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A Flower-Shape Geometry and Nonlinear Problems on Strip-Like Domains

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Abstract

In the present paper, we show how to define suitable subgroups of the orthogonal group $O(d-m)$ related to the unbounded part of a strip-like domain $\omega\times\mathbb{R}^{d-m}$ with $d \geq m+2$, in order to get "mutually disjoint" nontrivial subspaces of partially symmetric functions of $H_0^1(\omega \times \mathbb{R}^{d-m})$ which are compactly embedded in the associated Lebesgue spaces. As an application of the introduced geometrical structure, we prove (existence and) multiplicity results for semilinear elliptic problems set in a strip-like domain, in the presence of a nonlinearity which either satisfies the classical Ambrosetti–Rabinowitz condition or has a sublinear growth at infinity. The main theorems of this paper may be seen as an extension of existence and multiplicity results, already appeared in the literature, for nonlinear problems set in the entire space $\mathbb{R}^d$, as for instance, the ones due to Bartsch and Willem. The techniques used here are new.

Keywords Laplace equation · Variational methods · Critical points theory · Principle of Symmetric Criticality · Radial and non-radial solutions

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1 Introduction

1.1 Lack of Compactness and Symmetries on Unbounded Domains

Several important problems arising in many research fields such as physics and differential geometry lead to consider semilinear variational elliptic equations defined on unbounded domains of the Euclidean space and a great deal of work has been devoted to their study. From the mathematical point of view, probably the main interest relies on the fact that often the tools of nonlinear functional analysis, based on compactness arguments, cannot be used, at least in a straightforward way, and some new techniques have to be developed.

The seminal paper [15] by Lions has inspired a (nowadays usual) way to overcome the lack of compactness by exploiting symmetry. This approach is fruitful in the study of variational elliptic problems in presence of a suitable continuous action of a topological group on the Sobolev space where the solutions are being sought.

Along this direction, in the present paper, we exploit a group theoretical scheme, raised in the study of problems which are invariant with respect to the action of orthogonal subgroups, to show the existence of multiple solutions distinguished by their different symmetry properties. We emphasize that a wide class of nonlinear problems of this kind can be handled by constructing suitable subspaces, of "partially symmetric" functions, of the ambient Sobolev space, and by applying an appropriate version of the so-called Principle of Symmetric Criticality proved in the seminal paper [21] by Palais.

For instance, let $\mathbb{R}_0^+ = [0, \infty)$, let $\psi_1, \psi_2 : \mathbb{R}_0^+ \to \mathbb{R}$ be two functions that are bounded on bounded sets, with $\psi_1(t) < \psi_2(t)$ for every $t \in \mathbb{R}_0^+$ and consider the strip-like domain of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \geq 1$, given by

$$\Omega_\psi := \Big\{ q = (z, t) \in \mathbb{C}^n \times \mathbb{R} \ : \ \psi_1(|z|) < t < \psi_2(|z|) \Big\}.$$

The existence of weak solutions for subelliptic problems set on Ω_ψ has been investigated in [9,16,20] by employing symmetries. The main proofs, crucially based on the Palais Principle, are obtained by developing a suitable algebraic procedure on the unitary group $\mathbb{U}(n) := U(n) \times \{id\}$ that fits well with the approach developed along the present paper. This group acts continuously on the Folland–Stein space $HW_0^{1,2}(\Omega_\psi)$ by the action $\sharp : \mathbb{U}(n) \times HW_0^{1,2}(\Omega_\psi) \to HW_0^{1,2}(\Omega_\psi)$ pointwise defined by setting, for all $\widehat{\tau} := \tau \times id$ with $\tau \in U(n)$,

$$(\widehat{\tau}\sharp u)(q) = u(\tau^{-1}z, t) \quad \text{for a.e. } q = (z, t) \in \mathbb{H}^n.$$

For any choice, if any, of $\ell \geq 1$ and of ℓ-tuple $(n_1, \dots, n_\ell)$ such that for all $i \in \{1, \dots, \ell\}$, $n_i \geq 2$ and $\displaystyle\sum_{i=1}^{\ell} n_i = n$, denoting $T := U(n_1) \times \cdots \times U(n_\ell) \times \{id\}$, the set

$$Fix_T(HW_0^{1,2}(\Omega_\psi)) := \Big\{ u \in HW_0^{1,2}(\Omega_\psi) \ : \ \widehat{\tau}\sharp u = u \quad \text{for any } \widehat{\tau} \in T \Big\}$$

is a closed subspace of $HW_0^{1,2}(\Omega_\psi)$ which is compactly embedded in the Lebesgue space $L^\nu(\Omega_\psi)$ for any $\nu \in (2, 2_Q^*)$, where $2_Q^* := 2Q/(Q-2)$, and $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$; see [4, Theorem 1.1]. This fact allows to prove the existence of multiple T-symmetric solutions for subelliptic problems set on Ω_ψ.

Here, we are interested on problems settled in strip-like domains of the Euclidean space $\mathbb{R}^d$. Without loss of generality, fixed $m \in \mathbb{N}$ we consider a strip-like domain $\omega \times \mathbb{R}^{d-m}$, where $\omega \subset \mathbb{R}^m$ is an open-bounded Euclidean domain with smooth boundary $\partial\omega$. If $d \geq m+2$, we exploit some compact embedding of the space $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ of "cylindrically symmetric" functions of the Sobolev space $H_0^1(\omega \times \mathbb{R}^{d-m})$ into the Lebesgue space $L^\nu(\omega \times \mathbb{R}^{d-m})$ for all $\nu \in (2,\ 2^*)$, $2^* := 2d/(d-2)$. Subsequently, assuming that $d \geq m+4$, more partial symmetries (in addition to the so-called block radial symmetries) can be used and so more distinct subspaces of $H_0^1(\omega \times \mathbb{R}^{d-m})$ can be introduced on which one can recover compactness (see Proposition 2.2).

The proof of the main compactness result given in Proposition 2.2 somehow follows by [15, Théorème III.2], and it is crucially based on the use of the action induced by the orthogonal group $O(d-m)$ on $\mathbb{R}^{d-m}$, the unbounded part of the strip (see Sect. 2.1 below and [13, Subsection 2]). Actually, when $d = m+4$ or $d \geq m+6$, we set

$$\tau_{d,m} := (-1)^{d-m} + \left\lfloor \frac{d-m-3}{2} \right\rfloor \quad \text{and} \quad I_{d,m} := \left\{1, \dots, \tau_{d,m}\right\}, \tag{1.1}$$

(the symbol $\lfloor\cdot\rfloor$ denotes the integer value function), and, for any $i \in I_{d,m}$, we define

$$H_{d,m,i} := \begin{cases} O((d-m)/2) \times O((d-m)/2) & \text{if } i = \dfrac{d-m-2}{2} \\ O(i+1) \times O(d-m-2i-2) \times O(i+1) & \text{if } i \neq \dfrac{d-m-2}{2} \end{cases}$$

and

$$\widehat{H}_{d,m,i} := \{id_m\} \times H_{d,m,i} \subset \{id_m\} \times O(d-m) =: \widehat{O}(d-m).$$

Despite the fact that the $\tau_{d,m}$ sets $Fix_{\widehat{H}_{d,m,i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ of the functions in $H_0^1(\omega \times \mathbb{R}^{d-m})$ which are invariant with respect to the induced action $\sharp$ of $\widehat{H}_{d,m,i}$ on $H_0^1(\omega \times \mathbb{R}^{d-m})$ (for a precise definition see (2.3)), i.e.

$$Fix_{\widehat{H}_{d,m,i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) := \left\{u \in H_0^1(\omega \times \mathbb{R}^{d-m}) : \widehat{h}\sharp u = u \text{ for any } \widehat{h} \in \widehat{H}_{d,m,i}\right\},$$

are the so-called block-radial subspaces of $H_0^1(\omega \times \mathbb{R}^{d-m})$ and therefore, by [15, Théorème III.2], are compactly embedded in $L^\nu(\omega \times \mathbb{R}^{d-m})$ for all $\nu \in (2,\ 2^*)$, they are not "mutually disjoint". So, more effort must be done to get the multiplicity result.

To this aim, by adapting the arguments of [13, Theorem 2.2] introduced in the whole Euclidean space, we define on $\mathbb{R}^{d-m}$ the involution function $\eta_{d,m,i}$ (see (2.8) below) which allows us to construct subgroups $\widehat{H}_{d,m,\widehat{\eta}_i}$ of $\widehat{O}(d-m)$ (see (2.10) for the definition) such that the sets $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ of the functions

$u \in H_0^1(\omega \times \mathbb{R}^{d-m})$ which are invariant with respect to the action $\circledast_i$ (defined by (2.11) below) of $\widehat{H}_{d,m,\widehat{\eta}_i}$ on $H_0^1(\omega \times \mathbb{R}^{d-m})$, i.e.

$$
\begin{aligned}
& Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \\
& \quad := \left\{ u \in H_0^1(\omega \times \mathbb{R}^{d-m}) : \widehat{h} \circledast_i u = u \text{ for any } \widehat{h} \in \widehat{H}_{d,m,\widehat{\eta}_i} \right\},
\end{aligned}
$$

are nontrivial subspaces of $H_0^1(\omega \times \mathbb{R}^{d-m})$, still compactly embedded in $L^\nu(\omega \times \mathbb{R}^{d-m})$, (see Proposition 2.2) and with the property of being "mutually disjoint", i.e. their mutual intersection reduces to the trivial space, as proved in Proposition 2.3. For the precise statements and the related details see Sect. 2.2.

The new key results, of independent interest, given by Proposition 2.2 and Proposition 2.3 describe a sort of *flower-shape* geometry in $H_0^1(\omega \times \mathbb{R}^{d-m})$, whose $\tau_{d,m} + 1$ *petals* are the $\tau_{d,m}$ Sobolev spaces $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ introduced above plus the subspace of cylindrically symmetric functions $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$.

The advantage of this new type of symmetries in the study of nonlinear Dirichlet problems on strip-like domains has been investigated in Theorem 1.1 and Theorem 1.2 by using variational and topological methods. In particular, while, to get the existence result, we apply the variational argument to the space $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$, to obtain the multiplicity result, we use the same approach in each petal $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$, and so we must require that $\tau_{d,m} \geq 1$ (and, therefore, that $d = m + 4$ or $d \geq m + 6$) and that the nonlinear term appearing in the equation satisfies suitable symmetry assumptions to assure that the functional associated with the problem is invariant with respect to the action of the group $\widehat{H}_{d,m,\widehat{\eta}_i}$ (according to the statement in (3.34)).

1.2 Nonlinear Problems on Strip-Like Domains

In the present paper, we are interested in getting existence and multiplicity results of weak solutions to the following problem

$$
\begin{cases} -\Delta u = \lambda \alpha(x, y) f(u) & \text{in } \omega \times \mathbb{R}^{d-m} \\ u = 0 & \text{on } \partial\omega \times \mathbb{R}^{d-m}, \end{cases} \tag{P_λ}
$$

where λ is a positive parameter and $\omega \times \mathbb{R}^{d-m}$ is an unbounded strip of $\mathbb{R}^d$, being ω an open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$ and $d, m \in \mathbb{N}$, $d \geq m + 2$. Moreover, we assume that $\alpha : \omega \times \mathbb{R}^{d-m} \to \mathbb{R}$ verifies the following integrability, symmetry and sign conditions

$$
\alpha \in L^1(\omega \times \mathbb{R}^{d-m}) \cap L^\infty(\omega \times \mathbb{R}^{d-m}) \tag{α_1}
$$

$$\alpha(x, y) = \alpha(x, |y|) \ \text{ a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m} \tag{α_2}$$

$$\alpha \geq 0 \text{ a.e. in } \omega \times \mathbb{R}^{d-m} \quad \begin{array}{c}\text{and there exist } r > 0 \text{ and } \alpha_0 > 0 \text{ such that} \\ \operatorname{essinf}_{\omega \times B(0,r)} \alpha \geq \alpha_0 \,,\end{array} \tag{α_3}$$

where $B(0, r)$ is the ball in $\mathbb{R}^{d-m}$ centred at 0 with radius r, while on $f : \mathbb{R} \to \mathbb{R}$ we require the next hypotheses

$$f \text{ is continuous in } \mathbb{R} \tag{f_1}$$

$$f(t) = o(|t|) \text{ as } |t| \to 0 \tag{f_2}$$

$$\text{there exists } \sigma > 2 \text{ such that } 0 < \sigma F(t) \leq t f(t) \ \text{ for any } t \in \mathbb{R} \setminus \{0\}, \tag{f_3}$$

where F is the following antiderivative of the function f

$$F(t) = \int_0^t f(\tau)\, d\tau \,, \quad t \in \mathbb{R}\,, \tag{1.2}$$

$$\sup_{t \in \mathbb{R} \setminus \{0\}} \frac{|f(t)|}{|t| + |t|^{q-1}} < +\infty \ \text{ for some } q \in (2, 2^*), \tag{f_4}$$

where 2^* is the critical Sobolev exponent given by $2^* := 2d/(d-2)$. As a model for f we can consider, for fixed $q \in (2, 2^*)$, the function

$$f(t) = t|t|^{q-2}, \ \ t \in \mathbb{R}\,.$$

Assumption (f_3) is the well-known Ambrosetti–Rabinowitz condition, which is a **superlinear** assumption on the term f, namely a superquadratic one on its antiderivative F at infinity.

In this paper, we want to study Problem (P_λ) also under **sublinear** conditions at infinity on the nonlinearity f. More precisely, we shall also consider the case in which, instead of (f_3), the function f satisfies the following hypotheses

$$f(t) = o(|t|) \text{ as } |t| \to +\infty \tag{f_5}$$

$$\text{there exists } t_0 \in \mathbb{R}^+ \quad \text{such that } F(t_0) > 0 \text{ and } F(t) \geq 0 \text{ on } [0, t_0], \tag{f_6}$$

where F is given in (1.2). Note that when (f_5) is satisfied then, thanks to (f_2), condition (f_4) is also guaranteed. A prototype for f is given by the odd extension of the function f_r defined on $\mathbb{R}^0_+$ by setting

$$f_r(t) = \begin{cases} t|t|^{q-2} & \text{if } 0 \leq t \leq 1 \\ (\log 2 - 1)t + 2 - \log 2 & \text{if } 1 < t < 2 \\ \log t & \text{if } t \geq 2, \end{cases}$$

with fixed $q \in (2, 2^*)$.

Problem (P_λ) has a clear variational structure, indeed its solutions can be found as critical points of the following energy functional defined by setting for all $u \in H^1_0(\omega \times \mathbb{R}^{d-m})$

$$\mathcal{I}_\lambda(u) := \frac{1}{2} \int_{\omega \times \mathbb{R}^{d-m}} |\nabla u(x, y)|^2 \, dx \, dy - \lambda \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) F(u(x, y)) \, dx \, dy, \tag{1.3}$$

where F is given in (1.2).

Since the problem is set on the strip-like domain $\omega \times \mathbb{R}^{d-m}$, there is no compactness property which can be used with $\mathcal{I}_\lambda$ on the whole space. Hence, in order to find a weak solution to Problem (P_λ), we shall choose a suitable subspace of $H^1_0(\omega \times \mathbb{R}^{d-m})$ which allows us, from one side, to recover compactness and to get, by an application of the Mountain Pass Theorem (see [3]), a constrained critical point for the energy functional $\mathcal{I}_\lambda$ and, from the other side, to apply the Principle of Symmetric Criticality got by Palais in [21] (see also [27] for some applications) to show that the restriction to that subspace does not play any role.

