



# Spectral Estimates for Riemannian Submersions with Fibers of Basic Mean Curvature

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## Abstract

For Riemannian submersions with fibers of basic mean curvature, we compare the spectrum of the total space with the spectrum of a Schrödinger operator on the base manifold. Exploiting this concept, we study submersions arising from actions of Lie groups. In this context, we extend the state-of-the-art results on the bottom of the spectrum under Riemannian coverings. As an application, we compute the bottom of the spectrum and the Cheeger constant of connected, amenable Lie groups.

**Keywords** Riemannian submersion · Basic mean curvature · Riemannian principal bundle · Amenable Lie group · Bottom of spectrum · Discrete spectrum

**Mathematics Subject Classification** 58J50 · 35P15 · 53C99

## 1 Introduction

The study of the spectrum of the Laplacian on a Riemannian manifold has attracted much attention over the last years. In order to comprehend its relations with the geometry of the underlying manifold, it is reasonable to investigate its behavior under maps between Riemannian manifolds that respect the geometry of the manifolds to some extent. In this paper, we study the behavior of the spectrum under Riemannian submersions.

The notion of Riemannian submersion was introduced in the 1960s as a tool to express the geometry of a manifold in terms of the geometry of simpler components, namely, the base space and the fibers. Of course, by geometry of the fibers, we mean both their intrinsic and their extrinsic geometry as submanifolds of the total space. Bearing this in mind, it is natural to describe the spectrum of the total space in terms of the geometry and the spectrum of the base space and the fibers.

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To set the stage, let  $p : M_2 \rightarrow M_1$  be a Riemannian submersion and denote by  $F_x := p^{-1}(x)$  the fiber over  $x \in M_1$ . The spectrum of (the Laplacian on)  $M_2$  has been studied in the case where  $M_2$  is closed (that is, compact and without boundary) and the submersion has totally geodesic, or minimal fibers, or fibers of basic mean curvature (cf. for example the survey [6]). However, the situation is quite more complicated and yet unclear if  $M_2$  is not closed.

Recently, in [23], extending the result of [10], we established a lower bound for the bottom of the spectrum  $\lambda_0(M_2)$  of  $M_2$ , if the mean curvature of the fibers is bounded in a certain way. More precisely, according to [23, Theorem 1.1], if the (unnormalized) mean curvature of the fibers is bounded by  $\|H\| \leq C \leq 2\sqrt{\lambda_0(M_1)}$ , then the bottom of the spectrum of  $M_2$  satisfies

$$\lambda_0(M_2) \geq (\sqrt{\lambda_0(M_1)} - C/2)^2 + \inf_{x \in M_1} \lambda_0(F_x).$$

Moreover, if the equality holds and  $\lambda_0(M_1) \notin \sigma_{\text{ess}}(M_1)$  (that is,  $\lambda_0(M_1)$  is an isolated point of the spectrum of the Laplacian on  $M_1$ ), then  $\lambda_0(F_x)$  is equal to its infimum for almost any  $x \in M_1$ . Recall that, in general,  $\lambda_0(F_x)$  is only upper semi-continuous with respect to  $x \in M_1$  (cf. [23, Lemma 2.9]).

In the second part of [23], following [4], we studied Riemannian submersions with closed fibers. In this context, we introduced a Schrödinger operator on  $M_1$ , with potential determined by the volume of the fibers, and compared its spectrum with the spectrum of  $M_2$ . It should be noticed that if the submersion has fibers of infinite volume, then we are not able to define that operator, at least in the way we did in [23].

In this paper, motivated by the aforementioned results, we introduce a Schrödinger operator on the base space of a Riemannian submersion with fibers of basic mean curvature (see Sect. 2.1) and compare its spectrum with the spectrum of the total space. To be more specific, let  $p : M_2 \rightarrow M_1$  be a Riemannian submersion with fibers of basic mean curvature, and consider the Schrödinger operator

$$S = \Delta + \frac{1}{4} \|p_*H\|^2 - \frac{1}{2} \operatorname{div} p_*H \tag{1}$$

on  $M_1$ , where  $\Delta$  is the (non-negative) Laplacian on  $M_1$ . It is worth to point out that  $S$  is non-negative, that is,  $\lambda_0(S) \geq 0$ . Furthermore, it is evident that  $S$  coincides with the Laplacian, if the submersion has minimal fibers. Our first result relates the bottom of the spectrum of this operator with the bottom of the spectrum of  $M_2$ .

**Theorem 1.1** *Let  $p : M_2 \rightarrow M_1$  be a Riemannian submersion with fibers of basic mean curvature, and consider the Schrödinger operator  $S$  as above. Then*

$$\lambda_0(M_2) \geq \lambda_0(S) + \inf_{x \in M_1} \lambda_0(F_x).$$

*If, in addition, the equality holds and  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$ , then  $\lambda_0(F_x)$  is almost everywhere equal to its infimum.*

It should be emphasized that no assumptions on the geometry or the topology of the manifolds are required in this theorem. In particular, the manifolds do not have to be complete. This, together with the decomposition principle, allows us to derive a similar inequality involving the bottoms of the essential spectra, if the fibers are closed.

It is worth to mention that in some cases,  $\lambda_0(S)$  can be estimated in terms of  $\lambda_0(M_1)$ . For instance, if the mean curvature of the fibers is bounded by  $\|H\| \leq C \leq 2\sqrt{\lambda_0(M_1)}$ , then the bottoms of the spectra satisfy

$$\lambda_0(S) \geq (\sqrt{\lambda_0(M_1)} - C/2)^2.$$

Thus, Theorem 1.1 provides a sharper lower bound for  $\lambda_0(M_2)$  than [23, Theorem 1.1] in the case where both of them are applicable.

It is noteworthy that if the submersion has closed fibers, then the operator  $S$  defined in (1) coincides with the Schrödinger operator introduced in [23], and there is a remarkable relation with the work of Bordonì [5] on Riemannian submersions with fibers of basic mean curvature. Given such a submersion  $p: M_2 \rightarrow M_1$  with  $M_2$  closed, Bordonì considered the restrictions  $\Delta_c$  and  $\Delta_0$  of the Laplacian acting on lifted functions and on functions with zero average on any fiber, respectively, and showed in [5, Theorem 1.6] that the spectrum is written as  $\sigma(M_2) = \sigma(\Delta_c) \cup \sigma(\Delta_0)$ . In this setting, the spectrum of the operator  $S$  coincides with the spectrum of  $\Delta_c$ . It should be observed that expressing the latter one as the spectrum of an operator on  $M_1$  allows us to relate it more easily to the spectrum of  $M_1$ . For Riemannian submersions with closed fibers, we obtain the following consequence of Theorem 1.1 (compare with [23, Theorem 1.2]), where we denote by  $\lambda_0^{\text{ess}}$  the bottom of the essential spectrum of an operator.

**Corollary 1.2** *If  $p: M_2 \rightarrow M_1$  is a Riemannian submersion with closed fibers of basic mean curvature, then  $\lambda_0(M_2) = \lambda_0(S)$  and  $\lambda_0^{\text{ess}}(M_2) = \lambda_0^{\text{ess}}(S)$ . In particular,  $M_2$  has discrete spectrum if and only if the spectrum of  $S$  is discrete.*

This corollary generalizes [4, Theorem 1(ii)], which asserts that if  $p: M_2 \rightarrow M_1$  is a Riemannian submersion with closed and minimal fibers, then  $M_1$  has discrete spectrum if and only if  $M_2$  has discrete spectrum. This equivalence has been extended in [23, Corollary 1.4] under the weaker assumption that the fibers are closed and of bounded mean curvature. Corollary 1.2 characterizes the discreteness of the spectrum of  $M_2$  in terms of  $S$  instead of the Laplacian, which, nonetheless, is very natural. More precisely, for warped products of the form  $M \times_{\psi} F$  with  $F$  closed, this characterization coincides with [1, Theorem 3.3] of Baider.

If, in addition, the manifolds involved in Corollary 1.2 are complete, then we know from [23, Theorem 1.2] that the spectra and the essential spectra satisfy  $\sigma(S) \subset \sigma(M_2)$  and  $\sigma_{\text{ess}}(S) \subset \sigma_{\text{ess}}(M_2)$ . This, together with Theorem 1.1 and Corollary 1.2, shows that it is very reasonable to compare the spectrum of  $S$  with the spectrum of  $M_2$ , if the submersion has fibers of basic mean curvature.

In the second part of the paper, we study Riemannian principal bundles. To be more specific, let  $G$  be a possibly discrete Lie group acting smoothly, freely and properly on a Riemannian manifold  $M_2$  via isometries, where  $\dim G < \dim M_2$ . Such an action

induces on  $M_1 = M_2/G$  the structure of Riemannian manifold. If  $G$  is non-discrete, the projection  $p: M_2 \rightarrow M_1$  is a Riemannian submersion with fibers of basic mean curvature. We then say that  $p$  is a *Riemannian submersion arising from the action of  $G$* . The behavior of the spectrum under such submersions has been studied for instance in [13].

In the case where  $G$  is a discrete group, its action gives rise to a normal Riemannian covering. In this context, there are various results establishing relations between properties of the deck transformation group and the behavior of the spectrum. To be more precise, let  $q: M_2 \rightarrow M_1$  be a normal Riemannian covering with deck transformation group  $\Gamma$ . Then the bottoms of the spectra satisfy  $\lambda_0(M_2) \geq \lambda_0(M_1)$  (cf. for instance [2] and the references therein). Brooks was the first one to investigate when the equality holds. In [8], he showed that if  $M_1$  is closed, then  $\Gamma$  is amenable if and only if  $\lambda_0(M_2) = 0$ . It is apparent that in this setting, we also have that  $\lambda_0(M_1) = 0$ . In [2], we proved that if  $\Gamma$  is amenable, then  $\lambda_0(M_2) = \lambda_0(M_1)$ , without any assumptions on the topology or the geometry of  $M_1$ . It was established in [22] that if, in addition,  $M_1$  is complete, then  $\sigma(M_1) \subset \sigma(M_2)$ . Conversely, according to [21], if  $\lambda_0(M_2) = \lambda_0(M_1)$  and  $\lambda_0(M_1) \notin \sigma_{\text{ess}}(M_1)$ , then  $\Gamma$  is amenable.

If  $G$  is non-discrete, then from the above discussion, it makes sense to compare the spectrum of the Laplacian on the total space with the spectrum of the Schrödinger operator  $S$  on the base manifold, defined in (1). Theorem 1.1 implies that  $\lambda_0(M_2) \geq \lambda_0(S)$ . In the following theorem, we extend the aforementioned results to Riemannian submersions arising from Lie group actions, where we denote by  $G_0$  the connected component of the identity element of  $G$ .

**Theorem 1.3** *Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a Lie group  $G$ . Then*

- (i) *If  $G$  is amenable and  $G_0$  is unimodular, then  $\lambda_0(M_2) = \lambda_0(S)$ .*
- (ii) *If, in addition,  $M_1$  is complete, then  $\sigma(S) \subset \sigma(M_2)$ .*
- (iii) *Conversely, if  $\lambda_0(M_2) = \lambda_0(S)$  and  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$ , then  $G$  is amenable and  $G_0$  is unimodular.*

Recall that there exist connected Lie groups that are amenable but not unimodular (because any solvable group is amenable), and connected Lie groups that are unimodular but not amenable (since any connected, semisimple Lie group is unimodular).

It is notable that if  $G$  is compact, then Corollary 1.2 compares the spectra and the essential spectra of the operators. Even though Theorem 1.3 is formulated in terms of spectra, it also provides information about the essential spectra. This follows from the fact that if  $G$  is non-compact, then  $\sigma(M_2) = \sigma_{\text{ess}}(M_2)$  (cf. for example [22, Theorem 5.2]).

