On Holomorphic Curves Tangent to Real Hypersurfaces of Infinite Type

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Abstract

The purpose of this paper is to investigate the geometric properties of real hypersurfaces of D'Angelo infinite type in $\mathbb{C}^n$. In order to understand the situation of flatness of these hypersurfaces, it is natural to ask whether there exists a nonconstant holomorphic curve tangent to a given hypersurface to infinite order. A sufficient condition for this existence is given by using Newton polyhedra, which is an important concept in singularity theory. More precisely, equivalence conditions are given in the case of some model hypersurfaces.

Keywords Holomorphic curve · Real hypersurface · D'Angelo type · Bloom–Graham type · Infinite type

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1 Introduction

Let M be a (C^∞ smooth) real hypersurface in $\mathbb{C}^n$ and let p lie on M. Let r be a local defining function for M near p ($\nabla r \neq 0$ when $r = 0$). In [6,7], the following invariant is introduced:

$$\Delta_1(M, p) := \sup_{\gamma \in \Gamma} \frac{\text{ord}(r \circ \gamma)}{\text{ord}(\gamma - p)}, \tag{1.1}$$

where Γ denotes the set of (germs of) nonconstant holomorphic mappings $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$. (For a C^∞ mapping $h : \mathbb{C} \to \mathbb{C}$ or $\mathbb{C}^n$ such that $h(0) = 0$, let ord(h) denote the order of vanishing of h at 0.) The invariant $\Delta_1(M, p)$ is called the *D'Angelo type* of M at p. We say that M is *of finite type* at p if $\Delta_1(M, p) < \infty$

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and *of infinite type* at p otherwise (the latter case will be denoted by $\Delta_1(M, p) = \infty$). The class of finite type plays crucial roles in the study of the local regularity in the $\bar{\partial}$-Neumann problem over pseudoconvex domains Ω with smooth boundary $\partial\Omega$. Indeed, it was shown by Catlin [4,5] that $M = \partial\Omega$ is of finite type at p if and only if a local subelliptic estimate at p holds. From its importance, real hypersurfaces of finite type have been deeply investigated from various points of view.

On the other hand, to understand the geometric properties of real hypersurfaces of infinite type is also an interesting subject in the study of several complex variables. These hypersurfaces contain some kind of strong flatness. In order to describe the geometric structure of this flatness, the situation of contact of holomorphic curves with the respective hypersurface must be carefully observed. In this paper, we mainly consider the following question:

Question 1 When does there exist a nonconstant holomorphic curve γ_∞ tangent to M at p to infinite order?

Since the condition of the desired curve γ_∞ in Question 1 can be written as

$$(r \circ \gamma_\infty)(t) = O(t^N) \quad \text{as } t \in \mathbb{C} \to 0, \text{ for every } N \in \mathbb{N}, \tag{1.2}$$

the condition $\Delta_1(M, p) = \infty$ is necessary for the existence of the curve γ_∞. It has been shown in [7,11,19] that when M is real analytic, the above two conditions are equivalent (in this case, the curve γ_∞ is contained in M). Moreover, in the case of smooth M, this equivalence is also shown in the formal series sense in [8,16]. But, in general, the sufficient direction is not true. Indeed, the nonexistence of the curve γ_∞ in (1.2) is shown in the case of some real hypersurfaces constructed in [3,9,16,22]. Understanding flatness on hypersurfaces of infinite type has been recognized to be a delicate issue.

In this paper, in order to investigate the flatness of real hypersurfaces, we use not only holomorphic curves but also *Newton polyhedra* of defining functions, which plays important roles in singularity theory (cf. [1,2]). Approach from the viewpoint of singularity theory is useful in the study of types and there have been many works of the sort ([10,11,13,15,20], etc.).
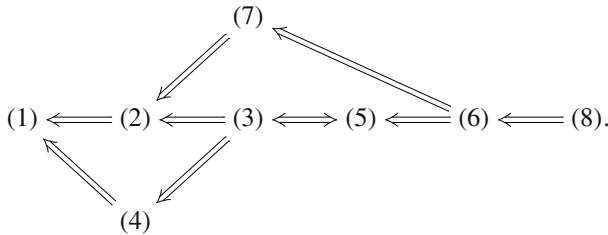
First, we consider the relationship among many kinds of infinite type hypersurfaces. Let $\Gamma^{\text{reg}} := \{\gamma \in \Gamma : \text{ord}(\gamma) = 1\}$ and define $\Delta_1^{\text{reg}}(M, p) := \sup_{\gamma \in \Gamma^{\text{reg}}}\{\text{ord}(r \circ \gamma)\}$, which is called *regular type* of M at p. Note that $\Delta_1^{\text{reg}}(M, p) \leq \Delta_1(M, p)$.