Finally, when $d = m + 4$ or $d \geq m + 6$ and the nonlinearity f is odd, by exploiting the flower-shape geometric structure in the Sobolev space $H^1_0(\omega \times \mathbb{R}^{d-m})$ described in Sect. 2, we get a multiplicity result for Problem (P_λ), using again variational and topological arguments.

More precisely, our main results for Problem (P_λ) are stated in Theorem 1.1 and Theorem 1.2 below, for the superlinear and, respectively, for the sublinear growth of the nonlinearity at infinity.

In the **superlinear** framework, our result reads as follows:

Theorem 1.1 (Superlinear setting) *Let $\omega \times \mathbb{R}^{d-m}$ be an unbounded strip of $\mathbb{R}^d$, with ω open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$, $d, m \in \mathbb{N}$, $d \geq m + 2$, and let λ be a positive parameter. Let α satisfy conditions (α_1), (α_2) and (α_3) and let f satisfy assumptions (f_1), (f_2), (f_3) and (f_4).*

Then,

(i) Existence: for any $\lambda > 0$, there exists a nontrivial weak solution u_λ of Problem (P_λ) in $H^1_0(\omega \times \mathbb{R}^{d-m})$ with cylindrical symmetry;

(ii) Multiplicity: if, in addition, $d = m + 4$ or $d \geq m + 6$ and f is odd, then for any $\lambda > 0$ Problem (P_λ) admits $s_{d,m}$ sequences of nontrivial weak solutions, with different symmetries, where $s_{d,m}$ is defined as follows

$$s_{d,m} = (-1)^{d-m} + \left\lfloor \frac{d-m-3}{2} \right\rfloor + 1\,. \tag{1.4}$$

We would remark that the number $s_{d,m}$ is equal to $\tau_{d,m} + 1$ (with $\tau_{d,m}$ given in (1.1)). Actually, the $s_{d,m}$ sequences of weak solutions to Problem (P_λ) found in Theorem 1.1 are characterized by different symmetries since they are found as critical points of the energy functional $\mathcal{I}_\lambda$ in $s_{d,m}$ subspaces of $H_0^1(\omega \times \mathbb{R}^{d-m})$ which are "'mutually disjoint" (in the sense that their mutual intersection reduces to the trivial space, see, for a precise statement, Proposition 2.3).

In the **sublinear** setting, our main result for Problem (P_λ) is stated here below.

Theorem 1.2 (Sublinear setting) *Let $\omega \times \mathbb{R}^{d-m}$ be an unbounded strip of $\mathbb{R}^d$, with ω open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$, $d, m \in \mathbb{N}$, $d \geq m + 2$, and let λ be a positive parameter. Let α satisfy conditions (α_1), (α_2) and (α_3) and let f satisfy (f_1), (f_2), (f_5) and (f_6).*

Then,

(i) there exists $\bar{\lambda} > 0$ such that for any $\lambda < \bar{\lambda}$ there are no nontrivial weak solutions for Problem (P_λ);
(ii) there exists $\lambda_E^* > 0$ such that for any $\lambda > \lambda_E^*$ there exist at least two nontrivial weak solutions of Problem (P_λ) in $H_0^1(\omega \times \mathbb{R}^{d-m})$ with cylindrical symmetry;
(iii) if, in addition, $d = m + 4$ or $d \geq m + 6$ and f is odd, then there exists $\lambda_M^* > 0$ such that for any $\lambda > \lambda_M^*$ Problem (P_λ) admits $s_{d,m}$ pairs of nontrivial weak solutions, with different symmetries ($s_{d,m}$ is defined by (1.4)).

It is an open problem to establish whether or not $\bar{\lambda}$ equals λ_E^* and if Problem (P_λ) has or not a nontrivial solution for $\lambda = \bar{\lambda}$ or $\lambda = \lambda_E^*$, as well as to check whether λ_E^* is exactly λ_M^* or not.

The proof of part (i) in Theorem 1.2 is a rather straightforward consequence of the definition of weak solution to (P_λ) and of the sublinear assumption (f_5) on f. While part (ii) and (iii) in Theorem 1.2 are just a byproduct of a more general existence and multiplicity result for the following problem, which actually can be used to study the stability of Problem (P_λ) with respect to changes of the nonlinearity:

$$\begin{cases} -\Delta u = \lambda\alpha(x, y) f(u) + \mu\beta(x, y) g(u) & \text{in } \omega \times \mathbb{R}^{d-m} \\ u = 0 & \text{on } \partial\omega \times \mathbb{R}^{d-m}, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where λ and μ are positive parameters, β verifies

$$\beta \in L^1(\omega \times \mathbb{R}^{d-m}) \cap L^\infty(\omega \times \mathbb{R}^{d-m}) \tag{β_1}$$

$$\beta(x, y) = \beta(x, |y|) \ \text{ a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m}\,, \tag{β_2}$$

and $g : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$g \text{ is continuous in } \mathbb{R} \tag{g_1}$$

$$\sup_{t \in \mathbb{R}\setminus\{0\}} \frac{|g(t)|}{|t| + |t|^{q-1}} < +\infty \text{ for some } q \in (2, 2^*). \tag{g_2}$$

With respect to Problem $(P_{\lambda,\mu})$, our main result reads as follows:

Theorem 1.3 *Let $\omega \times \mathbb{R}^{d-m}$ be an unbounded strip of $\mathbb{R}^d$, with ω open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$, $d, m \in \mathbb{N}$, $d \geq m + 2$, and let λ and μ be positive parameters. Let α satisfy (α_1), (α_2) and (α_3), and β satisfy (β_1) and (β_2). Let f satisfy (f_1), (f_2), (f_5) and (f_6) and g satisfy (g_1) and (g_2).*

Then,

- *(i) there exists $\lambda_E^{**} > 0$ such that for any $\lambda > \lambda_E^{**}$ there exists $\mu_{\lambda,E} > 0$ such that for any $\mu \in [0, \mu_{\lambda,E}]$ Problem $(P_{\lambda,\mu})$ admits at least two nontrivial weak solutions $u_{\lambda,\mu}$ and $\tilde{u}_{\lambda,\mu}$ in $H_0^1(\omega \times \mathbb{R}^{d-m})$ with cylindrical symmetry;*
- *(ii) if, in addition, $d = m + 4$ or $d \geq m + 6$ and f and g are odd, then there exists $\lambda_M^{**} > 0$ such that for any $\lambda > \lambda_M^{**}$ there exists $\mu_{\lambda,M} > 0$ such that for any $\mu \in [0, \mu_{\lambda,M}]$ Problem $(P_{\lambda,\mu})$ admits $s_{d,m}$ pairs of nontrivial weak solutions in $H_0^1(\omega \times \mathbb{R}^{d-m})$, with different symmetries ($s_{d,m}$ is given in (1.4)).*

The proof of Theorem 1.3 relies on an abstract critical points result due to Ricceri (see [25, Theorem 2] or Theorem 4.1 below) and again on the flower-shape geometric structure on the Sobolev space $H_0^1(\omega \times \mathbb{R}^{d-m})$ introduced in Sect. 2 and on the Principle of Symmetric Criticality.

We observe that, of course, λ_E^{**} and λ_M^{**} are greater than the constant $\overline{\lambda}$ in Theorem 1.2–(i). Moreover, Theorem 1.3 asserts that the existence result of solutions to Problem (P_λ) is stable with respect to small perturbations of the nonlinearity which are of superlinear and subcritical growth type.

The main theorems of this paper may be seen as an extension of existence and multiplicity results, already appeared in the literature, for nonlinear problems set in the entire space $\mathbb{R}^d$, as for instance, the ones obtained in the papers [5,6] due to Bartsch and Willem (see also [17]).

The techniques performed in this paper are new. Our abstract approach is in the spirit of the theoretical setting developed by Bartsch and Willem in [5,6], as well as of some recent contributions got in [13], see also the recent book [14] and references therein.

Several research perspective naturally arises exploiting the flower shape geometry constructed along the present paper: for instance, an interesting open problem is to investigate the existence of multiple solutions for nonlocal problems, as in [2], under the action of exterior topological groups (see, for instance, [12] for additional comments and related topics).

The present paper is organized as follows. In Sect. 2, we introduce the abstract setting which allows us to reveal a flower-shape geometry in the Sobolev space

$H_0^1(\omega \times \mathbb{R}^{d-m})$. In Sect. 3 we deal with the nonlinear Problem (P_λ) and we prove some existence, non-existence and multiplicity results for it, by using some classical theorems in critical points theory and the geometric construction given in Sect. 2. Section 4 is devoted to Problem ($P_{\lambda,\mu}$), which is a nonlinear perturbation of (P_λ) with sublinear growth, and to the proof of Theorem 1.3.

2 A Flower-Shape Geometry in Sobolev Spaces

In this section, we construct a flower-shape geometry in the Sobolev space $H_0^1(\omega \times \mathbb{R}^{d-m})$. Precisely, by using the orthogonal group in $\mathbb{R}^{d-m}$, $d, m \in \mathbb{N}$ with $d \geq m+2$, and its natural action on $H_0^1(\omega \times \mathbb{R}^{d-m})$, we define in $H_0^1(\omega \times \mathbb{R}^{d-m})$ a finite number of spaces, "mutually disjoint" and characterized by different symmetries, which are compactly embedded into the classical Lebesgue spaces. These properties will be crucial for getting the existence and multiplicity results for the nonlinear problems (P_λ) and ($P_{\lambda,\mu}$).

2.1 Preliminaries and Notations

In this subsection, we give some preliminaries and we introduce the notation used along the present paper. Here and in the sequel $H_0^1(\omega \times \mathbb{R}^{d-m})$ denotes the Sobolev space endowed with the inner product

$$\langle u, v\rangle_{H_0^1} := \int_{\omega\times\mathbb{R}^{d-m}} \nabla u(x,y)\cdot\nabla v(x,y)\,dx\,dy$$

and the norm

$$\|\cdot\|_{H_0^1} := \sqrt{\langle\cdot,\cdot\rangle_{H_0^1}}\,,$$

(indeed the Poincaré inequality holds in $H_0^1(\omega \times \mathbb{R}^{d-m})$, see, e.g. [7, Chapter IX, Remark 22] and [1, Chapter 6, 6.26]), while, given $1 \leq \nu \leq \infty$, $L^\nu(\omega \times \mathbb{R}^{d-m})$ is the classical Lebesgue space with norm defined as follows

$$\|u\|_\nu := \begin{cases} \left(\displaystyle\int_{\omega\times\mathbb{R}^{d-m}} |u(x,y)|^\nu\,dx\,dy\right)^{1/\nu} & \text{if } \nu \in [1,+\infty) \\ \inf\left\{C \geq 0 : |u(x,y)| \leq C \text{ a.e. } (x,y) \in \omega\times\mathbb{R}^{d-m}\right\} & \text{if } \nu = +\infty\,. \end{cases}$$

Since the embedding $H_0^1(\omega \times \mathbb{R}^{d-m}) \hookrightarrow L^\nu(\omega \times \mathbb{R}^{d-m})$ is continuous for any $\nu \in [2, 2^*]$, there exists $C_\nu > 0$ such that

$$\|u\|_\nu \leq C_\nu\|u\|_{H_0^1} \qquad \text{for any } u \in H_0^1(\omega \times \mathbb{R}^{d-m}). \tag{2.1}$$

Set $(O(d-m),\cdot)$ the orthogonal group in $\mathbb{R}^{d-m}$, we consider the group

$$\widehat{O}(d-m) := \{id_m\} \times O(d-m)$$

endowed with the natural multiplication law which maps any pair $(\widehat{g},\widehat{\tau}) \in \widehat{O}(d-m) \times \widehat{O}(d-m)$ into

$$\widehat{g}\cdot\widehat{\tau} := id_m \times (g\cdot\tau) \quad \text{for any } \widehat{g} = id_m \times g, \quad \widehat{\tau} = id_m \times \tau \in \widehat{O}(d-m)\,, \tag{2.2}$$

where $g\cdot\tau$ represents the product in $O(d-m)$ of g and τ. Here $\{id_m\}$ denotes the trivial group in $\mathbb{R}^m$ with the natural product and, from now on, in order to simplify the notation, we shall omit the $\cdot$ symbol.

The group $\widehat{O}(d-m)$ acts continuously and left-distributively on $\omega \times \mathbb{R}^{d-m}$ by the map

$$* : \widehat{O}(d-m) \times (\omega \times \mathbb{R}^{d-m}) \to \omega \times \mathbb{R}^{d-m}$$

defined by setting

$$\widehat{g} * (x,y) := (x, gy)\,,$$

for all $\widehat{g} = id_m \times g$ with $g \in O(d-m)$, and for $(x,y) \in \omega \times \mathbb{R}^{d-m}$.

The map $*$ induces the natural action

$$\sharp : \widehat{O}(d-m) \times H_0^1(\omega \times \mathbb{R}^{d-m}) \to H_0^1(\omega \times \mathbb{R}^{d-m})$$

of the group $\widehat{O}(d-m)$ on $H_0^1(\omega \times \mathbb{R}^{d-m})$, which maps any pair $(\widehat{g}, u) \in \widehat{O}(d-m) \times H_0^1(\omega \times \mathbb{R}^{d-m})$ into the function $\widehat{g}\sharp u \in H_0^1(\omega \times \mathbb{R}^{d-m})$ defined pointwise by setting for a.e. $(x,y) \in \omega \times \mathbb{R}^{d-m}$

$$\widehat{g}\sharp u(x,y) := u(x, g^{-1}y) \qquad \text{if } \widehat{g} = id_m \times g,\ g \in O(d-m), \tag{2.3}$$

i.e. in a more involved form,

$$\widehat{g}\sharp u(x,y) = u(\widehat{g^{-1}} * (x,y)).$$

Along the present paper we denote by $Fix_{\widehat{O}(d-m)}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ the set of points of $H_0^1(\omega \times \mathbb{R}^{d-m})$ which are fixed with respect to the action $\sharp$ of the group $\widehat{O}(d-m)$ on the space $H_0^1(\omega \times \mathbb{R}^{d-m})$, i.e.

$$\begin{aligned} &Fix_{\widehat{O}(d-m)}(H_0^1(\omega \times \mathbb{R}^{d-m})) \\ &\quad := \left\{ u \in H_0^1(\omega \times \mathbb{R}^{d-m}) : \widehat{g}\sharp u = u \text{ for any } \widehat{g} \in \widehat{O}(d-m) \right\}. \end{aligned} \tag{2.4}$$

We notice that $Fix_{\widehat{O}(d-m)}(H_0^1(\omega\times\mathbb{R}^{d-m}))$ is a linear subspace of $H_0^1(\omega\times\mathbb{R}^{d-m})$ and it is exactly the space $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ of cylindrically symmetric functions given by

$$\begin{aligned} &H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m}) \\ &:= \left\{u\in H_0^1(\omega\times\mathbb{R}^{d-m}) : u(x,y)=u(x,|y|) \text{ for a.e. } (x,y)\in\omega\times\mathbb{R}^{d-m}\right\}. \end{aligned} \tag{2.5}$$

We list below the following properties (see, the celebrated paper [10]) for the embedding of $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ into Lebesgue spaces:

$$H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m}) \hookrightarrow L^\nu(\omega\times\mathbb{R}^{d-m}) \text{ is continuous for any} \nu\in[2,2^*] \tag{2.6}$$

and

$$H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m}) \hookrightarrow L^\nu(\omega\times\mathbb{R}^{d-m}) \text{ is compact for any } \nu\in(2,2^*)\,. \tag{2.7}$$

Now, let either $d=m+4$ or $d\geq m+6$, so that the set $I_{d,m}$ defined by (1.1) is nonempty. Then, for any $i\in I_{d,m}$, by "grouping together" the $d-m$ variables of the unbounded part of the strip in blocks of at least two variables, we get $\tau_{d,m}=\mathrm{card}(I_{d,m})$ subgroups of $O(d-m)$

$$H_{d,m,i} := \begin{cases} O((d-m)/2)\times O((d-m)/2) & \text{if } i=\dfrac{d-m-2}{2} \\ O(i+1)\times O(d-m-2i-2)\times O(i+1) & \text{if } i\neq\dfrac{d-m-2}{2} \end{cases},$$

which define the subgroups

$$\widehat{H}_{d,m,i} := \{id_m\}\times H_{d,m,i}\subset\widehat{O}(d-m)\,.$$

The sets

$$Fix_{\widehat{H}_{d,m,i}}(H_0^1(\omega\times\mathbb{R}^{d-m})) := \left\{u\in H_0^1(\omega\times\mathbb{R}^{d-m}) \ : \ \widehat{g}\sharp u=u \text{ for all } \widehat{g}\in\widehat{H}_{d,m,i}\right\}$$

are known in the literature as the subspaces of block-radial (or block-cylindrical) functions of $H_0^1(\omega\times\mathbb{R}^{d-m})$ and they are compactly embedded into $L^\nu(\omega\times\mathbb{R}^{d-m})$ for every $\nu\in(2,2^*)$ (see [15, Théorème III.2]). Unfortunately, these $\tau_{d,m}$ subspaces are not "mutually disjoint" and, in term of our problems (P_λ) and $(P_{\lambda,\mu})$, this is an obstacle to get a multiplicity result for them.