As in the context of Riemannian coverings, it is plausible to wonder if the assumption  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$  can be weakened in Theorem 1.3(iii). We will construct a wide class of examples demonstrating that this assumption is essential. Namely, let  $M$  be any Riemannian manifold with  $\lambda_0(M) \in \sigma_{\text{ess}}(M)$ . We will show that there exists a Riemannian submersion  $p: M_2 \rightarrow M_1 := M$  with minimal fibers, arising from the action of a connected, non-unimodular Lie group  $G$ , such that  $\lambda_0(M_2) = \lambda_0(M_1)$ . Since the submersion has minimal fibers, it is clear that  $S$  coincides with the Laplacian on  $M_1$ .

In the case where the base manifold is closed, we derive another analog of Brooks' result, which is slightly different. This is because in Theorem 1.3, we investigate the validity of  $\lambda_0(M_2) = \lambda_0(S)$ , while the following corollary characterizes the stronger property  $\lambda_0(M_2) = 0$ .

**Corollary 1.4** *Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a Lie group  $G$ , where  $M_1$  is closed. Then  $G$  is unimodular and amenable if and only if  $\lambda_0(M_2) = 0$ .*

Finally, exploiting Theorems 1.1 and 1.3, we study quotients of Lie groups by normal subgroups. In this setting, we obtain some relations between the mean curvature of the subgroup and the bottom of the spectrum of the group, the subgroup, and the quotient.

**Theorem 1.5** *Let  $G$  be a connected Lie group endowed with a left-invariant metric and  $N$  be a closed (as a subset), connected, normal subgroup of  $G$  with mean curvature  $H$ . Then*

$$\lambda_0(G) \geq \lambda_0(G/N) + \lambda_0(N) - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H).$$

Moreover,  $N$  is unimodular and amenable if and only if

$$\lambda_0(G) = \lambda_0(G/N) - \frac{1}{4}\|H\|^2 + \frac{1}{2}\operatorname{tr}(\operatorname{ad} H).$$

As an application of this theorem, we compute the bottom of the spectrum and the Cheeger constant of connected, amenable Lie groups. This extends the result of [20] in various ways.

**Corollary 1.6** *Let  $G$  be a connected, amenable Lie group endowed with a left-invariant metric. Then the bottom of its spectrum and its Cheeger constant are given by*

$$\lambda_0(G) = \frac{1}{4}h(G)^2 = \frac{1}{4} \max_{X \in \mathfrak{g}, \|X\|=1} (\operatorname{tr}(\operatorname{ad} X))^2.$$

If  $G$  is not unimodular, then the maximum is achieved by the unit vector in the direction of the mean curvature (in  $G$ ) of the commutator subgroup  $[S, S]$  of the radical  $S$  of  $G$ .

The paper is organized as follows: In Sect. 2, we discuss some basic properties of Schrödinger operators, Riemannian submersions and Lie groups. In particular, we provide a spectral theoretic characterization for connected, amenable, and unimodular Lie groups, which is well known for simply connected Lie groups. In Sect. 3, we study Riemannian submersions with fibers of basic mean curvature and prove Theorem 1.1 and Corollary 1.2. In Sect. 4, we focus on submersions arising from Lie group actions and establish Theorem 1.3 and Corollary 1.4. In Sect. 5, we discuss some consequences of our results to Lie groups and show Theorem 1.5 and Corollary 1.6.

## 2 Preliminaries

Throughout this paper, manifolds are assumed to be connected and without boundary, unless otherwise stated, except for Lie groups. Consider a possibly non-connected Riemannian manifold  $M$ . A *Schrödinger operator* on  $M$  is an operator of the form  $S = \Delta + V$ , where  $\Delta$  is the Laplacian on  $M$  and  $V \in C^\infty(M)$ , such that there exists  $c \in \mathbb{R}$  satisfying

$$\langle Sf, f \rangle_{L^2(M)} \geq c \|f\|_{L^2(M)}^2$$

for any  $f \in C_c^\infty(M)$ . Then the operator

$$S: C_c^\infty(M) \subset L^2(M) \rightarrow L^2(M)$$

admits a Friedrichs extension, being densely defined, symmetric and bounded from below. It is worth to point out that if  $M$  is complete, then this operator is essentially self-adjoint; that is, its closure coincides with its Friedrichs extension (cf. [23, Proposition 2.4]).

The spectrum and the essential spectrum of (the Friedrichs extension of)  $S$  are denoted by  $\sigma(S)$  and  $\sigma_{\text{ess}}(S)$ , respectively, and their bottoms (that is, their minimums) by  $\lambda_0(S)$  and  $\lambda_0^{\text{ess}}(S)$ , respectively. In the case of the Laplacian (that is,  $V = 0$ ), we write  $\sigma(M)$ ,  $\sigma_{\text{ess}}(M)$  and  $\lambda_0(M)$ ,  $\lambda_0^{\text{ess}}(M)$  for these sets and quantities. We have by definition that  $\lambda_0^{\text{ess}}(S) = +\infty$  if  $S$  has *discrete spectrum*, which means that  $\sigma_{\text{ess}}(S)$  is empty. If  $\sigma_{\text{ess}}(M)$  is empty, we say that  $M$  has discrete spectrum.

The *Rayleigh quotient* of a non-zero  $f \in \text{Lip}_c(M)$  with respect to  $S$  is defined by

$$\mathcal{R}_S(f) := \frac{\int_M (\|\text{grad } f\|^2 + Vf^2)}{\int_M f^2}.$$

The Rayleigh quotient of  $f$  with respect to the Laplacian is denoted by  $\mathcal{R}(f)$ , or by  $\mathcal{R}_g(f)$  if the Riemannian metric  $g$  of  $M$  is not clear from the context. According to the next proposition, the bottom of the spectrum of  $S$  can be expressed as an infimum of Rayleigh quotients (cf. for example [21, Sect. 2] and the references therein).

**Proposition 2.1** *Let  $S$  be a Schrödinger operator on a Riemannian manifold  $M$ . Then the bottom of the spectrum of  $S$  satisfies*

$$\lambda_0(S) = \inf_f \mathcal{R}_S(f),$$

where the infimum is taken over all  $f \in C_c^\infty(M) \setminus \{0\}$ , or over all  $f \in \text{Lip}_c(M) \setminus \{0\}$ .

A remarkable property of the essential spectrum of  $S$  follows from the decomposition principle, which states that

$$\sigma_{\text{ess}}(S) = \sigma_{\text{ess}}(S, M \setminus K)$$

for any compact domain  $K$  of  $M$  with smooth boundary. This is well known in the case where  $M$  is complete (compare with [12, Proposition 2.1]), but also holds if  $M$  is non-complete (cf. for instance [3, Theorem A.17]). The next proposition summarizes the properties of the bottom of the essential spectrum that will be used in the sequel.

**Proposition 2.2** *Let  $S$  be a Schrödinger operator on a Riemannian manifold  $M$  and consider an exhausting sequence  $(K_n)_{n \in \mathbb{N}}$  of  $M$  consisting of compact domains with smooth boundary. Then the bottom of the essential spectrum of  $S$  is given by*

$$\lambda_0^{\text{ess}}(S) = \lim_n \lambda_0(S, M \setminus K_n).$$

*In particular, there exists  $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \setminus \{0\}$  with  $\text{supp } f_n$  pairwise disjoint, such that  $\mathcal{R}_S(f_n) \rightarrow \lambda_0^{\text{ess}}(S)$ . Furthermore, for any sequence  $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \setminus \{0\}$  with  $\text{supp } f_n$  pairwise disjoint, we have that*

$$\lambda_0^{\text{ess}}(S) \leq \liminf_n \mathcal{R}_S(f_n).$$

**Proof** The third assertion may be found for example in [23, Proposition 2.2]. From this and Proposition 2.1, it is not hard to see that

$$\lambda_0^{\text{ess}}(S) \leq \lim_n \lambda_0(S, M \setminus K_n),$$

while the decomposition principle gives that  $\lambda_0^{\text{ess}}(S) = \lambda_0^{\text{ess}}(S, M \setminus K_n) \geq \lambda_0(S, M \setminus K_n)$  for any  $n \in \mathbb{N}$ , as we wished. The proof is completed by the first part and Proposition 2.1. □

For  $\lambda \in \mathbb{R}$ , a sequence  $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \setminus \{0\}$  is called a *characteristic sequence* for  $S$  and  $\lambda$  if

$$\frac{\|(S - \lambda)f_n\|_{L^2(M)}}{\|f_n\|_{L^2(M)}} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

If  $M$  is complete, then  $S$  is essentially self-adjoint, which allows us to characterize the spectrum of  $S$  in terms of compactly supported smooth functions as follows.

**Proposition 2.3** *Let  $S$  be a Schrödinger operator on a complete Riemannian manifold  $M$  and consider  $\lambda \in \mathbb{R}$ . Then  $\lambda \in \sigma(S)$  if and only if there is a characteristic sequence for  $S$  and  $\lambda$ .*

Assume now that  $\varphi \in C^\infty(M)$  is a positive solution of  $S\varphi = \lambda\varphi$  for some  $\lambda \in \mathbb{R}$ . Denote by  $L_\varphi^2(M)$  the  $L^2$ -space of  $M$  with respect to the measure  $\varphi^2 d \text{vol}$ , where  $d \text{vol}$  is the volume element of  $M$  induced by its Riemannian metric. It is straightforward to verify that the isometric isomorphism  $m_\varphi : L_\varphi^2(M) \rightarrow L^2(M)$ , defined by  $m_\varphi f = \varphi f$ , intertwines  $S - \lambda$  with the diffusion operator

$$L := m_\varphi^{-1} \circ (S - \lambda) \circ m_\varphi = \Delta - 2 \text{grad } \ln \varphi.$$

The operator  $L$  is called *renormalization* of  $S$  with respect to  $\varphi$ . The Rayleigh quotient of a non-zero  $f \in C_c^\infty(M)$  with respect to  $L$  is defined by

$$\mathcal{R}_L(f) := \frac{\langle Lf, f \rangle_{L_\varphi^2(M)}}{\|f\|_{L_\varphi^2(M)}^2} = \frac{\int_M \|\text{grad } f\|^2 \varphi^2}{\int_M f^2 \varphi^2}.$$

**Lemma 2.4** *For any non-zero  $f \in C_c^\infty(M)$  and  $c \in \mathbb{R}$ , we have that*

- (i)  $\mathcal{R}_L(f) = \mathcal{R}_S(\varphi f) - \lambda$ ,
- (ii)  $\|(L - c)f\|_{L_\varphi^2(M)} = \|(S - \lambda - c)(\varphi f)\|_{L^2(M)}$ .

**Proof** Follows immediately from the definition of  $L$  and the fact that  $m_\varphi$  is an isometric isomorphism. □

### 2.1 Riemannian Submersions

Let  $M_1$  and  $M_2$  be Riemannian manifolds with  $\dim M_1 < \dim M_2$ . A surjective smooth map  $p: M_2 \rightarrow M_1$  is called a *submersion* if its differential is surjective at any point. The kernel of  $p_{*y}$  is called the *vertical space* at  $y \in M_2$ , and its orthogonal complement in  $T_y M_2$  is called the *horizontal space* at  $y$ . These spaces are denoted by  $(T_y M_2)^v$  and  $(T_y M_2)^h$ , respectively. It is evident that the fiber  $F_x := p^{-1}(x)$  over  $x \in M_1$  is a possibly non-connected submanifold of  $M_2$  and  $(T_y M_2)^v$  is the tangent space of  $F_x$  at  $y \in F_x$ . The submersion  $p$  is called *Riemannian submersion* if the restriction  $p_{*y}: (T_y M_2)^h \rightarrow T_{p(y)} M_1$  is an isometry for any  $y \in M_2$ . For more details, see [14] or [15].

Given a Riemannian submersion  $p: M_2 \rightarrow M_1$ , a smooth map  $s: U \rightarrow M_2$  defined on an open subset  $U$  of  $M_1$ , is called *section* if  $(p \circ s)(x) = x$  for any  $x \in U$ . We say that a section  $s: U \subset M_1 \rightarrow M_2$  is *extendible* if it can be extended to a section  $s': U' \subset M_1 \rightarrow M_2$  with  $\bar{U} \subset U'$ .