Proposition 1.1 *Let us consider the following eight conditions for a real hypersurface M at p:*

(1) $\Delta_1(M, p) = \infty$;
(2) $\Delta_1^{\text{reg}}(M, p) = \infty$;
(3) *There exists a $\gamma \in \Gamma^{\text{reg}}$ tangent to M at p to infinite order;*
(4) *There exists a $\gamma \in \Gamma$ tangent to M at p to infinite order;*
(5) *There exists a holomorphic coordinate $(z) = (z_1, \ldots, z_n)$ at p such that $p = 0$ and a defining function r for M on (z) is not convenient (see Sect. 2);*

(6) *There exists a holomorphic coordinate (z) at p such that $p = 0$ and the New-ton polyhedron of a defining function for M on (z) (see Sect. 2) takes the form $\mathcal{N}_+(r) = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}_+^n : \xi_n \geq 1\}$;*

(7) *The Bloom–Graham type of M at p is infinity (i.e. there are complex submanifolds of codimension one tangent to M at p to arbitrarily higher order);*

(8) *M is Levi-flat near p.*

Then, among the above eight conditions, the following implications hold:

$$(1) \Longleftarrow (2) \Longleftarrow (3) \Longleftrightarrow (5) \Longleftarrow (6) \Longleftarrow (8).$$

with (7) above connected by arrows to (2) and (5), and (4) below connected by arrows to (1) and (2).

The proof of the above proposition will be given in Sect. 4.1.

Remark 1.2 In the above proposition, for each implication with only one direction, its opposite direction is not true (see Remark 4.2 in Sect. 4 for details).

The following theorem gives a sufficient condition for the existence of the curve γ_∞ in Question 1. This condition is described by using Newton polyhedra of defining functions for real hypersurfaces.

Theorem 1.3 *If there exists an $\mathcal{N}$-canonical coordinate (z) for M at p, then the five conditions* (1)–(5) *are equivalent.*

The definition of $\mathcal{N}$-*canonical coordinates* will be given in Sect. 2 (Definition 2.3). The proof of the above theorem will be given in Sect. 4.2.

The following corollary can be directly obtained from Theorem 1.3.

Corollary 1.4 *Suppose that $\Delta_1(M, p) = \infty$. If there is no $\gamma \in \Gamma$ tangent to M at p to infinite order, then M does not admit any $\mathcal{N}$-canonical coordinates at p.*

It is easy to check that the examples of hypersurface constructed in [3,9,16,22] do not admit any $\mathcal{N}$-canonical coordinates. More exactly, we will give equivalence conditions in more restricted cases in Sects. 5 and 6.

Next, it is seen in [15] that when M is the boundary of pseudoconvex Rein-hardt domains, M always admits an $\mathcal{N}$-canonical coordinate. Therefore, Theorem 1.3 implies the following.

Corollary 1.5 *Let M be the boundary of pseudoconvex Reinhardt domains (with smooth boundary) and let p lie in M. If $\Delta_1(M, p) = \infty$, then there exists a $\gamma \in \Gamma^{\text{reg}}$ tangent to M at p to infinite order.*

This paper is organized as follows. In Sect. 2, we recall the concepts: Newton polyhedra, $\mathcal{N}$-nondegeneracy condition and $\mathcal{N}$-canonical coordinates, which were introduced in [15]. In Sect. 3, for the analysis later, we prepare appropriate coordinates on which hypersurfaces are expressed in clear form. Section 4 is devoted to the proof of results given in the Introduction. More precise results are given in the two dimensional case in Sect. 5 and the higher dimensional case under the Bloom–Graham infinity type assumption in Sect. 6. Since the Bloom–Graham type is the same as the D'Angelo type in the two-dimensional case, some results in Sect. 5 can be considered as special cases of those in Sect. 6. But, they are separately explained to make clear their difference. Lastly, we consider open problems in Sect. 7.

Notation, symbols and terminology.