In order to overcome this difficulty, for any $i \in I_{d,m} := \{1, ..., \tau_{d,m}\}$, we define the involution $\eta_{d,m,i} : \mathbb{R}^{d-m} \to \mathbb{R}^{d-m}$ as follows

$$\eta_{d,m,i}(y) := \begin{cases} (y_3, y_1) & \text{if } i = \dfrac{d-m-2}{2} \text{ and} y := (y_1, y_3) \in \mathbb{R}^{(d-m)/2} \times \mathbb{R}^{(d-m)/2} \\ (y_3, y_2, y_1) & \text{if} i \neq \dfrac{d-m-2}{2} \text{ and } y := (y_1, y_2, y_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{d-m-2i-2} \times \mathbb{R}^{i+1} \end{cases} \tag{2.8}$$

and we set

$$\widehat{\eta}_{d,m,i} := id_m \times \eta_{d,m,i}. \tag{2.9}$$

By (2.2) it is easily seen that for any $i \in I_{d,m}$

- $\widehat{\eta}_{d,m,i} \in \widehat{O}(d-m)$
- $\widehat{\eta}^2_{d,m,i} = \mathrm{id}_{\omega \times \mathbb{R}^{d-m}}$
- $\widehat{\eta}_{d,m,i} \notin \widehat{H}_{d,m,i}$
- $\widehat{\eta}_{d,m,i} \widehat{H}_{d,m,i} \widehat{\eta}^{-1}_{d,m,i} = \widehat{H}_{d,m,i}$,

where

$$\widehat{\eta}_{d,m,i} \widehat{H}_{d,m,i} \widehat{\eta}^{-1}_{d,m,i} := \left\{ \widehat{\eta}_{d,m,i} \widehat{h} \widehat{\eta}^{-1}_{d,m,i} : \widehat{h} \in \widehat{H}_{d,m,i} \right\}.$$

Finally, for every $i \in I_{d,m}$, we consider the compact subgroup of $\widehat{O}(d-m)$ given by

$$\widehat{H}_{d,m,\widehat{\eta}_i} := \langle \widehat{H}_{d,m,i}, \widehat{\eta}_{d,m,i} \rangle$$

(here, to short notation, $\widehat{\eta}_i$ stands for $\widehat{\eta}_{d,m,i}$), that is

$$\widehat{H}_{d,m,\widehat{\eta}_i} = \widehat{H}_{d,m,i} \cup \left(\widehat{\eta}_{d,m,i} \widehat{H}_{d,m,i} \right), \tag{2.10}$$

and the action

$$\circledast_i : \widehat{H}_{d,m,\widehat{\eta}_i} \times H^1_0(\omega \times \mathbb{R}^{d-m}) \to H^1_0(\omega \times \mathbb{R}^{d-m})$$

of $\widehat{H}_{d,m,\widehat{\eta}_i}$ on $H^1_0(\omega \times \mathbb{R}^{d-m})$ defined by setting

$$\widehat{h} \circledast_i u(x, y) := \begin{cases} u(x, h^{-1} y) & \text{if } \widehat{h} := id_m \times h \in \widehat{H}_{d,m,i} \\ -u(x, g^{-1} \eta^{-1}_{d,m,i} y) & \text{if } \widehat{h} = id_m \times \eta_{d,m,i} g \in \widehat{H}_{d,m,\widehat{\eta}_i} \setminus \widehat{H}_{d,m,i},\ g \in H_{d,m,i}, \end{cases} \tag{2.11}$$

for a.e. $(x, y) \in \omega \times \mathbb{R}^{d-m}$.

Bearing in mind (2.3) and fixing $i \in I_{d,m}$, the action $\circledast_i$ can be written, for a.e. $(x, y) \in \omega \times \mathbb{R}^{d-m}$, as follows

$$\widehat{h} \circledast_i u(x,y) := \begin{cases} \widehat{h}\sharp u(x,y) & \text{if } \widehat{h} \in \widehat{H}_{d,m,i} \\ -(\widehat{\eta}_{d,m,i}\widehat{g})\sharp u(x,y) & \text{if } \widehat{\eta}_{d,m,i}\widehat{g} \in \widehat{H}_{d,m,\widehat{\eta}_i} \setminus \widehat{H}_{d,m,i}. \end{cases} \tag{2.12}$$

We would observe that $\circledast_i$ is defined for every element of $\widehat{H}_{d,m,\widehat{\eta}_i}$. Indeed, if $\widehat{h} \in \widehat{H}_{d,m,\widehat{\eta}_i}$, then either $\widehat{h} \in \widehat{H}_{d,m,i}$ or $\widehat{h} = id_m \times \eta_{d,m,i} g \in \widehat{H}_{d,m,\widehat{\eta}_i} \setminus \widehat{H}_{d,m,i}$, with $g \in H_{d,m,i}$.

Now, we are ready to introduce, for any $i \in I_{d,m}$, the set $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$ of points of $H_0^1(\omega\times\mathbb{R}^{d-m})$ which are fixed with respect to the action $\circledast_i$ of the group $\widehat{H}_{d,m,\widehat{\eta}_i}$, i.e.

$$Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m})) := \left\{u \in H_0^1(\omega\times\mathbb{R}^{d-m}) : \widehat{h} \circledast_i u = u \text{ for any } \widehat{h} \in \widehat{H}_{d,m,\widehat{\eta}_i}\right\}. \tag{2.13}$$

It is easy to see that each set $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$ is a nontrivial linear subspace of $H_0^1(\omega\times\mathbb{R}^{d-m})$. In the next subsection, we prove some interesting properties of this space.

2.2 Compactness and Symmetries

In this subsection, we show that each one of the $\tau_{d,m}$ spaces $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$ is compactly embedded in the Lebesgue space $L^\nu(\omega\times\mathbb{R}^{d-m})$ for any $\nu \in (2, 2^*)$ and we prove some geometric properties for them.

Remark 2.1 We recall that "symmetry" allows to recover compactness when it involves at least two variables. So, any block of variables on which one asks for symmetry should be at least of dimension 2. Thus, the simplest possible setting is the block-radial symmetry in four dimensional Euclidean space with two 2-dimensional blocks. This justifies the requirement $d \geq m+4$ all along this subsection.

With respect to the compactness result, we get that (2.6) and (2.7) hold if we replace $H_{0,\mathrm{cyl}}^1(\omega\times\mathbb{R}^{d-m})$ with $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$. Precisely, our result reads as follows:

Proposition 2.2 *Let either $d = m+4$ or $d \geq m+6$, $d, m \in \mathbb{N}$. Let $i \in I_{d,m}$ and $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$ be defined as in (2.13).*

Then, the embedding

$$Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m})) \hookrightarrow L^\nu(\omega\times\mathbb{R}^{d-m})$$

- *is continuous for any $\nu \in [2, 2^*]$*
- *is compact for any $\nu \in (2, 2^*)$.*

Proof Let us fix $i \in I_{d,m}$. Since $\widehat{H}_{d,m,i} \subset \widehat{H}_{d,m,\widehat{\eta}_i}$, the first relation of (2.11) (or, equivalently, of (2.12)) and the continuity of the action $\circledast_i$ imply that $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ is a closed subspace of the space of block-radial functions $Fix_{\widehat{H}_{d,m,i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$.

Furthermore, the space $Fix_{\widehat{H}_{d,m,i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ is continuously embedded in $L^\nu(\omega \times \mathbb{R}^{d-m})$ for any $\nu \in [2, 2^*]$ and is compactly embedded in $L^\nu(\omega \times \mathbb{R}^{d-m})$ for any $\nu \in (2, 2^*)$ (see [15, Théorème III.2]). Hence, the embedding

$$Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \hookrightarrow L^\nu(\omega \times \mathbb{R}^{d-m})$$

is also continuous for any $\nu \in [2, 2^*]$ and compact for any $\nu \in (2, 2^*)$ and this ends the proof of Proposition 2.2. □

Now, we prove that the subspaces $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ are mutually disjoint, as stated here below. A key point along the proof of Proposition 2.3 is the transitive action of the subgroups $\langle H_{d,m,i}, H_{d,m,j}\rangle \subset O(d-m)$ on the Euclidean unit sphere $\mathbb{S}^{d-m-1} \subset \mathbb{R}^{d-m}$ and the structure of the action of $\widehat{H}_{d,m,\widehat{\eta}_i}$ on $H_0^1(\omega \times \mathbb{R}^{d-m})$ defined in (2.11).

Proposition 2.3 *Let either $d = m+4$ or $d \geq m+6$, $d, m \in \mathbb{N}$. Let $i \in I_{d,m}$ and $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ be defined as in (2.13).*

Then, the following statements hold:

(i) if $d = m+4$ or $d \geq m+6$, then

$$Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \cap H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) = \{0\}$$

for any $i \in I_{d,m}$;

(ii) if $d = m+6$ or $d \geq m+8$, then

$$Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \cap Fix_{\widehat{H}_{d,m,\widehat{\eta}_j}}(H_0^1(\omega \times \mathbb{R}^{d-m})) = \{0\}$$

for any $i, j \in I_{d,m}$ with $i \neq j$.

Proof Let us prove assertion (i). Fix $i \in I_{d,m}$ and let $u \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \cap H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$. Since u is $\widehat{H}_{d,m,\widehat{\eta}_i}$–invariant, taking into account (2.11) we have

$$u(x, y) = -u(x, g^{-1}\eta^{-1}_{d,m,i} y) \qquad \text{for a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m} \text{ and for all } g \in H_{d,m,i}. \tag{2.14}$$

Moreover, since u is radial in the second component, i.e. $u(x, y) = u(x, |y|)$, and $|y| = |g^{-1}\eta^{-1}_{d,m,i} y|$ for every $y \in \mathbb{R}^{d-m}$, by (2.14) we have that $u(x, y) = -u(x, y)$ for a.e. $(x, y) \in \omega \times \mathbb{R}^{d-m}$ and so u must be identically zero in $\omega \times \mathbb{R}^{d-m}$.

Now let us show assertion (ii). Let $d = m+6$ or $d \geq m+8$ so that $\tau_{d,m} \geq 2$. Then, fix $i, j \in I_{d,m}$, with $i < j$, and $u \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \cap Fix_{\widehat{H}_{d,m,\widehat{\eta}_j}}(H_0^1(\omega \times$

$\mathbb{R}^{d-m}$)). Since, as can be easily seen, the function u is both $\widehat{H}_{d,m,i}$–, and $\widehat{H}_{d,m,j}$–invariant, we deduce that u is also $\langle \widehat{H}_{d,m,i}, \widehat{H}_{d,m,j}\rangle$–invariant, that is

$$u(x, y) = u(\widehat{g}_{ij} \circledast_{ij} (x, y)) \tag{2.15}$$

for every $\widehat{g}_{ij} \in \langle \widehat{H}_{d,m,i}, \widehat{H}_{d,m,j}\rangle$ and for a.e. $(x, y) \in \omega \times \mathbb{R}^{d-m}$, where $\circledast_{ij}$ denotes the natural action of the group $\langle \widehat{H}_{d,m,i}, \widehat{H}_{d,m,j}\rangle$ on $\omega \times \mathbb{R}^{d-m}$ induced by $\circledast_i$ and $\circledast_j$.

Now, as proved in [13, Theorem 2.2 - Part (ii)], the group $\langle H_{d,m,i}, H_{d,m,j}\rangle$ acts transitively on the sphere $\mathbb{S}^{d-m-1}$. Hence, for any $(x, y) \in \omega \times \mathbb{R}^{d-m}$

$$\langle \widehat{H}_{d,m,i}, \widehat{H}_{d,m,j}\rangle(x, y) = \{x\} \times |y|\mathbb{S}^{d-m-1}.$$

As a consequence of (2.15), the function u is cylindrically symmetric, and we can apply (i) thus obtaining that u is identically zero in $\omega \times \mathbb{R}^{d-m}$. This concludes the proof of Proposition 2.3. □

We suggest the recent monograph [22] as a comprehensive reference for preliminaries and, in particular, for the main properties related to Sobolev spaces.

3 Dirichlet Problems on Strip-Like Domains

This section is devoted to the study of the nonlinear Problem (P_λ), under either **superlinear** assumption on the nonlinearity f at infinity (see (f_3)) or **sublinear** condition again at infinity (see (f_5)).

As already remarked, since the equation in (P_λ) has a variational nature, its weak solutions can be seen as critical points of the energy functional $\mathcal{I}_\lambda$ defined by (1.3). It is standard to see that, thanks to (α_1) and (f_2), (f_4) in the superlinear setting or (f_2), (f_5) (note that (f_2) and (f_5) imply (f_4)) in the sublinear framework, the functional $\mathcal{I}_\lambda$ is well defined on $H_0^1(\omega \times \mathbb{R}^{d-m})$, and that $\mathcal{I}_\lambda \in C^1(H_0^1(\omega \times \mathbb{R}^{d-m}))$ with

$$\begin{aligned}\langle \mathcal{I}_\lambda'(u), \varphi\rangle = &\int_{\omega\times\mathbb{R}^{d-m}} \nabla u(x, y)\nabla\varphi(x, y)\, dx\, dy \\ &-\lambda \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x, y) f(u(x, y))\varphi(x, y)\, dx\, dy\end{aligned} \tag{3.1}$$

for any $u, \varphi \in H_0^1(\omega \times \mathbb{R}^{d-m})$.