A vector field  $Y$  on  $M_2$  is called *horizontal* (*vertical*) if  $Y(y)$  belongs to the horizontal (vertical, respectively) space at  $y$  for any  $y \in M_2$ . It is easily checked that any vector field  $Y$  on  $M_2$  can be written as  $Y = Y^h + Y^v$  with  $Y^h$  horizontal and  $Y^v$  vertical. Moreover, any vector field  $X$  on  $M_1$  has a unique horizontal lift  $\tilde{X}$  on  $M_2$ ; that is,  $\tilde{X}$  is horizontal and  $p_* \tilde{X} = X$ . A vector field  $Y$  on  $M_2$  is called *basic* if  $Y = \tilde{X}$  for some vector field  $X$  on  $M_1$ .

The (unnormalized) *mean curvature vector* of the fibers is defined by

$$H(y) := \sum_{i=1}^k \alpha(X_i, X_i),$$

where  $\alpha(\cdot, \cdot)$  is the second fundamental form of the fiber  $F_{p(y)}$  and  $\{X_i\}_{i=1}^k$  is an orthonormal basis of  $(T_y M_2)^v$ . It should be observed that  $H$  is a horizontal vector field. We say that the Riemannian submersion  $p$  has *minimal fibers* or *fibers of basic mean curvature* if  $H = 0$  or  $H$  is basic, respectively.



We now discuss some basic examples of Riemannian submersions. It is worth to mention that the manifolds involved in following examples are not assumed to be compact.

**Example 2.5** (i) The *warped product*  $M_2 = M_1 \times_\psi F$  is the product manifold endowed with the Riemannian metric  $g_{M_1} \times \psi^2 g_F$ , where  $\psi \in C^\infty(M_1)$  is positive. Then the projection to the first factor  $p: M_2 \rightarrow M_1$  is a Riemannian submersion with fibers of basic mean curvature

$$H = -k \operatorname{grad} \ln \tilde{\psi},$$

where  $k := \dim F$ . It should be noticed that surfaces of revolution are warped products of the form  $\mathbb{R} \times_\psi S^1$ .

- (ii) Another generalization of surfaces of revolution was introduced by Bishop motivated by a result of Clairaut involving such surfaces. A Riemannian submersion  $p: M_2 \rightarrow M_1$  is called *Clairaut submersion* if there exists a positive  $f \in C^\infty(M_2)$ , such that for any geodesic  $c$  on  $M_2$ , the function  $(f \circ c) \sin \theta$  is constant, where  $\theta(t)$  is the angle between  $c'(t)$  and  $(T_{c(t)} M_2)^h$ . Bishop proved that a Riemannian submersion  $p: M_2 \rightarrow M_1$  of complete manifolds with connected fibers is a Clairaut submersion if and only if the fibers are totally umbilical with mean curvature

$$H = -k \operatorname{grad} \ln \tilde{\psi}$$

for some positive  $\psi \in C^\infty(M_1)$ , where  $k$  is the dimension of the fiber (cf. for instance [14, Theorem 1.7]).

- (iii) Let  $G$  be a Lie group acting smoothly, freely, and properly via isometries on a Riemannian manifold  $M_2$ , where  $\dim G < \dim M_2$ . Then  $M_1 := M_2/G$  is a Riemannian manifold and the projection  $p: M_2 \rightarrow M_1$  is a Riemannian submersion with fibers of basic mean curvature. In this case, we say that  $p: M_2 \rightarrow M_1$  is a Riemannian submersion *arising from the action of a Lie group*  $G$ .

Given a Riemannian submersion  $p: M_2 \rightarrow M_1$ , the *lift* of a function  $f \in C^\infty(M_1)$  on  $M_2$  is the smooth function  $\tilde{f} := f \circ p$ . The next lemma provides a simple expression for the Laplacian and the gradient of a lifted function.

**Lemma 2.6** *For any  $f \in C^\infty(M_1)$  and its lift  $\tilde{f}$  on  $M_2$ , we have that*

- (i)  $\operatorname{grad} \tilde{f} = \widetilde{\operatorname{grad} f}$ ,  
(ii)  $\Delta \tilde{f} = \Delta f + \langle \operatorname{grad} f, H \rangle$ .

**Proof** Both statements follow from elementary computations, which may be found for example in [4, Sect. 2.2].  $\square$

## 2.2 Lie Groups

In this subsection, we recall some basic definitions and results about Lie groups, and discuss some consequences of the Cheeger and Buser inequalities in this setting.

For a Borel subset  $A$  of a Riemannian manifold  $(M, g)$ , we denote the volume of  $A$  by  $|A|_g$ , or simply by  $|A|$  if the Riemannian metric of  $M$  is clear from the context. Similarly, for an  $m$ -dimensional submanifold  $N$  of  $M$ , we denote by  $|N|$  the  $m$ -dimensional volume of  $N$ . The Cheeger constant of a Riemannian manifold  $M$  is defined by

$$h(M) := \inf_K \frac{|\partial K|}{|K|},$$

where the infimum is taken over all compact domains  $K$  of  $M$  with smooth boundary. It is related to the bottom of the spectrum via the Cheeger inequality (cf. [11])

$$\lambda_0(M) \geq \frac{h(M)^2}{4}.$$

Buser [9] established an inverse inequality for complete manifolds with Ricci curvature bounded from below. In particular, if  $M$  is such a manifold, then  $\lambda_0(M) = 0$  if and only if  $h(M) = 0$ . For our purposes, we also need the following lemma from his work, where  $A^r$  stands for the  $r$ -tubular neighborhood of a subset  $A$  of  $M$ .

**Lemma 2.7** (Compare with [9, Lemma 7.2]; see also [22, Corollary 6.3]). *Let  $M$  be a non-compact, complete Riemannian manifold with Ricci curvature bounded from below. If  $h(M) = 0$ , then for any  $\varepsilon, r > 0$ , there exists an open, bounded  $W \subset M$  such that*

$$|(\partial W)^r| < \varepsilon |W \setminus (\partial W)^r|.$$

Throughout this paper, Lie groups are assumed to be non-discrete and possibly non-connected, unless otherwise stated. A possibly discrete Lie group  $G$  is called *amenable* if there exists a left-invariant mean on  $L^\infty(G)$ ; that is, a linear functional  $\mu: L^\infty(G) \rightarrow \mathbb{R}$  such that

$$\text{ess inf } f \leq \mu(f) \leq \text{ess sup } f \text{ and } \mu(f \circ L_x) = \mu(f),$$

for any  $f \in L^\infty(G)$  and  $x \in G$ , where  $L_x$  stands for multiplication from the left with an element  $x \in G$ . Here,  $L^\infty(G)$  is considered with respect to the Haar measure. If  $G$  is non-discrete, then its Haar measure is a constant multiple of the volume element of  $G$  induced from a left-invariant metric. If  $G$  is discrete, then its Haar measure is a constant multiple of the counting measure. For more details, see [16].

**Lemma 2.8** *If  $N$  is a normal subgroup of a possibly discrete Lie group  $G$ , then  $G$  is amenable if and only if  $N$  and  $G/N$  are amenable.*

It is not hard to verify that abelian and compact Lie groups are amenable. Therefore, so are compact extensions of solvable groups. As a matter of fact, a connected Lie group is amenable if and only if it is a compact extension of a solvable group (cf. for example [19, Lemma 2.2]).

Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$ . The *radical*  $\mathfrak{s}$  of  $\mathfrak{g}$  is the largest solvable ideal of  $\mathfrak{g}$ . The *radical*  $S$  of  $G$  is the connected subgroup with Lie algebra  $\mathfrak{s}$ . Then  $S$  is a closed, normal subgroup of  $G$  and the quotient  $G/S$  is semisimple. In this case, we have that  $G$  is amenable if and only if  $G/S$  is compact (cf. [17, p. 724f] and the references therein).

A Lie group is called *unimodular* if its Haar measure is also right-invariant. For a connected Lie group, this property may be reformulated in terms of its Lie algebra as follows.

**Lemma 2.9** [17, Proposition 1.2]. *A connected Lie group  $G$  is unimodular if and only if  $\text{tr}(\text{ad } X) = 0$  for any  $X$  in the Lie algebra of  $G$ .*

It is worth to point out that connected, nilpotent Lie groups are unimodular and amenable. In addition, compact extensions of connected, unimodular Lie groups are unimodular (cf. [18, Proposition 8]).

Although the aforementioned properties are group theoretic, they are characterized by the spectrum of the Lie group according to the next theorem, which is well known for simply connected Lie groups.

**Theorem 2.10** *A connected Lie group  $G$  is unimodular and amenable if and only if  $\lambda_0(G) = 0$  for some/any left-invariant metric on  $G$ .*

**Proof** We know from [17, Theorem 3.8] that a simply connected Lie group  $\tilde{G}$  is unimodular and amenable if and only if  $h(\tilde{G}) = 0$  with respect to some/any left-invariant metric. By the Cheeger and Buser inequalities, this gives the assertion for simply connected Lie groups. To show its validity for a connected Lie group  $G$ , let  $q: \tilde{G} \rightarrow G$  be the universal covering of  $G$ . It follows from Lemma 2.9 that  $\tilde{G}$  is unimodular if and only if  $G$  is unimodular, since their Lie algebras are isomorphic. Furthermore,  $\pi_1(G)$  is abelian and isomorphic to the kernel of  $q$  as a Lie group homomorphism. Therefore,  $\tilde{G}$  is an extension of  $G$  by an amenable group, and Lemma 2.8 yields that  $\tilde{G}$  is amenable if and only if  $G$  is amenable. Taking into account that  $\pi_1(G)$  is amenable, we conclude from [2, Theorem 1.2] that  $\lambda_0(\tilde{G}) = \lambda_0(G)$ .  $\square$

By virtue of Buser's lemma, we derive the following consequence of the preceding characterization.

**Corollary 2.11** *Let  $G$  be a non-compact, connected, unimodular, and amenable Lie group endowed with a left-invariant metric. Then for any  $\varepsilon, r > 0$ , there exists an open, bounded  $W \subset G$  such that*

$$|(\partial W)^r| < \varepsilon |W \setminus (\partial W)^r|.$$

### 3 Submersions with Fibers of Basic Mean Curvature

The aim of this section is to prove Theorem 1.1. Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion with fibers of basic mean curvature, and consider the Schrödinger operator

$$S = \Delta + \frac{1}{4} \|p_* H\|^2 - \frac{1}{2} \text{div } p_* H$$

on  $M_1$ . As in [4,5,23], the average of a function  $f \in C_c^\infty(M_2)$  is the smooth function

$$f_{\text{av}}(x) := \int_{F_x} f$$

on  $M_1$  with gradient given by

$$\langle \text{grad } f_{\text{av}}(x), X \rangle = \int_{F_x} \langle \text{grad } f - fH, \tilde{X} \rangle \tag{2}$$

for any  $x \in M_1$  and  $X \in T_x M_1$ , where  $\tilde{X}$  is the horizontal lift of  $X$  on  $F_x$ . The push-down of  $f$  is the function

$$h(x) := \sqrt{(f^2)_{\text{av}}(x)} = \left( \int_{F_x} f^2 \right)^{1/2}$$

on  $M_1$ . Then [23, Lemma 3.1] states that  $h \in \text{Lip}_c(M_1)$ . Hence, its gradient is defined almost everywhere and vanishes (if defined) in points where  $h$  is zero.

