_{\tilde{y}_1, y_2}(s) &\geq \cos d(\gamma_{\tilde{y}_1, y_2}(s), A_{y_1}) - C\delta^{1/2000n} \\ &\geq \cos s - C\delta^{1/2000n^2} \geq \frac{1}{\sqrt{2}} - C\delta^{1/2000n^2} \end{aligned}$$

for all $s \leq \min\{\pi/4, d(\tilde{y}_1, y_2)\}$. Therefore, we have

$$|\nabla^2 f_{y_1}|(\gamma_{\tilde{y}_1, y_2}(s)) \geq \frac{1}{\sqrt{n}} |\Delta f_{y_1}|(\gamma_{\tilde{y}_1, y_2}(s)) \geq \frac{n-p}{\sqrt{2n}} - C\delta^{1/2000n^2}$$

for all $s \leq \min\{\pi/4, d(\tilde{y}_1, y_2)\}$. Thus, (i) in Lemma 4.13 cannot occur, and so (ii) holds in the lemma.

Since we have $f_{y_1}(y_1) = |\tilde{\Psi}(y_1)|$, we get

$$|f_{y_1}(\tilde{y}_1) - 1| \leq C\delta^{1/1000n^2} \quad (56)$$

by Lemma 4.22 and $d(y_1, \tilde{y}_1) \leq C\delta^{1/100n}$. By (56) and Proposition 4.17 (iii), we have $|\nabla f_{y_1}|(\tilde{y}_1) \leq C\delta^{1/2000n^2}$. Thus, we get

$$|f_{y_1}(\gamma_{\tilde{y}_1, y_2}(s)) - \cos(|\dot{\gamma}_{\tilde{y}_1, y_2}^E|_s)| \leq C\delta^{1/2000n^2}$$

for all $s \in [0, d(\tilde{y}_1, y_2)]$ by Lemma 4.13. On the other hand, we have

$$|f_{y_1}(\gamma_{\tilde{y}_1, y_2}(s)) - \cos d_S(\Psi(y_1), \Psi(\gamma_{\tilde{y}_1, y_2}(s)))| \leq C\delta^{1/2000n^2}$$

for all $s \in [0, d(\tilde{y}_1, y_2)]$ by Proposition 4.17 (iv) and Corollary 4.28. Thus, we get the lemma. $\square$

Notation 4.31 We use the following notation:

- For any $y_1, y_2 \in M$ and $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, define

$$\begin{aligned} G_f^{y_1}(y_2) &:= \langle \dot{\gamma}_{y_2, y_1}(0), \nabla f(y_2) \rangle d(y_1, y_2) \sin d_S(\Psi(y_1), \Psi(y_2)) \\ &\quad + \left(\cos d(y_2, A_f) \cos d_S(\Psi(y_1), \Psi(y_2)) - \cos d(y_1, A_f) \right) d_S(\Psi(y_1), \Psi(y_2)). \end{aligned}$$

- For any $y_1, y_2 \in M$, define

$$H^{y_1}(y_2) := \begin{cases} 1 & d(y_1, y_2) \leq \pi, \\ 0 & d(y_1, y_2) > \pi. \end{cases}$$

- For any $y_1, y_2 \in M$ and $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, define

$$\begin{aligned} C_f^{y_1}(y_2) &:= \left\{ y_3 \in M : \gamma_{y_2, y_3}(s) \in I_{y_1} \setminus \{y_1\} \text{ for almost all } s \in [0, d(y_2, y_3)], \text{ and} \right. \\ &\quad \left. \int_0^{d(y_2, y_3)} |G_f^{y_1} H^{y_1}(\gamma_{y_2, y_3}(s))| ds \leq \delta^{1/12000n^2} \right\}, \\ P_f^{y_1} &:= \{y_2 \in M : \text{Vol}(M \setminus C_f^{y_1}(y_2)) \leq \delta^{1/12000n^2} \text{Vol}(M)\}. \end{aligned}$$

Pinching condition on $G_f^{y_1}$ plays a crucial role for our purpose. Let us estimate $G_f^{y_1}$.

Lemma 4.32 Take $\eta > 0$ with $\eta \geq \delta^{1/2000n}$, $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, $y_1 \in Q_f$ and $y_2 \in D_f(y_1)$. Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis of T^*M along γ_{y_1, y_2} in Lemma 4.13 for f .

$$\|\dot{\gamma}_{y_1, y_2}^E |d(y_1, y_2) - d_S(\Psi(y_1), \Psi(y_2))\| \leq \eta,$$

then $|G_f^{y_1}(y_2)| \leq C\eta$.

Proof We have

$$\begin{aligned} & \left| f(y_1) - f(y_2) \cos(|\dot{\gamma}_{y_1, y_2}^E| d(y_1, y_2)) \right. \\ & \quad \left. - \frac{1}{|\dot{\gamma}_{y_1, y_2}^E|} \langle \nabla f(y_2), \dot{\gamma}_{y_2, y_1}(0) \rangle \sin(|\dot{\gamma}_{y_1, y_2}^E| d(y_1, y_2)) \right| \leq C\delta^{1/250} \end{aligned}$$

by Lemma 4.13. Thus, by Proposition 4.17 (iv), we get

$$\begin{aligned} & \left| |\dot{\gamma}_{y_1, y_2}^E| \cos d(y_1, A_f) - |\dot{\gamma}_{y_1, y_2}^E| \cos d(y_2, A_f) \cos(|\dot{\gamma}_{y_1, y_2}^E| d(y_1, y_2)) \right. \\ & \quad \left. - \langle \nabla f(y_2), \dot{\gamma}_{y_2, y_1}(0) \rangle \sin(|\dot{\gamma}_{y_1, y_2}^E| d(y_1, y_2)) \right| \leq C\delta^{1/2000n}, \end{aligned}$$

and so we get the lemma. $\square$

The quantity $|\dot{\gamma}_{y_1, y_2}^E|$ in the above lemma is slightly different from that of Lemma 4.30. Comparing these two quantity, we get the following:

Corollary 4.33 Take $\eta > 0$ with $\eta \geq \delta^{1/2000n}$, $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, $y_1 \in M$, $\tilde{y}_1 \in D_{f_{y_1}}(p_{y_1}) \cap R_{f_{y_1}} \cap Q_{f_{y_1}} \cap Q_f$ with $d(y_1, \tilde{y}_1) \leq C\delta^{1/100n}$ and $y_2 \in D_{f_{y_1}}(\tilde{y}_1) \cap D_f(\tilde{y}_1)$. Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis of T^*M along $\gamma_{\tilde{y}_1, y_2}$ in Lemma 4.13 for f_{y_1} . If

$$|\dot{\gamma}_{\tilde{y}_1, y_2}^E| d(\tilde{y}_1, y_2) - d_S(\Psi(\tilde{y}_1), \Psi(y_2)) \leq \eta,$$

then $|G_f^{\tilde{y}_1}(y_2)| \leq C\eta$.

Proof Let $\{\tilde{E}^1, \dots, \tilde{E}^n\}$ be a parallel orthonormal basis of T^*M along $\gamma_{\tilde{y}_1, y_2}$ in Lemma 4.13 for f (if (i) holds, then we can assume that $\tilde{E}^i = E^i$ for all i). We show that $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| - |\dot{\gamma}_{\tilde{y}_1, y_2}^{\tilde{E}}| \leq C\delta^{1/50}$. Then, we immediately get the corollary by Lemma 4.32.

We first suppose that Assumption 4.2 holds. We have $|\omega(y_2) - E^{n-p+1} \wedge \dots \wedge E^n| \leq C\delta^{1/25}$ by Lemmas 4.13 and 4.30. Since $|\dot{\gamma}_{\tilde{y}_1, y_2}^E|^2 = 1 - |\iota(\dot{\gamma}_{\tilde{y}_1, y_2})(E^{n-p+1} \wedge \dots \wedge E^n)|^2$, we get

$$|\dot{\gamma}_{\tilde{y}_1, y_2}^E|^2 - \left(1 - |\iota(\dot{\gamma}_{\tilde{y}_1, y_2})\omega|^2(y_2)\right) \leq C\delta^{1/25}. \quad (57)$$

Similarly, we get

$$|\dot{\gamma}_{\tilde{y}_1, y_2}^{\tilde{E}}|^2 - \left(1 - |\iota(\dot{\gamma}_{\tilde{y}_1, y_2})\omega|^2(y_2)\right) \leq C\delta^{1/25}. \quad (58)$$

By (57) and (58), we get $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| - |\dot{\gamma}_{\tilde{y}_1, y_2}^{\tilde{E}}| \leq C\delta^{1/50}$.

We next suppose that Assumption 4.3 holds. Similarly, we have

$$\begin{aligned} |\dot{\gamma}_{\tilde{y}_1, y_2}^E|^2 - |\iota(\dot{\gamma}_{\tilde{y}_1, y_2})\xi|^2(y_2) &\leq C\delta^{1/25}, \\ |\dot{\gamma}_{\tilde{y}_1, y_2}^{\tilde{E}}|^2 - |\iota(\dot{\gamma}_{\tilde{y}_1, y_2})\xi|^2(y_2) &\leq C\delta^{1/25}, \end{aligned}$$

and so $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| - |\dot{\gamma}_{\tilde{y}_1, y_2}^{\tilde{E}}| \leq C\delta^{1/50}$.

By the above two cases, we get the corollary. $\square$

Let us show the integral pinching condition.

Lemma 4.34 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$, $y_1 \in M$ and $\tilde{y}_1 \in D_{f_{y_1}}(p_{y_1}) \cap R_{f_{y_1}} \cap Q_{f_{y_1}} \cap Q_f$ with $d(y_1, \tilde{y}_1) \leq C\delta^{1/100n}$. Then, $\|G_f^{\tilde{y}_1} H_{\tilde{y}_1}\|_1 \leq C\delta^{1/4000n^2}$ and $\text{Vol}(M \setminus P_f^{\tilde{y}_1}) \leq C\delta^{1/12000n^2}$.

Proof Take arbitrary $y_2 \in D_f(\tilde{y}_1) \cap D_{f_{y_1}}(\tilde{y}_1)$. Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis of T^*M along $\gamma_{\tilde{y}_1, y_2}$ in Lemma 4.13 for f_{y_1} . Then, we have $\|\dot{\gamma}_{\tilde{y}_1, y_2}^E\| |d(\tilde{y}_1, y_2) - d_S(\Psi(\tilde{y}_1), \Psi(y_2))| \leq C\delta^{1/4000n^2}$, if $d(\tilde{y}_1, y_2) \leq \pi$ by Lemmas 4.25 and 4.30. Thus, by Corollary 4.33, we have

$$\sup_{D_f(\tilde{y}_1) \cap D_{f_{y_1}}(\tilde{y}_1)} |G_f^{\tilde{y}_1} H^{\tilde{y}_1}| \leq C\delta^{1/4000n^2}.$$

Since $\text{Vol}(M \setminus (D_f(\tilde{y}_1) \cap D_{f_{y_1}}(\tilde{y}_1))) \leq C\delta^{1/100}\text{Vol}(M)$ and $\|G_f^{\tilde{y}_1} H^{\tilde{y}_1}\|_\infty \leq C$, we get $\|G_f^{\tilde{y}_1} H^{\tilde{y}_1}\|_1 \leq C\delta^{1/4000n^2}$. By the segment inequality (Theorem 4.6), we get the remaining part of the lemma. $\square$

Notation 4.35 We use the following notation.

$$\eta_0 = \delta^{1/12000n^3}, \quad \eta_1 = \eta_0^{1/26}, \quad \eta_2 = \eta_1^{1/78} \text{ and } L = \eta_2^{1/150}.$$

We use Lemma 4.34 to give the almost Pythagorean theorem for the special case (see Lemma 4.43). For the general case, we need to estimate $\|G_f^{\tilde{y}_1}\|_1$. To do this, we show that $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| |d(\tilde{y}_1, y_2) \leq \pi + L$ under the assumption of Lemma 4.30 in Lemma 4.45. Then, we can estimate $\|G_f^{\tilde{y}_1}\|_1$ similarly to Lemma 4.34. After proving that, we use Lemma 4.38 again to give the almost Pythagorean theorem for the general case. The following lemma, which guarantees that an almost shortest pass from a point in M to A_f almost corresponds to a geodesic in S^{n-p} through Ψ under some assumptions, is the first step to achieve these objectives.

Lemma 4.36 Take

- $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$,
- $u \in S^{n-p}$ with $f = \sum_{i=1}^{n-p+1} u_i f_i$,
- $x, y \in M$,
- $\eta > 0$ with $\eta_0 \leq \eta \leq L^{1/3n}$.

Suppose

- $d(y, A_f) \leq C\eta$,
- $|d(x, A_f) - d(x, y)| \leq C\eta$.

Then, we have the following for all $s, s' \in [0, d(x, y)]$:

- $|d(\gamma_{y,x}(s), A_f) - s| \leq C\eta$,
- $||s - s'| - d_S(\Psi(\gamma_{y,x}(s)), \Psi(\gamma_{y,x}(s')))| \leq C\eta$,
- If in addition $d(x, A_f) \geq \frac{1}{C}\eta^{1/26}$, there exists $v \in S^{n-p}$ such that $u \cdot v = 0$ and

$$d_S(\Psi(\gamma_{y,x}(s)), \gamma_v(s)) \leq C\eta^{3/13}$$

for all $s \in [0, d(x, y)]$, where we define $\gamma_v(s) := (\cos s)u + (\sin s)v \in S^{n-p}$.

Proof We first prove (i). We have $d(\gamma_{y,x}(s), A_f) \leq s + C\eta$ and

$$d(x, y) - C\eta \leq d(x, A_f) \leq d(\gamma_{y,x}(s), A_f) + d(x, y) - s.$$

Thus, we get (i).

We next prove (ii). By Lemma 4.26, we have $d_S(\Psi(y), u) \leq C\eta$ and $|d_S(\Psi(\gamma_{y,x}(s)), u) - d(\Psi(\gamma_{y,x}(s)), A_f)| \leq C\delta^{1/2000n^2}$, and so we get

$$|s - d_S(\Psi(\gamma_{y,x}(s)), \Psi(y))| \leq C\eta \quad (59)$$

for all $s \in [0, d(x, y)]$ by (i). Take arbitrary $s, s' \in [0, d(x, y)]$ with $s < s'$. Then,

$$\begin{aligned} s' - s &= d(\gamma_{y,x}(s), \gamma_{y,x}(s')) \geq d(\gamma_{y,x}(s), A_{\gamma_{y,x}(s')}) - d(\gamma_{y,x}(s'), A_{\gamma_{y,x}(s')}) \\ &\geq d_S(\Psi(\gamma_{y,x}(s)), \Psi(\gamma_{y,x}(s'))) - C\delta^{1/2000n^2} \end{aligned} \quad (60)$$

by Corollaries 4.28 and 4.29. On the other hand, we have

$$\begin{aligned} s' - C\eta &\leq d_S(\Psi(\gamma_{y,x}(s')), \Psi(y)) \\ &\leq d_S(\Psi(\gamma_{y,x}(s)), \Psi(\gamma_{y,x}(s'))) + d_S(\Psi(\gamma_{y,x}(s)), \Psi(y)) \\ &\leq d_S(\Psi(\gamma_{y,x}(s)), \Psi(\gamma_{y,x}(s'))) + s + C\eta \end{aligned}$$

by (59), and so

$$s' - s \leq d_S(\Psi(\gamma_{y,x}(s)), \Psi(\gamma_{y,x}(s'))) + C\eta. \quad (61)$$

By (60) and (61), we get (ii).