We shall prove in Sect. 3.1 and in Sect. 3.2, respectively, the existence and multiplicity results stated in Theorem 1.1 and in Theorem 1.2.

3.1 Problem (P_λ) with Superlinear Growth at Infinity

In this subsection, we study the semilinear equation (P_λ), when f satisfies the Ambrosetti–Rabinowitz condition (f_3). The main tools are given by the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [3,23]), the Principle of Symmetric

Criticality of Palais (see [21]) and the flower-shape geometry in the Sobolev space $H_0^1(\omega\times\mathbb{R}^{d-m})$ introduced in Sect. 2.

This subsection is devoted to the proof of Theorem 1.1: in particular in Sect. 3.1.1 we prove the existence result (i), while in Sect. 3.1.2 we prove the multiplicity result (ii) of nontrivial weak solutions to Problem (P_λ).

3.1.1 A Mountain Pass Existence Result for Problem (P_λ)

In this subsection, we prove the existence result stated in Theorem 1.1, by applying the Mountain Pass Theorem to the energy functional $\mathcal{I}_\lambda$ defined in (1.3).

As it is well known, in order to follow this strategy, it is necessary to have some compactness properties on the functional, and so we shall exploit (2.7) by working, with fixed $\lambda>0$, with the functional $\mathcal{J}_\lambda$ defined as the restriction of $\mathcal{I}_\lambda$ to the space $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$, i.e.

$$\mathcal{J}_\lambda(u):=(\mathcal{I}_\lambda)_{|\,H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})}(u),\ \ u\in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})\,.$$

The main ingredients of our proof are the application of the following results:

- the Mountain Pass Theorem by Ambrosetti and Rabinowitz (see [3]) to get a critical point $u_\lambda\in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ for the functional $\mathcal{J}_\lambda$;
- the Principle of Symmetric Criticality by Palais (see [21]) to prove that $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ is a natural constraint for the functional $\mathcal{I}_\lambda$, i.e. critical points of $\mathcal{I}_\lambda$ constrained on $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ are actually critical points of $\mathcal{I}_\lambda$ in $H_0^1(\omega\times\mathbb{R}^{d-m})$.

First of all, let us show that $\mathcal{J}_\lambda$ satisfies the geometric Mountain Pass structure. For this, note that by conditions (f_2) and (f_4), it is standard to see that for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that for any $t\in\mathbb{R}$

$$|f(t)|\le\varepsilon|t|+\delta(\varepsilon)|t|^{q-1} \tag{3.2}$$

and, as a consequence, such that

$$|F(t)|\le\frac{\varepsilon}{2}|t|^2+\frac{\delta(\varepsilon)}{q}|t|^q\,. \tag{3.3}$$

Now, let us proceed by steps.

Claim 3.1.1 *There exist $\rho>0$ and $\gamma_\rho>0$ such that $\mathcal{J}_\lambda(u)\ge\gamma_\rho$ for any $u\in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ with $\|u\|_{H_0^1}=\rho$.*

Proof Let u be a function in $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$. By ($\alpha_1$), ($\alpha_3$), (2.6), (3.3) and the positivity of λ, we get that for any $\varepsilon > 0$

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geq \frac{1}{2}\|u\|^2_{H^1_0} - \frac{\varepsilon\lambda}{2}\|\alpha\|_\infty\|u\|_2^2 - \frac{\delta(\varepsilon)\lambda}{q}\|\alpha\|_\infty\|u\|_q^q \\ &\geq \frac{1}{2}\Big(1 - \varepsilon\lambda C_2^2\|\alpha\|_\infty\Big)\|u\|^2_{H^1_0} - \frac{\delta(\varepsilon)\lambda C_q^q}{q}\|\alpha\|_\infty\|u\|^q_{H^1_0} \\ &= \|u\|^2_{H^1_0}\left[\frac{1}{2}\Big(1 - \varepsilon\lambda C_2^2\|\alpha\|_\infty\Big) - \frac{\delta(\varepsilon)\lambda C_q^q}{q}\|\alpha\|_\infty\|u\|^{q-2}_{H^1_0}\right],\end{aligned} \tag{3.4}$$

where the constant C_2 (resp. C_q) is the constant C_ν in (2.1) with $\nu = 2$ (resp. $\nu = q$).

By choosing $\varepsilon > 0$ small enough to have $\varepsilon\lambda C_2^2\|\alpha\|_\infty < 1$, we get that there exist suitable positive constants $\bar{\kappa}$ and $\tilde{\kappa}$ such that

$$\inf_{\substack{u \in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m}) \\ \|u\|_{H^1_0} = \rho}} \mathcal{J}_\lambda(u) \geq \rho^2\big(\bar{\kappa} - \tilde{\kappa}\rho^{q-2}\big) =: \gamma_\rho > 0\,, \tag{3.5}$$

provided ρ is sufficiently small (i.e. ρ such that $\bar{\kappa} - \tilde{\kappa}\rho^{q-2} > 0$). Hence, Claim 3.1.1 is proved. □

Claim 3.1.2 *There exists a strictly positive function $\overline{u} \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ such that $\|\overline{u}\|_{H^1_0} > \rho$ and $\mathcal{J}_\lambda(\overline{u}) < \gamma_\rho$, where ρ and γ_ρ are given in Claim 3.1.1.*

Proof First of all, note that as a consequence of (f_1) and (f_3), we easily have that there exist two positive constants a_1 and a_2 such that

$$F(t) \geq a_1|t|^\sigma - a_2 \qquad \text{for any } t \in \mathbb{R}\,. \tag{3.6}$$

Let $u \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ be such that $\|u\|_{H^1_0} = 1$ and $u > 0$ a.e. in $\omega \times \mathbb{R}^{d-m}$ and let $s > 0$. By (α_1), (α_3) and (3.6), we have, since $\lambda > 0$, that

$$\begin{aligned}\mathcal{J}_\lambda(su) &= \frac{s^2}{2}\|u\|^2_{H^1_0} - \lambda\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y)F(su(x,y))\,dx\,dy \\ &\leq \frac{s^2}{2} - \lambda a_1\alpha_0 s^\sigma\|u\|^\sigma_\sigma + \lambda a_2\|\alpha\|_1\,.\end{aligned} \tag{3.7}$$

Since $\sigma > 2$, passing to the limit as $s \to +\infty$, we get that $\mathcal{J}_\lambda(su) \to -\infty$, so that Claim 3.1.2 follows taking $\overline{u} = \bar{s}u$, with $\bar{s}$ sufficiently large. □

Claim 3.1.3 *The functional $\mathcal{J}_\lambda$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$, that is for any sequence $(u_k)_k$ in $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ such that, as $k \to +\infty$,*

$$\mathcal{J}_\lambda(u_k) \to c \tag{3.8}$$

and

$$\sup\left\{\left|\langle\,\mathcal{J}_\lambda'(u_k),\varphi\,\rangle\right|\;:\;\varphi\in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})\,,\,\|\varphi\|_{H^1_0}=1\right\}\to 0\,, \tag{3.9}$$

there exists $u_\infty\in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ *such that, up to a subsequence,*

$$\|u_k-u_\infty\|_{H^1_0}\to 0\qquad as\ k\to+\infty\,. \tag{3.10}$$

Proof Let $(u_k)_k$ be a Palais–Smale sequence for $\mathcal{J}_\lambda$, i.e. a sequence satisfying (3.8) and (3.9) for some fixed $c\in\mathbb{R}$. First of all, let us prove that $(u_k)_k$ is bounded in $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$. At this purpose, note that, by (3.8) and (3.9), it easily follows that

$$\mathcal{J}_\lambda(u_k)-\frac{1}{\sigma}\langle\mathcal{J}_\lambda'(u_k),u_k\rangle\le\kappa\left(1+\|u_k\|_{H^1_0}\right)\qquad\text{for any }k\in\mathbb{N}, \tag{3.11}$$

for a suitable positive constant κ, where σ is the constant in (f_3).

Moreover, thanks to (α_3) and (f_3), we get that, for any $k\in\mathbb{N}$,

$$\begin{aligned}\mathcal{J}_\lambda(u_k)-\frac{1}{\sigma}\langle\mathcal{J}_\lambda'(u_k),u_k\rangle&=\left(\frac12-\frac1\sigma\right)\|u_k\|^2_{H^1_0}\\&\quad-\frac{\lambda}{\sigma}\int_{\omega\times\mathbb{R}^{d-m}}\alpha(x,y)\Big(\sigma F(u_k(x,y))-f(u_k(x,y))\,u_k(x,y)\Big)\,dx\,dy\\&\ge\left(\frac12-\frac1\sigma\right)\|u_k\|^2_{H^1_0}\,.\end{aligned} \tag{3.12}$$

So, by combining (3.11) and (3.12) we get, for a suitable positive constant κ_*, that

$$\|u_k\|^2_{H^1_0}\le\kappa_*\left(1+\|u_k\|_{H^1_0}\right)\qquad\text{for any }k\in\mathbb{N}\,.$$

Hence, the sequence $(u_k)_k$ is bounded in $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ and so, by definition of $\mathcal{J}_\lambda$ and (3.9), we have that, as $k\to+\infty$,

$$0\leftarrow\langle\mathcal{J}_\lambda'(u_k),u_k\rangle=\|u_k\|^2_{H^1_0}-\lambda\int_{\omega\times\mathbb{R}^{d-m}}\alpha(x,y)f(u_k(x,y))u_k(x,y)\,dx\,dy\,. \tag{3.13}$$

Since $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ is a reflexive space, we also get, up to a subsequence, still denoted by $(u_k)_k$, that there exists $u_\infty\in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ such that

$$u_k\to u_\infty\quad\text{weakly in }H^1_0(\omega\times\mathbb{R}^{d-m})\qquad\text{as }k\to+\infty\,. \tag{3.14}$$

Moreover, by applying the compact embedding (2.7), we get, again up to a subsequence still denoted by $(u_k)_k$, that,

$$u_k\to u_\infty\quad\text{in }L^\nu(\omega\times\mathbb{R}^{d-m})\qquad\text{as }k\to+\infty\qquad\text{for any }\nu\in(2,\,2^*)\,, \tag{3.15}$$

and, as a consequence, that

$$u_k \to u_\infty \quad \text{a.e. in } \omega \times \mathbb{R}^{d-m} \qquad \text{as } k \to +\infty\,, \tag{3.16}$$

while, by using the continuous embedding (2.6), we deduce that there exist two positive constants κ_2 and κ_{2^*} such that

$$\|u_k\|_2 \le \kappa_2 \text{ and } \|u_k\|_{2^*} \le \kappa_{2^*} \qquad \text{for any } k \in \mathbb{N}\,. \tag{3.17}$$

Now, we claim that, as $k \to +\infty$,

$$\begin{aligned}&\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y) f(u_k(x,y)) u_\infty(x,y)\,dx\,dy \\ &\quad \to \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y) f(u_\infty(x,y)) u_\infty(x,y)\,dx\,dy.\end{aligned} \tag{3.18}$$

Indeed, by (f_1) and (3.16), we get that

$$f(u_k(\cdot)) \to f(u_\infty(\cdot)) \quad \text{a.e. in } \omega \times \mathbb{R}^{d-m} \qquad \text{as } k \to +\infty. \tag{3.19}$$

Moreover, since α satisfies condition (α_1), it is easy to see (since $|\alpha|^\nu = |\alpha|^{\nu-1}|\alpha| \le \|\alpha\|_\infty^{\nu-1}|\alpha|$ in $\omega \times \mathbb{R}^{d-m}$) that

$$\alpha \in L^\nu(\omega \times \mathbb{R}^{d-m}) \text{ for any } \nu \in [1, +\infty]. \tag{3.20}$$

Now, by (2.1), (3.2) with $\varepsilon = 1$ and set $\delta := \delta(1)$, by (3.20) and the Hőlder Inequality, we have that, set $q' := q/(q-1)$, the conjugate exponent of q,

$$\begin{aligned}&\int_{\omega\times\mathbb{R}^{d-m}} \Big|\alpha(x,y) f(u_k(x,y))\Big|^{q'}\,dx\,dy \\ &\le \int_{\omega\times\mathbb{R}^{d-m}} \big|\alpha(x,y)\big|^{q'} \Big(|u_k(x,y)| + \delta|u_k(x,y)|^{q-1}\Big)^{q'}\,dx\,dy \\ &\le 2^{q'-1}\Big(\int_{\omega\times\mathbb{R}^{d-m}} \big|\alpha(x,y)\big|^{q'} |u_k(x,y)|^{q'}\,dx\,dy \\ &\quad + \delta^{q'} \int_{\omega\times\mathbb{R}^{d-m}} \big|\alpha(x,y)\big|^{q'} |u_k(x,y)|^{q}\,dx\,dy\Big) \\ &\le 2^{1/(q-1)}\|\alpha\|_{q/(q-2)}^{q'}\|u_k\|_q^{q'} + 2^{1/(q-1)}\delta^{q'}\|\alpha\|_\infty^{q'}\|u_k\|_q^q \\ &\le 2^{1/(q-1)} C_q^{q'}\|\alpha\|_{q/(q-2)}^{q'}\|u_k\|_{H_0^1}^{q'} + 2^{1/(q-1)}\delta^{q'} C_q^q\|\alpha\|_\infty^{q'}\|u_k\|_{H_0^1}^q\end{aligned} \tag{3.21}$$

for any $k \in \mathbb{N}$, where C_q is the constant given in (2.1) with $\nu = q$. Since $(u_k)_k$ is bounded in $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$, by (3.21) we deduce that the sequence $\Big(\alpha(\cdot) f(u_k(\cdot))\Big)_k$

is bounded in $L^{q'}(\omega \times \mathbb{R}^{d-m})$, which, together with (3.19), yields that

$$\alpha(\cdot) f(u_k(\cdot)) \to \alpha(\cdot) f(u_\infty(\cdot)) \text{ weakly in } L^{q'}(\omega \times \mathbb{R}^{d-m}) \tag{3.22}$$

as $k \to +\infty$. Then, we get (3.18) by testing this weak convergence with u_∞.

Now, we claim that, as $k \to +\infty$,

$$\int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_k(x, y)) \Big(u_k(x, y) - u_\infty(x, y) \Big) \, dx \, dy \to 0 \,. \tag{3.23}$$

Indeed, by (3.2), the Hőlder Inequality and (3.17), we have that

$$\begin{aligned} &\Big| \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_k(x, y)) \Big(u_k(x, y) - u_\infty(x, y) \Big) \, dx \, dy \Big| \\ &\quad \le \varepsilon \|\alpha\|_\infty \int_{\omega \times \mathbb{R}^{d-m}} |u_k(x, y)| \, |u_k(x, y) - u_\infty(x, y)| \, dx \, dy \\ &\qquad + \delta(\varepsilon) \|\alpha\|_\infty \int_{\omega \times \mathbb{R}^{d-m}} |u_k(x, y)|^{q-1} |u_k(x, y) - u_\infty(x, y)| \, dx \, dy \\ &\quad \le \varepsilon \|\alpha\|_\infty \|u_k\|_2 \|u_k - u_\infty\|_2 + \delta(\varepsilon) \|\alpha\|_\infty \|u_k\|_q^{q-1} \|u_k - u_\infty\|_q \\ &\quad \le \varepsilon \|\alpha\|_\infty \kappa_2 (\kappa_2 + \|u_\infty\|_2) + \delta(\varepsilon) \|\alpha\|_\infty \|u_k\|_q^{q-1} \|u_k - u_\infty\|_q \,. \end{aligned} \tag{3.24}$$

By (3.15), (3.24) and the arbitrariness of ε, we get (3.23).