**Proposition 3.1** *Let  $h \in \text{Lip}_c(M_1)$  be the push-down of a function  $f \in C_c^\infty(M_2)$  with  $\|f\|_{L^2(M_2)} = 1$ . Then their Rayleigh quotients are related by*

$$\mathcal{R}(f) \geq \mathcal{R}_S(h) + \int_{M_1} \lambda_0(F_x) h^2(x) dx.$$

**Proof** For any  $x \in M_1$  with  $h(x) > 0$ , we readily see from formula (2) that

$$\begin{aligned} \text{grad } h(x) &= \frac{1}{2h(x)} \int_{F_x} (p_* \text{grad } f^2 - f^2 p_* H) \\ &= \frac{1}{h(x)} \int_{F_x} f p_* \text{grad } f - \frac{1}{2} h(x) p_* H(x). \end{aligned}$$

In view of this, the fact that  $\|h\|_{L^2(M_1)} = 1$ , the divergence formula, the Cauchy-Schwarz inequality and that

$$\begin{aligned} \frac{1}{2} \langle \text{grad } h^2(x), p_* H(x) \rangle &= h(x) \langle \text{grad } h(x), p_* H(x) \rangle \\ &= \int_{F_x} f \langle \text{grad } f, H \rangle - \frac{1}{2} h^2(x) \|p_* H(x)\|^2 \end{aligned}$$

for any  $x \in M_1$ , we compute

$$\begin{aligned} \mathcal{R}_S(h) &= \int_{M_1} \left( \|\text{grad } h\|^2 + \frac{1}{4} \|p_* H\|^2 h^2 - \frac{1}{2} h^2 \text{div } p_* H \right) \\ &= \int_{M_1} \left( \frac{1}{h^2} \left\| \int_{F_x} f p_* \text{grad } f \right\|^2 + \frac{1}{4} h^2 \|p_* H\|^2 - \int_{F_x} f \langle \text{grad } f, H \rangle \right) \end{aligned}$$

$$\begin{aligned}
 &+ \int_{M_1} \left( \frac{1}{4} h^2 \|p_* H\|^2 + \frac{1}{2} \langle \text{grad } h^2, p_* H \rangle \right) \\
 \leq &\int_{M_1} \int_{F_x} \|(\text{grad } f)^h\|^2 = \int_{M_2} \|(\text{grad } f)^h\|^2.
 \end{aligned} \tag{3}$$

Since at any point of  $M_2$ , the tangent space of  $M_2$  splits into the orthogonal sum of the horizontal and the vertical space, it is easily checked that (cf. [23, Formula (6)])

$$\begin{aligned}
 \mathcal{R}(f) &= \int_{M_2} \|(\text{grad } f)^h\|^2 + \int_{M_2} \|(\text{grad } f)^v\|^2 \\
 &\geq \int_{M_2} \|(\text{grad } f)^h\|^2 + \int_{M_1} \lambda_0(F_x) h^2(x).
 \end{aligned}$$

The proof is now completed by formula (3) and Proposition 2.1. □

**Proof of Theorem 1.1** From Propositions 2.1 and 3.1, it is immediate verify the asserted inequality. Suppose now that the equality holds. Then there exists  $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2)$  with  $\|f_n\|_{L^2(M_2)} = 1$  and  $\mathcal{R}(f_n) \rightarrow \lambda_0(M_2)$ , as follows from Proposition 2.1. Denote by  $h_n \in \text{Lip}_c(M_1)$ , the push-down of  $f_n$ ,  $n \in \mathbb{N}$ . Arguing as in the proof of [23, Theorem 1.1], using Proposition 3.1 instead of [23, Proposition 3.2], we obtain that

$$\mathcal{R}_S(h_n) \rightarrow \lambda_0(S) \text{ and } \int_{M_1} (\lambda_0(F_x) - \inf_{y \in M_1} \lambda_0(F_y)) h_n^2(x) dx \rightarrow 0. \tag{4}$$

Since  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$ , we deduce from [21, Proposition 3.5] that after passing to a subsequence if necessary, we may assume that  $h_n \rightarrow \varphi$  in  $L^2(M_1)$  for some function  $\varphi \in C^\infty(M_1)$  with  $S\varphi = \lambda_0(S)\varphi$ . Then  $\varphi$  is positive, by [21, Proposition 3.7]. Arguing as in the proof of [23, Theorem 1.1], we conclude from (4) that

$$\lambda_0(F_x) = \inf_{y \in M_1} \lambda_0(F_y)$$

for almost any  $x \in M_1$ . □

**Proof of Corollary 1.2** If the submersion has closed fibers of basic mean curvature, then  $S$  is written as follows:

$$S = \Delta - \frac{\Delta \sqrt{V}}{\sqrt{V}},$$

where  $V(x)$  is the volume of  $F_x$  (cf. [23, Sect. 4]). Thus, we may consider the renormalization  $L$  of  $S$  with respect to  $\sqrt{V}$ . Then Lemmas 2.4 and 2.6 imply that for any non-zero  $f \in C_c^\infty(M_1)$ , its lift  $\tilde{f} \in C_c^\infty(M_2)$  satisfies

$$\mathcal{R}(\tilde{f}) = \frac{\int_{M_2} \|\text{grad } \tilde{f}\|^2}{\int_{M_2} \tilde{f}^2} = \frac{\int_{M_1} \|\text{grad } f\|^2 V}{\int_{M_1} f^2 V} = \mathcal{R}_L(f) = \mathcal{R}_S(f\sqrt{V}). \tag{5}$$

We derive from Proposition 2.1 that  $\lambda_0(M_2) \leq \lambda_0(S)$ , while the inverse inequality is a consequence of Theorem 1.1.

About the second statement, choose an exhausting sequence  $(K_n)_{n \in \mathbb{N}}$  of  $M_1$  consisting of compact domains with smooth boundary. Then  $(p^{-1}(K_n))_{n \in \mathbb{N}}$  is an exhausting sequence of  $M_2$  consisting of compact domains with smooth boundary, because the submersion has closed fibers. Applying Theorem 1.1 to the restriction of  $p: M_2 \setminus p^{-1}(K_n) \rightarrow M_1 \setminus K_n$  over any connected component of  $M_1 \setminus K_n$ , together with Proposition 2.2, gives the estimate

$$\lambda_0^{\text{ess}}(M_2) = \lim_n \lambda_0(M_2 \setminus p^{-1}(K_n)) \geq \lim_n \lambda_0(S, M_1 \setminus K_n) = \lambda_0^{\text{ess}}(S).$$

From Proposition 2.2, there exists a sequence  $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_1) \setminus \{0\}$  with  $\text{supp } f_n$  pairwise disjoint, such that  $\mathcal{R}_S(f_n) \rightarrow \lambda_0^{\text{ess}}(S)$ . It is immediate to verify that the lifts  $\tilde{h}_n$  of  $h_n := f_n/\sqrt{V}$  also have pairwise disjoint supports. Then Proposition 2.2 and formula (5) yield that

$$\lambda_0^{\text{ess}}(M_2) \leq \liminf_n \mathcal{R}(\tilde{h}_n) = \liminf_n \mathcal{R}_S(f_n) = \lambda_0^{\text{ess}}(S),$$

as we wished. □

It should be noticed that if the submersion has minimal fibers, then  $S$  coincides with the Laplacian on  $M_1$ . Therefore, [23, Example 3.3] is an example of a Riemannian submersion  $p: M_2 \rightarrow M_1$  with minimal fibers, where  $M_1$  is closed and  $M_2$  is complete, such that

$$0 = \lambda_0(M_2) = \lambda_0(S) + \inf_{x \in F_x} \lambda_0(F_x)$$

and there is  $x \in M_1$  with  $\lambda_0(F_x) > 0$ . It is evident that  $(\lambda_0(S) \notin \sigma_{\text{ess}}(S), M_1)$  being closed. Hence, in general, the asserted equality in the second part of Theorem 1.1 holds almost everywhere, but not everywhere.

According to the next lemma, the Schrödinger operator  $S$  defined in (1) is always non-negative. Moreover, it demonstrates that Theorem 1.1 provides a sharper lower bound for  $\lambda_0(M_2)$  than [23, Theorem 1.1] in the case where both of them are applicable.

**Lemma 3.2** *Let  $X$  be a smooth vector field on a Riemannian manifold  $M$ . Then the operator*

$$S = \Delta + \frac{1}{4} \|X\|^2 - \frac{1}{2} \text{div } X$$

*is non-negative. Furthermore, if  $\|X\| \leq C \leq 2\sqrt{\lambda_0(M)}$ , then*

$$\lambda_0(S) \geq (\sqrt{\lambda_0(M)} - C/2)^2.$$

**Proof** For any  $f \in C_c^\infty(M)$  with  $\|f\|_{L^2(M)} = 1$ , observe that its Rayleigh quotient is given by

$$\begin{aligned}\mathcal{R}_S(f) &= \int_M \left( \|\operatorname{grad} f\|^2 + \frac{1}{4}\|X\|^2 f^2 + \langle \operatorname{grad} f, fX \rangle \right) \\ &= \int_M \left\| \operatorname{grad} f + \frac{f}{2}X \right\|^2,\end{aligned}\tag{6}$$

where we used the divergence formula. From Proposition 2.1, we readily see that  $S$  is non-negative.

Suppose now that  $\|X\| \leq C \leq 2\sqrt{\lambda_0(M)}$  and let  $f \in C_c^\infty(M)$  with  $\|f\|_{L^2(M)} = 1$ . An elementary calculation shows that

$$\begin{aligned}\mathcal{R}_S(f) &\geq \int_M \left( \|\operatorname{grad} f\| - \frac{|f|}{2}\|X\| \right)^2 \\ &= \int_M \left( \|\operatorname{grad} f\|^2 + \frac{f^2}{4}\|X\|^2 - \|\operatorname{grad} f\||f|\|X\| \right) \\ &\geq \mathcal{R}(f) + \frac{1}{4} \int_M f^2\|X\|^2 - \mathcal{R}(f)^{1/2} \left( \int_M f^2\|X\|^2 \right)^{1/2} \\ &= \left( \sqrt{\mathcal{R}(f)} - \frac{1}{2} \left( \int_M f^2\|X\|^2 \right)^{1/2} \right)^2.\end{aligned}\tag{7}$$

By the assumption that  $\|X\| \leq C \leq 2\sqrt{\lambda_0(M)}$ , the fact that  $\|f\|_{L^2(M)} = 1$  and Proposition 2.1, we obtain that

$$\sqrt{\mathcal{R}(f)} - \frac{1}{2} \left( \int_M f^2\|X\|^2 \right)^{1/2} \geq \sqrt{\lambda_0(M_1)} - C/2 > 0.$$

The proof is completed by Proposition 2.1 and formula (7).  $\square$

We end this section by discussing the application of Theorem 1.1 and Corollary 1.2 to the submersions described in Examples 2.5.

**Example 3.3** (i) Consider the warped product  $M_2 = M_1 \times_\psi F$  and the projection to the first factor  $p: M_2 \rightarrow M_1$ . In this case, the operator  $S$  defined in (1) is written as follows:

$$S = \Delta - \frac{\Delta\psi^{k/2}}{\psi^{k/2}},$$

and Theorem 1.1 says that

$$\lambda_0(M_2) \geq \lambda_0(S) + \inf_{x \in M_1} \lambda_0(F_x) = \lambda_0(S) + \lambda_0(F) \inf_{x \in M_1} \psi^{-2}(x).$$

If, in addition,  $F$  is closed, then we deduce from Corollary 1.2 that  $\lambda_0(M_2) = \lambda_0(S)$  and  $\lambda_0^{\text{ess}}(M_2) = \lambda_0^{\text{ess}}(S)$ . In particular,  $M_2$  has discrete spectrum if and only if the spectrum of  $S$  is discrete (compare with [1, Theorem 3.3]). It is not difficult to establish analogous statements for Clairaut submersions.

- (ii) Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a Lie group  $G$ . In view of Theorem 1.1, the bottoms of the spectra are related by  $\lambda_0(M_2) \geq \lambda_0(S)$ . According to Corollary 1.2, if  $G$  is compact, then  $\lambda_0(M_2) = \lambda_0(S)$ .