Finally, we prove (iii). Since $d(x, A_f) \geq \frac{1}{C}\eta^{1/26}$, there exists $s_0 \in [0, d(x, y)]$ such that $\frac{1}{C}\eta^{1/26} \leq d(z, y) \leq \pi - \frac{1}{C}\eta^{1/26}$, where we put $z = \gamma_{y,x}(s_0)$. Then, there exists $v \in S^{n-p}$ with $u \cdot v = 0$ and $t_1 \in [0, \pi]$ such that $\Psi(z) = (\cos t_1)u + (\sin t_1)v$. We have

$$\begin{aligned} |\cos t_1 - \cos d(z, y)| &= |\cos d_S(\Psi(z), u) - \cos s_0| \\ &\leq |\cos d(z, A_f) - \cos s_0| + C\delta^{1/2000n^2} \leq C\eta \end{aligned}$$

by Lemma 4.26 and (i). This gives

$$|t_1 - d(z, y)| \leq C\eta^{1/2}. \quad (62)$$

Take arbitrary $s \in [0, d(x, y)]$. Then, there exist $w \in S^{n-p}$ and $x_1, x_2, x_3 \in \mathbb{R}$ such that $w \perp \text{Span}_{\mathbb{R}}\{u, v\}$, $x_1^2 + x_2^2 + x_3^2 = 1$ and $\Psi(\gamma_{y,x}(s)) = x_1u + x_2v + x_3w$. Since we have $|s - d_S(\Psi(\gamma_{y,x}(s)), u)| \leq C\eta$ by (i) and Lemma 4.26, and $\cos d_S(\Psi(\gamma_{y,x}(s)), u) = x_1$, we get

$$|\cos s - x_1| \leq C\eta. \quad (63)$$

We have

$$|d(z, y) - s| - d_S(\Psi(\gamma_{y,x}(s)), \Psi(z))| \leq C\eta$$

by (ii). Since $\cos d_S(\Psi(\gamma_{y,x}(s)), \Psi(z)) = x_1 \cos t_1 + x_2 \sin t_1$, we get

$$|\cos(d(z, y) - s) - x_1 \cos d(z, y) - x_2 \sin d(z, y)| \leq C\eta^{1/2} \quad (64)$$

by (62). By (63) and (64), we have $|\sin d(z, y) - \sin s - x_2| \leq C\eta^{1/2}$. By the assumption, we have $\sin d(z, y) \geq \frac{1}{C}\eta^{1/26}$, and so we get

$$|\sin s - x_2| \leq C\eta^{6/13}. \quad (65)$$

By (63) and (65), we get

$$|\cos d_S(\Psi(\gamma_{y,x}(s)), \gamma_v(s)) - 1| = |x_1 \cos s + x_2 \sin s - 1| \leq C\eta^{6/13}.$$

Thus, we get (iii). $\square$

The following lemma asserts that the differential of an almost shortest pass from a point in M to A_f is in the direction of ∇f under some assumptions.

Lemma 4.37 *Take*

- $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$,
- $x \in D_f(p_f) \cap Q_f \cap R_f$,
- $y \in D_f(x) \cap D_f(p_f) \cap Q_f \cap R_f$,
- $\eta > 0$ with $\eta_0 \leq \eta \leq L^{1/3n}$.

Suppose

- $d(x, A_f) \geq \frac{1}{C}\eta^{1/26}$,
- $d(y, A_f) \leq C\eta$,
- $|d(x, A_f) - d(x, y)| \leq C\eta$.

Let $\{E^1, \dots, E^n\}$ be a parallel orthonormal basis of T^*M along $\gamma_{x,y}$ in Lemma 4.13 for f . Then, we have the following for all $s \in [0, d(x, y)]$:

- (i) $|\dot{\gamma}_{x,y}^E - 1| \leq C\eta^{6/13}$,
- (ii) $|\nabla f(\gamma_{y,x}(s)) + (\sin s)\dot{\gamma}_{y,x}(s)| \leq C\eta^{3/26}$.

Proof We first note that we have

$$d(x, y) \leq \pi + C\eta \quad (66)$$

by the assumption and Proposition 4.17 (iv).

Let us prove (i). By $d(y, A_f) \leq C\eta$, we have $\cos d(y, A_f) \geq 1 - C\eta^2$. Thus, we have

$$|1 - f(y)| \leq C\eta^2 \quad (67)$$

by Proposition 4.17 (iv). By Proposition 4.17 (iii), we get $|\nabla f|(y) \leq C\eta$. Thus, we have

$$|f(x) - \cos(|\dot{\gamma}_{x,y}^E|d(x, y))| \leq C\eta \quad (68)$$

by Lemma 4.13, and so $||\dot{\gamma}_{x,y}^E|d(x, y) - d(x, A_f)| \leq C\eta^{1/2}$ by Proposition 4.17 (iv) and (66). By the assumptions, we get (i).

We next prove (ii). By Proposition 4.17, we have $||\nabla f|^2(x) - \sin^2 d(x, A_f)| \leq C\delta^{1/2000n}$, and so $||\nabla f|(x) - |\sin d(x, A_f)|| \leq C\delta^{1/4000n}$. Since $\sin d(x, A_f) \geq -C\delta^{1/100n}$ by Proposition 4.17 (iv), we have $||\nabla f|(x) - \sin d(x, A_f)| \leq C\delta^{1/4000n}$. Thus, we get

$$||\nabla f|(x) - \sin d(x, y)| \leq C\eta \quad (69)$$

by the assumption. On the other hand, by (i) and Lemma 4.13, we have $|f(y) - f(x) \cos d(x, y) - \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \sin d(x, y)| \leq C\eta^{6/13}$, and so

$$|\sin^2 d(x, y) - \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \sin d(x, y)| \leq C\eta^{6/13} \quad (70)$$

by (67) and (68).

We consider the following two cases:

- $d(x, y) \leq \pi - \eta^{3/13}$,
- $d(x, y) > \pi - \eta^{3/13}$.

We first suppose that $d(x, y) \leq \pi - \eta^{3/13}$. We get $|\sin d(x, y) - \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle| \leq C\eta^{3/13}$ by the assumption and (70). By (69), we get

$$|\nabla f|(x) - \langle \nabla f(x), \dot{\gamma}_{x,y}(0) \rangle \leq C\eta^{3/13}. \quad (71)$$

We next suppose that $d(x, y) > \pi - \eta^{3/13}$. Then, we have $\cos d(x, A_f) \leq -1 + C\eta^{6/13}$, and so $|\nabla f|(x) \leq C\eta^{3/13}$ by Proposition 4.17 (iii) and (iv). Thus, we also get (71) for this case.

By (i), (48) and Lemma 4.13, we have

$$\int_0^{d(x,y)} \left| \frac{d}{ds} \left(|\nabla f|^2(\gamma_{x,y}(s)) - \langle \nabla f(\gamma_{x,y}(s)), \dot{\gamma}_{x,y}(s) \rangle^2 \right) \right| ds \leq C\eta^{6/13}.$$

Thus, we get

$$|\nabla f|^2(\gamma_{x,y}(s)) - \langle \nabla f(\gamma_{x,y}(s)), \dot{\gamma}_{x,y}(s) \rangle^2 \leq C\eta^{3/13} \quad (72)$$

for all $s \in [0, d(x, y)]$ by (71). Since

$$\begin{aligned} & |\nabla f(\gamma_{x,y}(s)) - \langle \nabla f(\gamma_{x,y}(s)), \dot{\gamma}_{x,y}(s) \rangle \dot{\gamma}_{x,y}(s)|^2 \\ &= |\nabla f|^2(\gamma_{x,y}(s)) - \langle \nabla f(\gamma_{x,y}(s)), \dot{\gamma}_{x,y}(s) \rangle^2, \end{aligned}$$

we get $|\nabla f(\gamma_{x,y}(s)) - \langle \nabla f(\gamma_{x,y}(s)), \dot{\gamma}_{x,y}(s) \rangle \dot{\gamma}_{x,y}(s)| \leq C\eta^{3/26}$ by (72). Since we have

$$|\langle \nabla f(\gamma_{x,y}(s)), \dot{\gamma}_{x,y}(s) \rangle + \cos d(x, y) \sin s - \sin d(x, y) \cos s| \leq C\eta^{3/13}$$

by (68), (69), (71), (i) and Lemma 4.13, we get

$$|\nabla f(\gamma_{x,y}(s)) - \sin(d(x, y) - s) \dot{\gamma}_{x,y}(s)| \leq C\eta^{3/26}$$

This gives (ii). $\square$

The following lemma is crucial to show the almost Pythagorean theorem.

Lemma 4.38 *Take*

- $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$,
- $x \in D_f(p_f) \cap Q_f \cap R_f$,
- $y \in D_f(x) \cap D_f(p_f) \cap Q_f \cap R_f$,
- $z \in M$,
- $\eta > 0$ with $\eta_0 \leq \eta \leq L^{1/3n}$ and $T \in [0, d(x, y)]$.

Suppose

- $d(y, A_f) \leq C\eta$,
- $|d(x, A_f) - d(x, y)| \leq C\eta$,
- $\gamma_{y,x}(s) \in I_z \setminus \{z\}$ for almost all $s \in [T, d(x, y)]$,
- $\int_T^{d(x,y)} |G_f^z(\gamma_{y,x}(s))| ds \leq C\eta^{3/26}$.

Then, we have

$$\left| d(z, x)^2 - d_S(\Psi(z), \Psi(x))^2 - d(z, \gamma_{y,x}(T))^2 + d_S(\Psi(z), \Psi(\gamma_{y,x}(T)))^2 \right| \leq C\eta^{1/26}.$$

Proof If $d(x, A_f) \leq \eta^{1/26}$, then $d(x, y) \leq C\eta^{1/26}$, and so $d(x, \gamma_{y,x}(T)) \leq C\eta^{1/26}$. Thus, we immediately get the lemma by Lemma 4.25 if $d(x, A_f) \leq \eta^{1/26}$. In the following, we assume that $d(x, A_f) \geq \eta^{1/26}$. Take $u \in S^{n-p}$ with $f = \sum_{i=1}^{n-p+1} u_i f_i$, and $v \in S^{n-p}$ of Lemma 4.36 (iii). Define

$$r(s) := d_S(\Psi(z), \gamma_v(s)).$$

Then, by the triangle inequality and Lemma 4.36 (iii), we have

$$|r(s) - d_S(\Psi(z), \Psi(\gamma_{y,x}(s)))| \leq C\eta^{3/13}. \quad (73)$$

There exist $w \in S^{n-p}$ and $x_1, x_2, x_3 \in \mathbb{R}$ such that $w \perp \text{Span}_{\mathbb{R}}\{u, v\}$, $x_1^2 + x_2^2 + x_3^2 = 1$ and $\Psi(z) = x_1 u + x_2 v + x_3 w$. Then,

$$\cos r(s) = x_1 \cos s + x_2 \sin s \quad (74)$$

by the definition of γ_v in Lemma 4.36 (iii), and so

$$-x_1 \sin s + x_2 \cos s = \frac{d}{ds} \cos r(s) = -r'(s) \sin r(s).$$

Thus, we get

$$-r'(s) \sin r(s) \sin s = -x_1 \sin^2 s + x_2 \sin s \cos s = \cos r(s) \cos s - x_1 \quad (75)$$

by (74). Since $x_1 = \Psi(z) \cdot u$ and $f(z) = \tilde{\Psi}(z) \cdot u$, we have

$$|x_1 - \cos d(z, A_f)| \leq C\delta^{1/1000n^2} \quad (76)$$

by Proposition 4.17 (iv) and Lemma 4.22. By Lemma 4.36, (73), (75) and (76), we get

$$\begin{aligned} & \left| \left(\cos d(\gamma_{y,x}(s), A_f) \cos d_S(\Psi(z), \Psi(\gamma_{y,x}(s))) \right. \right. \\ & \quad \left. \left. - \cos d(z, A_f) \right) d_S(\Psi(z), \Psi(\gamma_{y,x}(s))) \right. \\ & \quad \left. + r'(s)r(s) \sin r(s) \sin s \right| \leq C\eta^{3/13}. \end{aligned} \quad (77)$$

Define

$$l(s) := d(z, \gamma_{y,x}(s)).$$

Then, we have $l'(s) = \langle \dot{\gamma}_{z, \gamma_{y,x}(s)}(l(s)), \dot{\gamma}_{y,x}(s) \rangle$ for all $s \in [0, d(x, y)]$ with $\gamma_{y,x}(s) \in I_z \setminus \{z\}$, and so $|l'(s) \sin s + \langle \dot{\gamma}_{z, \gamma_{y,x}(s)}(l(s)), \nabla f(\gamma_{y,x}(s)) \rangle| \leq C\eta^{3/26}$ by Lemma 4.37 (ii). Thus, for almost all $s \in [T, d(x, y)]$, we have

$$\begin{aligned} & \left| \langle \dot{\gamma}_{\gamma_{y,x}(s), z}(0), \nabla f(\gamma_{y,x}(s)) \rangle l(s) \sin d_S(\Psi(z), \Psi(\gamma_{y,x}(s))) \right. \\ & \quad \left. - l'(s)l(s) \sin r(s) \sin s \right| \leq C\eta^{3/26} \end{aligned} \quad (78)$$

by (73). By the definition of G_f^z , (77) and (78), for almost all $s \in [T, d(x, y)]$, we have

$$\left| G_f^z(\gamma_{y,x}(s)) - l'(s)l(s) \sin r(s) \sin s + r'(s)r(s) \sin r(s) \sin s \right| \leq C\eta^{3/26}.$$

Thus, by the assumption, we get

$$\int_T^{d(x,y)} \left| \left(\frac{d}{ds} (l(s)^2 - r(s)^2) \right) \sin r(s) \sin s \right| ds \leq C\eta^{3/26}. \quad (79)$$

Define

$$I := \{s \in [T, d(x, y)] : \eta^{1/26} \leq s \leq \pi - \eta^{1/26} \text{ and } \eta^{1/26} \leq r(s) \leq \pi - \eta^{1/26}\}$$

$$II := [T, d(x, y)] \setminus I.$$

Then, we have

$$\int_I \left| \frac{d}{ds} (l(s)^2 - r(s)^2) \right| ds \leq C\eta^{1/26} \quad (80)$$

by (79). Let us estimate $H^1(II)$, where H^1 denotes the 1-dimensional Hausdorff measure. Suppose that

$$\{s \in [T, d(x, y)] : r(s) < \eta^{1/26} \text{ or } r(s) > \pi - \eta^{1/26}\} \neq \emptyset,$$

and take arbitrary $s \in [T, d(x, y)]$ such that $r(s) < \eta^{1/26}$ or $r(s) > \pi - \eta^{1/26}$. Then, we have

$$||\cos r(s)| - 1| \leq C\eta^{1/13}. \quad (81)$$

Note that we have $r(s) \leq \pi$ by $\text{diam}(S^{n-p}) = \pi$. By (74), we get

$$1 - C\eta^{1/13} \leq (x_1^2 + x_2^2)^{1/2} \leq 1. \quad (82)$$

Take $s_1 \in [0, 2\pi]$ such that

$$\cos s_1 = \frac{x_1}{(x_1^2 + x_2^2)^{1/2}},$$

$$\sin s_1 = \frac{x_2}{(x_1^2 + x_2^2)^{1/2}}.$$

Then, we get $||\cos(s - s_1)| - 1| \leq C\eta^{1/13}$ by (74), (81) and (82). Thus, there exists $n \in \mathbb{Z}$ such that $|s - s_1 - n\pi| \leq C\eta^{1/26}$. Then, we have $|n| \leq 2$, and so

$$H^1\left(\{s \in [T, d(x, y)] : r(s) < \eta^{1/26} \text{ or } r(s) > \pi - \eta^{1/26}\}\right) \leq C\eta^{1/26}.$$

Note that we have $d(x, y) \leq d(x, A_f) + C\eta \leq \pi + C\eta$ by the assumption and Proposition 4.17 (iv). Since we have

$$H^1\left(\{s \in [T, d(x, y)] : s < \eta^{1/26} \text{ or } s > \pi - \eta^{1/26}\}\right) \leq C\eta^{1/26},$$

we get $H^1(II) \leq C\eta^{1/26}$. Since $\left|\frac{d}{ds}(l(s)^2 - r(s)^2)\right| \leq C$ for almost all $s \in [T, d(x, y)]$, we get

$$\int_{II} \left| \frac{d}{ds} (l(s)^2 - r(s)^2) \right| ds \leq C\eta^{1/26}. \quad (83)$$

By (80) and (83), we get

$$\int_T^{d(x,y)} \left| \frac{d}{ds} (l(s)^2 - r(s)^2) \right| ds \leq C\eta^{1/26}.$$

Thus, we have $|l(d(x, y))^2 - r(d(x, y))^2 - l(T)^2 + r(T)^2| \leq C\eta^{1/26}$. By (73) and the definition of l , we get the lemma. $\square$

Definition 4.39 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$. By Lemma 4.34 and the Bishop-Gromov inequality, for any triple $(x_1, x_2, x_3) \in M \times M \times M$, we can take points $\tilde{x}_1 \in D_{f_{x_1}}(p_{x_1}) \cap Q_{f_{x_1}} \cap R_{f_{x_1}} \cap Q_f$, $\tilde{x}_2 \in D_f(p_f) \cap Q_f \cap R_f \cap P_f^{\tilde{x}_1}$ and $\tilde{x}_3 \in D_f(\tilde{x}_2) \cap D_f(p_f) \cap Q_f \cap R_f \cap C_f^{\tilde{x}_1}(\tilde{x}_2)$ such that $d(x_1, \tilde{x}_1) \leq C\delta^{1/100n}$, $d(x_2, \tilde{x}_2) \leq C\eta_0$, $d(x_3, \tilde{x}_3) \leq C\eta_0$. We call the triple $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ a “ Π -triple for (x_1, x_2, x_3, f) ”.