Finally, (3.18) and (3.23) yield that, as $k \to +\infty$,

$$\begin{aligned} &\int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_k(x, y)) u_k(x, y) \, dx \, dy \\ &\qquad \to \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_\infty(x, y)) u_\infty(x, y) \, dx \, dy \,. \end{aligned} \tag{3.25}$$

Now, we are in position to conclude our proof. Indeed, as a consequence of (3.13) and (3.25) we deduce that

$$\|u_k\|_{H_0^1}^2 \to \lambda \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_\infty(x, y)) u_\infty(x, y) \, dx \, dy \qquad \text{as } k \to +\infty \,. \tag{3.26}$$

Furthermore,

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}_\lambda'(u_k), u_\infty \rangle = &\int_{\omega \times \mathbb{R}^{d-m}} \nabla u_k(x, y) \nabla u_\infty(x, y) \, dx \, dy \\ &- \lambda \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_k(x, y)) u_\infty(x, y) \, dx \, dy \end{aligned} \tag{3.27}$$

as $k \to +\infty$. So, by using (3.14) and (3.18) in (3.27), we obtain

$$\|u_\infty\|_{H_0^1}^2 = \lambda \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) f(u_\infty(x, y)) u_\infty(x, y) \, dx \, dy \,. \tag{3.28}$$

Therefore, (3.26) and (3.28) state that

$$\|u_k\|_{H_0^1} \to \|u_\infty\|_{H_0^1} \qquad \text{as } k \to +\infty. \tag{3.29}$$

Finally, thanks to (3.14) and (3.29), we have that

$$\begin{aligned} \|u_k - u_\infty\|^2_{H_0^1} &= \|u_k\|^2_{H_0^1} + \|u_\infty\|^2_{H_0^1} - 2\int_{\omega\times\mathbb{R}^{d-m}} \nabla u_k(x,y)\nabla u_\infty(x,y)\,dx\,dy \\ &\to 2\|u_\infty\|^2_{H_0^1} - 2\|u_\infty\|^2_{H_0^1} = 0 \qquad \text{as } k \to +\infty\,, \end{aligned} \tag{3.30}$$

and this concludes the proof of Claim 3.1.3. □

Now, we are ready to provide the proof of the existence result stated in Theorem 1.1.

Proof of Theorem 1.1 Let $\lambda > 0$ be fixed. Thanks to Claim 3.1.1, Claim 3.1.2 and Claim 3.1.3 we get, by applying the Mountain Pass Theorem, the existence of $u_\lambda \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$, $u_\lambda \not\equiv 0$ (indeed, by (3.5) $\mathcal{J}_\lambda(u_\lambda) \geq \gamma_\rho > 0 = \mathcal{J}_\lambda(0)$), which is a critical point for the functional $\mathcal{J}_\lambda$, i.e. such that

$$\begin{aligned} &\int_{\omega\times\mathbb{R}^{d-m}} \nabla u_\lambda(x,y)\nabla\varphi(x,y)\,dx\,dy \\ &\quad -\lambda\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y)f(u_\lambda(x,y))\varphi(x,y)\,dx\,dy = 0 \end{aligned}$$

for any $\varphi \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$. Hence, u_λ is a constrained critical point of $\mathcal{I}_\lambda$ on $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$.

Finally, it remains to prove that $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ is a natural constraint for $\mathcal{I}_\lambda$. This is an easy consequence of the Principle of Symmetric Criticality by Palais. Indeed, thanks to the fact that $O(d-m)$ is the orthogonal group in $\mathbb{R}^{d-m}$, it is easy to see that the action $\sharp$, defined in (2.3), of the group $\widehat{O}(d-m)$ is an *isometry* on $H^1_0(\omega\times\mathbb{R}^{d-m})$. Indeed, for any $\widehat{g} \in \widehat{O}(d-m)$, with $\widehat{g} = id_m \times g$, $g \in O(d-m)$, and for any $u \in H^1_0(\omega \times \mathbb{R}^{d-m})$, we have (by changing the integration variable)

$$\begin{aligned} \|\widehat{g}\sharp u\|^2_{H_0^1} &= \int_{\omega\times\mathbb{R}^{d-m}} |\nabla u(x, g^{-1}y)|^2\,dx\,dy \\ &= \int_{\omega\times\mathbb{R}^{d-m}} |\nabla u(x,y)|^2\,dx\,dy = \|u\|^2_{H_0^1}. \end{aligned} \tag{3.31}$$

Moreover, by using (α_2) and again the fact that $O(d-m)$ is the orthogonal group in $\mathbb{R}^{d-m}$, we get that $\mathcal{I}_\lambda$ is *invariant* with respect to $\widehat{O}(d-m)$. Indeed, since $|y| = |gy|$ for all $y \in \mathbb{R}^{d-m}$ and $g \in O(d-m)$, by (α_2), we get that

$$\alpha(x,y) = \alpha(x,|y|) = \alpha(x,gy) \ \text{ a.e. } (x,y) \in \omega\times\mathbb{R}^{d-m} \ \text{ and for any } g \in O(d-m)\,, \tag{3.32}$$

and so, as a consequence, we deduce that, for any $\widehat{g} \in \widehat{O}(d-m)$, with $\widehat{g} = id_m \times g$, $g \in O(d-m)$, and for any $u \in H_0^1(\omega \times \mathbb{R}^{d-m})$,

$$\begin{aligned}\mathcal{I}_\lambda(\widehat{g}\sharp u) &= \frac{1}{2}\int_{\omega\times\mathbb{R}^{d-m}} |\nabla u(x, g^{-1}y)|^2\,dx\,dy \\ &\quad - \lambda\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y)F(u(x,g^{-1}y))\,dx\,dy \\ &= \frac{1}{2}\int_{\omega\times\mathbb{R}^{d-m}} |\nabla u(x,y')|^2\,dx\,dy' \\ &\quad - \lambda\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,gy')F(u(x,y'))\,dx\,dy' \\ &= \frac{1}{2}\int_{\omega\times\mathbb{R}^{d-m}} |\nabla u(x,y)|^2\,dx\,dy \\ &\quad - \lambda\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y)F(u(x,y))\,dx\,dy \\ &= \mathcal{I}_\lambda(u).\end{aligned} \tag{3.33}$$

Hence, by (3.31) and (3.33) we obtain, by the Principle of Symmetric Criticality, that u_λ is a critical point of $\mathcal{I}_\lambda$. Then, we have shown the existence of a nontrivial weak solution u_λ to Problem (P_λ), with cylindrical symmetry, concluding the proof of Theorem 1.1–(i). □

3.1.2 A Multiplicity Result for Problem (P_λ)

This subsection is devoted to the proof of the multiplicity result stated in Theorem 1.1. Fixed $\lambda > 0$ and $i \in I_{d,m}$, our strategy consists in arguing as in Sect. 3.1.1, just by replacing the space $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ with the space $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ defined by (2.13) and $\mathcal{J}_\lambda$ with the restriction $\mathcal{J}_{\lambda,i}$ of $\mathcal{I}_\lambda$ to $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$, i.e.

$$\mathcal{J}_{\lambda,i}(u) := (\mathcal{I}_\lambda)_{|\,Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))}(u), \quad u \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))\,.$$

Furthermore, as usual when dealing with odd nonlinearities, we apply the Symmetric Mountain Pass Theorem due to Ambrosetti–Rabinowitz (see again [3]) to our functional.

Now, we give the following claims, stated for fixed $\lambda > 0$ and $i \in I_{d,m}$.

Claim 3.1.4 *There exist $\rho_i > 0$ and $\gamma_{\rho_i} > 0$ such that $\mathcal{J}_{\lambda,i}(u) \geq \gamma_{\rho_i}$ for any $u \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ with $\|u\|_{H_0^1} = \rho_i$.*

Proof The claim follows verbatim the proof of Claim 3.1.1. □

As for the geometry required by the Symmetric Mountain Pass Theorem, we need the next property on $\mathcal{J}_{\lambda,i}$:

Claim 3.1.5 *For any finite dimensional subspace $\mathbb{F}$ of $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$, there exists $R > \rho_i$ such that $\mathcal{J}_{\lambda,i}(u) \le 0$ for any $u \in \mathbb{F}$ with $\|u\|_{H_0^1} \ge R$, where ρ_i is given in Claim 3.1.4.*

Proof Let $\mathbb{F}$ be a finite dimensional subspace of $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ and let $u \in \mathbb{F}$. By using the same arguments considered in Claim 3.1.2, we have, see (3.7), that

$$\mathcal{J}_{\lambda,i}(u) \le \frac{1}{2}\|u\|_{H_0^1}^2 - \lambda a_1 \alpha_0 \|u\|_\sigma^\sigma + \lambda a_2 \|\alpha\|_1 ,$$

and so, by taking into account that in $\mathbb{F}$, all the norms are equivalent and that $\sigma > 2$, we get that

$$\mathcal{J}_{\lambda,i}(u) \to -\infty \qquad \text{as } \|u\|_{H_0^1} \to +\infty,$$

and this concludes the proof of Claim 3.1.5. □

Proof of Theorem 1.1 Let us fix $\lambda > 0$ and $i \in I_{d,m}$, see (1.1). As we already said, we can argue as in Sect. 3.1.1, just by replacing $H_{0,\mathrm{cyl}}^1(\omega \times \mathbb{R}^{d-m})$ with $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ and $\mathcal{J}_\lambda$ with $\mathcal{J}_{\lambda,i}$. By taking into account Proposition 2.2 and using the same arguments considered in the proof of Claim 3.1.3, we easily have that the functional $\mathcal{J}_{\lambda,i}$ satisfies the Palais–Smale (compactness) condition at any level $c \in \mathbb{R}$. In addition, it fulfills the geometric conditions stated in Claim 3.1.4 and Claim 3.1.5.

Now, since f is odd, by the Symmetric Mountain Pass Theorem (see [3] and also the version given in [24, Chapter 1]) applied to the functional $\mathcal{J}_{\lambda,i}$, we obtain the existence of an unbounded sequence $(u_{\lambda,k}^{(i)})_k$ of critical points $u_{\lambda,k}^{(i)}$ of $\mathcal{J}_{\lambda,i}$, constrained on $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$, i.e. such that

$$\begin{aligned}&\int_{\omega\times\mathbb{R}^{d-m}} \nabla u_{\lambda,k}^{(i)}(x,y)\nabla\varphi(x,y)\,dx\,dy\\ &\quad -\lambda\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y) f(u_{\lambda,k}^{(i)}(x,y))\varphi(x,y)\,dx\,dy = 0\end{aligned}$$

for any $\varphi \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ and for any $k \in \mathbb{N}$.

Now, we claim that $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ is a natural constraint for $\mathcal{I}_\lambda$, i.e. $u_{\lambda,k}^{(i)}$ is a critical point of $\mathcal{I}_\lambda$ for any $k \in \mathbb{N}$. Indeed, not only the action $\circledast_i$, defined by (2.11), of the group $\widehat{H}_{d,m,\widehat{\eta}_i}$ on the space $H_0^1(\omega \times \mathbb{R}^{d-m})$ is an isometry, but also $\mathcal{I}_\lambda$ is invariant with respect to the action $\circledast_i$ of the group $\widehat{H}_{d,m,\widehat{\eta}_i}$. Indeed, since f is odd (and so F is even) and $\widehat{H}_{d,m,\widehat{\eta}_i}$ is a subgroup of the group $O(d-m)$, by (3.32), we have that

$$\mathcal{I}_\lambda(\widehat{h} \circledast_i u) = \mathcal{I}_\lambda(u) \qquad \text{for all } \widehat{h} \in \widehat{H}_{d,m,\widehat{\eta}_i},\ u \in H_0^1(\omega \times \mathbb{R}^{d-m}). \tag{3.34}$$

Then, by applying the Principle of Symmetric Criticality of Palais to $\mathcal{I}_\lambda$, we get that each $u^{(i)}_{\lambda,k} \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H^1_0(\omega\times\mathbb{R}^{d-m}))$ is a nontrivial weak solution for Problem (P_λ) for any $k \in \mathbb{N}$.

Finally, we note that, by Proposition 2.3–(i), for any $i \in I_{d,m}$ and any $k \in \mathbb{N}$, $u^{(i)}_{\lambda,k}$ is distinct from the critical point $u_\lambda \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ provided in the existence part of Theorem 1.1. Moreover, Proposition 2.3–(ii) yields that (when $d = m + 6$ or $d \geq m + 8$) $u^{(i)}_{\lambda,h} \neq u^{(j)}_{\lambda,k}$ for any $i, j \in I_{d,m}$ with $i \neq j$, and any $h, k \in \mathbb{N}$. Hence, by introducing the constant sequence $(u^{(0)}_{\lambda,k})_k$ of constant value the critical point $u_\lambda \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$, we get $\mathrm{card}(I_{d,m}) + 1$ distinct sequences of weak solutions to (P_λ). Since, by (1.1), $\mathrm{card}(I_{d,m}) = \tau_{d,m} = s_{d,m} - 1$, we conclude the proof of Theorem 1.1–(ii). □

Remark 3.1 In order to assure the invariance (see (3.34)) of the functional $\mathcal{I}_\lambda$ with respect to the action $\circledast_i$ of the group $\widehat{H}_{d,m,\widehat{\eta}_i}$, for any $i \in I_{d,m}$, it is not enough, (as instead happens for the analogous property (3.33)), to assume just the cylindrical symmetry property on the weight α (see condition (α_2)). Indeed, by (2.12), we have

$$\mathcal{I}_\lambda(\widehat{h} \circledast_i u) := \begin{cases} \mathcal{I}_\lambda(\widehat{h}\sharp u) & \text{if} \widehat{h} \in \widehat{H}_{d,m,i} \\ \mathcal{I}_\lambda\left(-(\widehat{\eta}_{d,m,i}\widehat{g})\sharp u\right) & \text{if } \widehat{\eta}_{d,m,i}\widehat{g} \in \widehat{H}_{d,m,\widehat{\eta}_i} \setminus \widehat{H}_{d,m,i}, \end{cases} \tag{3.35}$$

and the presence of the minus sign in the second case makes the evenness of $\mathcal{I}_\lambda$ necessary to get the invariance (3.34) and this justifies the oddness requirement on f while getting the multiplicity result. (The same will be true while getting the invariance of the functional $\mathcal{I}_{\lambda,\mu}$ associated to Problem ($P_{\lambda,\mu}$), and this justifies the oddness requirement on both f and g).

3.2 Problem (P_λ) with Sublinear Growth at Infinity

In this subsection we consider the semilinear Problem (P_λ) in the case in which the term f satisfies sublinear growth assumptions at infinity, namely condition (f_5) (and (f_6)) instead of (f_3) and we prove Theorem 1.2. As already remarked, assumptions (f_2) and (f_5) imply (f_4).