## 4 Submersions Arising from Lie Group Actions

Throughout this section, we consider a Riemannian submersion  $p: M_2 \rightarrow M_1$  arising from the action of a Lie group  $G$ . For convenience of the reader, we provide a brief outline of the section and the proof of Theorem 1.3.

In Sect. 4.1, we show that identifying the fiber with  $G$  along a section of the submersion gives rise to a smooth family of left-invariant metrics on  $G$ . This remark plays a quite important role in our discussion. More precisely, from this and Theorem 1.1, we obtain Theorem 1.3(iii).

The other assertions of Theorem 1.3 are first proved in the case where  $G$  is connected. If  $G$  is compact, then they follow from Corollary 1.2 and [23, Theorem 1.2]. Thus, we have to focus on the case where  $G$  is non-compact and connected. In Sect. 4.2, we construct cut-off functions on such  $G$  closely related to the open sets  $W$  from Corollary 2.11. In terms of these functions, for a section  $s: U \subset M_1 \rightarrow M_2$ , we define cut-off functions in  $p^{-1}(U)$  with uniformly (that is, independently from the corresponding  $W$ ) bounded gradient and Laplacian.

We begin Sect. 4.3 with the proposition that establishes this auxiliary result. The main idea is that given an  $f \in C_c^\infty(M_1)$ , we may write it as a sum of functions supported in domains admitting sections. Using cut-off functions as above, we are able to pull up these functions, and for suitable choices of  $W$ , the sum of these pulled up functions coincides with the lift of  $f$  in a relatively large part of its support. In the rest of its support, its gradient and its Laplacian are bounded independently from  $W$ .

The proof of Theorem 1.3 is completed after observing that such a submersion  $p$  is expressed as the composition of the submersion arising from the action of  $G_0$  with the covering arising from the action of  $G/G_0$ .

### 4.1 Induced Metrics on the Lie Group

Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a (possibly non-connected) Lie group  $G$ . Given a section  $s: U \subset M_1 \rightarrow M_2$ , it is easily checked that the map  $\Phi: G \times U \rightarrow p^{-1}(U)$  defined by  $\Phi(x, y) := xs(y)$  is a diffeomorphism, and so is its restriction  $\Phi_y := \Phi(\cdot, y): G \rightarrow F_y$ . Denote by  $g_{s(y)}$  the metric induced on  $G$  via  $\Phi_y$ , that is, the pullback via  $\Phi_y$  of the restriction of the metric of  $M_2$  on  $F_y$ . It is straightforward to see that the metric  $g_{s(y)}$  depends only on  $s(y)$  and not on the behavior of  $s$  in a neighborhood of  $y$ .



**Proposition 4.1** *Let  $s : U \subset M_1 \rightarrow M_2$  be a section. Then the Riemannian metric  $g_{s(y)}$  is left-invariant and depends smoothly on  $y \in U$ .*

**Proof** For  $x_1, x_2 \in G$ , it is immediate to verify that

$$x_1 \Phi_y(x_2) = x_1 x_2 s(y) = \Phi_y(x_1 x_2),$$

and therefore,  $x_{1*} \Phi_{y*} = \Phi_{y*} L_{x_1*}$ . Bearing in mind that  $G$  acts on  $M_2$  via isometries, given  $x \in G$  and  $X, Y \in T_e G$ , where  $e$  is the neutral element of  $G$ , it is now elementary to compute

$$\begin{aligned} g_{s(y)}(L_{x*} X, L_{x*} Y)(x) &= \langle \Phi_{y*} L_{x*} X, \Phi_{y*} L_{x*} Y \rangle_{\Phi_y(x)} = \langle x_* \Phi_{y*} X, x_* \Phi_{y*} Y \rangle_{x \Phi_y(e)} \\ &= \langle \Phi_{y*} X, \Phi_{y*} Y \rangle_{\Phi_y(e)} = g_{s(y)}(X, Y)(e), \end{aligned}$$

which yields that the induced metric on  $G$  is left-invariant.

Choose a left-invariant frame field  $\{X_i\}_{i=1}^k$  on  $G$ . After endowing  $G$  with a left-invariant metric and  $G \times U$  with the product metric, it is evident that the projection to the first factor  $q : G \times U \rightarrow G$  is a Riemannian submersion. Consider the horizontal lift  $\tilde{X}_i$  of  $X_i$  on  $G \times U$ . Notice that  $\{\tilde{X}_i\}_{i=1}^k$  is a  $G$ -invariant, smooth frame field, and hence, so is  $\{\Phi_* \tilde{X}_i\}_{i=1}^k$ . Then for  $y \in U$  and  $x \in G$ , we deduce that

$$g_{s(y)}(X_i, X_j)(x) = g_{s(y)}(X_i, X_j)(e) = \langle \Phi_* \tilde{X}_i, \Phi_* \tilde{X}_j \rangle_{s(y)}.$$

Since  $\langle \Phi_* \tilde{X}_i, \Phi_* \tilde{X}_j \rangle_z$  is a smooth function (with respect to  $z$ ) in  $p^{-1}(U)$ , so is its composition with  $s$ , as we wished.  $\square$

**Corollary 4.2** *Let  $s : U \subset M_1 \rightarrow M_2$  be a section and fix a left-invariant metric  $g$  on  $G$ . Then there exists  $V_s \in C^\infty(U)$  such that for any  $y \in U$ , the volume element of the induced metric satisfies*

$$d\text{vol}_{g_{s(y)}} = V_s(y) d\text{vol}_g.$$

**Proof** Follows immediately from Proposition 4.1 and the local expression of the volume element.  $\square$

For  $y \in M_1$  and  $z_1, z_2 \in F_y$ , consider the diffeomorphisms  $\Phi_i : G \rightarrow F_y$  defined by  $\Phi_i(x) = x z_i$ , and the induced metrics  $g_i := g_{z_i}$  on  $G$ ,  $i = 1, 2$ . Because  $G$  acts transitively on  $F_y$ , there exists  $x_0 \in G$  such that  $x_0 z_1 = z_2$ . Then it is apparent that

$$\Phi_2(x) = x z_2 = x x_0 z_1 = (\Phi_1 \circ R_{x_0})(x).$$

In particular, if  $G$  is unimodular, then we have that

$$d\text{vol}_{g_2} = \Phi_2^*(d\text{vol}_{F_y}) = R_{x_0}^*(\Phi_1^*(d\text{vol}_{F_y})) = R_{x_0}^*(d\text{vol}_{g_1}) = d\text{vol}_{g_1}, \quad (8)$$

where  $d\text{vol}_{F_y}$  is the volume element of  $F_y$  with respect to the induced metric from  $M_2$ . This implies that the function  $V_s$  from Corollary 4.2 is independent from the section  $s$  and can be defined globally.

**Corollary 4.3** *Suppose that  $G$  is unimodular and fix a left-invariant metric  $g$  on  $G$ . Then there exists  $V \in C^\infty(M_1)$  such that for any section  $s: U \subset M_1 \rightarrow M_2$  and  $y \in U$ , the volume element of the induced metric satisfies*

$$d\text{vol}_{g_{s(y)}} = V(y)d\text{vol}_g.$$

Moreover, the gradient of  $V$  is given by

$$\text{grad } V = -Vp_*H.$$

**Proof** The existence of the function  $V$  is a consequence of Corollary 4.2 and formula (8). About the second statement, let  $y \in M_1$  and  $s: U \subset M_1 \rightarrow M_2$  be a section defined in a neighborhood  $U$  of  $y$  that is horizontal at  $y$ , which means that  $s_*T_yM_1$  is the horizontal space of  $M_2$  at  $s(y)$ . Let  $X \in T_yM_1$  and  $c: (-\varepsilon, \varepsilon) \rightarrow M_1$  be a smooth curve with  $c(0) = y$  and  $c'(0) = X$ . Denote by  $F: (-\varepsilon, \varepsilon) \times G \rightarrow M_2$  the smooth variation of the isometric immersion  $F(0, \cdot): (G, g_{s(y)}) \rightarrow M_2$  defined by  $F(t, x) = xs(c(t))$ , and observe that its variational vector field is the horizontal lift  $\tilde{X}$  of  $X$  on  $F_y$ . The asserted equality follows now from the first variational formula.  $\square$

It is well known that if  $p: M_2 \rightarrow M_1$  is a Riemannian submersion and  $M_2$  is complete, then so is  $M_1$ . According to the next corollary, if the submersion arises from the action of a Lie group, the converse implication is also true.

**Corollary 4.4** *Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a Lie group  $G$ . If  $M_1$  is complete, then  $M_2$  is complete.*

**Proof** Fix a left-invariant metric  $g$  on  $G$  and let  $(z_n)_{n \in \mathbb{N}} \subset M_2$  be a Cauchy sequence. Then  $(p(z_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $M_1$ , and hence,  $p(z_n) \rightarrow y$  for some  $y \in M_1$ . Let  $s: U \subset M_1 \rightarrow M_2$  be a section defined in a neighborhood  $U$  of  $y$ , and consider the corresponding diffeomorphism  $\Phi: G \times U \rightarrow M_2$ , as in the beginning of this subsection. Without loss of generality, we may assume that  $z_n \in p^{-1}(U)$  for any  $n \in \mathbb{N}$ . Writing  $z_n = \Phi(x_n, p(z_n))$ , it remains to prove that  $(x_n)_{n \in \mathbb{N}}$  converges. Given a precompact neighborhood  $U_y$  of  $y$  with  $\bar{U}_y \subset U$ , it is simple to see that for any sufficiently small  $\varepsilon > 0$ , there is  $n_0 \in \mathbb{N}$ , such that for any  $n, m \geq n_0$ , there exists a smooth curve  $c_{n,m}$  from  $z_n$  to  $z_m$  of length less than  $\varepsilon$ , with image contained in  $p^{-1}(U_y)$ . Denoting by  $q: G \times U \rightarrow G$  the projection to the first factor, it is clear that  $\hat{c}_{n,m} := q \circ \Phi^{-1} \circ c_{n,m}$  is a smooth curve from  $x_n$  to  $x_m$ . Since  $U_y$  is precompact, we derive from Proposition 4.1 that there exists  $C > 0$  such that  $\ell_g(\hat{c}_{n,m}) \leq C\ell(c_{n,m})$  for any  $n, m \geq n_0$ , where  $\ell(\cdot)$  stands for the length of a curve. This shows that  $(x_n)_{n \in \mathbb{N}}$  is Cauchy in  $(G, g)$  and, thus, converges.  $\square$

### 4.2 Cut-Off Functions

The aim of this subsection is to construct some special functions on the Lie group that will be used in the sequel to obtain cut-off functions on  $M_2$ . Throughout this subsection, we consider a non-compact, connected Lie group  $G$  endowed with a left-invariant

metric. Given  $r > 0$ , choose a sequence  $(x_n)_{n \in \mathbb{N}} \subset G$  such that  $d(x_n, x_m) \geq r$  for any  $n \neq m$  and the open balls  $B(x_n, r)$  cover  $G$ .