Lemma 4.40 Take

- $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$,
- $x, y, z \in M$,
- $\eta > 0$ with $\eta_0 \leq \eta \leq L^{1/3n}$ and $T \in [0, d(x, y)]$.

Take a Π -triple $(\tilde{z}, \tilde{x}, \tilde{y})$ for (z, x, y, f) . Suppose

- $d(y, A_f) \leq C\eta$,
- $|d(x, A_f) - d(x, y)| \leq C\eta$,
- $d(\tilde{z}, \gamma_{\tilde{y}, \tilde{x}}(s)) \leq \pi$ for all $s \in [T, d(\tilde{x}, \tilde{y})]$.

Then, we have

$$\left| d(\tilde{z}, \tilde{x})^2 - d_S(\Psi(\tilde{z}), \Psi(\tilde{x}))^2 - d(\tilde{z}, \gamma_{\tilde{y}, \tilde{x}}(T))^2 + d_S(\Psi(\tilde{z}), \Psi(\gamma_{\tilde{y}, \tilde{x}}(T)))^2 \right| \leq C\eta^{1/26}.$$

Proof We have $(G_f^{\tilde{z}} H^{\tilde{z}})(\gamma_{\tilde{y}, \tilde{x}}(s)) = G_f^{\tilde{z}}(\gamma_{\tilde{y}, \tilde{x}}(s))$ for all $s \in [T, d(\tilde{x}, \tilde{y})]$. Thus, we get the lemma immediately by the definition of $C_f^{\tilde{z}}(\tilde{x})$ and Lemma 4.38. $\square$

The following lemma guarantees that if the images of two points in M under Φ_f are close to each other in $S^{n-p} \times A_f$, then their distance in M are close to each other under some assumptions.

Lemma 4.41 Take

- $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$,
- $x, y, z \in M$,
- $\eta > 0$ with $\eta_0 \leq \eta \leq L^{1/3n}$.

Suppose

- $d(x, A_f) \leq \pi - \frac{1}{C}\eta^{1/78}$ and $d(z, A_f) \leq \pi - \frac{1}{C}\eta^{1/78}$,
- $d(y, A_f) \leq C\eta$,
- $|d(x, A_f) - d(x, y)| \leq C\eta$ and $|d(z, A_f) - d(z, y)| \leq C\eta$

- $d_S(\Psi(x), \Psi(z)) \leq C\eta$.

Then, we have $d(x, z) \leq C\eta^{1/52}$.

Proof We first show the following claim.

Claim 4.42 *If $x, y, z \in M$ satisfies:*

- $d(x, A_f) \leq \frac{1}{2}\pi - \frac{1}{C}\eta^{1/2}$ and $d(z, A_f) \leq \frac{1}{2}\pi - \frac{1}{C}\eta^{1/2}$,
- $d(y, A_f) \leq C\eta$,
- $|d(x, A_f) - d(x, y)| \leq C\eta$ and $|d(z, A_f) - d(z, y)| \leq C\eta$,
- $d_S(\Psi(x), \Psi(z)) \leq C\eta^{1/52}$.

Then, we have $d(x, z) \leq C\eta^{1/52}$.

Proof of Claim 4.42 Take $u \in S^{n-p}$ with $f = \sum_{i=1}^{n-p+1} u_i f_i$. By the assumptions and Lemma 4.26, we have

$$\begin{aligned} d_S(u, \Psi(y)) &\leq C\eta, \\ |d_S(\Psi(z), u) - d(z, A_f)| &\leq C\delta^{1/2000n^2}. \end{aligned}$$

Since we have $|d(z, A_f) - d(z, y)| \leq C\eta$ by the assumptions, we get

$$|d_S(\Psi(z), \Psi(y)) - d(z, y)| \leq C\eta. \quad (84)$$

Take a Π -triple $(\tilde{z}, \tilde{x}, \tilde{y})$ for (z, x, y, f) . Then, we have

$$d(\tilde{z}, \gamma_{\tilde{y}, \tilde{x}}(s)) \leq d(z, y) + d(y, x) + C\eta_0 \leq \pi - \frac{1}{C}\eta^{1/2} + C\eta \leq \pi$$

for all $s \in [0, d(\tilde{x}, \tilde{y})]$, and so

$$\left| d(z, x)^2 - d_S(\Psi(z), \Psi(x))^2 - d(z, y)^2 + d_S(\Psi(z), \Psi(y))^2 \right| \leq C\eta^{1/26}$$

by Lemmas 4.25 and 4.40. Thus, we get $d(x, z) \leq C\eta^{1/52}$ by (84). $\square$

Let us suppose that $x, y, z \in M$ satisfies the assumptions of the lemma. Take $u \in S^{n-p}$ with $f = \sum_{i=1}^{n-p+1} u_i f_i$. By the assumptions and Lemma 4.26, we have

$$|d(x, A_f) - d(z, A_f)| \leq |d_S(\Psi(x), u) - d_S(\Psi(z), u)| + C\delta^{1/2000n^2} \leq C\eta \quad (85)$$

Thus, if either $d(x, A_f) \leq \eta^{1/26}$ or $d(z, A_f) \leq \eta^{1/26}$ holds, then the lemma is trivial. In the following, we assume $d(x, A_f) \geq \eta^{1/26}$ and $d(z, A_f) \geq \eta^{1/26}$. Take a Π -triple $(\tilde{z}, \tilde{x}, \tilde{y})$ for (z, x, y, f) . By Lemma 4.36 (iii), we can take $v_1, v_2 \in S^{n-p}$ such that $u \cdot v_i = 0$ ($i = 1, 2$),

$$d_S(\Psi(\gamma_{\tilde{y}, \tilde{x}}(s)), \gamma_{v_1}(s)) \leq C\eta^{3/13} \quad (86)$$

for all $s \in [0, d(\tilde{y}, \tilde{x})]$ and

$$d_S(\Psi(\gamma_{\tilde{y}, \tilde{z}}(s)), \gamma_{v_2}(s)) \leq C\eta^{3/13} \quad (87)$$

for all $s \in [0, d(\tilde{y}, \tilde{z})]$, where $\gamma_{v_i}(s) := (\cos s)u + (\sin s)v_i \in S^{n-p}$ ($i = 1, 2$). By the assumptions and (85), we get

$$|d(\tilde{y}, \tilde{x}) - d(\tilde{y}, \tilde{z})| \leq C\eta, \quad (88)$$

and so

$$\begin{aligned} \sin d(\tilde{y}, \tilde{x})|v_1 - v_2| &\leq Cd_S(\gamma_{v_1}(d(\tilde{y}, \tilde{x})), \gamma_{v_2}(d(\tilde{y}, \tilde{x}))) \\ &\leq Cd_S(\Psi(\tilde{x}), \Psi(\tilde{z})) + C\eta^{3/13} \leq C\eta^{3/13} \end{aligned}$$

by (86) and (87). By $\eta^{1/26} \leq d(x, A_f) \leq \pi - \frac{1}{C}\eta^{1/78}$, we have $\sin d(\tilde{y}, \tilde{x}) \geq \frac{1}{C}\eta^{1/26}$. Thus, we get $|v_1 - v_2| \leq C\eta^{1/26}$. This gives

$$d_S(\gamma_{v_1}(s), \gamma_{v_2}(s)) \leq C\eta^{1/26}. \quad (89)$$

for all $s \in \mathbb{R}$.

Put $a := \gamma_{\tilde{y}, \tilde{x}}(d(\tilde{y}, \tilde{x})/2)$ and $b := \gamma_{\tilde{y}, \tilde{z}}(d(\tilde{y}, \tilde{z})/2)$. By (86), (87), (88) and (89), we have $d_S(\Psi(a), \Psi(b)) \leq C\eta^{1/26}$. Moreover, other assumptions of Claim 4.42 hold for the pair (a, y, b) by Lemma 4.36 (i), and so $d(a, b) \leq C\eta^{1/52}$. Therefore, we have

$$\begin{aligned} d(\tilde{z}, \gamma_{\tilde{y}, \tilde{x}}(s)) &\leq d(\tilde{z}, b) + d(a, b) + d(\gamma_{\tilde{y}, \tilde{x}}(s), a) \\ &\leq \frac{1}{2}d(\tilde{x}, \tilde{y}) + \frac{1}{2}d(\tilde{z}, \tilde{y}) + C\eta^{1/52} \leq \pi \end{aligned}$$

for all $s \in [0, d(\tilde{y}, \tilde{x})]$, and so $d(\tilde{x}, \tilde{z}) \leq C\eta^{1/52}$ similarly to Claim 4.42. Thus, we get the lemma. $\square$

Let us show the almost Pythagorean theorem for the special case. Recall that we defined $\eta_1 := \eta_0^{1/26}$.

Lemma 4.43 *Take*

- $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$,
- $x, y, z, w \in M$,
- $\eta > 0$ with $\eta_1 \leq \eta \leq L^{1/3n}$.

Suppose

- $d(x, z) \leq C\eta$,
- $d(x, A_f) \leq \pi - \frac{1}{C}\eta^{1/2}$ and $d(z, A_f) \leq \pi - \frac{1}{C}\eta^{1/2}$,
- $d(y, A_f) \leq C\eta_0$ and $d(w, A_f) \leq C\eta_0$,
- $|d(x, A_f) - d(x, y)| \leq C\eta_0$ and $|d(z, A_f) - d(z, w)| \leq C\eta_0$.

Then, we have

$$|d(x, z)^2 - d_S(\Psi(x), \Psi(z))^2 - d(y, w)^2| \leq C\eta_1.$$

Proof By Lemma 4.26, we have

$$d_S(\Psi(y), \Psi(w)) \leq d(y, A_f) + d(w, A_f) + C\delta^{1/2000n^2} \leq C\eta_0. \quad (90)$$

Put $a_0 := x$ and $b_0 := z$. In the following, we define $a_i, b_i \in M$ ($i = 1, 2, 3$) so that

- (i) $d(a_i, b_i) \leq C\eta^{1/2}$,
- (ii) $|d(a_i, A_f) - d(a_i, y)| \leq C\eta_0$ and $|d(b_i, A_f) - d(b_i, w)| \leq C\eta_0$,
- (iii) $d(a_i, A_f) \leq \frac{3-i}{3}\pi + C\eta_0$ and $d(b_i, A_f) \leq \frac{3-i}{3}\pi + C\eta_0$,
- (iv) $|d(a_{i+1}, b_{i+1})^2 - d_S(\Psi(a_{i+1}), \Psi(b_{i+1}))^2 - d(a_i, b_i)^2 + d_S(\Psi(a_i), \Psi(b_i))^2| \leq C\eta_0^{1/26}$ ($i = 0, 1, 2$),
- (v) $d(y, a_3) \leq C\eta_0$ and $d(w, b_3) \leq C\eta_0$.

If we succeed in defining such a_i and b_i , we have

$$|d(x, z)^2 - d_S(\Psi(x), \Psi(z))^2 - d(y, w)^2 + d_S(\Psi(y), \Psi(w))^2| \leq C\eta_0^{1/26} = C\eta_1$$

by (iv) and (v), and so we get the lemma by (90).

Take arbitrary $i \in \{0, 1, 2\}$ and suppose that we have chosen $a_i, b_i \in M$ such that (i), (ii) and (iii) hold if $i \geq 1$. Let us define $a_{i+1}, b_{i+1} \in M$ that satisfy our properties. Take a Π -triple $(\tilde{b}_i, \tilde{a}_i, \tilde{y}_i)$ for (b_i, a_i, y, f) . Define

$$a_{i+1} := \gamma_{\tilde{y}_i, \tilde{a}_i} \left(\frac{2-i}{3-i} d(\tilde{y}_i, \tilde{a}_i) \right).$$

Since

$$d(\tilde{b}_i, \gamma_{\tilde{y}_i, \tilde{a}_i}(s)) \leq d(\tilde{a}_i, \tilde{b}_i) + d(\tilde{a}_i, \gamma_{\tilde{y}_i, \tilde{a}_i}(s)) \leq \frac{\pi}{3} + C\eta^{1/2}$$

for all $s \in \left[\frac{2-i}{3-i} d(\tilde{y}_i, \tilde{a}_i), d(\tilde{y}_i, \tilde{a}_i) \right]$ by the assumptions, we get

$$|d(a_{i+1}, b_i)^2 - d_S(\Psi(a_{i+1}), \Psi(b_i))^2 - d(a_i, b_i)^2 + d_S(\Psi(a_i), \Psi(b_i))^2| \leq C\eta_0^{1/26} \quad (91)$$

by Lemmas 4.25 and 4.40. Take a Π -triple $(\bar{a}_{i+1}, \bar{b}_i, \bar{w}_i)$ for (a_{i+1}, b_i, w, f) . Define

$$b_{i+1} := \gamma_{\bar{w}_i, \bar{b}_i} \left(\frac{2-i}{3-i} d(\bar{w}_i, \bar{b}_i) \right).$$

Since

$$d(\bar{a}_{i+1}, \gamma_{\bar{w}_i, \bar{b}_i}(s)) \leq d(\bar{a}_{i+1}, a_i) + d(a_i, \bar{b}_i) + d(\bar{b}_i, \gamma_{\bar{w}_i, \bar{b}_i}(s)) \leq \frac{2}{3}\pi + C\eta^{1/2}$$

for all $s \in \left[\frac{2-i}{3-i} d(\bar{w}_i, \bar{b}_i), d(\bar{w}_i, \bar{b}_i) \right]$ by the assumptions, we get

$$\begin{aligned} & |d(a_{i+1}, b_{i+1})^2 - d_S(\Psi(a_{i+1}), \Psi(b_{i+1}))^2 \\ & - d(a_{i+1}, b_i)^2 + d_S(\Psi(a_{i+1}), \Psi(b_i))^2| \leq C\eta_0^{1/26} \end{aligned} \quad (92)$$

by Lemmas 4.25 and 4.40. By (91) and (92), we get (iv).

By the assumptions and Lemma 4.36, we get (ii) for a_{i+1} and b_{i+1} .

By the assumptions, we have

$$\begin{aligned} d(a_{i+1}, A_f) & \leq d(a_{i+1}, \tilde{y}_i) + d(y, A_f) + C\eta_0 \\ & = \frac{2-i}{3-i} d(\tilde{a}_i, \tilde{y}_i) + C\eta_0 \leq \frac{2-i}{3}\pi + C\eta_0. \end{aligned}$$

Similarly, we have $d(b_{i+1}, A_f) \leq \frac{2-i}{3}\pi + C\eta_0$. Thus, we get (iii) for a_{i+1} and b_{i+1} .

By definition, we have $a_3 = \tilde{y}_3$ and $b_3 = \bar{w}_3$. Thus, we get (v).