- We denote $\mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.
- The multi-indices are used as follows. For $z = (z_1, \ldots, z_n)$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$, $\in \mathbb{C}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$, define

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \bar{z}^\beta := \bar{z}_1^{\beta_1} \ldots \bar{z}_n^{\beta_n},$$

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \alpha! := \alpha_1! \cdots \alpha_n!, \quad 0! := 1,$$

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \quad \bar{D}^\beta := \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}}.$$

- We always consider smooth functions, mappings, real hypersurfaces and complex curves as their respective germs without any mentioning. The following rings of germs of $\mathbb{C}$-valued functions are considered:
  - $C_0^\infty(\mathbb{C}^n)$ is the ring of germs of C^∞ functions at the origin in $\mathbb{C}^n$.
  - $\mathcal{O}_0(\mathbb{C})$ is the ring of germs of holomorphic functions at the origin in $\mathbb{C}$.

- We always take a *good* parametrization for corves $\gamma \in \Gamma$ without any mentioning. That is to say, a point on the curve, defined by $t \mapsto \gamma(t)$, corresponds to only one value of t. For example, (t, t^2) is good, but (t^2, t^4) is not good.
- We use the words *pure terms* for any harmonic polynomial and *mixed terms* for any sum of monomials that are neither holomorphic nor anti-holomorphic.

2 Newton Polyhedra for Real Hypersurfaces

Let us define the Newton polyhedron of a real-valued smooth function F defined near the origin in $\mathbb{C}^n$. The Taylor series expansion of F at the origin is

$$F(z, \bar{z}) \sim \sum_{\alpha, \beta \in \mathbb{Z}_+^n} C_{\alpha\beta} z^\alpha \bar{z}^\beta \quad \text{with } C_{\alpha\beta} = \frac{1}{\alpha!\beta!} D^\alpha \bar{D}^\beta F(0, 0). \qquad (2.1)$$

The *Newton polyhedron* of F is defined by

$$\mathcal{N}_+(F) = \text{The convex hull of} \left(\bigcup_{\alpha+\beta \in S(F)} (\alpha + \beta + \mathbb{R}^n_+) \right),$$

where $S(F) = \{\alpha + \beta \in \mathbb{Z}^n_+ : C_{\alpha\beta} \neq 0\}$. The *Newton diagram* $\mathcal{N}(F)$ of F is defined to be the union of the compact faces of $\mathcal{N}_+(F)$. We use coordinates $(\xi) = (\xi_1, \ldots, \xi_n)$ for points in the plane containing the Newton polyhedron. The following classes of functions F simply characterized by using their Newton polyhedra often appear in this paper:

- F is called to be *flat* at 0 if $\mathcal{N}_+(F)$ is an empty set.
- F is called to be *convenient* at 0 if $\mathcal{N}_+(F)$ meets every coordinate axis.

Let $(z) = (z_1, \ldots, z_n)$ be a holomorphic coordinate around p such that $p = 0$. Let r be a local defining function for M near p on the coordinate (z). For a given tuple $(M, p; (z))$, we define a quantity $\rho_1(M, p; (z)) \in \mathbb{Z}_+ \cup \{\infty\}$ as follows. If r is convenient, then let

$$\rho_1(M, p; (z)) := \max\{\rho_j(r) : j = 1, \ldots, n\},$$

where $\rho_j(r)$ is the coordinate of the point at which the Newton diagram $\mathcal{N}(r)$ intersects the ξ_j-axis. Otherwise, let $\rho_1(M, p; (z)) := \infty$. We remark that $\rho_1(M, p; (z))$ depends on the chosen coordinate (z), but it is independent of the choice of defining functions after fixing a coordinate. Considering the curves $\gamma_j(t) = (0, \ldots, \overset{(j)}{t}, \ldots, 0) \in \Gamma^{\text{reg}}$ for $j = 1, \ldots, n$, we can see that the inequality $\rho_1(M, p; (z)) \leq \Delta_1^{\text{reg}}(M, p)$ always holds.

Next, let us introduce an important concept "$\mathcal{N}$-nondegeneracy condition" on a smooth function F defined near the origin in $\mathbb{C}^n$.