**Lemma 4.5** *There exists  $n(r) \in \mathbb{N}$  such that any  $x \in G$  lies in at most  $n(r)$  of the balls  $B(x_n, 2r)$ , with  $n \in \mathbb{N}$ .*

**Proof** Let  $x \in G$  and set  $E_x := \{n \in \mathbb{N} : x \in B(x_n, 2r)\}$ . Notice that for  $n \in E_x$ , we have that  $B(x_n, r/2) \subset B(x, 5r/2)$  and the balls  $B(x_n, r/2)$  are disjoint. Bearing in mind that  $G$  is a homogeneous space, we compute

$$|B(x, 5r/2)| \geq \sum_{n \in E_x} |B(x_n, r/2)| = |E_x| |B(x, r/2)|,$$

where  $|E_x|$  is the cardinality of  $E_x$ . Since the Ricci curvature of  $G$  is bounded from below (say by  $(k - 1)C$ , where  $k$  is the dimension of  $G$ ), the Bishop-Gromov volume comparison theorem gives the estimate

$$|E_x| \leq \frac{|B(x, 5r/2)|}{|B(x, r/2)|} \leq \frac{|B_{5r/2}|}{|B_{r/2}|} =: n(r),$$

where  $B_\rho$  is a ball of radius  $\rho$  in the  $k$ -dimensional space form of sectional curvature  $C$ . □

Fix  $\psi_e \in C_c^\infty(G)$  with  $0 \leq \psi_e \leq 1$ ,  $\text{supp } \psi_e \subset B(e, 3r/2)$  and  $\psi_e = 1$  in  $B(e, r)$ . For  $n \in \mathbb{N}$ , the function  $\psi_n := \psi_e \circ L_{x_n}^{-1}$  satisfies  $0 \leq \psi_n \leq 1$ ,  $\text{supp } \psi_n \subset B(x_n, 3r/2)$  and  $\psi_n = 1$  in  $B(x_n, r)$ . We know from Lemma 4.5 that the cover  $\{B(x_n, 3r/2)\}_{n \in \mathbb{N}}$  is locally finite, which implies that the function  $\psi := \sum_{n \in \mathbb{N}} \psi_n$  is well defined and smooth. It is evident that  $\psi \geq 1$ ,  $G$  being covered by the balls  $B(x_n, r)$ . The smooth partition of unity on  $G$  consisting of the functions  $\zeta_n := \psi_n/\psi$  with  $n \in \mathbb{N}$  is called a *partition of unity corresponding to  $r$* . Clearly, any point of  $G$  lies in at most  $n(r)$  of the supports of  $\zeta_n$ , where  $n(r)$  is the constant from Lemma 4.5. The cut-off function corresponding to a subset  $E$  of  $\mathbb{N}$  is defined by

$$\chi_E := \sum_{n \in E} \zeta_n.$$

Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a non-compact, connected Lie group  $G$ . Consider a relatively compact, open domain  $U \subset M_1$  that admits an extensible section  $s: U \rightarrow M_2$ , and the corresponding diffeomorphism  $\Phi: G \times U \rightarrow p^{-1}(U)$  defined by  $\Phi(x, y) := xs(y)$ . For a function  $f: G \rightarrow \mathbb{R}$ , we denote by  $f_s: p^{-1}(U) \rightarrow \mathbb{R}$  the function satisfying

$$f_s(\Phi(x, y)) := f(x)$$

for any  $x \in G$  and  $y \in U$ . Given a left-invariant metric on  $G$  and  $r > 0$ , we consider a partition of unity on  $G$  corresponding to  $r$  and the functions  $\chi_E$  for  $E \subset \mathbb{N}$ .

**Lemma 4.6** *Let  $s : U \rightarrow M_2$  be an extensible section defined on a precompact domain  $U$  of  $M_1$ . Then there exists a constant  $C$  independent from  $E \subset \mathbb{N}$ , such that*

$$|\Delta(\chi_E)_s(z)| \leq C \text{ and } \|\text{grad}(\chi_E)_s(z)\| \leq C$$

for any  $z \in p^{-1}(U)$ .

**Proof** Since  $U$  is precompact and  $s$  is extensible, it is easily checked that the Laplacian and the gradient of  $(\psi_e)_s$  are bounded. Since  $(\psi_n)_s(z) = (\psi_e)_s(x_n^{-1}z)$  for any  $n \in \mathbb{N}$  and  $z \in p^{-1}(U)$ , we obtain uniform estimates for the Laplacian and the gradient of  $(\psi_n)_s$  for all  $n \in \mathbb{N}$ . Then Lemma 4.5 yields a uniform bound for the Laplacian and the gradient of the functions  $\sum_{n \in E} (\psi_n)_s$  for all subsets  $E \subset \mathbb{N}$ . The proof is completed after observing that

$$(\chi_E)_s = \frac{\sum_{n \in E} (\psi_n)_s}{\sum_{n \in \mathbb{N}} (\psi_n)_s}$$

and that  $\sum_{n \in \mathbb{N}} (\psi_n)_s \geq 1$ . □

The purpose of considering this partition of unity becomes more clear in the next proposition, where we combine this construction with Corollary 2.11 in the case where  $G$  is unimodular and amenable.

**Proposition 4.7** *Let  $G$  be a non-compact, connected, unimodular and amenable Lie group endowed with a left-invariant metric. Consider  $r > 0$  and a partition of unity  $\{\zeta_n\}_{n \in \mathbb{N}}$  corresponding to  $r/2$ . Then for any  $\varepsilon > 0$ , there exists an open, bounded  $W \subset G$  and a finite  $E \subset \mathbb{N}$ , such that  $\chi_E = 1$  in  $W \setminus (\partial W)^r$ ,  $\text{supp } \chi_E \subset W^{r/2}$  and*

$$|(\partial W)^{2r}| < \varepsilon |W \setminus (\partial W)^{2r}|.$$

**Proof** As a consequence of Corollary 2.11, for any  $\varepsilon > 0$ , there exists an open, bounded  $W \subset G$  such that the desired inequality for the volumes holds. Consider the finite set  $E := \{n \in \mathbb{N} : x_n \in W \setminus (\partial W)^{r/4}\}$ . It is elementary to verify that if  $x \in W \setminus (\partial W)^r$ , then  $n \in E$  for any  $n \in \mathbb{N}$  with  $x \in B(x_n, 3r/4)$ , and therefore,  $\chi_E = 1$  in  $W \setminus (\partial W)^r$ . From the fact that  $\text{supp } \zeta_n \subset B(x_n, 3r/4)$ , it follows that  $\text{supp } \chi_E \subset W^{r/2}$ . □

### 4.3 Pulling Up

Suppose now that  $G$  is unimodular and let  $V$  be the function from Corollary 4.3. A straightforward calculation shows that

$$S = \Delta + \frac{1}{4} \|p_* H\|^2 - \frac{1}{2} \text{div } p_* H = \Delta - \frac{\Delta \sqrt{V}}{\sqrt{V}}.$$

This allows us to consider the renormalization

$$L = m^{-1}_{\sqrt{V}} \circ S \circ m_{\sqrt{V}} = \Delta - \text{grad } \ln V = \Delta + p_* H$$

of  $S$  with respect to  $\sqrt{V}$ , where we used again Corollary 4.3. According to Lemma 2.6, the Laplacian of the lift  $\tilde{f}$  of any  $f \in C^\infty(M_1)$  is given by

$$\Delta \tilde{f} = \tilde{L}f. \tag{9}$$

**Proposition 4.8** *Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a non-compact, connected, unimodular, and amenable Lie group  $G$ . Then for any  $\lambda \in \mathbb{R}$ ,  $\varepsilon > 0$  and  $f \in C_c^\infty(M_1) \setminus \{0\}$ , there exists  $h \in C_c^\infty(M_2) \setminus \{0\}$ , such that*

$$\frac{\|(\Delta - \lambda)h\|_{L^2(M_2)}^2}{\|h\|_{L^2(M_2)}^2} \leq \frac{\|(L - \lambda)f\|_{L^2_{\sqrt{V}}(M_1)}^2}{\|f\|_{L^2_{\sqrt{V}}(M_1)}^2} + \varepsilon.$$

**Proof** Cover  $\text{supp } f$  with finitely many open, precompact domains  $U_i$  that admit extensible sections  $s_i: U_i \rightarrow M_2, i = 1, \dots, k$ , and choose non-negative  $\varphi_i \in C_c^\infty(U_i)$  with  $\sum_{i=1}^k \varphi_i = 1$  in  $\text{supp } f$ . Denote by  $x_{ij}: U_i \cap U_j \rightarrow G$  the transition maps defined by  $s_j(y) = x_{ij}(y)s_i(y)$  for all  $y \in U_i \cap U_j$ , and by  $\Phi_i: G \times U_i \rightarrow p^{-1}(U_i)$  the diffeomorphisms defined by  $\Phi_i(x, y) = xs_i(y)$ .

Fix a left-invariant metric  $g$  on  $G$ . Since  $U_i$  is precompact and  $s_i$  is extensible, notice that there exists  $r > 0$  such that  $x_{ij}(U_i \cap U_j) \subset B_g(e, r)$  for any  $i, j = 1, \dots, k$ . Let  $\{\zeta_n\}_{n \in \mathbb{N}}$  be a partition of unity on  $G$  corresponding to  $r/2$ , as in Sect. 4.2. For a finite subset  $E$  of  $\mathbb{N}$ , consider the compactly supported, smooth function

$$h_i := (\chi_E)_{s_i} \tilde{\varphi}_i \tilde{f}$$

in  $p^{-1}(U_i), i = 1, \dots, k$ . Setting  $h = \sum_{i=1}^k h_i$ , we derive from Lemma 4.6 that there exists a constant  $C$  independent from  $E$ , such that  $|(\Delta - \lambda)h(z)| \leq C$  for any  $z \in M_2$ .

We know from Proposition 4.7 that for any  $\varepsilon > 0$ , there exists an open, bounded  $W \subset G$  and a finite  $E \subset \mathbb{N}$ , such that  $\chi_E = 1$  in  $W \setminus (\partial W)'$ ,  $\text{supp } \chi_E \subset W^{r/2}$  and

$$\frac{|W'_0|_g}{|W_0|_g} < \frac{\|f\|_{L^2_{\sqrt{V}}(M_1)}^2}{C^2 \int_{\text{supp } f} V}, \tag{10}$$

where  $W_0 := W \setminus (\partial W)^{2r}, W'_0 := (\partial W)^{2r}$  and tubular neighborhoods are considered with respect to the background metric  $g$ . Denote by  $W_i(y)$  and  $W'_i(y)$  the images of  $W_0$  and  $W'_0$  via  $\Phi_i(\cdot, y)$ , respectively. Bearing in mind that

$$\Phi_i(x, y) = \Phi_j(xx_{ji}(y), y)$$

for any  $y \in U_i \cap U_j$  and  $x \in G$ , it is not difficult to see that  $h(z) = \tilde{f}(z)$  for any  $z \in W_i(y)$  and that  $\text{supp } h \cap F_y \subset W_i(y) \cup W'_i(y)$  for any  $y \in U_i, i = 1, \dots, k$ . In

view of Corollary 4.3, it is now simple to compute

$$\begin{aligned} \|h\|_{L^2(M_2)}^2 &= \sum_{i=1}^k \int_{M_2} \tilde{\varphi}_i h^2 \geq \sum_{i=1}^k \int_{U_i} \int_{W_i(y)} \tilde{\varphi}_i h^2 dy \\ &= \sum_{i=1}^k \int_{U_i} \varphi_i(y) f^2(y) |W_0|_{g_{S_i(y)}} dy \\ &= |W_0|_g \sum_{i=1}^k \int_{U_i} \varphi_i f^2 V = |W_0| \|f\|_{L^2_{\sqrt{V}}(M_1)}^2. \end{aligned}$$

Furthermore, it is apparent that

$$\begin{aligned} \|(\Delta - \lambda)h\|_{L^2(M_2)}^2 &= \sum_{i=1}^k \int_{M_2} \tilde{\varphi}_i ((\Delta - \lambda)h)^2 \\ &= \sum_{i=1}^k \int_{U_i} \int_{W_i(y)} \tilde{\varphi}_i ((\Delta - \lambda)h)^2 dy \\ &\quad + \sum_{i=1}^k \int_{U_i} \int_{W'_i(y)} \tilde{\varphi}_i ((\Delta - \lambda)h)^2 dy. \end{aligned}$$