In the following, we prove (i) for a_{i+1} and b_{i+1} . If $d(a_i, y) \leq \eta_0^{1/26}$, then we have

$$d(b_i, w) \leq d(b_i, A_f) + C\eta_0 \leq d(a_i, A_f) + C\eta^{1/2} \leq C\eta^{1/2},$$

and so $d(y, w) \leq C\eta^{1/2}$, $d(a_{i+1}, y) \leq C\eta^{1/2}$ and $d(b_{i+1}, w) \leq C\eta^{1/2}$. Then, we have $d(a_{i+1}, b_{i+1}) \leq C\eta^{1/2}$. Similarly, if $d(b_i, w) \leq \eta_0^{1/26}$, then $d(a_{i+1}, b_{i+1}) \leq C\eta^{1/2}$. Thus, in the following, we assume that $d(a_i, y) \geq \eta_0^{1/26}$ and $d(b_i, w) \geq \eta_0^{1/26}$. By Lemma 4.36, we can take $u, v_1, v_2 \in S^{n-p}$ such that $f = \sum_{j=1}^{n-p+1} u_j f_j$, $u \cdot v_k = 0$ ($k = 1, 2$),

$$d_S(\Psi(\gamma_{\tilde{y}_i, \tilde{a}_i}(s)), \gamma_{v_1}(s)) \leq C\eta_0^{3/13} \quad (93)$$

for all $s \in [0, d(\tilde{a}_i, \tilde{y}_i)]$ and

$$d_S(\Psi(\gamma_{\bar{w}_i, \bar{b}_i}(s)), \gamma_{v_2}(s)) \leq C\eta_0^{3/13} \quad (94)$$

for all $s \in [0, d(\bar{b}_i, \bar{w}_i)]$, where $\gamma_{v_k}(s) := (\cos s)u + (\sin s)v_k \in S^{n-p}$ ($k = 1, 2$). Since

$$|d(\tilde{a}_i, \tilde{y}_i) - d(\bar{b}_i, \bar{w}_i)| \leq |d(a_i, A_f) - d(b_i, A_f)| + C\eta_0 \leq d(a_i, b_i) + C\eta_0,$$

we have

$$|d_S(\Psi(\tilde{a}_i), \Psi(\bar{b}_i)) - d_S(\gamma_{v_1}(l_i), \gamma_{v_2}(l_i))| \leq d(a_i, b_i) + C\eta_0^{3/13} \quad (95)$$

and

$$\left| d_S(\Psi(a_{i+1}), \Psi(b_{i+1})) - d_S\left(\gamma_{v_1}\left(\frac{2-i}{3-i}l_i\right), \gamma_{v_2}\left(\frac{2-i}{3-i}l_i\right)\right) \right| \leq d(a_i, b_i) + C\eta_0^{3/13} \quad (96)$$

by (93) and (94), where we put $l_i := d(\tilde{a}_i, \tilde{y}_i)$. By (95) and Lemma 4.25, we get

$$|v_1 - v_2| \sin l_i \leq Cd_S(\gamma_{v_1}(l_i), \gamma_{v_2}(l_i)) \leq Cd(a_i, b_i) + C\eta_0^{3/13}. \quad (97)$$

We first suppose that $d(a_i, y) \leq \pi/6$. Since $l_i \leq \pi/2$, we have

$$\sin\left(\frac{2-i}{3-i}l_i\right) \leq \sin l_i,$$

and so

$$\begin{aligned} d_S(\Psi(a_{i+1}), \Psi(b_{i+1})) &\leq d_S\left(\gamma_{v_1}\left(\frac{2-i}{3-i}l_i\right), \gamma_{v_2}\left(\frac{2-i}{3-i}l_i\right)\right) + C\eta^{1/2} \\ &\leq C|v_1 - v_2| \sin\left(\frac{2-i}{3-i}l_i\right) + C\eta^{1/2} \\ &\leq C|v_1 - v_2| \sin l_i + C\eta^{1/2} \\ &\leq Cd_S(\Psi(\tilde{a}_i), \Psi(\tilde{b}_i)) + C\eta^{1/2} \leq C\eta^{1/2} \end{aligned}$$

by (95), (96) and $d(a_i, b_i) \leq C\eta^{1/2}$. Thus, we get $d(a_{i+1}, b_{i+1}) \leq C\eta^{1/2}$ by (iv).

We next suppose that $\pi/6 \leq d(a_i, y) \leq 5\pi/6$. By (97) and $d(a_i, b_i) \leq C\eta^{1/2}$, we have $|v_1 - v_2| \leq C\eta^{1/2}$. Thus, we get $d_S(\Psi(a_{i+1}), \Psi(b_{i+1})) \leq C\eta^{1/2}$ by (96). Thus, we get $d(a_{i+1}, b_{i+1}) \leq C\eta^{1/2}$ by (iv).

If $i \geq 1$, we have $d(a_i, y) \leq 5\pi/6$, and so we get $d(a_{i+1}, b_{i+1}) \leq C\eta^{1/2}$ by the above two cases.

Finally, we suppose that $i = 0$ and $d(x, y) \geq 5\pi/6$. By (97) and $d(a_0, b_0) \leq C\eta$, we have $|v_1 - v_2| \sin l_0 \leq C\eta$. By the definition of l_0 , we have $|l_0 - d(x, y)| \leq C\eta_0$. Thus, we have $\sin l_0 \geq \frac{1}{C}(\pi - l_0) \geq \frac{1}{C}\eta^{1/2}$, and so we get $|v_1 - v_2| \leq C\eta^{1/2}$. This gives $d_S(\Psi(a_{i+1}), \Psi(b_{i+1})) \leq C\eta^{1/2}$ by (96). Thus, $d(a_{i+1}, b_{i+1}) \leq C\eta^{1/2}$ by (iv).

Therefore, we have (i) for all cases, and we get the lemma. $\square$

Let us show that the map $\Phi_f: M \rightarrow S^{n-p} \times A_f$, $x \mapsto (\Psi(x), a_f(x))$ is almost surjective.

Proposition 4.44 *Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$. For any $(v, a) \in S^{n-p} \times A_f$, there exists $x \in M$ such that $d(\Phi_f(x), (v, a)) \leq C\eta_1^{1/2}$ holds.*

Proof Take arbitrary $(v, a) \in S^{n-p} \times A_f$. Take $u \in S^{n-p}$ with $f = \sum_{i=1}^{n-p+1} u_i f_i$. Since there exists $\tilde{v} \in S^{n-p}$ such that $d_S(u, \tilde{v}) \leq \pi - \eta_1^{1/2}$ and $d_S(v, \tilde{v}) \leq \eta_1^{1/2}$, it is enough to prove the proposition assuming $d_S(u, v) \leq \pi - \eta_1^{1/2}$.

Put $F_v := \sum_{i=1}^{n-p+1} v_i f_i$. Then, $|F_v(p_{F_v}) - 1| \leq C\delta^{1/800n}$ and $A_{F_v} = \{x \in M : |F_v(x) - 1| \leq \delta^{1/900n}\}$ by Proposition 4.17. In the following, we show that $a_v := a_{F_v}(a) \in A_{F_v}$ has the desired property. By Lemma 4.26, we get

$$\begin{aligned} d_S(\Psi(a), u) &\leq C\delta^{1/2000n^2}, \\ d_S(\Psi(a_v), v) &\leq C\delta^{1/2000n^2}. \end{aligned} \quad (98)$$

Thus, by Lemma 4.26, we get

$$\begin{aligned} |d(a, a_v) - d(a_f(a_v), a_v)| &= |d(a, A_{F_v}) - d(a_v, A_f)| \\ &\leq |d_S(\Psi(a), v) - d_S(\Psi(a_v), u)| + C\delta^{1/2000n^2} \\ &\leq C\delta^{1/2000n^2} \leq \eta_0 \end{aligned}$$

and

$$d(a_v, A_f) \leq d_S(\Psi(a_v), u) + C\delta^{1/2000n^2} \leq d_S(u, v) + C\delta^{1/2000n^2} \leq \pi - \frac{1}{2}\eta_1^{1/2}.$$

Since we have $d(a_v, A_f) = d(a_v, a_f(a_v))$, we get

$$|d(a_v, A_f) - d(a_v, a)| \leq |d(a_v, A_f) - d(a_v, a_f(a_v))| + \eta_0 = \eta_0,$$

and so we get

$$d(a, a_f(a_v)) \leq C\eta_1^{1/2} \quad (99)$$

by Lemma 4.43 putting $x = z = a_v$, $y = a$ and $w = a_f(a_v)$.

By (98) and (99), putting $x = a_v$, we get the proposition. $\square$

Now, we are in position to show $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| d(\tilde{y}_1, y_2) \leq \pi + L$ under the assumption of Lemma 4.30. Note that we defined $\eta_2 = \eta_1^{1/78}$ and $L = \eta_2^{1/150}$.

Lemma 4.45 Take $y_1 \in M$, $\tilde{y}_1 \in D_{f_{y_1}}(p_{y_1}) \cap R_{f_{y_1}} \cap Q_{f_{y_1}}$ with $d(y_1, \tilde{y}_1) \leq C\delta^{1/100n}$ and $y_2 \in D_{f_{y_1}}(\tilde{y}_1)$. Let $\{E_1, \dots, E_n\}$ be a parallel orthonormal basis of TM along $\gamma_{\tilde{y}_1, y_2}$ in Lemma 4.13 for f_{y_1} . Then, $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| d(\tilde{y}_1, y_2) \leq \pi + L$ and

$$||\dot{\gamma}_{\tilde{y}_1, y_2}^E| d(\tilde{y}_1, y_2) - d_S(\Psi(y_1), \Psi(y_2))| \leq CL.$$

Proof We immediately get the second assertion by the first assertion and Lemma 4.30.

Let us show the first assertion by contradiction. Suppose that $|\dot{\gamma}_{\tilde{y}_1, y_2}^E| d(\tilde{y}_1, y_2) > \pi + L$. Put

$$f := -f_{y_1}, \gamma := \gamma_{\tilde{y}_1, y_2}, s_0 := \frac{1}{|\dot{\gamma}^E|} \eta_2^{1/104} \text{ and } s_1 := \frac{1}{|\dot{\gamma}^E|} (\pi + L).$$

Take $k \in \mathbb{N}$ to be $(s_1 - s_0)/\eta_2^{-1} < k \leq (s_1 - s_0)/\eta_2^{-1} + 1$, and put $t_j := s_0 + (s_1 - s_0)j/k$ for each $j \in \{0, \dots, k\}$. Note that we have $t_0 = s_0$, $t_k = s_1$ and

$$\frac{1}{C}\eta_2^{-1} \leq k \leq C\eta_2^{-1}. \quad (100)$$

For all $s \in [s_0, s_1]$, we have

$$\cos d_S(\Psi(y_1), \Psi(\gamma(s))) \leq \cos(|\dot{\gamma}^E|s) + C\delta^{1/2000n^2} \leq 1 - \frac{1}{C}\eta_2^{1/52}$$

for all $s \in [s_0, s_1]$ by Lemma 4.30. Since $f(\gamma(s)) = -|\tilde{\Psi}|(\gamma(s)) \cos d_S(\Psi(y_1), \Psi(\gamma(s)))$ by the definitions of f_{y_1} and f , we get $f(\gamma(s)) \geq -1 + \frac{1}{C}\eta_2^{1/52}$ for all $s \in [s_0, s_1]$ by Lemma 4.22. This gives

$$d(\gamma(s), A_f) \leq \pi - \frac{1}{C}\eta_2^{1/104} \quad (101)$$

$s \in [s_0, s_1]$ by Proposition 4.17. By the definition of t_j and (101), we have

$$\begin{aligned} d(\gamma(t_j), \gamma(t_{j+1})) &\leq \eta_2, \\ d(\gamma(t_{j+\sigma}), A_f) &\leq \pi - \frac{1}{C}\eta_2^{1/104} \leq \pi - \eta_2^{1/2} \end{aligned} \quad (102)$$

for all $j \in \{0, \dots, k-1\}$ and $\sigma \in \{0, 1\}$, and so we get

$$\begin{aligned} &|d(\gamma(t_j), \gamma(t_{j+1}))^2 - d_S(\Psi(\gamma(t_j)), \Psi(\gamma(t_{j+1})))^2 \\ &\quad - d(a_f(\gamma(t_j)), a_f(\gamma(t_{j+1})))^2| \leq C\eta_1 \end{aligned} \quad (103)$$

by Lemma 4.43. In particular, we get

$$d(a_f(\gamma(t_j)), a_f(\gamma(t_{j+1}))) \leq C\eta_2 \quad (104)$$

by (102).

Take $j_0 \in \{1, \dots, k-1\}$ to be $|\dot{\gamma}^E|t_{j_0} < \pi \leq |\dot{\gamma}^E|t_{j_0+1}$. Since

$$||\dot{\gamma}^E|s - d_S(\Psi(y_1), \Psi(\gamma(s)))| \leq C\delta^{1/4000n^2}$$

for all $s \in \left[0, \frac{1}{|\dot{\gamma}^E|}\pi\right]$ by Lemma 4.30, we get

$$\begin{aligned} d_S(\Psi(\gamma(t_j)), \Psi(\gamma(t_{j+1}))) &\geq d_S(\Psi(y_1), \Psi(\gamma(t_{j+1}))) - d_S(\Psi(y_1), \Psi(\gamma(t_j))) \\ &\geq |\dot{\gamma}^E|(t_{j+1} - t_j) - C\delta^{1/4000n^2} \end{aligned} \quad (105)$$

for all $j \in \{0, \dots, j_0-1\}$. Since

$$|2\pi - |\dot{\gamma}^E|s - d_S(\Psi(y_1), \Psi(\gamma(s)))| \leq C\delta^{1/4000n^2}$$

for all $s \in \left[\frac{1}{|\dot{\gamma}^E|} \pi, s_1 \right]$ by Lemma 4.30, we get

$$d_S(\Psi(\gamma(t_j)), \Psi(\gamma(t_{j+1}))) \geq |\dot{\gamma}^E|(t_{j+1} - t_j) - C\delta^{1/4000n^2} \quad (106)$$

for all $j \in \{j_0 + 1, \dots, k - 1\}$. By (103), (105) and (106), we get

$$d(a_f(\gamma(t_j)), a_f(\gamma(t_{j+1})))^2 \leq d(\gamma(t_j), \gamma(t_{j+1}))^2 - |\dot{\gamma}^E|^2(t_{j+1} - t_j)^2 + C\eta_1 \quad (107)$$

for all $j \in \{0, \dots, k - 1\} \setminus \{j_0\}$.

Since we have

$$d_S(\Psi(\gamma(s_l)), \Psi(p_f)) \leq d(\gamma(s_l), A_f) + C\delta^{1/2000n^2} \leq \pi - \frac{1}{C}\eta_2^{1/104}$$

for each $l = 0, 1$ by Lemma 4.26, Corollary 4.27 and (101), we can take a curve $\beta: [0, K] \rightarrow S^{n-p}$ in S^{n-p} with unit speed (K is some constant) such that

$$\begin{aligned} \beta(0) &= \Psi(\gamma(s_0)), \\ \beta(K) &= \Psi(\gamma(s_1)), \\ |d_S(\Psi(\gamma(s_0)), \Psi(\gamma(s_1))) - K| &\leq C\eta_2^{1/104}, \\ d_S(\beta(s), \Psi(p_f)) &\leq \pi - \frac{1}{C}\eta_2^{1/104} \end{aligned}$$

for all $s \in [0, K]$. Note that we can find such β by taking an almost shortest pass in $\left\{ u \in S^{n-p} : d(u, \Psi(p_f)) \leq \pi - \frac{1}{C}\eta_2^{1/104} \right\}$. By Proposition 4.44, there exists $x_j \in M$ such that

$$d\left(\Phi_f(x_j), \left(\beta\left(\frac{j}{k}K\right), a_f(\gamma(t_j))\right)\right) \leq C\eta_1^{1/2} \quad (108)$$

for each $j \in \{0, \dots, k\}$. Moreover, we can take $x_0 := \gamma(s_0)$ and $x_k := \gamma(s_1)$. By (100), (104), (108), Lemma 4.26 and Corollary 4.27, we have

$$\begin{aligned} d(a_f(x_j), a_f(x_{j+1})) &\leq C\eta_2, \\ d_S(\Psi(x_j), \Psi(x_{j+1})) &\leq \frac{1}{k}K + C\eta_1^{1/2} \leq C\eta_2, \end{aligned} \quad (109)$$

$$\begin{aligned} d(x_j, A_f) &\leq d_S(\Psi(x_j), \Psi(p_f)) + C\delta^{1/2000n^2} \\ &\leq d_S\left(\beta\left(\frac{j}{k}K\right), \Psi(p_f)\right) + C\eta_1^{1/2} \leq \pi - \frac{1}{C}\eta_2^{1/104} \end{aligned} \quad (110)$$

for all j , and so

$$d(x_j, x_{j+1}) \leq C\eta_2^{1/52} \quad (111)$$

by Lemma 4.41 putting $x = x_j$, $y = a_f(x_j)$, $z = x_{j+1}$ and $\eta = \eta_2$. By (110), (111) and Lemma 4.43 putting $x = x_j$, $y = a_f(x_j)$, $z = x_{j+1}$, $w = a_f(x_{j+1})$ and $\eta = \eta_2^{1/52}$, we get

$$|d(x_j, x_{j+1})^2 - d_S(\Psi(x_j), \Psi(x_{j+1}))^2 - d(a_f(x_j), a_f(x_{j+1}))^2| \leq C\eta_1 \quad (112)$$

for all $j \in \{0, \dots, k-1\}$. By (107), (109) and (112), we have

$$\begin{aligned} d(x_j, x_{j+1})^2 &\leq \frac{1}{k^2} K^2 + d(a_f(x_j), a_f(x_{j+1}))^2 + C\eta_1^{1/2} \\ &\leq \frac{1}{k^2} K^2 + d(\gamma(t_j), \gamma(t_{j+1}))^2 - |\dot{\gamma}^E|^2(t_{j+1} - t_j)^2 + C\eta_1^{1/2} \end{aligned} \quad (113)$$

for all $j \in \{0, \dots, k-1\} \setminus \{j_0\}$. Since $K \leq \pi + C\eta_2^{1/104}$, we have

$$\frac{1}{k^2} K^2 \leq \frac{\pi^2}{k^2} + \frac{C}{k^2} \eta_2^{1/104}. \quad (114)$$