Proof of Theorem 1.2 Let us start with assertion (i). First of all, note that conditions (f_1), (f_2), (f_5) and the Weierstrass Theorem yield that there exists a positive constant κ_f, depending on f, such that

$$|f(t)| \leq \kappa_f |t| \quad \text{for any } t \in \mathbb{R}\,. \tag{3.36}$$

Now, we argue by contradiction and we assume that there exists a sequence $(\lambda_k)_k$ in $\mathbb{R}^+_0$ such that

$$\lambda_k \to 0 \qquad \text{as } k \to +\infty, \tag{3.37}$$

and such that Problem (P_λ) with $\lambda = \lambda_k$ admits a nontrivial weak solution $u_k \in H^1_0(\omega \times \mathbb{R}^{d-m})$ for any $k \in \mathbb{N}$. Thus, by taking u_k as a test function in the equation and by using (3.36), we get that

$$
\begin{aligned}
\|u_k\|^2_{H^1_0} &= \lambda_k \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x, y) f(u_k(x, y))u_k(x, y)\, dx\, dy \\
&\le \lambda_k \|\alpha\|_\infty \int_{\omega\times\mathbb{R}^{d-m}} |f(u_k(x, y))u_k(x, y)|\, dx\, dy \\
&\le \lambda_k \kappa_f \|\alpha\|_\infty \|u_k\|_2^2 \\
&\le \lambda_k \kappa_f C_2^2 \|\alpha\|_\infty \|u_k\|^2_{H^1_0}\,,
\end{aligned}
\tag{3.38}
$$

for any $k \in \mathbb{N}$, where C_2 is the constant given in (2.1) with $\nu = 2$. So, unless $u_k \equiv 0$ for all large enough k, we would deduce $\lambda_k \ge (\kappa_f C_2^2 \|\alpha\|_\infty)^{-1}$ in contradiction with (3.37) for infinitely many values of k. Hence, the non-existence result stated in (i) is proved.

Finally, for what concerns assertions (ii) and (iii), here we just observe that Problem (P_λ) is a particular case of Problem $(P_{\lambda,\mu})$, with $\mu = 0$. So, assertions (ii) and (iii) are a consequence of Theorem 1.3 (whose proof will be provided in Sect. 4). This concludes the proof of Theorem 1.2.

4 A Nonlinear Perturbation of Problem (P_λ) with Sublinear Growth

In this section we deal with Problem $(P_{\lambda,\mu})$, which can be seen as a nonlinear perturbation of Problem (P_λ). Precisely, here we prove the existence and multiplicity results stated in Theorem 1.3.

Weak solutions to Problem $(P_{\lambda,\mu})$ can be found as critical points of the energy functional $\mathcal{I}_{\lambda,\mu} : H^1_0(\omega \times \mathbb{R}^{d-m}) \to \mathbb{R}$ naturally associated with it and defined by setting, for any $u \in H^1_0(\omega \times \mathbb{R}^{d-m})$,

$$
\begin{aligned}
\mathcal{I}_{\lambda,\mu}(u) := \frac{1}{2} \int_{\omega\times\mathbb{R}^{d-m}} |\nabla u(x, y)|^2\, dx\, dy - \lambda \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x, y) F(u(x, y))\, dx\, dy \\
- \mu \int_{\omega\times\mathbb{R}^{d-m}} \beta(x, y) G(u(x, y))\, dx\, dy\,,
\end{aligned}
$$

where F is the function defined in (1.2) and G is analogously given by

$$
G(t) = \int_0^t g(\tau)\, d\tau, \quad t \in \mathbb{R}\,. \tag{4.1}
$$

Under the assumptions (f_1), (f_2), (f_5), (g_1), (g_2), (α_1), (β_1) and thanks to the embeddings in (2.1), it is standard to check that $\mathcal{I}_{\lambda,\mu}$ is well defined on $H^1_0(\omega \times \mathbb{R}^{d-m})$

and that $\mathcal{I}_{\lambda,\mu} \in C^1(H_0^1(\omega \times \mathbb{R}^{d-m}))$ with

$$\begin{aligned}\langle \mathcal{I}'_{\lambda,\mu}(u), \varphi\rangle = &\int_{\omega\times\mathbb{R}^{d-m}} \nabla u(x,y)\nabla\varphi(x,y)\,dx\,dy \\ &- \lambda \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y) f(u(x,y))\varphi(x,y)\,dx\,dy \\ &- \mu \int_{\omega\times\mathbb{R}^{d-m}} \beta(x,y) g(u(x,y))\varphi(x,y)\,dx\,dy\end{aligned}$$

for any $u, \varphi \in H_0^1(\omega \times \mathbb{R}^{d-m})$.

For the proof of Theorem 1.3, the main tools are the following ones:

- the abstract critical points result stated in [25, Theorem 2] (see also Theorem 4.1 below), which assures the existence of multiple critical points for a suitable functional;
- the Principle of Symmetric Criticality due to Palais (see [21]);
- the flower-shape geometry in the Sobolev space $H_0^1(\omega \times \mathbb{R}^{d-m})$ described in Sect. 2.

4.1 Existence of At Least Two Nontrivial Weak Solutions

This subsection is devoted to the proof of the existence of at least two nontrivial weak solutions of Problem ($P_{\lambda,\mu}$) in $H_0^1(\omega \times \mathbb{R}^{d-m})$ with cylindrical symmetry. In order to do this, we use the abstract critical points result [25, Theorem 2] due to Ricceri, stated here below for the reader's convenience.

Theorem 4.1 [25, Theorem 2] *Let $(X, \|\cdot\|)$ be a real, separable and reflexive Banach space. Let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse in the dual of X and such that*

$$\begin{gathered}\textit{any sequence } (x_k)_k \subset X \textit{ such that } x_k \to x \textit{ weakly in } X \textit{ and } \liminf_{k\to\infty} \Phi(x_k) \le \Phi(x) \\ \textit{admits a strongly converging subsequence}.\end{gathered} \tag{4.2}$$

Let $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that

$$\Phi \textit{ has a strict local minimum } x_0 \textit{ with } \Phi(x_0) = J(x_0) = 0\,. \tag{4.3}$$

Finally, set

$$a := \max\left\{0,\ \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)},\ \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\right\} \tag{4.4}$$

$$b := \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(x)}{\Phi(x)}, \tag{4.5}$$

assume that $a < b$.

Then, for each compact interval $I \subset]b^{-1}, a^{-1}[$ (with the conventions $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$) there exists $r > 0$ with the following property: for every $\lambda \in I$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x) \tag{4.6}$$

has at least three solutions whose norms are less than r.

By looking at the functional $\mathcal{I}_{\lambda,\mu}$, we shall apply Theorem 4.1 by taking $X = H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ and

$$\Phi(u) := \frac{1}{2} \int_{\omega \times \mathbb{R}^{d-m}} |\nabla u(x, y)|^2 \, dx \, dy = \frac{1}{2} \|u\|^2_{H^1_0} \tag{4.7}$$

$$J(u) := \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x, y) F(u(x, y)) \, dx \, dy \tag{4.8}$$

and

$$\Psi(u) := \int_{\omega \times \mathbb{R}^{d-m}} \beta(x, y) G(u(x, y)) \, dx \, dy \,, \tag{4.9}$$

so that, since

$$\mathcal{I}_{\lambda,\mu}(u) = \Phi(u) - \lambda J(u) - \mu \Psi(u) \qquad \text{for any } u \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) \,,$$

solutions to (4.6) give critical points of $\mathcal{I}_{\lambda,\mu}$ constrained on $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$.

Then, by using the Principle of Symmetric Criticality by Palais, we get at least three solutions to Problem ($P_{\lambda,\mu}$).

Proof of Theorem 1.3-(i): The proof consists simply in showing that all requirements in Theorem 4.1 are fulfilled by the space $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ and by the three introduced functionals Φ, J and Ψ, in determining the function x_0 in (4.3) and in checking the inequality $a < b$ between the constants a and b defined by (4.4) and (4.5), respectively.

(1) *The space X*: $X = H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ is a real, separable and reflexive Banach space;
(2) *The functional* Φ: since $\Phi = 1/2\|\cdot\|^2_{H^1_0}$, the requirements of Theorem 4.1 are trivially satisfied. In particular, for what concerns property (4.2), let $(x_k)_k$ be a sequence in X such that $x_k \to x$ weakly in X as $k \to +\infty$ and $\liminf_{k\to\infty} \Phi(x_k) \leq \Phi(x)$. Since Φ is sequentially weakly lower semicontinuous, the assumption $\liminf_{k\to\infty} \Phi(x_k) \leq \Phi(x)$ gives

$$\liminf_{k\to\infty} \|x_k\|^2_{H^1_0} \leq \|x\|^2_{H^1_0} \leq \liminf_{k\to\infty} \|x_k\|^2_{H^1_0} \,,$$

so that there exists a subsequence of $(x_k)_k$, still denoted by x_k, such that $\|x_k\|_{H_0^1} \to \|x\|_{H_0^1}$ as $k \to +\infty$. This and the weak convergence of $(x_k)_k$ imply that $x_k \to x$ in X as $k \to +\infty$. Therefore, (4.2) holds;

(3) *The functional* J: we shall prove the fulfilment of the requirements on J in Lemma 4.2 below;
(4) *The functional* Ψ: we shall prove the fulfilment of the requirements on Ψ in Lemma 4.3 below;
(5) *The assumption* (4.3): since $\Phi = 1/2\|\cdot\|^2_{H_0^1}$ and F is defined by (1.2), (4.3) is trivially true with $x_0 = 0$;
(6) *The inequality* $a < b$ *between the constants defined in* (4.4) *and* (4.5): by Claim 4.1.1 and Claim 4.1.2 we get that $a = 0$, while, by Claim 4.1.3 we prove that $b > 0$.

Then, set $\lambda_E^{**} := b^{-1}$, we get that, for any $\lambda > \lambda_E^{**}$, there exists $\mu_{\lambda,E} > 0$ such that for any $\mu \in [0, \mu_{\lambda,E}]$ the functional $\mathcal{I}_{\lambda,\mu}$ admits two nontrivial critical points $u_{\lambda,\mu}$ and $\tilde{u}_{\lambda,\mu}$ constrained on $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$.

Finally, thanks to (α_2) and (β_2), we can apply the Principle of Symmetric Criticality by Palais (arguing as in (3.31) and (3.33)) and deduce that $u_{\lambda,\mu}$ and $\tilde{u}_{\lambda,\mu}$ are critical points of $\mathcal{I}_{\lambda,\mu}$ in $H_0^1(\omega \times \mathbb{R}^{d-m})$, i.e. these critical points are solutions to Problem ($P_{\lambda,\mu}$). This ends the proof of Theorem 1.3–(i). □

Now, the remaining part of this subsection will be devoted to state and prove the lemmas and claims used in the proof of Theorem 1.3–(i).

First of all, we start by proving the required compactness property of the functionals J and Ψ, as stated in the following lemmas, in which the compactness of the embedding $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) \hookrightarrow L^\nu(\omega \times \mathbb{R}^{d-m})$ for any $\nu \in (2, 2^*)$ (see (2.7)) plays a crucial role.

Lemma 4.2 *Assume* (f_1), (f_2), (f_5), (α_1) *and* (α_2). *Then, the functional* $J \in C^1(H_0^1(\omega \times \mathbb{R}^{d-m}))$ *and* J' *is compact in* $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$.

Proof The proof of this assertion is quite standard: we repeat it here just for the reader's convenience.

First of all, note that $J \in C^1(H_0^1(\omega \times \mathbb{R}^{d-m}))$, thanks to the assumptions (f_1), (f_2), (f_5) on f and to the fact that α satisfies (α_1) and since (2.1) holds. Moreover, it is easy to see that

$$\langle J'(u), \varphi\rangle = \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x, y) f(u(x, y))\varphi(x, y)\, dx\, dy \tag{4.10}$$

for any $u, \varphi \in H_0^1(\omega \times \mathbb{R}^{d-m})$.

Now, let $(u_k)_k \subset H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ be a bounded sequence. Then, due to (2.7), up to a subsequence, still denoted by $(u_k)_k$, there exists $u_\infty \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ such that

$$\begin{aligned} &u_k \to u_\infty \quad \text{weakly in } H_0^1(\omega \times \mathbb{R}^{d-m})\\ &u_k \to u_\infty \quad \text{in } L^q(\omega \times \mathbb{R}^{d-m}) \text{ for any } q \in (2, 2^*)\\ &u_k \to u_\infty \quad \text{a.e. in } \omega \times \mathbb{R}^{d-m}\end{aligned} \tag{4.11}$$

as $k \to +\infty$ and, furthermore there exists $\ell \in L^q(\omega \times \mathbb{R}^{d-m})$ such that

$$|u_k(x,y)| \le \ell(x,y) \qquad \text{for a.e. } (x,y) \in \omega \times \mathbb{R}^{d-m}, \text{ for any } k \in \mathbb{N}. \tag{4.12}$$

Thus, since (f_1), (3.36), (4.11) and (4.12) hold, the Dominated Convergence Theorem yields that

$$f(u_k) \to f(u_\infty) \quad \text{in } L^q(\omega \times \mathbb{R}^{d-m}) \qquad \text{as } k \to +\infty. \tag{4.13}$$

Now, by (α_1) (which assures that $\alpha \in L^\nu(\omega \times \mathbb{R}^{d-m})$ for any $\nu \in [1,+\infty]$, see (3.20)), the Hőlder Inequality, (2.1) (applied with $\nu = 2$) and (4.13), (set $q' := q/(q-1)$ the conjugate exponent of q) we have that, for all $\varphi \in H^1_{0,\text{cyl}}(\omega \times \mathbb{R}^{d-m})$, $\|\varphi\|_{H^1_0} = 1$,

$$\begin{aligned}
&\left| \int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y)\Big(f(u_k(x,y)) - f(u_\infty(x,y))\Big)\varphi(x,y)\,dx\,dy \right| \\
&\le \left(\int_{\omega\times\mathbb{R}^{d-m}} \left|\alpha(x,y)\Big(f(u_k(x,y)) - f(u_\infty(x,y))\Big)\right|^{q'} dx\,dy \right)^{1/q'} \|\varphi\|_q \\
&\le \|\alpha\|^{q'}_{q/(q-2)} \|f(u_k) - f(u_\infty)\|^{q'}_q \|\varphi\|_q \\
&\le C_q \|\alpha\|^{q'}_{q/(q-2)} \|f(u_k) - f(u_\infty)\|^{q'}_q \to 0 \qquad \text{as } k \to +\infty,
\end{aligned} \tag{4.14}$$

where C_q is the constant given in (2.1) with $\nu = q$. As a consequence of (4.10) and (4.14), we obtain that $\left\| J'(u_k) - J'(u_\infty) \right\| \to 0$ as $k \to +\infty$. Hence, J' is compact in $H^1_{0,\text{cyl}}(\omega \times \mathbb{R}^{d-m})$ and the proof of Lemma 4.2 is complete. □

Lemma 4.3 *Assume (g_1), (g_2), (β_1) and (β_2). Then, the functional $\Psi \in C^1(H^1_0(\omega \times \mathbb{R}^{d-m}))$ and Ψ' is compact in $H^1_{0,\text{cyl}}(\omega \times \mathbb{R}^{d-m})$.*

Proof In order to get that the functional $\Psi \in C^1(H^1_0(\omega \times \mathbb{R}^{d-m}))$ we can argue as in the proof of Lemma 4.2, by taking into account assumptions (β_1), (g_1) and (g_2), while the proof of the compactness of Ψ' is a more delicate question since we can not use the Dominated Convergence Theorem as in the previous lemma (indeed, the function g is not necessarily sublinear and so it does not need to satisfy a relation analogous to (3.36)).