By virtue of Corollary 4.3 and formula (9), we deduce that

$$\begin{aligned} \sum_{i=1}^k \int_{U_i} \int_{W_i(y)} \tilde{\varphi}_i ((\Delta - \lambda)h)^2 dy &= \sum_{i=1}^k \int_{U_i} \int_{W_i(y)} \tilde{\varphi}_i ((\Delta - \lambda)\tilde{f})^2 dy \\ &= \sum_{i=1}^k \int_{U_i} \varphi_i(y) ((L - \lambda)f(y))^2 |W_0|_{g_{S_i(y)}} dy \\ &= |W_0|_g \int_{M_1} ((L - \lambda)f)^2 V \\ &= |W_0|_g \| (L - \lambda)f \|_{L^2_{\sqrt{V}}(M_1)}^2. \end{aligned}$$

Finally, Corollary 4.3 implies that

$$\begin{aligned} \sum_{i=1}^k \int_{U_i} \int_{W'_i(y)} \tilde{\varphi}_i ((\Delta - \lambda)h)^2 &\leq C^2 \sum_{i=1}^k \int_{\text{supp } f \cap U_i} \varphi_i(y) |W'_0|_{g_{S_i(y)}} dy \\ &= C^2 |W'_0|_g \int_{\text{supp } f} V. \end{aligned}$$

From the above estimates, we conclude that

$$\frac{\|(\Delta - \lambda)h\|_{L^2(M_2)}^2}{\|h\|_{L^2(M_2)}^2} \leq \frac{\|(L - \lambda)f\|_{L^2(\sqrt{V}(M_1))}^2}{\|f\|_{L^2(\sqrt{V}(M_1))}^2} + \frac{|W'_0|_g C^2 \int_{\text{supp } f} V}{|W_0|_g \|f\|_{L^2(\sqrt{V}(M_1))}^2},$$

which, together with (10), completes the proof.  $\square$

Similarly, exploiting the second inequality of Lemma 4.6, it is not hard to show the following:

**Proposition 4.9** *Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a non-compact, connected, unimodular, and amenable Lie group  $G$ . Then for any  $\varepsilon > 0$  and  $f \in C_c^\infty(M_1) \setminus \{0\}$ , there exists  $h \in C_c^\infty(M_2) \setminus \{0\}$  such that  $\mathcal{R}(h) \leq \mathcal{R}_L(f) + \varepsilon$ .*

Before proceeding to the proof of Theorem 1.3, we establish a part of it in the case where  $G$  is a connected Lie group.

**Proposition 4.10** *Let  $p: M_2 \rightarrow M_1$  be a Riemannian submersion arising from the action of a connected Lie group  $G$ . If  $G$  is unimodular and amenable, then the bottoms of the spectra satisfy  $\lambda_0(M_2) = \lambda_0(S)$ . If, in addition,  $M_1$  is complete, then  $\sigma(S) \subset \sigma(M_2)$ .*

**Proof** According to Corollary 1.2, if  $G$  is compact, then  $\lambda_0(M_2) = \lambda_0(S)$ . If, in addition,  $M_1$  is complete, then Corollary 4.4 asserts that so is  $M_2$ , and the second statement is a consequence of [23, Theorem 1.2].

Suppose now that  $G$  is non-compact, unimodular, and amenable. Then for any  $\varepsilon > 0$ , there exists a non-zero  $f \in C_c^\infty(M_1)$  such that  $\mathcal{R}_S(f) < \lambda_0(S) + \varepsilon/2$ , by Proposition 2.1. From Propositions 2.4 and 4.9, it follows that there exists a non-zero  $h \in C_c^\infty(M_2)$  with

$$\mathcal{R}(h) \leq \mathcal{R}_L(f/\sqrt{V}) + \varepsilon/2 = \mathcal{R}_S(f) + \varepsilon/2 < \lambda_0(S) + \varepsilon.$$

The proof of the first assertion is completed by Proposition 2.1,  $\varepsilon > 0$  being arbitrary.

Assume now that, in addition,  $M_1$  is complete and notice that  $M_2$  is also complete, from Corollary 4.4. Then Proposition 2.3 yields that for any  $\lambda \in \sigma(S)$ , there exists a characteristic sequence  $(f_n)_{n \in \mathbb{N}}$  for  $S$  and  $\lambda$ . In view of Proposition 4.8 and Lemma 2.4, for any  $n \in \mathbb{N}$ , there exists  $h_n \in C_c^\infty(M_2) \setminus \{0\}$  satisfying

$$\begin{aligned} \frac{\|(\Delta - \lambda)h_n\|_{L^2(M_2)}^2}{\|h_n\|_{L^2(M_2)}^2} &\leq \frac{\|(L - \lambda)(f_n/\sqrt{V})\|_{L^2(\sqrt{V}(M_1))}^2}{\|f_n/\sqrt{V}\|_{L^2(\sqrt{V}(M_1))}^2} + \frac{1}{n} \\ &= \frac{\|(S - \lambda)f_n\|_{L^2(M_1)}^2}{\|f_n\|_{L^2(M_1)}^2} + \frac{1}{n} \rightarrow 0, \end{aligned}$$

as  $n \rightarrow +\infty$ . That is,  $(h_n)_{n \in \mathbb{N}}$  is a characteristic sequence for  $\Delta$  (on  $M_2$ ) and  $\lambda$ , and hence,  $\lambda \in \sigma(M_2)$ , from Proposition 2.3.  $\square$

Consider now a Riemannian submersion  $p: M_2 \rightarrow M_1$  arising from the action of a Lie group  $G$ . Denote by  $p_1: M_2 \rightarrow M$  the Riemannian submersion arising from the action of the connected component  $G_0$  of  $G$ . Then the action of  $G$  on  $M_2$  descends to a properly discontinuous action of  $G/G_0$  on  $M$ , which gives rise to a Riemannian covering  $p_2: M \rightarrow M_1$ , and the original submersion is decomposed as  $p = p_2 \circ p_1$ . It is immediate to verify that the Schrödinger operator

$$S_M := \Delta + \frac{1}{4} \|p_{1*}H\|^2 - \frac{1}{2} \operatorname{div} p_{1*}H$$

on  $M$ , defined as in (1), is the lift of the corresponding Schrödinger operator  $S$  on  $M_1$ .

**Proof of Theorem 1.3** Write  $p = p_2 \circ p_1$  as above, and suppose that  $G$  is amenable and  $G_0$  is unimodular. Then Lemma 2.8 states that  $G_0$  and  $G/G_0$  are also amenable. From Proposition 4.10 and [2, Theorem 1.2], we obtain that

$$\lambda_0(M_2) = \lambda_0(S_M) = \lambda_0(S).$$

If, in addition,  $M_1$  is complete, then so is  $M$ , and the spectra are related by

$$\sigma(S) \subset \sigma(S_M) \subset \sigma(M_2),$$

where the first inclusion follows from [22, Corollaries 4.21 and 4.22] and the second one from Proposition 4.10.

Conversely, assume that  $\lambda_0(M_2) = \lambda_0(S) \notin \sigma_{\text{ess}}(S)$ . By virtue of Theorem 1.1, we have that  $\lambda_0(F_x) = 0$  for almost any  $x \in M_1$ . Recall that  $F_x$  is isometric to  $G$  endowed with a left-invariant metric, from Lemma 4.1. Taking into account that  $\lambda_0(G) = \lambda_0(G_0)$ , we derive from Theorem 2.10 that  $G_0$  is unimodular and amenable. Moreover, Theorem 1.1 and [2, Theorem 1.1] show that

$$\lambda_0(M_2) \geq \lambda_0(S_M) \geq \lambda_0(S),$$

and thus,  $\lambda_0(S_M) = \lambda_0(S)$ . Since  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$ , we conclude from [21, Theorem 1.2] that  $p_2$  is an amenable covering, or equivalently,  $G/G_0$  is amenable. The proof is completed by Lemma 2.8.  $\square$

**Proof of Corollary 1.4** Suppose first that  $G$  is unimodular and amenable, and fix a left-invariant metric on  $G$ . By formula (6), it is easily checked that  $\mathcal{R}_S(\sqrt{V}) = 0$  for the positive  $V \in C^\infty(M_1)$  from Corollary 4.3, which together with Theorem 1.3, Proposition 2.1 and Lemma 3.2, implies that  $\lambda_0(M_2) = \lambda_0(S) = 0$ .

Conversely, assume that  $\lambda_0(M_2) = 0$  and write  $p = p_2 \circ p_1$  as above. We readily see from Theorem 1.1 that  $\lambda_0(S) = 0$ . Then  $G$  is amenable and  $G_0$  is unimodular, from Theorem 1.3, because  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$ ,  $M_1$  being closed. We know from Corollary 4.3 that there exists  $V \in C^\infty(M)$  with  $p_{1*}H = -\operatorname{grad} \ln V$ , such that for any section



$s : U \subset M \rightarrow M_2$ , the volume elements of the induced metrics on  $G_0$  (and on  $G$ ) satisfy

$$d\text{vol}_{g_{s(y)}} = V(y)d\text{vol}_g,$$

where  $g$  is a fixed left-invariant metric on  $G$ .

Observe that there exists a positive  $\varphi \in C^\infty(M_1)$  with  $S\varphi = 0$ , from [21, Proposition 3.7] and the fact that  $\lambda_0(S) = 0 \notin \sigma_{\text{ess}}(S)$ . Then  $\mathcal{R}_S(\varphi) = 0$ , which together with formula (6), gives that  $p_*H = -2 \text{grad} \ln \varphi$ . It is now clear that  $V$  is a constant multiple of the lift  $\tilde{\varphi}^2$  of  $\varphi^2$  on  $M$ , and in particular,  $G/G_0$ -invariant.

Given  $z \in M_2$  and  $x \in G$ , using formula (8), the definition and the  $G/G_0$ -invariance of  $V$ , we compute

$$\begin{aligned} R_x^*(d \text{vol}_{g_z}) &= d \text{vol}_{g_{xz}} = V(p_1(xz))d \text{vol}_g = V([x]p_1(z))d \text{vol}_g \\ &= V(p_1(z))d \text{vol}_g = d \text{vol}_{g_z}, \end{aligned}$$

where  $[x]$  stands for the class of  $x$  in  $G/G_0$ . Therefore,  $g_z$  is a left-invariant metric on  $G$  with right-invariant volume element, which means that  $G$  is unimodular.  $\square$

We end this section with a class of examples demonstrating that the assumption  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$  in Theorem 1.3(iii) cannot be dropped, even if the manifolds are complete and the fibers are minimal.

**Example 4.11** Let  $G$  be the simply connected Lie group with Lie algebra spanned by two vectors  $X, Y$  such that  $[X, Y] = Y$ . Given  $c > 0$ , define the left-invariant metric  $g_c$  on  $G$  by  $g_c(X, X) = c^{-1}$ ,  $g_c(X, Y) = 0$  and  $g_c(Y, Y) = c$ . It is obvious that

$$\langle \nabla_{X_1} X_2, X_3 \rangle = \frac{1}{2} (\langle [X_1, X_2], X_3 \rangle - \langle [X_2, X_3], X_1 \rangle + \langle [X_3, X_1], X_2 \rangle)$$

for any left-invariant vector fields  $X_1, X_2, X_3$  on  $G$ , where the inner products are with respect to  $g_c$  and  $\nabla$  stands for the Levi-Civita connection of  $g_c$ . From this, it is easy to see that  $(G, g_c)$  has constant sectional curvature  $-c$ . Thus,  $(G, g_c)$  is isometric to the 2-dimensional space form of sectional curvature  $-c$ , and in particular, the bottom of its spectrum is given by

$$\lambda_0(G, g_c) = \frac{c^2}{4}. \tag{11}$$

Bearing in mind that  $G$  is solvable, observe that  $G$  is not unimodular, from Theorem 2.10.

Let  $M$  be a Riemannian manifold with  $\lambda_0(M) \in \sigma_{\text{ess}}(M)$ . For a positive function  $\psi \in C^\infty(M)$ , endow the product manifold  $M_2 := M \times G$  with the Riemannian metric  $g(y, x) = g_M(y) \times g_{\psi(y)}(x)$ . It is evident that  $G$  acts smoothly, freely, and properly via isometries on  $M_2$ , and the Riemannian submersion arising from this action is the projection to the first factor  $p : M_2 \rightarrow M$ . It is noteworthy that  $p$  has minimal fibers,

since the volume element of  $g_c$  does not depend on  $c$ . Hence, the operator  $S$  defined as in (1) coincides with the Laplacian on  $M$ .