Since

$$|\dot{\gamma}^E|(t_{j+1} - t_j) = \frac{|\dot{\gamma}^E|}{k}(s_1 - s_0) = \frac{1}{k}(\pi + L - \eta_2^{1/104}) \geq \frac{1}{k} \left(\pi + \frac{1}{2}L \right),$$

we have

$$|\dot{\gamma}^E|^2(t_{j+1} - t_j)^2 \geq \frac{\pi^2}{k^2} + \frac{1}{k^2}L \quad (115)$$

for all $j \in \{0, \dots, k-1\}$. By (114) and (115), we get

$$|\dot{\gamma}^E|^2(t_{j+1} - t_j)^2 - \frac{1}{k^2} K^2 \geq \frac{1}{k^2}L - \frac{C}{k^2} \eta_2^{1/104} \geq \frac{1}{2k^2}L$$

for all $j \in \{0, \dots, k-1\}$. Thus, by (113), we have

$$d(x_j, x_{j+1})^2 \leq d(\gamma(t_j), \gamma(t_{j+1}))^2 - \frac{1}{2k^2}L + C\eta_1^{1/2} \leq d(\gamma(t_j), \gamma(t_{j+1}))^2 - \frac{1}{4k^2}L$$

for all $j \in \{0, \dots, k-1\} \setminus \{j_0\}$. Since $d(\gamma(t_j), \gamma(t_{j+1})) + d(x_j, x_{j+1}) \leq 1$, we get

$$\frac{1}{4k^2}L \leq d(\gamma(t_j), \gamma(t_{j+1}))^2 - d(x_j, x_{j+1})^2 \leq d(\gamma(t_j), \gamma(t_{j+1})) - d(x_j, x_{j+1}) \quad (116)$$

$j \in \{0, \dots, k-1\} \setminus \{j_0\}$. By (100), (111) and (116), we get

$$\begin{aligned} d(x_0, x_k) &\leq \sum_{i=0}^{k-1} d(x_j, x_{j+1}) \leq \sum_{i=0}^{k-1} d(\gamma(t_j), \gamma(t_{j+1})) - \frac{k-1}{4k^2}L + d(x_{j_0}, x_{j_0+1}) \\ &\leq d(x_0, x_k) - \frac{1}{8k}L. \end{aligned}$$

This is a contradiction. Thus, we get the lemma. $\square$

Notation 4.46 For all $y_1, y_2 \in M$, define

$$\begin{aligned} \overline{C}_f^{y_1}(y_2) &= \left\{ y_3 \in M : \gamma_{y_2, y_3}(s) \in I_{y_1} \setminus \{y_1\} \text{ for almost all } s \in [0, d(y_2, y_3)], \text{ and} \right. \\ &\quad \left. \int_0^{d(y_2, y_3)} |G_f^{y_1}|(\gamma_{y_2, y_3}(s)) ds \leq L^{1/3} \right\}, \\ \overline{P}_f^{y_1} &= \{y_2 \in M : \text{Vol}(M \setminus \overline{C}_f^{y_1}(y_2)) \leq L^{1/3} \text{Vol}(M)\}. \end{aligned}$$

Let us complete the Gromov-Hausdorff approximation.

Theorem 4.47 Take $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p+1}\}$ with $\|f\|_2^2 = 1/(n-p+1)$. Then, the map $\Phi_f: M \rightarrow S^{n-p} \times A_f$ is a $CL^{1/156n}$ -Hausdorff approximation map. In particular, we have $d_{GH}(M, S^{n-p} \times A_f) \leq CL^{1/156n}$.

Proof Take arbitrary $y_1 \in M$ and $\tilde{y}_1 \in D_{f_{y_1}}(p_{y_1}) \cap R_{f_{y_1}} \cap Q_{f_{y_1}} \cap Q_f$ with $d(y_1, \tilde{y}_1) \leq C\delta^{1/100n}$. By Lemmas 4.25, 4.45 and Corollary 4.33, we have $|G_f^{\tilde{y}_1}|(y_2) \leq CL$ for all $y \in D_f(\tilde{y}_1) \cap D_{f_{y_1}}(\tilde{y}_1)$. Since $\text{Vol}(M \setminus (D_f(\tilde{y}_1) \cap D_{f_{y_1}}(\tilde{y}_1))) \leq C\delta^{1/100} \text{Vol}(M)$ and $\|G_f^{\tilde{y}_1}\|_{\infty} \leq C$, we get $\|G_f^{\tilde{y}_1}\|_1 \leq CL$. Thus, by the segment inequality, we get $\text{Vol}(M \setminus \overline{P}_f^{\tilde{y}_1}) \leq CL^{1/3}$.

Take arbitrary $x, z \in M$. By the Bishop-Gromov inequality, there exist $\tilde{z} \in D_{f_z}(p_z) \cap Q_{f_z} \cap R_{f_z} \cap Q_f$, $\tilde{x} \in D_f(p_f) \cap Q_f \cap R_f \cap \overline{P}_f^{\tilde{z}}$ and $\tilde{y} \in D_f(\tilde{x}) \cap D_f(p_f) \cap Q_f \cap R_f \cap \overline{C}_f^{\tilde{z}}(\tilde{x})$ such that $d(z, \tilde{z}) \leq C\delta^{1/100n}$, $d(x, \tilde{x}) \leq CL^{1/3n}$ and $d(a_f(x), \tilde{y}) \leq CL^{1/3n}$. Here, we used the estimate $\text{Vol}(M \setminus \overline{P}_f^{\tilde{z}}) \leq CL^{1/3}$. Then, we get

$$\left| d(\tilde{z}, \tilde{x})^2 - d_S(\Psi(\tilde{z}), \Psi(\tilde{x}))^2 - d(\tilde{z}, \tilde{y})^2 + d_S(\Psi(\tilde{z}), \Psi(\tilde{y}))^2 \right| \leq CL^{1/78n}$$

by Lemma 4.38. Thus, we get

$$\left| d(z, x)^2 - d_S(\Psi(z), \Psi(x))^2 - d(z, a_f(x))^2 + d_S(\Psi(z), \Psi(a_f(x)))^2 \right| \leq CL^{1/78n} \quad (117)$$

by Lemma 4.25. Similarly, we have

$$\begin{aligned} & \left| d(a_f(x), z)^2 - d_S(\Psi(a_f(x)), \Psi(z))^2 - d(a_f(x), a_f(z))^2 \right. \\ & \quad \left. + d_S(\Psi(a_f(x)), \Psi(a_f(z)))^2 \right| \\ & \leq CL^{1/78n}. \end{aligned} \quad (118)$$

Since we have $d_S(\Psi(a_f(x)), \Psi(a_f(z))) \leq C\delta^{1/2000n^2}$ by Lemma 4.26, we get

$$\left| d(z, x)^2 - d_S(\Psi(z), \Psi(x))^2 - d(a_f(x), a_f(z))^2 \right| \leq CL^{1/78n}.$$

by (117) and (118). This gives

$$\begin{aligned} & |d(z, x) - d(\Phi_f(z), \Phi_f(x))| \\ & = \left| d(z, x) - \left(d_S(\Psi(z), \Psi(x))^2 + d(a_f(x), a_f(z))^2 \right)^{1/2} \right| \leq CL^{1/156n}. \end{aligned}$$

Combining this and Proposition 4.43, we get the theorem. $\square$

By the above theorem, we get Main Theorem 2 except for the orientability, which is proved in Sect. 4.7.

4.6 Further Inequalities

In this subsection, we assume that Assumption 4.2 holds, and prepare two lemmas to prove the remaining part of main theorems.

Lemma 4.48 *For any $f \in \text{Span}_{\mathbb{R}}\{f_1, \dots, f_k\}$, we have*

$$\left\| \sum_{i=1}^n e^i \otimes (\nabla_{e_i} df + f e^i) \wedge \omega \right\|_2 \leq C\delta^{1/8} \|f\|_2.$$

Proof We have

$$\begin{aligned} & \left| \sum_{i=1}^n e^i \otimes (\nabla_{e_i} df + f e^i) \wedge \omega \right|^2 \\ & = |\nabla^2 f|^2 |\omega|^2 - \frac{1}{n-p} (\Delta f)^2 |\omega|^2 + 2\Delta f \left(\frac{1}{n-p} \Delta f - f \right) |\omega|^2 \\ & \quad - (n-p) \left(\left(\frac{\Delta f}{n-p} \right)^2 - f^2 \right) |\omega|^2 - \left| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right|^2 \\ & \quad - 2 \sum_{i=1}^n f \langle \omega, e^i \wedge \iota(\nabla_{e_i} \nabla f) \omega \rangle. \end{aligned} \quad (119)$$

By the assumption, we have

$$\left\| \Delta f \left(\frac{1}{n-p} \Delta f - f \right) |\omega|^2 \right\|_1 \leq C \delta^{1/2} \|f\|_2^2, \quad (120)$$

$$\left\| \left(\left(\frac{\Delta f}{n-p} \right)^2 - f^2 \right) |\omega|^2 \right\|_1 \leq C \delta^{1/2} \|f\|_2^2. \quad (121)$$

By Lemma 3.6 (iv) and Lemma 4.10 (ii), we have

$$\left\| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right\|_2 \leq \|\nabla(\iota(\nabla f) \omega)\|_2 + C \delta^{1/2} \|f\|_2 \leq C \delta^{1/4} \|f\|_2, \quad (122)$$

and so

$$\left\| \sum_{i=1}^n f \langle \omega, e^i \wedge \iota(\nabla_{e_i} \nabla f) \omega \rangle \right\|_1 \leq C \|f\|_2 \left\| \sum_{i=1}^n e^i \otimes \iota(\nabla_{e_i} \nabla f) \omega \right\|_2 \leq C \delta^{1/4} \|f\|_2^2. \quad (123)$$

By Lemma 4.10, (119), (120), (121), (122) and (123), we get the lemma. $\square$

Lemma 4.49 Define $G = G(f_1, \dots, f_k)$ by

$$G := \left\{ x \in M : |f_i^2 + |\nabla f_i|^2 - 1|(x) \leq \delta^{1/1600n} \text{ for all } i = 1, \dots, k, \text{ and } \left| \frac{1}{2}(f_i \pm f_j)^2 + \frac{1}{2}|\nabla f_i \pm \nabla f_j|^2 - 1 \right|(x) \leq \delta^{1/1600n} \text{ for all } i \neq j \right\}.$$

Then, we have the following properties.

- (i) We have $\text{Vol}(M \setminus G) \leq C \delta^{1/1600n} \text{Vol}(M)$.
- (ii) For all $x \in G$ and i, j with $i \neq j$, we have $|f_i f_j + \langle \nabla f_i, \nabla f_j \rangle|(x) \leq \delta^{1/1600n}$.

Proof By Proposition 4.17 (iii), we have

$$\|f_i^2 + |\nabla f_i|^2 - 1\|_1 \leq C \delta^{1/800n},$$

$$\left\| \frac{1}{2}(f_i \pm f_j)^2 + \frac{1}{2}|\nabla f_i \pm \nabla f_j|^2 - 1 \right\|_1 \leq C \delta^{1/800n}$$

for all $i \neq j$. Therefore, we get

$$\begin{aligned} & \text{Vol} \left(\left\{ x \in M : |f_i^2 + |\nabla f_i|^2 - 1|(x) > \delta^{1/1600n} \right\} \right) \\ & \leq \delta^{-1/1600n} \int_M |f_i^2 + |\nabla f_i|^2 - 1| d\mu_g \leq C \delta^{1/1600n} \text{Vol}(M) \end{aligned}$$

for all i . Similarly, we have

$$\begin{aligned} & \text{Vol} \left(\left\{ x \in M : \left| \frac{1}{2}(f_i \pm f_j)^2 + \frac{1}{2}|\nabla f_i \pm \nabla f_j|^2 - 1 \right| (x) > \delta^{1/1600n} \right\} \right) \\ & \leq C\delta^{1/1600n} \text{Vol}(M) \end{aligned}$$

for all $i \neq j$. Thus, we get (i).

For all $x \in G$ and i, j with $i \neq j$, we have

$$\begin{aligned} & |f_i f_j + \langle \nabla f_i, \nabla f_j \rangle| (x) \\ &= \frac{1}{2} \left| \frac{1}{2}(f_i + f_j)^2 + \frac{1}{2}|\nabla f_i + \nabla f_j|^2 - \frac{1}{2}(f_i - f_j)^2 - \frac{1}{2}|\nabla f_i - \nabla f_j|^2 \right| (x) \\ &\leq \delta^{1/1600n}. \end{aligned}$$

Thus, we get (ii). $\square$

4.7 Orientability

The goal of this subsection is to show the orientability of the manifold under the assumption of Main Theorem 2.

Theorem 4.50 *If Assumption 4.1 for $k = n - p + 1$ and Assumption 4.2 hold, then M is orientable.*

Proof To prove the theorem, we use the following claim:

Claim 4.51 *Define*

$$\lambda_1(\Delta_{C,n}) := \inf \left\{ \frac{\|\nabla \eta\|_2^2}{\|\eta\|_2^2} : \eta \in \Gamma \left(\bigwedge^n T^*M \right) \text{ with } \eta \neq 0 \right\}.$$

If $\lambda_1(\Delta_{C,n}) < n(n - p - 1)/(n - 1)$ holds, then M is orientable.

Proof of Claim 4.51 Suppose that M is not orientable. Take the two-sheeted oriented Riemannian covering $\pi : (\tilde{M}, \tilde{g}) \rightarrow (M, g)$. Since we have $\text{Ric}_{\tilde{g}} \geq (n - p - 1)\tilde{g}$, we get

$$\lambda_1(\Delta_{C,n}, g) \geq \lambda_2(\Delta_{C,n}, \tilde{g}) = \lambda_1(\tilde{g}) \geq \frac{n}{n-1}(n-p-1)$$

by the Lichnerowicz estimate (note that $\lambda_1(\Delta_{C,n}, \tilde{g}) = \lambda_0(\tilde{g}) = 0$). This gives the claim. $\square$

Put

$$V := \sum_{i=1}^{n-p+1} (-1)^{i-1} f_i df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \in \Gamma \left(\bigwedge^n T^*M \right).$$

In the following, we show that $\|\nabla V\|_2^2/\|V\|_2^2 < n(n-p+1)/(n-1)$.