Let $(u_k)_k \subset H^1_{0,\text{cyl}}(\omega \times \mathbb{R}^{d-m})$ be a bounded sequence. First of all, let us show that

$$\Big(\Psi'(u_k)\Big)_k \text{ is bounded in the dual space of } H^1_{0,\text{cyl}}(\omega \times \mathbb{R}^{d-m}). \tag{4.15}$$

For this purpose, note that by (g_2) there exists a positive constant $C > 0$ such that

$$|g(t)| \le C(|t| + |t|^{q-1}) \qquad \text{for all } t \in \mathbb{R} \tag{4.16}$$

so, by (β_1), (2.1) (twice applied with $\nu = 2$ and $\nu = q$), (2.6) and the Hőlder Inequality, we have that, for any $k \in \mathbb{N}$,

$$\begin{aligned}
\left\|\Psi'(u_k)\right\| &= \sup_{\substack{\varphi \in H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m}) \\ \|\varphi\|_{H^1_0}=1}} \int_{\omega\times\mathbb{R}^{d-m}} \left|\beta(x,y)g(u_k(x,y))\varphi(x,y)\right| dx\,dy \\
&\leq C\|\beta\|_\infty \left(\int_{\omega\times\mathbb{R}^{d-m}} |u_k(x,y)|\,|\varphi(x,y)|\,dx\,dy\right. \\
&\quad \left. + \int_{\omega\times\mathbb{R}^{d-m}} |u_k(x,y)|^{q-1}|\varphi(x,y)|\,dx\,dy\right) \\
&\leq C\|\beta\|_\infty \left(\|u_k\|_2\|\varphi\|_2 + \|u_k\|_q^{q-1}\|\varphi\|_q\right) \\
&\leq C\|\beta\|_\infty \left(C_2^2\|u_k\|_{H^1_0} + C_q^q\|u_k\|_{H^1_0}^{q-1}\right) \\
&\leq \tilde{C}\,,
\end{aligned} \tag{4.17}$$

where $\tilde{C} > 0$ is a suitable constant which is independent of $k \in \mathbb{N}$, since $(u_k)_k$ is bounded in $H^1_0(\omega \times \mathbb{R}^{d-m})$. Hence, (4.15) is proved.

As a consequence of (4.15), there exists $\mathcal{H}$ in the dual space of $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ such that, as $k \to +\infty$,

$$\Psi'(u_k) \to \mathcal{H} \text{ weakly in the dual space of } H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m}). \tag{4.18}$$

In order to complete the proof, we need to prove that

$$\|\Psi'(u_k) - \mathcal{H}\| \to 0 \qquad \text{as } k \to +\infty\,. \tag{4.19}$$

To get this goal, we argue by contradiction and we suppose that there exists $\delta > 0$ and $k^* \in \mathbb{N}$ such that $\delta - \frac{1}{k^*} > 0$, and

$$\|\Psi'(u_k) - \mathcal{H}\| > \delta \qquad \text{for all } k > k^*\,.$$

Then, for $k > k^*$ there exists $\varphi_k \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ such that $\|\varphi_k\|_{H^1_0} = 1$ and

$$\langle \Psi'(u_k) - \mathcal{H}, \varphi_k\rangle > \delta - \frac{1}{k^*}\,. \tag{4.20}$$

Since $(\varphi_k)_k$ is bounded in $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ and (2.7) holds, up to a subsequence, still denoted by $(\varphi_k)_k$, there exists $\varphi_\infty \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$ such that, as $k \to +\infty$,

$$\varphi_k \to \varphi_\infty \quad \text{weakly in } H^1_0(\omega\times\mathbb{R}^{d-m}) \tag{4.21}$$

and

$$\varphi_k \to \varphi_\infty \quad \text{in } L^q(\omega\times\mathbb{R}^{d-m}). \tag{4.22}$$

Now, by (β_1) (which yields that $\beta \in L^\nu(\omega \times \mathbb{R}^{d-m})$ for any $\nu \in [1, +\infty]$, see the analogous argument (3.20) for the weight α), (g_2), the Hőlder Inequality and (4.16), we have that

$$
\begin{aligned}
&\Big|\langle \Psi'(u_k), \varphi_k - \varphi_\infty\rangle\Big| \\
&\quad \le \int_{\omega\times\mathbb{R}^{d-m}} \Big|\beta(x,y) g(u_k(x,y))\big(\varphi_k(x,y) - \varphi_\infty(x,y)\big)\Big|\, dx\, dy \\
&\quad \le C \int_{\omega\times\mathbb{R}^{d-m}} |\beta(x,y)|\, |u_k(x,y)|\, |\varphi_k(x,y) - \varphi_\infty(x,y)|\, dx\, dy \\
&\qquad + C \int_{\omega\times\mathbb{R}^{d-m}} |\beta(x,y)|\, |u_k(x,y)|^{q-1} |\varphi_k(x,y) - \varphi_\infty(x,y)|\, dx\, dy \\
&\quad \le C \Big(\int_{\omega\times\mathbb{R}^{d-m}} |\beta(x,y) u_k(x,y)|^{q'}\, dx\, dy\Big)^{(q-1)/q} \|\varphi_k - \varphi_\infty\|_q \\
&\qquad + C \|\beta\|_\infty \|u_k\|_q^{q-1} \|\varphi_k - \varphi_\infty\|_q \\
&\quad \le C \|\beta\|_{q/(q-2)}^{q'} \|u_k\|_q^{q'} \|\varphi_k - \varphi_\infty\|_q + C\|\beta\|_\infty \|u_k\|_q^{q-1} \|\varphi_k - \varphi_\infty\|_q \\
&\quad \le C C_q^{q'} \|\beta\|_{q/(q-2)}^{q'} \|u_k\|_{H_0^1}^{q'} \|\varphi_k - \varphi_\infty\|_q \\
&\qquad + C C_q^{q-1} \|\beta\|_\infty \|u_k\|_{H_0^1}^{q-1} \|\varphi_k - \varphi_\infty\|_q \to 0
\end{aligned}
\tag{4.23}
$$

as $k \to +\infty$, thanks to (4.22) and to the boundedness of $(u_k)_k$ in $H_0^1(\omega \times \mathbb{R}^{d-m})$. Here C is the positive constant in (4.16) and C_q is the Sobolev embedding constant given in (2.1) with $\nu = q$ and $q' := q/(q-1)$, the conjugate exponent of q.

Moreover, due to (4.21) and, respectively, to (4.18), we have that

$$
\langle \mathcal{H}, \varphi_k - \varphi_\infty\rangle \to 0 \qquad \text{and} \qquad \langle \Psi'(u_k) - \mathcal{H}, \varphi_\infty\rangle \to 0
$$

as $k \to +\infty$. Therefore, by (4.23), we have that

$$
\begin{aligned}
\langle \Psi'(u_k) - \mathcal{H}, \varphi_k\rangle &= \langle \Psi'(u_k), \varphi_k - \varphi_\infty\rangle - \langle \mathcal{H}, \varphi_k - \varphi_\infty\rangle \\
&\quad + \langle \Psi'(u_k) - \mathcal{H}, \varphi_\infty\rangle \to 0 \quad \text{as } k \to +\infty,
\end{aligned}
\tag{4.24}
$$

which contradicts (4.20), since $\delta - \frac{1}{k} > \delta - \frac{1}{k^*} > 0$ for all $k > k^*$. Hence, (4.19) holds and this, as already said, ends the proof of Lemma 4.3. □

Now, we state the claims concerning the functional Φ and J, used in the proof of Theorem 1.3–(i) in order to get the inequality $a < b$ between the constants a and b defined by (4.4) and (4.5), respectively.

Claim 4.1.1 *Assume (f_1), (f_2), (f_5) and (α_1). Then, the following inequality holds*

$$
\limsup_{\|u\|_{H_0^1} \to +\infty} \frac{J(u)}{\Phi(u)} \le 0\,.
$$

Proof First of all, let us observe that by (f_1), (f_2), (f_5) and the Weierstrass Theorem, we get that for any $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that for any $t \in \mathbb{R}$

$$|f(t)| \le \varepsilon|t| + K_\varepsilon\,,$$

so that

$$|F(t)| \le \frac{\varepsilon}{2}|t|^2 + K_\varepsilon|t|\,. \tag{4.25}$$

As a consequence of (3.20), (4.25) and the Hőlder Inequality, we have that, for any $u \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) \setminus \{0\}$,

$$\begin{aligned}\frac{J(u)}{\Phi(u)} &= \frac{2\displaystyle\int_{\omega\times\mathbb{R}^{d-m}} \alpha(x,y)F(u(x,y))\,dx\,dy}{\|u\|^2_{H^1_0}} \\ &\le \frac{\varepsilon\|\alpha\|_\infty\|u\|_2^2 + 2K_\varepsilon\|\alpha\|_2\|u\|_2}{\|u\|^2_{H^1_0}} \\ &\le \varepsilon\|\alpha\|_\infty C_2^2 + \frac{2K_\varepsilon C_2\|\alpha\|_2}{\|u\|_{H^1_0}}\end{aligned} \tag{4.26}$$

thanks to (2.6) (here C_2 is the Sobolev embedding constant given in (2.1), with $\nu = 2$).

Passing to the limsup as $\|u\|_{H^1_0} \to +\infty$ in (4.26), we get that

$$\limsup_{\|u\|_{H^1_0}\to+\infty} \frac{J(u)}{\Phi(u)} \le \varepsilon\|\alpha\|_\infty C_2^2\,.$$

Now, the arbitrariness of ε gives the assertion of Claim 4.1.1. □

Claim 4.1.2 *Assume (f_1), (f_2), (f_5) and (α_1). Then, the following inequality holds*

$$\limsup_{\|u\|_{H^1_0}\to 0} \frac{J(u)}{\Phi(u)} \le 0\,.$$

Proof We can argue similarly to Claim 4.1.1, just replacing (4.25) with the next inequality

$$|F(t)| \le \frac{\varepsilon}{2}|t|^2 + K_\varepsilon|t|^\nu, \quad t \in \mathbb{R}\,, \tag{4.27}$$

where $2 < \nu < 2^*$. By (2.1), (2.6) and (4.27), we get that for any $u \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) \setminus \{0\}$

$$\begin{aligned}\frac{J(u)}{\Phi(u)} &\le \frac{\varepsilon\|\alpha\|_\infty\|u\|_2^2 + 2K_\varepsilon\|\alpha\|_\infty\|u\|_\nu^\nu}{\|u\|^2_{H^1_0}} \\ &\le \varepsilon\|\alpha\|_\infty C_2^2 + 2K_\varepsilon\|\alpha\|_\infty C_\nu^\nu\|u\|^{\nu-2}_{H^1_0}\,.\end{aligned}$$

Thus, passing to the limsup as $\|u\|_{H^1_0} \to 0$ in the above inequality and by taking into account that $\nu > 2$ and the arbitrariness of ε, we get the assertion, concluding the proof of Claim 4.1.2. □

In the next claim, assumptions (α_3) and (f_6) are essential.

Claim 4.1.3 *Assume* (f_1), (f_2), (f_5), (f_6), (α_1) *and* (α_3). *Then, the following inequality holds*

$$\sup\left\{\frac{J(u)}{\Phi(u)} : u \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) \setminus \{0\}\right\} > 0\,.$$

Proof It is enough to show that there exists $\bar{u} \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m}) \setminus \{0\}$ such that

$$\frac{J(\bar{u})}{\Phi(\bar{u})} > 0\,. \tag{4.28}$$

At this purpose, let t_0 be as in (f_6) and $r > 0$ be as in (α_3). Let us fix r_1 and r_2 with $0 < r_1 < r_2 < r$ and, for any $\varepsilon \in \big(0, (r_2 - r_1)/2\big)$, define the function $v_\varepsilon : \mathbb{R}^{d-m} \to \mathbb{R}$ as follows

$$v_\varepsilon(y) := \begin{cases} 0 & \text{if } |y| < r_1 \\ \frac{t_0}{\varepsilon}\Big(|y| - r_1\Big) & \text{if } r_1 \le |y| \le r_1 + \varepsilon \\ t_0 & \text{if } r_1 + \varepsilon < |y| < r_2 - \varepsilon \\ \frac{t_0}{\varepsilon}\Big(r_2 - |y|\Big) & \text{if } r_2 - \varepsilon \le |y| \le r_2 \\ 0 & \text{if } |y| > r_2. \end{cases}$$

Note that

$$\begin{gathered} \operatorname{supp} v_\varepsilon \subset \Big\{y \in \mathbb{R}^{d-m} : r_1 \le |y| \le r_2\Big\} \\ 0 \le v_\varepsilon(y) \le t_0 \text{ for any } y \in \mathbb{R}^{d-m}\,. \end{gathered} \tag{4.29}$$

Now, let K and ω' be two open sets in $\mathbb{R}^m$ with

$$K \Subset \bar{\omega}' \subset \omega \text{ and Lebesgue measure } \mathcal{L}(K) > 0 \tag{4.30}$$

and let $\tilde{\varphi} \in C^\infty_0(\omega')$ be a positive function such that $\|\tilde{\varphi}\|_\infty = 1$ and $\tilde{\varphi} \equiv 1$ on K. Let $\varphi \in C^\infty_0(\omega)$ be the natural extension of $\tilde{\varphi}$ on ω, given by

$$\varphi(x) = \begin{cases} \tilde{\varphi}(x) & \text{if } x \in \omega' \\ 0 & \text{if } x \in \omega \setminus \omega', \end{cases} \tag{4.31}$$

and define the function $u_\varepsilon : \omega \times \mathbb{R}^{d-m} \to \mathbb{R}$ as follows

$$u_\varepsilon(x, y) = \varphi(x) v_\varepsilon(y) \qquad \text{for a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m}\,. \tag{4.32}$$

It is easy to check that, for any $\varepsilon \in \big(0, (r_2 - r_1)/2\big)$, $u_\varepsilon \in H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$. Furthermore, taking into account (4.29), we get that

$$\begin{aligned} \operatorname{supp} u_\varepsilon \subseteq \overline{\omega} \times \Big\{ y \in \mathbb{R}^{d-m} : r_1 \le |y| \le r_2 \Big\} \\ u_\varepsilon(x, y) = \varphi(x) v_\varepsilon(y) \le v_\varepsilon(y) \in [0, t_0] \text{ a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m}. \end{aligned} \tag{4.33}$$

Condition (f_6), (the first requirement in) (α_3) and (4.33) yield that

$$F(u_\varepsilon(x, y)) \ge 0 \text{ and } \alpha(x, y) F(u_\varepsilon(x, y)) \ge 0 \text{ for a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m}. \tag{4.34}$$

Then, by (4.31), (4.32), (4.33), (4.34), (the second requirement in) (α_3), by the choice of r_1 and r_2 (and by the rough estimate $F(t) \ge -2 \max_{[-r,r]} |F|$ for all $|t| \le r$), we get that

$$\begin{aligned} J(u_\varepsilon) &= \int_{\omega \times \{y \in \mathbb{R}^{d-m} : r_1 \le |y| \le r_2\}} \alpha(x, y) F(u_\varepsilon(x, y))\, dx\, dy \\ &\ge \alpha_0 \Bigg(\int_{K \times \{y \in \mathbb{R}^{d-m} : r_1 \le |y| \le r_1 + \varepsilon\}} F\Big(\frac{t_0}{\varepsilon}\big(|y| - r_1\big)\Big)\, dx\, dy \\ &\quad + \int_{K \times \{y \in \mathbb{R}^{d-m} : r_1 + \varepsilon \le |y| \le r_2 - \varepsilon\}} F(t_0)\, dx\, dy \\ &\quad + \int_{K \times \{y \in \mathbb{R}^{d-m} : r_2 - \varepsilon \le |y| \le r_2\}} F\Big(\frac{t_0}{\varepsilon}\big(r_2 - |y|\big)\Big)\, dx\, dy \Bigg) \\ &\ge \alpha_0 \Big\{ F(t_0)\, \mathcal{L}(K)\, \omega_{d-m} \Big[(r_2 - \varepsilon)^{d-m} - (r_1 + \varepsilon)^{d-m} \Big] \\ &\quad -2 \max_{|t| \le r} |F(t)|\, \mathcal{L}(K)\, \omega_{d-m} \Big[(r_1 + \varepsilon)^{d-m} - r_1^{d-m} + r_2^{d-m} - (r_2 - \varepsilon)^{d-m} \Big] \Big\} \\ &\quad \to \alpha_0 F(t_0)\, \mathcal{L}(K)\, \omega_{d-m} \big(r_2^{d-m} - r_1^{d-m} \big) > 0 \qquad \text{as } \varepsilon \to 0, \end{aligned} \tag{4.35}$$

where, as usual, ω_{d-m} is the volume of the unit ball in $\mathbb{R}^{d-m}$.