By Proposition 2.2, there exists a sequence  $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \setminus \{0\}$  such that  $\mathcal{R}(f_n) \rightarrow \lambda_0^{\text{ess}}(M) = \lambda_0(M)$  and  $\text{supp } f_n \subset U_n$  for some precompact, open domains  $U_n$  with  $\bar{U}_n$  pairwise disjoint. Clearly, we may choose a positive  $\psi \in C^\infty(M)$  with  $\psi = c_n < 1/n$  in  $U_n$  for any  $n \in \mathbb{N}$ . Then  $p^{-1}(U_n)$  is isometric to the Riemannian product  $U_n \times G$ , where  $G$  is endowed with the Riemannian metric  $g_{c_n}$ . In view of Proposition 2.1 and formula (11), it follows that for any  $n \in \mathbb{N}$ , there exists  $h_n \in C_c^\infty(G) \setminus \{0\}$  with  $\mathcal{R}_{g_{c_n}}(h_n) < 1/(4n^2)$ . Setting  $\tilde{h}_n(y, x) = h_n(x)$  and  $\tilde{f}_n(y, x) = f_n(y)$ , we have that  $\tilde{h}_n \tilde{f}_n \in C_c^\infty(M_2)$  and a straightforward calculation implies that

$$\mathcal{R}(\tilde{h}_n \tilde{f}_n) = \mathcal{R}_{g_{c_n}}(h_n) + \mathcal{R}(f_n) \rightarrow \lambda_0(M),$$

as  $n \rightarrow +\infty$ . From this, together with Theorem 1.1 and Proposition 2.1, we deduce that  $\lambda_0(M_2) = \lambda_0(M) = \lambda_0(S)$ , while  $G$  is not unimodular.

### 5 Bottom of Spectrum of Lie Groups

In this section, we discuss some applications of our results to Lie groups. We begin by establishing Theorem 1.5.

**Proof of Theorem 1.5** Clearly, the projection  $p: G \rightarrow G/N$  is the Riemannian submersion arising from the (left) action of  $N$  on  $G$ , and the fiber over  $p(z)$  is written as is  $F_{p(z)} = Nz = zN$  for any  $z \in G$ ,  $N$  being normal. Since multiplication  $L_x$  from the left with an element  $x \in G$  maps isometrically  $F_{p(z)}$  to  $F_{p(xz)}$  for any  $z \in G$ , it is evident that the mean curvature  $H$  of the fibers is left-invariant, and so is  $p_*H$  on  $G/N$ . Then the operator  $S$  on  $G/N$  defined as in (1) is of the form  $S = \Delta + c$  for some  $c \in \mathbb{R}$ , and the bottom of its spectrum is  $\lambda_0(S) = \lambda_0(G/N) + c$ .

To determine this constant, let  $\{X_i\}_{i=1}^m$  be an orthonormal basis of  $T_eG$  with  $\{X_i\}_{i=1}^k$  spanning  $T_eN$ . Considering the left-invariant extension of  $X_i$  (also denoted by  $X_i$ ), it is easily checked that

$$\begin{aligned} \|H\|^2 &= \sum_{i=1}^k \langle \nabla_{X_i} X_i, H \rangle = - \sum_{i=1}^k \langle \nabla_{X_i} H, X_i \rangle \\ &= - \sum_{i=1}^m \langle \nabla_{X_i} H, X_i \rangle + \sum_{i=k+1}^m \langle \nabla_{X_i} H, X_i \rangle \\ &= \sum_{i=1}^m \langle [H, X_i], X_i \rangle + \sum_{i=k+1}^m \langle \nabla_{p_*X_i} p_*H, p_*X_i \rangle \\ &= \text{tr}(\text{ad } H) + \text{div } p_*H, \end{aligned} \tag{12}$$

and the operator  $S$  is written as follows:

$$S = \Delta - \frac{1}{4} \|H\|^2 + \frac{1}{2} \operatorname{tr}(\operatorname{ad} H).$$

The first statement follows from Theorem 1.1, after noticing that  $\lambda_0(F_y) = \lambda_0(N)$  for any  $y \in G/N$ ,  $F_y$  being isometric to  $N$ . If  $N$  is unimodular and amenable, then Theorem 1.3 establishes the asserted equality. Conversely, as a consequence of Theorem 1.1, if

$$\lambda_0(G) = \lambda_0(G/N) - \frac{1}{4} \|H\|^2 + \frac{1}{2} \operatorname{tr}(\operatorname{ad} H),$$

then the infimum of  $\lambda_0(F_y)$  with  $y \in G/N$  is zero. Then  $\lambda_0(N) = 0$ , since  $F_y$  is isometric to  $N$  (endowed with the induced left-invariant metric from  $G$ ) and Theorem 2.10 yields that  $N$  is unimodular and amenable.  $\square$

It is worth to point out that in the above setting, the assumption  $\lambda_0(S) \notin \sigma_{\text{ess}}(S)$  involved in Theorem 1.3(iii) is not satisfied in general. Indeed, if  $G/N$  is non-compact, then  $\sigma(S) = \sigma_{\text{ess}}(S)$ ,  $S$  being invariant under multiplication from the left with elements of  $G/N$  (cf. for instance [22, Theorem 5.2]). However, the conclusion of Theorem 1.3(iii) holds because the fibers are isometric.

**Corollary 5.1** *Let  $G$  be a connected, unimodular, and amenable Lie group endowed with a left-invariant metric and  $N$  be a closed (as a subset), connected, normal subgroup of  $G$  with mean curvature  $H$ . Then*

$$\lambda_0(G/N) = \frac{1}{4} \|H\|^2.$$

*In particular,  $G/N$  is also unimodular (and amenable) if and only if  $N$  is minimal.*

**Proof** Since  $G$  is unimodular, we obtain from Lemma 2.9 that  $\operatorname{tr}(\operatorname{ad} H) = 0$  and that  $N$  is also unimodular. According to Lemma 2.8, since  $G$  is amenable, so are  $N$  and  $G/N$ . The proof is completed by Theorems 1.5 and 2.10.  $\square$

Recall that, in general, the quotient of a unimodular and amenable Lie group does not have to be unimodular. The next example demonstrates this fact.

**Example 5.2** Let  $G$  be the simply connected, solvable Lie group with Lie algebra  $\mathfrak{g}$  generated by  $X, Y, Z$  satisfying  $[X, Y] = Y$ ,  $[X, Z] = -Z$  and  $[Y, Z] = 0$ . It is obvious that  $\operatorname{tr}(\operatorname{ad} X') = 0$  for any  $X' \in \mathfrak{g}$ , and we deduce from Lemma 2.9 that  $G$  is unimodular. Let  $N$  be the closed (as a subset), connected, normal subgroup of  $G$  with Lie algebra the ideal generated by  $Z$ . Denoting by  $p: G \rightarrow G/N$  the projection, it is elementary to verify that  $\operatorname{tr}(\operatorname{ad} p_*X) = 1$ . We conclude from Lemma 2.9 that  $G/N$  is not unimodular, while  $G$  is unimodular and amenable.

Before proceeding to the proof of Corollary 1.6, we need some auxiliary results. The next proposition provides a standard way of estimating the Cheeger constant of a Riemannian manifold.

**Proposition 5.3** *Let  $X$  be a smooth vector field on a Riemannian manifold  $M$  with  $\|X\| \leq 1$  and  $\operatorname{div} X \geq c$  for some  $c \in \mathbb{R}$ . Then the Cheeger constant of  $M$  is bounded by  $h(M) \geq c$ .*

**Proof** Using the divergence formula, for any compact domain  $K$  of  $M$  with smooth boundary, we compute

$$c|K| \leq \int_K \operatorname{div} X = \int_{\partial K} \langle X, \nu \rangle \leq |\partial K|,$$

where  $\nu$  is the outward pointing unit normal to  $\partial K$ . □

**Corollary 5.4** *Let  $G$  be a connected Lie group endowed with a left-invariant metric. Then the Cheeger constant of  $G$  satisfies*

$$h(G) \geq \max_{X \in \mathfrak{g}, \|X\|=1} \operatorname{tr}(\operatorname{ad} X).$$

**Proof** A straightforward calculation shows that  $\operatorname{tr}(\operatorname{ad} X) = -\operatorname{div} X$  for any  $X \in \mathfrak{g}$ , and the assertion is a consequence of Proposition 5.3. □

**Proposition 5.5** *Let  $G$  be a connected, amenable Lie group endowed with a left-invariant metric. Suppose that its radical  $S$  is not abelian and denote by  $H$  the mean curvature (in  $G$ ) of the commutator subgroup  $[S, S]$ . Then*

$$\lambda_0(G) = \frac{1}{4} \|H\|^2 = \frac{1}{4} \operatorname{tr}(\operatorname{ad} H).$$

**Proof** Consider the universal covering  $q: \tilde{S} \rightarrow S$ . Since  $\tilde{S}$  is simply connected and solvable, it is known that its commutator subgroup  $[\tilde{S}, \tilde{S}]$  is closed (as a subset of  $\tilde{S}$ ) and nilpotent (cf. for instance [17, Proposition 1.6] and the references therein). This yields that the commutator subgroup  $N := [S, S] = q([\tilde{S}, \tilde{S}])$  is a connected, closed (as a subset), normal, and nilpotent subgroup of  $G$ . Since connected, nilpotent groups are unimodular and amenable, Theorem 2.10 gives that

$$\lambda_0(G) = \lambda_0(G/N) - \frac{1}{4} \|H\|^2 + \frac{1}{2} \operatorname{tr}(\operatorname{ad} H).$$

Bearing in mind that  $G$  is a compact extension of  $S$ , it is evident that  $G/N$  is a compact extension of the abelian group  $S/N$ . In particular,  $G/N$  is unimodular and amenable, and hence,  $\lambda_0(G/N) = 0$ , from Theorem 2.10. Let  $\{X_i\}_{i=1}^m$  be an orthonormal basis of  $\mathfrak{g}$  with  $\{X_i\}_{i=1}^k$  spanning the Lie algebra of  $N$ . Then formula (12) yields that

$$\|H\|^2 = \operatorname{tr}(\operatorname{ad} H) - \operatorname{tr}(\operatorname{ad} p_* H).$$

We derive from Lemma 2.9 that  $\operatorname{tr}(\operatorname{ad} p_* H) = 0$ ,  $G/N$  being unimodular, as we wished. □

**Proof of Corollary 1.6** If  $G$  is unimodular, then the statement follows from Lemma 2.9, Theorem 2.10 and the Cheeger inequality. Suppose now that  $G$  is not unimodular and observe that  $S$  is not abelian, since  $G$  is a compact extension of  $S$ . It follows from Theorem 2.10 that  $\lambda_0(G) > 0$ , and thus, the mean curvature (in  $G$ )  $H$  of the commutator subgroup  $N := [S, S]$  of the radical  $S$  of  $G$  is non-zero, from Proposition 5.5. In view of Corollary 5.4, Proposition 5.5 and the Cheeger inequality, we conclude that

$$\frac{1}{4}h(G)^2 \geq \frac{1}{4}(\operatorname{tr}(\operatorname{ad} H_0))^2 = \frac{1}{4}\operatorname{tr}(\operatorname{ad} H) = \lambda_0(G) \geq \frac{1}{4}h(G)^2,$$

where  $H_0 := \|H\|^{-1}H$ . □

According to [7], if the Cheeger constant coincides with the exponential volume growth, then the equality holds in the Cheeger inequality. However, this fails in Corollary 1.6, since there exist unimodular and amenable Lie groups of exponential volume growth (cf. [20, p. 1525] and the references therein).

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