Define a vector bundle $E := T^*M \oplus \mathbb{R}e$, where $\mathbb{R}e$ denotes the trivial bundle of rank 1 with a nowhere vanishing section e . We consider an inner product $\langle \cdot, \cdot \rangle$ on E defined by $\langle \alpha + fe, \beta + he \rangle := \langle \alpha, \beta \rangle + fh$ for all $\alpha, \beta \in \Gamma(T^*M)$ and $f, h \in C^\infty(M)$. Put

$$S_i := df_i + f_i e \in \Gamma(E)$$

for each i , and

$$\alpha := S_1 \wedge \cdots \wedge S_{n-p+1} \in \Gamma\left(\bigwedge^{n-p+1} E\right).$$

Then, we have $\alpha \wedge \omega = e \wedge V$, and so

$$|\alpha \wedge \omega| = |V|. \quad (124)$$

For each $k = 1, \dots, n-p+1$, we have

$$\begin{aligned} & \left\| \langle S_k \wedge \cdots \wedge S_{n-p+1} \wedge \omega, (\iota(S_{k-1}) \cdots \iota(S_1)\alpha) \wedge \omega \rangle \right. \\ & \quad \left. - \langle S_{k+1} \wedge \cdots \wedge S_{n-p+1} \wedge \omega, (\iota(S_k) \cdots \iota(S_1)\alpha) \wedge \omega \rangle \right\|_1 \\ &= \left\| \langle S_{k+1} \wedge \cdots \wedge S_{n-p+1} \wedge \omega, (\iota(S_{k-1}) \cdots \iota(S_1)\alpha) \wedge \iota(df_k)\omega \rangle \right\|_1 \\ &\leq C \|\iota(df_k)\omega\|_2 \leq C\delta^{1/4} \end{aligned}$$

by Lemma 4.10 (i). By induction, we get

$$\| |\alpha \wedge \omega|^2 - |\alpha|^2 |\omega|^2 \|_1 \leq C\delta^{1/4}. \quad (125)$$

In particular, we have

$$\left| \|\alpha \wedge \omega\|_2^2 - \| |\alpha|^2 |\omega|^2 \|_1 \right| \leq C\delta^{1/4}. \quad (126)$$

Since we have $|\langle S_i(x), S_j(x) \rangle - \delta_{ij}| \leq \delta^{1/1600n}$ for all $x \in G = G(f_1, \dots, f_{n-p+1})$ and i, j by Lemma 4.49 (ii), we get $||\alpha|^2(x) - 1| \leq C\delta^{1/1600n}$ for all $x \in G$. Thus, we get

$$\begin{aligned} & \left| \frac{1}{\text{Vol}(M)} \int_M (|\alpha|^2 |\omega|^2 - 1) d\mu_g \right| \\ &= \left| \frac{1}{\text{Vol}(M)} \int_G (|\alpha|^2 - 1) |\omega|^2 d\mu_g \right. \\ & \quad \left. + \frac{1}{\text{Vol}(M)} \int_{M \setminus G} (|\alpha|^2 - 1) |\omega|^2 d\mu_g + \frac{1}{\text{Vol}(M)} \int_M (|\omega|^2 - 1) d\mu_g \right| \\ &\leq C\delta^{1/1600n} \end{aligned} \quad (127)$$

by Lemmas 3.5 and 4.49 (i). By (124), (126) and (127), we get

$$\|V\|_2^2 - 1 \leq C\delta^{1/1600n}. \quad (128)$$

We next estimate $\|\nabla V\|_2^2$. We have

$$\begin{aligned} \nabla V &= \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \\ &+ \sum_{j < i}^n \sum_{k=1}^n (-1)^{i-1} (-1)^{j-1} f_i e^k \otimes (\nabla_{e_k} df_j) \wedge df_1 \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \\ &+ \sum_{i < j}^n \sum_{k=1}^n (-1)^{i-1} (-1)^j f_i e^k \otimes (\nabla_{e_k} df_j) \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \\ &+ \sum_{i=1}^{n-p+1} \sum_{k=1}^n (-1)^{i-1} f_i e^k \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \nabla_{e_k} \omega. \end{aligned}$$

Thus, we get

$$\begin{aligned} &\left\| \nabla V - \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \right\|_2 \\ &\leq \left\| \sum_{j < i}^n \sum_{k=1}^n (-1)^{i-1} (-1)^{j-1} f_i f_j e^k \otimes e^k \wedge df_1 \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \right. \\ &\quad \left. + \sum_{i < j}^n \sum_{k=1}^n (-1)^{i-1} (-1)^j f_i f_j e^k \otimes e^k \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \right\|_2 \\ &\quad + C \sum_{i=1}^{n-p+1} \left\| \sum_{k=1}^n e^k \otimes (\nabla_{e_k} df_i + f_i e^k) \wedge \omega \right\|_2 + C \|\nabla \omega\|_2 \\ &\leq C\delta^{1/8} \end{aligned} \quad (129)$$

by Lemma 4.48.

Similarly to (125), we have

$$\begin{aligned} &\left\| \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \right\|_2^2 \\ &- \left\| \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \right\|_2^2 \|\omega\|_1^2 \\ &\leq C\delta^{1/4}. \end{aligned} \quad (130)$$

Since we have $df_1 \wedge \cdots \wedge df_{n-p+1} \wedge \omega = 0$, we get

$$\begin{aligned} &\| |df_1 \wedge \cdots \wedge df_{n-p+1}|^2 |\omega|^2 \|_1 \\ &= \| |df_1 \wedge \cdots \wedge df_{n-p+1}|^2 |\omega|^2 - |df_1 \wedge \cdots \wedge df_{n-p+1} \wedge \omega|^2 \|_1 \leq C\delta^{1/4} \end{aligned}$$

$$(131)$$

similarly to (125). By (6), we get

$$\begin{aligned} & \left| \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \right|^2 \\ &= (n-p+1) |df_1 \wedge \cdots \wedge df_{n-p+1}|^2. \end{aligned} \quad (132)$$

By (131) and (132), we get

$$\left\| \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \right\|_1^2 \leq C\delta^{1/4}. \quad (133)$$

By (130) and (133), we have

$$\left\| \sum_{i=1}^{n-p+1} (-1)^{i-1} df_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p+1} \wedge \omega \right\|_2^2 \leq C\delta^{1/4}. \quad (134)$$

By (129) and (134), we get

$$\|\nabla V\|_2 \leq C\delta^{1/8}. \quad (135)$$

By (128) and (135), we get $\lambda_1(\Delta_{C,n}) \leq C\delta^{1/4}$, and so we get the theorem by Claim 4.51. $\square$

Combining Theorems 4.47 and 4.50, we get Main Theorem 2.

4.8 Almost Parallel $(n-p)$ -form II

In this subsection, we show that the assumption “ $\lambda_{n-p}(g)$ is close to $n-p$ ” implies the condition “ $\lambda_{n-p+1}(g)$ is close to $n-p$ ” under the assumption $\lambda_1(\Delta_{C,n-p}) \leq \delta$.

Lemma 4.52 *Suppose that Assumption 4.1 for $k = n-p$ and Assumption 4.3 hold. Put $F := \langle df_1 \wedge \cdots \wedge df_{n-p}, \xi \rangle \in C^\infty(M)$. Then, we have*

$$\left| \|F\|_2^2 - \frac{1}{n-p+1} \right| \leq C\delta^{1/1600n}, \quad \left| \|\nabla F\|_2^2 - \frac{n-p}{n-p+1} \right| \leq C\delta^{1/1600n}$$

and

$$\left| \frac{1}{\text{Vol}(M)} \int_M f_i F d\mu_g \right| \leq C\delta^{1/2}$$

for all $i = 1, \dots, n-p$.

Proof If M is not orientable, we take the two-sheeted oriented Riemannian covering $\pi: (\tilde{M}, \tilde{g}) \rightarrow (M, g)$, and put $\tilde{F} := F \circ \pi$ and $\tilde{f}_i := f_i \circ \pi$. Then, we have $\|F\|_2 = \|\tilde{F}\|_2$, $\|\nabla F\|_2 = \|\nabla \tilde{F}\|_2$,

$$\frac{1}{\text{Vol}(\tilde{M})} \int_{\tilde{M}} \tilde{f}_i \tilde{F} d\mu_{\tilde{g}} = \frac{1}{\text{Vol}(M)} \int_M f_i F d\mu_g$$

and $\tilde{F} = \langle d\tilde{f}_1 \wedge \dots \wedge d\tilde{f}_{n-p}, \pi^*\xi \rangle$. Thus, it is enough to consider the case when M is orientable. In the following, we assume that M is orientable, and we fix an orientation of M .

Put $\omega := *\xi \in \Gamma(\wedge^p T^*M)$. Let $V_g \in \Gamma(\wedge^n T^*M)$ be the volume form of (M, g) . Then, we have

$$FV_g = df_1 \wedge \dots \wedge df_{n-p} \wedge \omega. \quad (136)$$

Define a vector bundle $E := T^*M \oplus \mathbb{R}e$ and an inner product $\langle \cdot, \cdot \rangle$ on it as in the proof of Theorem 4.50. Put

$$S_i := df_i + f_i e \in \Gamma(E)$$

for each i , and

$$\beta := S_1 \wedge \dots \wedge S_{n-p} \in \Gamma(\wedge^{n-p} E).$$

Since we have $|F| = |FV_g|$, we get $\| |F|^2 - |df_1 \wedge \dots \wedge df_{n-p}|^2 |\omega|^2 \|_1 \leq C\delta^{1/4}$ similarly to (125) by (136), and so

$$\left\| \|F\|_2^2 - \left\| |df_1 \wedge \dots \wedge df_{n-p}|^2 |\omega|^2 \right\|_1 \right\| \leq C\delta^{1/4} \quad (137)$$

By Lemma 4.48 and (136), we have

$$\begin{aligned} & \left\| \nabla(FV_g) + \sum_{i=1}^{n-p} \sum_{k=1}^n (-1)^{i-1} f_i e^k \otimes e^k \wedge df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_{n-p} \wedge \omega \right\|_2 \\ & \leq C\delta^{1/8}. \end{aligned}$$

Since $|\nabla(FV_g)| = |\nabla F|$, we get

$$\begin{aligned} & \left\| \|\nabla F\|_2^2 - \left\| \sum_{i=1}^{n-p} \sum_{k=1}^n (-1)^{i-1} f_i e^k \otimes e^k \wedge df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_{n-p} \wedge \omega \right\|_1^2 \right\|_1 \\ & \leq C\delta^{1/8}. \end{aligned} \quad (138)$$

We have

$$\begin{aligned} & \left| \sum_{i=1}^{n-p} \sum_{k=1}^n (-1)^{i-1} f_i e^k \otimes e^k \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \wedge \omega \right|^2 \\ &= \left| \sum_{i=1}^{n-p} (-1)^{i-1} f_i df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \wedge \omega \right|^2. \end{aligned} \quad (139)$$

Similarly to (125), we have

$$\begin{aligned} & \left\| \sum_{i=1}^{n-p} (-1)^{i-1} f_i df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \wedge \omega \right\|^2 \\ & - \left\| \sum_{i=1}^{n-p} (-1)^{i-1} f_i df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \right\|^2 \|\omega\|^2 \Big\|_1 \leq C\delta^{1/4}. \end{aligned}$$

Since we have

$$\iota(e)\beta = \sum_{i=1}^{n-p} (-1)^{i-1} f_i df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p},$$

we get

$$\left\| \sum_{i=1}^{n-p} (-1)^{i-1} f_i df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \wedge \omega \right\|^2 - |\iota(e)\beta|^2 \|\omega\|^2 \Big\|_1 \leq C\delta^{1/4}. \quad (140)$$

By (138), (139) and (140), we get

$$\left| \|\nabla F\|_2^2 - \|\iota(e)\beta\|^2 \|\omega\|^2 \right|_1 \leq C\delta^{1/8}. \quad (141)$$

We have

$$|\beta|^2 = |df_1 \wedge \cdots \wedge df_{n-p}|^2 + |\iota(e)\beta|^2. \quad (142)$$

We calculate $\sum_{k=1}^n |e^k \wedge \beta|^2$ in two ways. We have

$$\begin{aligned} \sum_{k=1}^n |e^k \wedge \beta|^2 &= (p+1)|\beta|^2 - |e \wedge \beta|^2 \\ &= (p+1)|\beta|^2 - |df_1 \wedge \cdots \wedge df_{n-p}|^2 = p|\beta|^2 + |\iota(e)\beta|^2 \end{aligned} \quad (143)$$

by (142). For all $\eta \in \Gamma(T^*M)$, we have

$$\begin{aligned} |\eta \wedge \beta|^2 &= |\eta|^2 |\beta|^2 - \langle \iota(\eta)\beta, \iota(\eta)\beta \rangle \\ &= |\eta|^2 |\beta|^2 - \sum_{i,j=1}^{n-p} (-1)^{i+j} \langle \eta, df_i \rangle \langle \eta, df_j \rangle \\ &\quad \langle S_1 \wedge \cdots \wedge \widehat{S}_i \wedge \cdots \wedge S_{n-p}, S_1 \wedge \cdots \wedge \widehat{S}_j \wedge \cdots \wedge S_{n-p} \rangle, \end{aligned}$$

and so we get

$$\begin{aligned} \sum_{k=1}^n |e^k \wedge \beta|^2 &= n |\beta|^2 - \sum_{i,j=1}^{n-p} (-1)^{i+j} \langle df_i, df_j \rangle \\ &\quad \langle S_1 \wedge \cdots \wedge \widehat{S}_i \wedge \cdots \wedge S_{n-p}, S_1 \wedge \cdots \wedge \widehat{S}_j \wedge \cdots \wedge S_{n-p} \rangle. \end{aligned} \quad (144)$$

By (143) and (144), we get

$$\begin{aligned} |\iota(e)\beta|^2 &= (n-p) |\beta|^2 - \sum_{i,j=1}^{n-p} (-1)^{i+j} \langle df_i, df_j \rangle \\ &\quad \langle S_1 \wedge \cdots \wedge \widehat{S}_i \wedge \cdots \wedge S_{n-p}, S_1 \wedge \cdots \wedge \widehat{S}_j \wedge \cdots \wedge S_{n-p} \rangle \end{aligned} \quad (145)$$

Since we have $|\langle S_i, S_j \rangle(x) - \delta_{ij}| \leq C\delta^{1/1600n}$ for all $x \in G = G(f_1, \dots, f_{n-p})$ by Lemma 4.49 (ii), we have

$$\begin{aligned} &\left\| \sum_{i=1}^{n-p} |df_i|^2 - \sum_{i,j=1}^{n-p} (-1)^{i+j} \langle df_i, df_j \rangle \right. \\ &\quad \left. \langle S_1 \wedge \cdots \wedge \widehat{S}_i \wedge \cdots \wedge S_{n-p}, S_1 \wedge \cdots \wedge \widehat{S}_j \wedge \cdots \wedge S_{n-p} \rangle |\omega|^2 \right\|_1 \leq C\delta^{1/1600n} \end{aligned} \quad (146)$$

and

$$\left| \left\| |\beta|^2 |\omega|^2 \right\|_1 - 1 \right| \leq C\delta^{1/1600n} \quad (147)$$

by Lemmas 3.5 and 4.49 (i). By the assumption, we have

$$\left| \sum_{i=1}^{n-p} \|df_i\|_2^2 - \frac{(n-p)^2}{n-p+1} \right| \leq C\delta^{1/2}. \quad (148)$$

By (145), (146), (147) and (148), we get

$$\left\| |\iota(e)\beta|^2 |\omega|^2 \right\|_1 - \frac{n-p}{n-p+1} \leq C\delta^{1/1600n}, \quad (149)$$

and so

$$\left\| |df_1 \wedge \cdots \wedge df_{n-p}|^2 |\omega|^2 \right\|_1 - \frac{1}{n-p+1} \leq C\delta^{1/1600n} \quad (150)$$

by (142) and (147). By (137) and (150), we get

$$\left| \|F\|_2^2 - \frac{1}{n-p+1} \right| \leq C\delta^{1/1600n}.$$

By (141) and (149), we get

$$\left| \|\nabla F\|_2^2 - \frac{n-p}{n-p+1} \right| \leq C\delta^{1/1600n}.$$

Let us show the remaining assertion. Since we have

$$\begin{aligned} f_i F V_g &= \frac{1}{2} (-1)^{i-1} d \left(f_i^2 df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \wedge \omega \right) \\ &\quad - \frac{1}{2} (-1)^{i-1} (-1)^{n-p-1} f_i^2 df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{n-p} \wedge d\omega, \end{aligned}$$

we get

$$\left| \frac{1}{\text{Vol}(M)} \int_M f_i F d\mu_g \right| \leq C \|\nabla \omega\|_2 \leq C\delta^{1/2}$$

by the Stokes theorem. $\square$

By applying the min-max principle

$$\begin{aligned} &\lambda_{n-p+1}(g) \\ &= \inf \left\{ \sup_{f \in V \setminus \{0\}} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} : V \text{ is an } (n-p+1)\text{-dimensional subspace of } C^\infty(M) \right\} \end{aligned}$$

to the subspace $\text{Span}_{\mathbb{R}}\{f_1, \dots, f_{n-p}, F\}$, we immediately get the following corollary:

Corollary 4.53 *If Assumption 4.1 for $k = n - p$ and Assumption 4.3 hold, then we have $\lambda_{n-p+1}(g) \leq n - p + C\delta^{1/1600n}$.*

Combining Theorem 4.47 and Corollary 4.53, we get Main Theorem 4.

Finally, we investigate the Gromov-Hausdorff limit of the sequence of the Riemannian manifolds that satisfy our pinching condition.

Theorem 4.54 *Take $n \geq 5$ and $2 \leq p < n/2$. Let $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence of n -dimensional closed Riemannian manifolds with $\text{Ric}_{g_i} \geq (n - p - 1)g_i$ that satisfies one of the following:*

- (i) $\lim_{i \rightarrow \infty} \lambda_{n-p+1}(g_i) = n - p$ and $\lim_{i \rightarrow \infty} \lambda_1(\Delta_{C,p}, g_i) = 0$,
- (ii) $\lim_{i \rightarrow \infty} \lambda_{n-p}(g_i) = n - p$ and $\lim_{i \rightarrow \infty} \lambda_1(\Delta_{C,n-p}, g_i) = 0$.

If $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ converges to a geodesic space X , then there exists a geodesic space Y such that X is isometric to $S^{n-p} \times Y$.