Thus, since $\Phi = 1/2\| \cdot \|^2_{H^1_0}$, we obtain (4.28) by (4.35) by taking $\bar{u} = u_\varepsilon$ with ε small enough. This ends the proof of Claim 4.1.3. □

4.2 A Multiplicity Result

In this subsection we provide the multiplicity result stated in Theorem 1.3–(ii). In order to get this goal, the main idea consists in applying, for any $i \in I_{d,m}$ (see (1.1)), Theorem 4.1 with $X = Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H^1_0(\omega \times \mathbb{R}^{d-m}))$ to the three functionals Φ_i, J_i and Ψ_i, which are the respective restrictions of the functionals Φ, J and Ψ (see (4.7), (4.8) and (4.9)) to the space $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H^1_0(\omega \times \mathbb{R}^{d-m}))$, i.e.

$$\Phi_i(u) := \Phi(u)_{|\, Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))}(u)$$
$$J_i(u) := J(u)_{|\, Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))}(u)$$
$$\Psi_i(u) := \Psi(u)_{|\, Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))}(u),$$

for any $u \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$.

Proof of Theorem 1.3-(ii): Let us fix $i \in I_{d,m}$. By arguing as in the Sect. 4.1, we are able to check that all requirements of Theorem 4.1 asked to the space and to the functionals are fulfilled by $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$ and Φ_i, J_i and Ψ_i, respectively.

In particular, defining a_i and b_i according to (4.4) and (4.5) by replacing Φ and J with Φ_i and J_i, respectively, and $H^1_{0,\mathrm{cyl}}(\omega\times\mathbb{R}^{d-m})$ with $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega\times\mathbb{R}^{d-m}))$, we have that $a_i = 0$. Moreover, by Claim 4.2.1 below, we prove that $b_i > 0$.

Then, set

$$\lambda_i^* := b_i^{-1} \qquad \text{and} \qquad \lambda_M^* := \max_{i\in I_{d,m}} \{\lambda_E^*,\ \lambda_i^*\},$$

we get that for any $\lambda > \lambda_M^*$ there exists $\mu_{\lambda,M} > 0$ such that for any $\mu \in [0, \mu_{\lambda,M}]$, the functional $\mathcal{I}_{\lambda,\mu}$ admits two nontrivial critical points $u_{\lambda,\mu,i}$ and $\tilde{u}_{\lambda,\mu,i}$ constrained on $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$.

Then, thanks to (α_2) and (β_2) and the oddness assumption on f and g, we can apply the Principle of Symmetric Criticality by Palais, (see Remark 3.1, (3.31) and (3.33) also with α and f replaced by β and g, respectively) and deduce that these critical points are solutions to Problem ($P_{\lambda,\mu}$).

Note that Proposition 2.3–(i) allows us to say that, for any $i \in I_{d,m}$, the solutions $u_{\lambda,\mu,i}$ and $\tilde{u}_{\lambda,\mu,i}$ are distinct from both $u_{\lambda,\mu}$ and $\tilde{u}_{\lambda,\mu}$ got in $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$. Moreover when $d = m + 6$ or $d \geq m + 8$ by Proposition 2.3–(ii), we get that $u_{\lambda,\mu,i}$, $\tilde{u}_{\lambda,\mu,i} \notin \{u_{\lambda,\mu,j},\ \tilde{u}_{\lambda,\mu,j}\}$ for any $i, j \in I_{d,m}$ with $i \neq j$. Since we have added to the pair $(u_{\lambda,\mu},\ \tilde{u}_{\lambda,\mu})$, got in $H^1_{0,\mathrm{cyl}}(\omega \times \mathbb{R}^{d-m})$, $\tau_{d,m} = s_{d,m} - 1$ (see (1.1) and (1.4)) pairs of solutions $(u_{\lambda,\mu,i},\ \tilde{u}_{\lambda,\mu,i})$ got in the "petals" $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$, we deduce that Problem ($P_{\lambda,\mu}$) admits $s_{d,m}$ pairs of nontrivial weak solutions, with different symmetries, provided λ is sufficiently large and μ is small enough. The proof of Theorem 1.3–(ii) is then complete. □

So, we conclude this subsection just by proving the natural counterpart of Claim 4.1.3 in $Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m}))$, where assumption ($\alpha_3$) and ($f_6$) plays a crucial rule.

Claim 4.2.1 *Assume (f_1), (f_2), (f_5), (f_6), (α_1) and (α_3). Then, the following inequality holds*

$$\sup\left\{\frac{J_i(u)}{\Phi_i(u)} : u \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \setminus \{0\}\right\} > 0 \qquad \textit{for all } i \in I_{d,m}\,.$$

Proof Let us fix $i \in I_{d,m}$ and let any $y \in \mathbb{R}^{d-m}$ be decomposed as follows

$$y := \begin{cases} (y_1, y_3) \in \mathbb{R}^{(d-m)/2} \times \mathbb{R}^{(d-m)/2} & \text{if } i = \dfrac{d-m-2}{2} \\ (y_1, y_2, y_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{d-m-2i-2} \times \mathbb{R}^{i+1} & \text{if } i \neq \dfrac{d-m-2}{2} \,. \end{cases} \tag{4.36}$$

As in Claim 4.1.3, we have to construct a function $\bar{u}_i \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H_0^1(\omega \times \mathbb{R}^{d-m})) \setminus \{0\}$ such that

$$\frac{J_i(\bar{u}_i)}{\Phi_i(\bar{u}_i)} > 0\,. \tag{4.37}$$

At this purpose, we follow [13] and we adapt to our setting the function introduced therein. Let r be as in (α_3) and let us fix r_1 and r_2 with $0 < r_1 < r_2 < r$ and such that $(5 + 4\sqrt{2})r_1 > r_2$. For any $\varepsilon \in (0, 1]$, let us consider the following subset of $\mathbb{R}^{d-m}$

$$B^{1,3}_{\varepsilon,i} := \begin{cases} \left\{(y_1, y_3) \in \mathbb{R}^{(d-m)/2} \times \mathbb{R}^{(d-m)/2} : (y_1, y_3) \text{ satisfies (4.38)}\right\} & \text{if } i = \frac{d-m-2}{2} \\ \left\{(y_1, y_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : (y_1, y_3) \text{ satisfies (4.38)}\right\} & \text{if } i \neq \frac{d-m-2}{2} \end{cases}$$

and

$$B^{3,1}_{\varepsilon,i} := \begin{cases} \left\{(y_1, y_3) \in \mathbb{R}^{(d-m)/2} \times \mathbb{R}^{(d-m)/2} : (y_3, y_1) \text{ satisfies (4.38)}\right\} & \text{if } i = \frac{d-m-2}{2} \\ \left\{(y_1, y_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : (y_3, y_1) \text{ satisfies (4.38)}\right\} & \text{if } i \neq \frac{d-m-2}{2}\,, \end{cases}$$

where

$$\left(|y_1| - \frac{r_2 + 3r_1}{4}\right)^2 + |y_3|^2 \leq \varepsilon^2 \left(\frac{r_2 - r_1}{4}\right)^2, \tag{4.38}$$

and the set

$$B^2_{\varepsilon,i} := \left\{y_2 \in \mathbb{R}^{d-m-2i-2} : y_2 \text{ satisfies (4.39)}\right\} \text{ when } i \neq \frac{d-m-2}{2}\,,$$

where

$$|y_2| \leq \varepsilon \frac{r_2 - r_1}{4}\,. \tag{4.39}$$

Note that, if $i = \frac{d-m-2}{2}$, i.e. when no coordinate block y_2 can be defined, we are assuming that (4.39) is, of course, satisfied. We also observe that the assumptions on r_1 and r_2 give that $B^{1,3}_{1,i}$ and $B^{3,1}_{1,i}$ are disjoint. As a consequence, being $\varepsilon \leq 1$, we get that

$$\begin{gathered} B^{1,3}_{\varepsilon,i} \cap B^{3,1}_{\varepsilon,i} = \emptyset \\ B^{1,3}_{\varepsilon,i} \cap B^{3,1}_{1,i} = B^{1,3}_{1,i} \cap B^{3,1}_{\varepsilon,i} = \emptyset\,. \end{gathered} \tag{4.40}$$

Moreover, let us define

$$S_{\varepsilon,i} := \begin{cases} B^{1,3}_{\varepsilon,i} \cup B^{3,1}_{\varepsilon,i} & \text{if } i = \frac{d-m-2}{2} \\ \Big\{(y_1, y_2, y_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{d-m-2i-2} \times \mathbb{R}^{i+1} : \\ (y_1, y_3) \in B^{1,3}_{\varepsilon,i} \cup B^{3,1}_{\varepsilon,i} \text{and } y_2 \in B^2_{\varepsilon,i}\Big\} & \text{if } i \neq \frac{d-m-2}{2}\,. \end{cases} \tag{4.41}$$

Now, let t_0 be as in (f_6) and, for any $\varepsilon > 0$, let $v_{\varepsilon,i} : \mathbb{R}^{d-m} \to \mathbb{R}$ be the function given by

$$\begin{aligned} v_{\varepsilon,i}(y) := \Bigg[& \left(\frac{r_2 - r_1}{4} - \max\left\{\sqrt{\left(|y_1| - \frac{r_2 + 3r_1}{4}\right)^2 + |y_3|^2}, \varepsilon\frac{r_2 - r_1}{4}\right\}\right)^+ \\ & - \left(\frac{r_2 - r_1}{4} - \max\left\{\sqrt{\left(|y_3| - \frac{r_2 + 3r_1}{4}\right)^2 + |y_1|^2}, \varepsilon\frac{r_2 - r_1}{4}\right\}\right)^+\Bigg] \\ & \cdot \left(\frac{r_2 - r_1}{4} - \max\left\{|y_2|, \varepsilon\frac{r_2 - r_1}{4}\right\}\right)^+ \frac{16t_0}{(r_2 - r_1)^2(1-\varepsilon)^2}\,. \end{aligned}$$

By direct computations and taking into account (4.40), it is easy to see that

$$\begin{aligned} 0 \le v_{\varepsilon,i}(y) \le t_0 \;\; & \text{for any } y \in \mathbb{R}^{d-m} \\ |v_{\varepsilon,i}(y)| = t_0 \quad & \text{for any} \quad y \in S_{\varepsilon,i}\,. \end{aligned} \tag{4.42}$$

Now, let K and ω' be two open sets in $\mathbb{R}^m$ satisfying (4.30), let φ be the cut-off function given in (4.31) and define the function $u_{\varepsilon,i} : \omega \times \mathbb{R}^{d-m} \to \mathbb{R}$ by setting

$$u_{\varepsilon,i}(x, y) := \varphi(x)v_{\varepsilon,i}(y) \qquad \text{for a.e. } (x, y) \in \omega \times \mathbb{R}^{d-m}.$$

First of all, note that, as in [13], it is easily seen that

$$\mathcal{L}((K \times S_{1,i}) \setminus (K \times S_{\varepsilon,i})) \to 0 \text{ as } \varepsilon \to 1^-\,, \tag{4.43}$$

while direct computations ensure that $u_{\varepsilon,i} \in H^1(\omega\times\mathbb{R}^{d-m})$. Moreover, since supp $u_{\varepsilon,i}$ is a compact subset of $\omega \times \mathbb{R}^{d-m}$, by [7, Lemma IX.5] it follows that $u_{\varepsilon,i} \in H^1_0(\omega \times \mathbb{R}^{d-m})$. Arguing as in (3.34) (see also Remark 3.1), it is easy to check also that $u_{\varepsilon,i} \in Fix_{\widehat{H}_{d,m,\widehat{\eta}_i}}(H^1_0(\omega \times \mathbb{R}^{d-m}))$.

Finally, the definition of $v_{\varepsilon,i}$ and the choice of r_1 and r_2 give that

$$\begin{aligned} \operatorname{supp} u_{\varepsilon,i} & \subset \overline{\omega} \times S_{1,i} \\ & \subseteq \left\{(x, y) \in \overline{\omega} \times \mathbb{R}^{d-m} : r_1 \le |y| \le r_2\right\} \\ & \subseteq \left\{(x, y) \in \overline{\omega} \times \mathbb{R}^{d-m} : |y| \le r\right\}. \end{aligned} \tag{4.44}$$

In addition, by (4.31) and (4.42), we get that

$$
\begin{aligned}
\|u_{\varepsilon,i}\|_\infty &\le t_0 \\
|u_{\varepsilon,i}(x, y)| &= t_0 \text{ for a.e. } (x, y) \in K \times S_{\varepsilon,i} \,.
\end{aligned}
\tag{4.45}
$$

All in all, arguing as in the proof of Claim 4.1.3, (α_3), (f_6), (4.43), (4.44) and (4.45) yield that

$$
\begin{aligned}
J_i(u_{\varepsilon,i}) &= \int_{\omega\times S_{1,i}} \alpha(x, y) F(u_{\varepsilon,i}(x, y))\, dx\, dy \\
&\ge \alpha_0 \int_{K\times S_{1,i}} F(u_{\varepsilon,i}(x, y))\, dx\, dy \\
&\ge \alpha_0 \left(F(t_0)\mathcal{L}(K \times S_{\varepsilon,i}) - 2 \max_{|t|\le r} |F(t)| \mathcal{L}((K \times S_{1,i}) \setminus (K \times S_{\varepsilon,i})) \right) \\
&\to \alpha_0\, F(t_0) \lim_{\varepsilon\to 1^-} \mathcal{L}(K \times S_{\varepsilon,i}) > 0
\end{aligned}
$$

as $\varepsilon \to 1^-$. As a consequence of this, by choosing $\bar{u}_i = u_{\varepsilon,i}$ with ε sufficiently close to 1, we obtain (4.37), and so Claim 4.2.1 is proved. □

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