Proof By Main Theorems 2 and 4, we get that there exist a sequence of positive real numbers $\{\epsilon_i\}$ and compact metric spaces $\{Y_i\}$ such that $\lim_{i \rightarrow \infty} \epsilon_i = 0$ and $d_{GH}(M_i, S^{n-p} \times Y_i) \leq \epsilon_i$. Then, $\{S^{n-p} \times Y_i\}$ converges to X in the Gromov-Hausdorff topology, and so $\{Y_i\}$ is pre-compact in the Gromov-Hausdorff topology by [20, Theorem 11.1.10]. Thus, there exists a subsequence that converges to some compact metric space Y . Therefore, we get that X is isometric to $S^{n-p} \times Y$. Since X is a geodesic space, Y is also a geodesic space. $\square$

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Appendix A. Limit Spaces and Unorientability

In this appendix, we show the stability of unorientability under the non-collapsing Gromov-Hausdorff convergence assuming the two-sided bound on the Ricci curvature. Similarly to Claim 4.51, we have the following.

Lemma A.1 *For any n -dimensional unorientable closed Riemannian manifold (M, g) with $\text{Ric} \geq -Kg$ and $\text{diam}(M) \leq D$ ($K, D > 0$) we have $\lambda_1(\Delta_{C,n}, g) \geq$*

$C_1(n, K, 2D)$, where $C_1(n, K, D)$ is defined by

$$C_1(n, K, D) := \frac{1}{(n-1)D^2 \exp\left(1 + \sqrt{1 + 4(n-1)KD^2}\right)}.$$

Note that we have $\lambda_1(g_1) \geq C_1(n, K, D)$ for any n -dimensional closed Riemannian manifold (N_1, g_1) with $\text{Ric}_{g_1} \geq -Kg_1$ and $\text{diam}(N_1) \leq D$ by the Li-Yau estimate [22, p.116].

We immediately get the following corollary.

Corollary A.2 *Let (M, g) be an n -dimensional closed Riemannian manifold with $\text{Ric} \geq -Kg$ and $\text{diam}(M) \leq D$ ($K, D > 0$). If $\lambda_1(\Delta_{C,n}, g) < C_1(n, K, 2D)$, then M is orientable.*

The following theorem is the main result of this section.

Theorem A.3 *Take real numbers $K_1, K_2 \in \mathbb{R}$ and positive real numbers $D > 0$ and $v > 0$. Let $\{(M_i, g_i)\}$ be a sequence of n -dimensional unorientable closed Riemannian manifolds with $K_1g_i \leq \text{Ric}_{g_i} \leq K_2g_i$, $\text{diam}(M) \leq D$ and $\text{Vol}(M) \geq v$. Suppose that $\{(M_i, g_i)\}$ converges to a limit space X in the Gromov-Hausdorff sense. Then, X is not orientable in the sense of Honda [16] (see also the definition below).*

Note that Honda [16, Theorem 1.3] showed the stability of orientability without assuming the upper bound on the Ricci curvature.

Before proving Theorem A.3, we fix our notation and recall definitions about limit spaces.

Notation A.4 Take real numbers $K_1, K_2 \in \mathbb{R}$ and positive real numbers $D > 0$ and $v > 0$. Let $\mathcal{M} = \mathcal{M}(n, K_1, K_2, D, v)$ be the set of isometry classes of n -dimensional closed Riemannian manifolds (M, g) with $K_1g \leq \text{Ric}_g \leq K_2g$, $\text{diam}(M) \leq D$ and $\text{Vol}(M) \geq v$. Let $\overline{\mathcal{M}} = \overline{\mathcal{M}}(n, K_1, K_2, D, v)$ be the closure of $\mathcal{M}$ in the Gromov-Hausdorff topology.

If $X_i \in \overline{\mathcal{M}}$ ($i \in \mathbb{N}$) converges to $X \in \overline{\mathcal{M}}$ in the Gromov-Hausdorff topology, then there exist a sequence of positive real numbers $\{\epsilon_i\}_{i \in \mathbb{N}}$ with $\lim_{i \rightarrow \infty} \epsilon_i = 0$, and a sequence of ϵ_i -Hausdorff approximation maps $\phi_i: X_i \rightarrow X$. Fix such a sequence. We say a sequence $x_i \in X_i$ converges to $x \in X$ if $\lim_{i \rightarrow \infty} \phi_i(x_i) = x$ (denote it by $x_i \xrightarrow{GH} x$). By the volume convergence theorem [8, Theorem 5.9], (X_i, H^n) converges to (X, H^n) in the measured Gromov-Hausdorff sense, i.e., for all $r > 0$ and all sequence $x_i \in X_i$ that converges to $x \in X$, we have $\lim_{i \rightarrow \infty} H^n(B_r(x_i)) = H^n(B_r(x))$, where H^n denotes the n -dimensional Hausdorff measure.

For all $X \in \overline{\mathcal{M}}$, we can consider the cotangent bundle $\pi: T^*X \rightarrow X$ with a canonical inner product by [5] and [9] (see also [15, Section 2] for a short review). We have $H^n(X \setminus \pi(T^*X)) = 0$ and $T_x^*X := \pi^{-1}(x)$ is an n -dimensional vector space for all $x \in \pi(T^*X)$. For all Lipschitz function f on X , we can define $df(x) \in T_x^*X$ for almost all $x \in X$, and we have $df \in L^\infty(T^*X)$.

Let us recall definitions of functional spaces on limit spaces briefly. Note that we can define such functional spaces on more general spaces than our assumption. Some of the following functional spaces are first introduced by Gigli [11].

Definition A.5 Let $X \in \overline{\mathcal{M}}$.

(i) Let $\text{LIP}(X)$ be the set of the Lipschitz functions on X . For all $f \in \text{LIP}(X)$, we define $\|f\|_{H^{1,2}}^2 = \|f\|_2^2 + \|df\|_2^2$. Let $H^{1,2}(X)$ be the completion of $\text{LIP}(X)$ with respect to this norm.

(ii) Define

$$\mathcal{D}^2(\Delta, X) := \left\{ f \in H^{1,2}(X) : \text{there exists } F \in L^2(X) \text{ such that} \right. \\ \left. \int_X \langle df, dh \rangle dH^n = \int_X Fh dH^n \text{ for all } h \in H^{1,2}(X) \right\}.$$

For any $f \in \mathcal{D}^2(\Delta, X)$, the function $F \in L^2(X)$ is uniquely determined. Thus, we define $\Delta f := F$.

(iii) Define

$$\text{Test}F(X) := \left\{ f \in \mathcal{D}^2(\Delta, X) \cap \text{LIP}(X) : \Delta f \in H^{1,2}(X) \right\}, \\ \text{TestForm}_p(X) := \left\{ \sum_{i=1}^N f_{0,i} df_{1,i} \wedge \dots \wedge df_{p,i} : N \in \mathbb{N}, f_{j,i} \in \text{Test}F(X) \right\}$$

for all $p \in \{1, \dots, n\}$.

(vi) The operator $\nabla : \text{TestForm}_p(X) \rightarrow L^2(T^*X \otimes \bigwedge^p T^*X)$ is defined by

$$\nabla \sum_{i=1}^N f_{0,i} df_{1,i} \wedge \dots \wedge df_{p,i} \\ := \sum_{i=1}^N \left(df_{0,i} \otimes df_{1,i} \wedge \dots \wedge df_{p,i} \right. \\ \left. + \sum_{j=1}^p f_{0,i} df_{1,i} \wedge \dots \wedge \nabla^2 f_{j,i} \wedge \dots \wedge df_{p,i} \right),$$

where ∇^2 denotes the Hessian Hess defined in [11, Definition 3.3.1] or [14].

(v) For any $\omega \in \text{TestForm}_p(X)$, we define $\|\omega\|_{H_C^{1,2}}^2 := \|\omega\|_2^2 + \|\nabla\omega\|_2^2$. Let

$H_C^{1,2}(\bigwedge^p T^*X)$ be the completion of $\text{TestForm}_p(X)$ with respect to this norm.

(vi) Define

$$\mathcal{D}^2(\Delta_{C,p}, X) := \left\{ \omega \in H_C^{1,2}(\bigwedge^p T^*X) : \text{there exists } \hat{\omega} \in L^2(\bigwedge^p T^*X) \text{ such that} \right. \\ \left. \int_X \langle \nabla\omega, \nabla\eta \rangle dH^n = \int_X \langle \hat{\omega}, \eta \rangle dH^n \text{ for all } \eta \in H_C^{1,2}(\bigwedge^p T^*X) \right\}.$$

For any $\omega \in \mathcal{D}^2(\Delta_{C,p}, X)$, the form $\hat{\omega} \in L^2(\bigwedge^p T^*X)$ is uniquely determined.

Thus, we put $\Delta_{C,p}\omega := \hat{\omega}$.

(viii) For all $k \in \mathbb{Z}_{>0}$, we define

$$\lambda_k(\Delta_{C,p}, X) := \inf \left\{ \sup_{\omega \in \mathcal{E}_k \setminus \{0\}} \frac{\|\nabla \omega\|_2^2}{\|\omega\|_2^2} : \mathcal{E}_k \subset H_C^{1,2}(\bigwedge^p T^*X) \text{ is a } k\text{-dimensional subspace} \right\}.$$

Similarly to the smooth case, there exists a complete orthonormal system of eigenforms of the connection Laplacian $\Delta_{C,p}$ in $L^2(\bigwedge^p T^*M)$, and each eigenform is an element of $\mathcal{D}^2(\Delta_{C,p}, X)$ (see [17, Theorem 4.17]).

Honda [17] showed the following theorem:

Theorem A.6 ([17]) *Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence in $\overline{\mathcal{M}}$ and let $X \in \overline{\mathcal{M}}$ be its Gromov-Hausdorff limit. Then, we have $\lim_{i \rightarrow \infty} \lambda_k(\Delta_{C,p}, X_i) = \lambda_k(\Delta_{C,p}, X)$ for all $p \in \{0, \dots, n\}$ and $k \in \mathbb{Z}_{>0}$.*

Definition A.7 (Orientation [16]) Let $X \in \overline{\mathcal{M}}$. We say that X is orientable if there exists $\omega \in L^\infty(\bigwedge^n T^*X)$ such that $|\omega|(z) = 1$ for almost all $z \in X$ and that $\langle \omega, \eta \rangle \in H^{1,2}(X)$ for any $\eta \in \text{TestForm}_n(X)$. We call ω an orientation of X .

Lemma A.8 *Let $X \in \overline{\mathcal{M}}$. Then, X is orientable if and only if $\lambda_1(\Delta_{C,n}, X) = 0$.*

Proof We first suppose that X is orientable and show $\lambda_1(\Delta_{C,n}, X) = 0$. Let $\omega \in L^\infty(\bigwedge^n T^*X)$ be the orientation of X . By [16, Proposition 6.5], for almost all $z \in X$, ω is differentiable at z and $\nabla^{g_X} \omega(z) = 0$, where ∇^{g_X} denotes the Levi-Civita connection defined in [14]. By Proposition 4.5 and Remark 4.7 in [17], we have $\omega \in H_C^{1,2}(\bigwedge^n T^*X)$. By [18, Corollary 7.10], we have $\nabla \omega(z) = \nabla^{g_X} \omega(z) = 0$ for almost all $z \in X$. Thus, we get $\lambda_1(\Delta_{C,n}, X) = 0$ by the definition of $\lambda_1(\Delta_{C,n}, X)$.

We next suppose $\lambda_1(\Delta_{C,n}, X) = 0$ and show that X is orientable. Let $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence in $\mathcal{M}$ that converges to X in the Gromov-Hausdorff topology. Then, we have $\lim_{i \rightarrow \infty} \lambda_1(\Delta_{C,n}, g_i) = 0$ by Theorem A.6. Thus, by Corollary A.2, we get that M_i is orientable for sufficiently large i , and so X is orientable by the stability of orientability [16, Theorem 1.3]. $\square$

Proof of Theorem A.3 Let $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence in $\mathcal{M}$ and let X be its Gromov-Hausdorff limit. Suppose that each M_i is not orientable. Then, we have $\lambda_1(\Delta_{C,n}, g_i) \geq C_1(n, K_1, 2D)$ by Lemma A.1. By Theorem A.6, we get $\lambda_1(\Delta_{C,n}, X) \geq C_1(n, K_1, 2D)$. Thus, by Lemma A.8, we get the theorem. $\square$

Theorem A.9 *Let $X \in \overline{\mathcal{M}}$. If X is not orientable, then we have $\lambda_1(\Delta_{C,n}, X) \geq C_1(n, K_1, 2D)$.*

Proof Let $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence in $\mathcal{M}$ that converges to X in the Gromov-Hausdorff topology. By Lemma A.8, we have $\lambda_1(\Delta_{C,n}, X) > 0$, and so we get $\lambda_1(\Delta_{C,n}, g_i) > 0$ for sufficiently large i by Theorem A.6. Thus, M_i is not orientable and $\lambda_1(\Delta_{C,n}, g_i) \geq C_1(n, K_1, 2D)$ for sufficiently large i by Lemma A.1. By Theorem A.6, we get the theorem. $\square$

We immediately get the following corollaries:

Corollary A.10 Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence in $\overline{\mathcal{M}}$ and let $X \in \overline{\mathcal{M}}$ be its Gromov-Hausdorff limit. If X_i is not orientable for each i , then X is not orientable.

Corollary A.11 Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence in $\overline{\mathcal{M}}$ and let $X \in \overline{\mathcal{M}}$ be its Gromov-Hausdorff limit. Then, the following two conditions are mutually equivalent.

- (i) X_i is orientable for sufficiently large i .
- (ii) X is orientable.

By Corollary A.11, we have that if $X_1 \in \overline{\mathcal{M}}$ is orientable and $X_2 \in \overline{\mathcal{M}}$ is unorientable, then X_1 and X_2 belong to different connected components in $\overline{\mathcal{M}}$ with respect to the Gromov-Hausdorff topology.

Appendix B. Eigenvalue Estimate for L^2 Almost Kähler Manifolds

In this section, we consider L^2 almost Kähler manifolds, i.e., we assume that there exists a 2-form ω which satisfies that $\|\nabla\omega\|_2$ and $\|J_\omega^2 + \text{Id}\|_1$ are small, where $J_\omega \in \Gamma(T^*M \otimes TM)$ is defined so that $\omega = g(J_\omega \cdot, \cdot)$. The main goal is to give the almost version of (1).

Notation B.1 Let (M, g) be a Riemannian manifold. For each 2-form $\omega \in \Gamma(\wedge^2 T^*M)$, let $J_\omega \in \Gamma(T^*M \otimes TM)$ denotes the anti-symmetric tensor that satisfies $\omega = g(J_\omega \cdot, \cdot)$.

We first show the following easy lemmas.

Lemma B.2 Let (M, g) be an n -dimensional closed Riemannian manifold. If there exists a 2-form ω such that $\|J_\omega^2 + \text{Id}\|_1 < 1$ holds, then n is an even integer.

Proof There exists a point $x \in M$ such that $|J_\omega^2(x) + \text{Id}_{T_x M}| < 1$. For any $v \in T_x M$ with $|v| = 1$, we have $|J_\omega^2(x)(v) + v| < 1$, and so $|J_\omega^2(x)(v)| > 0$. Thus, $J_\omega(x)$ is non-degenerate. Therefore, $(T_x M, \omega_x)$ is a symplectic vector space. This implies the lemma. $\square$

Lemma B.3 Given integers $n \geq 2$, $1 \leq p \leq n - 1$, and positive real numbers $K > 0$, $D > 0$, there exists $\delta_0(n, p, K, D) > 0$ such that if (M, g) is an n -dimensional closed Riemannian manifold with $\text{Ric} \geq -Kg$ and $\text{diam}(M) \leq D$, then we have $\lambda_{\alpha(n,p)+1}(\Delta_{C,p}) \geq \delta_0(n, p, K, D)$, where we defined $\alpha(n, p) := n!/(p!(n-p)!)$.

Proof Put $\delta := \lambda_{\alpha(n,p)+1}(\Delta_{C,p})$. If $\delta \geq 1$, we get the lemma. Thus, we assume that $\delta < 1$. Let $\omega_i \in \Gamma(\wedge^p T^*M)$ denotes the i -th eigenform of the connection Laplacian $\Delta_{C,p}$ acting on p -forms with $\|\omega_i\|_2 = 1$.

We have

$$\|\langle \omega_i, \omega_j \rangle\|_2^2 \leq \frac{1}{\lambda_1(g)} \|\nabla \langle \omega_i, \omega_j \rangle\|_2^2 \leq C(n, p, K, D)\delta \quad (151)$$

for each $i, j = 1, \dots, \alpha(n, p) + 1$ with $i \neq j$ by the Li-Yau estimate [22, p.116] and Lemma 3.7. By Lemma 3.5 and (151), we have

$$\|\langle \omega_i, \omega_j \rangle\|_1 \leq C(n, p, K, D)\delta^{1/2} \quad (i, j = 1, \dots, \alpha(n, p) + 1 \text{ with } i \neq j),$$

$$\| |\omega_i|^2 - 1 \|_1 \leq C(n, p, K, D) \delta^{1/2} \quad (i = 1, \dots, \alpha(n, p) + 1).$$

Put

$$G := \left\{ x \in M : |\omega_i|^2 - 1(x) \leq \delta^{1/4} \text{ for all } i = 1, \dots, \alpha(n, p) + 1, \text{ and } \right. \\ \left. |\langle \omega_i, \omega_j \rangle|(x) \leq \delta^{1/4} \text{ for all } i, j = 1, \dots, \alpha(n, p) + 1 \text{ with } i \neq j \right\}.$$

Then, we have $\text{Vol}(M \setminus G) \leq C_1(n, p, K, D) \delta^{1/4} \text{Vol}(M)$ for some positive constant $C_1(n, p, K, D)$ depending only on n, p, K and D similarly to Lemma 4.49.

Let us show $\delta \geq \min \{1/C_1(n, p, K, D)^4, 1/(\alpha(n, p) + 1)^4\}$ by contradiction. Suppose that $\delta < \min \{1/C_1(n, p, K, D)^4, 1/(\alpha(n, p) + 1)^4\}$. Then, we have $G \neq \emptyset$, and so we can take a point $x_0 \in G$. We show that $\omega_1(x_0), \dots, \omega_{\alpha(n, p)+1}(x_0) \in \bigwedge^p T_{x_0}^* M$ are linearly independent. Take arbitrary $a_1, \dots, a_{\alpha(n, p)+1} \in \mathbb{R}$ with $a_1 \omega_1(x_0) + \dots + a_{\alpha(n, p)+1} \omega_{\alpha(n, p)+1}(x_0) = 0$. Take i with $|a_i| = \max\{|a_1|, \dots, |a_{\alpha(n, p)+1}|\}$. Since we have $\langle a_1 \omega_1(x_0) + \dots + a_k \omega_{\alpha(n, p)+1}(x_0), \omega_i(x_0) \rangle = 0$, we get

$$0 \geq |a_i| |\omega_i(x_0)|^2 - \sum_{i \neq j} |a_j \langle \omega_i(x_0), \omega_j(x_0) \rangle| \geq |a_i| \left(1 - (\alpha(n, p) + 1) \delta^{1/4}\right).$$

Thus, $|a_i| = 0$, and so $a_1 = \dots = a_k = 0$. This implies the linearly independence of $\omega_1(x_0), \dots, \omega_{\alpha(n, p)+1}(x_0)$. This contradicts to $\dim \left(\bigwedge^p T_{x_0}^* M\right) = \alpha(n, p)$. Thus, we get $\lambda_{\alpha(n, p)+1}(\Delta_{C, p}) = \delta \geq \min \{1/C_1(n, p, K, D)^4, 1/(\alpha(n, p) + 1)^4\}$. $\square$

Lemma B.4 *Let (M, g) be an n -dimensional closed Riemannian manifold. Suppose that a 2-form ω satisfies*

- (i) $\|\nabla \omega\|_2^2 \leq \delta \|\omega\|_2^2$,
- (ii) $\|J_\omega^2 + \text{Id}\|_1 \leq \delta^{1/4} \|\omega\|_2^2$

for some $0 < \delta \leq 1/4$. Let ω_α be its image of the orthogonal projection

$$P_\delta : L^2 \left(\bigwedge^2 T^* M \right) \rightarrow \bigoplus_{\lambda_i(\Delta_{C, 2}) \leq \delta^{1/2}} \mathbb{R} \omega_i,$$

where ω_i denotes the i -th eigenform of the connection Laplacian $\Delta_{C, 2}$ with $\|\omega_i\|_2 = 1$ ($\omega_\alpha := P_\delta(\omega)$). Then, we have

- $\|\nabla \omega_\alpha\|_2^2 \leq 2\delta \|\omega_\alpha\|_2^2$,
- $\|J_{\omega_\alpha}^2 + \text{Id}\|_1 \leq 10\delta^{1/4} \|\omega_\alpha\|_2^2$.

Proof Put $\omega_\beta := \omega - \omega_\alpha$. Then, we have $\|\omega\|_2^2 = \|\omega_\alpha\|_2^2 + \|\omega_\beta\|_2^2$. By the assumption (i), we have

$$\delta \|\omega\|_2^2 \geq \|\nabla \omega\|_2^2 = \|\nabla \omega_\alpha\|_2^2 + \|\nabla \omega_\beta\|_2^2 \geq \|\nabla \omega_\alpha\|_2^2 + \delta^{1/2} \|\omega_\beta\|_2^2.$$

Thus, we get

$$\|\nabla\omega_\alpha\|_2^2 \leq \delta\|\omega\|_2^2, \quad (152)$$

$$\|\omega_\beta\|_2^2 \leq \delta^{1/2}\|\omega\|_2^2, \quad (153)$$

and so

$$\|\omega_\alpha\|_2^2 = \|\omega\|_2^2 - \|\omega_\beta\|_2^2 \geq (1 - \delta^{1/2})\|\omega\|_2^2 \geq \frac{1}{2}\|\omega\|_2^2. \quad (154)$$

By the definitions of the norms, we have $|J_\omega|^2 = 2|\omega|^2$ and $|J_{\omega_\alpha}|^2 = 2|\omega_\alpha|^2$. Since we have $J_\omega^2 - J_{\omega_\alpha}^2 = J_\omega(J_\omega - J_{\omega_\alpha}) + (J_\omega - J_{\omega_\alpha})J_{\omega_\alpha}$, we get $|J_\omega^2 - J_{\omega_\alpha}^2| \leq 2(|\omega| + |\omega_\alpha|)|\omega_\beta|$. Therefore, we have

$$\|J_\omega^2 - J_{\omega_\alpha}^2\|_1 \leq 4\|\omega\|_2\|\omega_\beta\|_2 \leq 4\delta^{1/4}\|\omega\|_2^2$$

by (153), and so

$$\|J_{\omega_\alpha}^2 + \text{Id}\|_1 \leq \|J_\omega^2 + \text{Id}\|_1 + \|J_\omega^2 - J_{\omega_\alpha}^2\|_1 \leq 5\delta^{1/4}\|\omega\|_2^2 \leq 10\delta^{1/4}\|\omega_\alpha\|_2^2 \quad (155)$$

by (154). By (152) and (155), we get the lemma. $\square$

Let us show the orientability for L^2 almost Kähler manifolds.

Proposition B.5 *For any integer $n \geq 2$ and positive real numbers $K > 0$, $D > 0$, there exists a constant $\delta_1(n, K, D) > 0$ such that the following property holds. Let (M, g) be an n -dimensional closed Riemannian manifold with $\text{Ric} \geq -Kg$ and $\text{diam}(M) \leq D$. If there exists a 2-form ω such that*

- (i) $\|\nabla\omega\|_2^2 \leq \delta_1\|\omega\|_2^2$,
- (ii) $\|J_\omega^2 + \text{Id}\|_1 \leq \delta_1^{1/4}\|\omega\|_2^2$,

then M is orientable.

Proof By Lemma B.2, we have that $n = 2m$ is an even integer. We first assume that $\delta_1 < \min\{1/4m^2, \delta_0(n, 2, K, D)^2\}$. Since J_ω is anti-symmetric, we have $|J_\omega|^2 \leq \sqrt{2m}|J_\omega^2|$. Thus, we get

$$\sqrt{\frac{2}{m}}\|\omega\|_2^2 = \frac{1}{\sqrt{2m}}\|J_\omega\|_2^2 \leq \sqrt{2m} + \delta_1^{1/4}\|\omega\|_2^2$$

by $|\text{Id}| = \sqrt{2m}$. This and $\delta_1^{1/4} \leq \frac{1}{2}\sqrt{\frac{2}{m}}$ imply that $\|\omega\|_2 \leq \sqrt{2m}$. Put $\omega_\alpha := P_{\delta_1}(\omega)$. Note that we have that $\|\omega_\alpha\|_2 \leq \|\omega\|_2 \leq \sqrt{2m}$ and that $\|\omega_\alpha\|_\infty \leq C(n, K, D)$ by Lemmas 3.7 and B.3.

We first fix $x \in M$, and consider the $\mathbb{C}$ -linear map

$$J_{\omega_\alpha}(x): T_x M \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T_x M \otimes_{\mathbb{R}} \mathbb{C}.$$

Let us extend the Riemannian metric $\langle \cdot, \cdot \rangle$ to $T_x M \otimes_{\mathbb{R}} \mathbb{C}$ so that

$$\langle u_1 + i v_1, u_2 + i v_2 \rangle = (\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) + i(\langle v_1, u_2 \rangle - \langle u_1, v_2 \rangle)$$

for all $u_1, u_2, v_1, v_2 \in T_x M$. Since $J_{\omega_\alpha}(x)$ is anti-symmetric, there exist eigenvalues $\{\lambda_1, \bar{\lambda}_1, \dots, \lambda_m, \bar{\lambda}_m\}$ of $J_{\omega_\alpha}(x)$ and an orthogonal basis $\{E_1, \bar{E}_1, \dots, E_m, \bar{E}_m\}$ of $T_x M \otimes_{\mathbb{R}} \mathbb{C}$ such that $J_{\omega_\alpha}(x)E_i = \lambda_i E_i$, where the overline denotes the complex conjugate. Note that each λ_i is a pure imaginary number. Let $\{E^1, \bar{E}^1, \dots, E^m, \bar{E}^m\} \subset T_x^* M \otimes_{\mathbb{R}} \mathbb{C} \cong (T_x M \otimes_{\mathbb{R}} \mathbb{C})^*$ be the dual basis of $\{E_1, \bar{E}_1, \dots, E_m, \bar{E}_m\}$. If we extend $\omega_\alpha(x)$ to a complex bilinear form, then we have $\omega_\alpha(x) = \sum_{i=1}^m \lambda_i E^i \wedge \bar{E}^i$. Thus, we get $\omega_\alpha^m(x) = m! \lambda_1 \cdots \lambda_m E^1 \wedge \bar{E}^1 \wedge \cdots \wedge E^m \wedge \bar{E}^m$, and so $|\omega_\alpha^m(x)| = m! |\lambda_1| \cdots |\lambda_m|$. Since we have $|\lambda_i|^2 = |(J_{\omega_\alpha}^2 + \text{Id})E_i - E_i|$, we get $|1 - |\lambda_i|^2| \leq |J_{\omega_\alpha}^2 + \text{Id}|(x)$ and $|\lambda_i| \leq C(n, K, D)$. Therefore, we get

$$\left| |\omega_\alpha^m|^2 - (m!)^2 \right| \leq C |J_{\omega_\alpha}^2 + \text{Id}|,$$

and so $\left| \|\omega_\alpha^m\|_2^2 - (m!)^2 \right| \leq C \delta_1^{1/4}$ by Lemma B.4. Since we have $\|\nabla(\omega_\alpha^m)\|_2^2 \leq C \delta_1$ by Lemma B.4, we get the proposition taking δ_1 sufficiently small by Corollary A.2 (ii). $\square$

The following theorem is the goal of this section.

Theorem B.6 *For any integer $n \geq 2$, there exists a constant $C(n) > 0$ such that the following property holds. Let (M, g) be an n -dimensional closed Riemannian manifold with $\text{Ric} \geq (n-1)g$. If there exists a 2-form ω such that*

- (i) $\|\nabla \omega\|_2^2 \leq \delta \|\omega\|_2^2$,
- (ii) $\|J_\omega^2 + \text{Id}\|_1 \leq \delta^{1/4} \|\omega\|_2^2$,

for some $\delta > 0$, then we have $\lambda_1(g) \geq 2(n-1) - C(n)\delta^{1/2}$.

Remark B.1 It is enough to prove the theorem when δ is small. Thus, we can assume that $n = 2m$ is an even integer by Lemma B.2. If $n = 2$, then $\lambda_1(g) \geq 2(n-1)$ is the original Lichnerowicz estimate. If $n = 4$, the conclusion of the theorem can also be deduced from Main Theorem 1.

Proof We first assume that $\delta < \min\{1/4m^2, \delta_0(n, 2, K, D)^2\}$. Put $\omega_\alpha := P_\delta(\omega) = \sum_{i=1}^k a_i \omega_i$. Here, ω_i is the i -th eigenform of the connection Laplacian $\Delta_{C,2}$ with $\|\omega_i\|_2 = 1$ corresponding to the eigenvalue $\lambda_i(\Delta_{C,2}) \leq \delta^{1/2}$ for each $i = 1, \dots, k$. Similarly to Proposition B.5, we have $\|\omega_\alpha\|_\infty \leq C$.

Let $f \in C^\infty(M)$ be the first eigenfunction of the Laplacian with $\|f\|_2 = 1$. If $\lambda_1(g) \geq 2(n-1) + 1$, we get the theorem. Thus, we assume that $\lambda_1(g) \leq 2(n-1) + 1$. Then, we have $\|f\|_\infty \leq C$ and $\|\nabla f\|_\infty \leq C$ by Lemma 3.7. By Lemma 3.6 (i) and (iii), we have

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \left| \int_M \langle \iota(\nabla f) \omega_\alpha, \iota(\nabla f) \omega_\alpha \rangle - \lambda_1(g) \langle \iota(\nabla f) \omega_\alpha, \iota(\nabla f) \omega_\alpha \rangle d\mu_g \right| \\ & \leq C \delta^{1/2} \|\omega_\alpha\|_2^2 \end{aligned} \quad (156)$$

and

$$\|d^*(\iota(\nabla f)\omega_\alpha)\|_2^2 \leq C\delta\|\omega_\alpha\|_2^2. \quad (157)$$

By (4), (156), (157) and the Bochner formula, we get

$$\begin{aligned} & \frac{n-1}{\text{Vol}(M)} \int_M \langle J_{\omega_\alpha} \nabla f, J_{\omega_\alpha} \nabla f \rangle d\mu_g \\ & \leq \frac{1}{\text{Vol}(M)} \int_M \text{Ric}(J_{\omega_\alpha} \nabla f, J_{\omega_\alpha} \nabla f) d\mu_g \\ & = \frac{1}{\text{Vol}(M)} \int_M \langle \Delta(\iota(\nabla f)\omega_\alpha), \iota(\nabla f)\omega_\alpha \rangle d\mu_g - \frac{1}{\text{Vol}(M)} \int_M |\nabla(\iota(\nabla f)\omega_\alpha)|^2 d\mu_g \\ & \leq \frac{\lambda_1(g)}{2\text{Vol}(M)} \int_M \langle \iota(\nabla f)\omega_\alpha, \iota(\nabla f)\omega_\alpha \rangle d\mu_g + C\delta^{1/2} \\ & = \frac{\lambda_1(g)}{2\text{Vol}(M)} \int_M \langle J_{\omega_\alpha} \nabla f, J_{\omega_\alpha} \nabla f \rangle d\mu_g + C\delta^{1/2}. \end{aligned} \quad (158)$$

Since J_{ω_α} is anti-symmetric, we have

$$\begin{aligned} & \frac{1}{\text{Vol}(M)} \left| \int_M \langle J_{\omega_\alpha} \nabla f, J_{\omega_\alpha} \nabla f \rangle d\mu_g - \int_M \langle \nabla f, \nabla f \rangle d\mu_g \right| \\ & \leq \frac{1}{\text{Vol}(M)} \left| \int_M \langle (J_{\omega_\alpha}^2 + \text{Id}) \nabla f, \nabla f \rangle d\mu_g \right| \\ & \leq C \|J_{\omega_\alpha}^2 + \text{Id}\|_1 \leq C\delta^{1/4} \end{aligned}$$

by Lemma B.4. Thus, taking δ sufficiently small, we get

$$\|J_{\omega_\alpha} \nabla f\|_2^2 \geq \|\nabla f\|_2^2 - C\delta^{1/4} = \lambda_1(g) - C\delta^{1/4} \geq n - C\delta^{1/4} \geq \frac{n}{2} \quad (159)$$

by the Lichnerowicz estimate. By (158) and (159), we get the theorem. $\square$

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