Let κ be a compact face of $\mathcal{N}_+(F)$. The κ-*part of* F is the polynomial defined by

$$F_\kappa(z, \bar{z}) = \sum_{\alpha+\beta \in \kappa} C_{\alpha\beta} z^\alpha \bar{z}^\beta. \tag{2.2}$$

The set of holomorphic curves Γ_κ is defined by

$$\Gamma_\kappa := \{(c_1 t^{a_1}, \ldots, c_n t^{a_n}) : c \in (\mathbb{C} \setminus \{0\})^n, t \in \mathbb{C}, \, a \in \mathbb{N}^n \text{ determines } \kappa \},$$

where $c = (c_1, \ldots, c_n) \in (\mathbb{C} \setminus \{0\})^n$, $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and "$a \in \mathbb{N}^n$ determines κ" means that the set $\{\xi \in \mathcal{N}_+(F) : \sum_{j=1}^n a_j \xi_j = l\}$ coincides with the face κ for some $l \in \mathbb{N}$.

Definition 2.1 Let κ be a compact face of $\mathcal{N}_+(F)$. The κ-part F_κ of F is said to be $\mathcal{N}$-*nondegenerate* if

$$F_\kappa \circ \gamma \not\equiv 0 \quad \text{for any } \gamma \in \Gamma_\kappa.$$

A function F is said to be $\mathcal{N}$-*nondegenerate*, if F_κ is $\mathcal{N}$-nondegenerate for every compact face κ of $\mathcal{N}_+(F)$.

The above concept is analogous to the nondegeneracy condition introduced by Kouchnirenko [18], which plays important roles in the singularity theory. Detailed properties of this condition are explained in [15].

Definition 2.2 A holomorphic coordinate (z) at p is said to be $\mathcal{N}$-*canonical* for M at p if a local defining function r for M on (z) is $\mathcal{N}$-nondegenerate.

The following relationship

$$\Delta_1(M, p) \geq \Delta_1^{\text{reg}}(M, p) \geq \rho_1(M, p; (z)) \tag{2.3}$$

is always established for every coordinate (z) at p. The following theorem shows that the equalities in (2.3) are satisfied under the $\mathcal{N}$-nondegeneracy condition.

Theorem 2.3 [15] *If there exists an $\mathcal{N}$-canonical coordinate (z) at p, then the following equalities hold:*

$$\Delta_1(M, p) = \Delta_1^{\text{reg}}(M, p) = \rho_1(M, p; (z)). \tag{2.4}$$

Note that the above theorem is valid for the infinite type case. From the above theorem, the existence of $\mathcal{N}$-canonical coordinates implies that both values of $\Delta_1(M, p)$ and $\Delta_1^{\text{reg}}(M, p)$ can be directly seen from geometrical Newton data of M at p.

3 Standard Coordinates

Let M be a real hypersurface in $\mathbb{C}^{n+1}$ ($n \geq 1$) and let p lie in M.

It follows from Taylor's formula that there exists a holomorphic coordinate $(z, w) := (z_1, \ldots, z_n, w)$ at p on which a local defining function r for M is expressed near the origin as in the following form:

$$r(z, w, \bar{z}, \bar{w}) = 2\text{Re}(w) + F(z, \bar{z}) + R_1(z, \bar{z}) \cdot \text{Im}(w) + R_2(z, w, \bar{z}, \bar{w}), \quad (3.1)$$

where

(i) $F \in C_0^\infty(\mathbb{C}^n)$ satisfies that $F(0, 0) = 0$ and $|\nabla F(0, 0)| = 0$;
(ii) $R_1 \in C_0^\infty(\mathbb{C}^n)$ and $R_2 \in C_0^\infty(\mathbb{C}^{n+1})$. Moreover, R_1, R_2 satisfies that $\text{ord}(R_1) \geq 1$ and $|R_2| \leq C|\text{Im}(w)|^2$ near $(z, w) = 0$ where C is a positive constant independent of (z, w).

Of course, there may be many such coordinates, which are said to be *standard* for M at p.

Furthermore, if there exists a holomorphic coordinate $(z, w) := (z_1, \ldots, z_n, w)$ around p on which a local defining function r for M is expressed near the origin as in the *model* form:

$$r(z, w, \bar{z}, \bar{w}) = 2\text{Re}(w) + F(z, \bar{z}), \tag{3.2}$$

where F is as in (3.1), then (z, w) is called a *good (standard) coordinate* for M at p. Note that good coordinates do not always exist for all hypersurfaces.

4 Proofs of Results in the Introduction

Let M be a real hypersurface in $\mathbb{C}^n$ and let p lie in M.

From Theorem 2.3, under the $\mathcal{N}$-nondegeneracy condition, a defining function for M is convenient if and only if M is of finite type at p. Roughly speaking, when M does not satisfy the convenience condition, M contains a flat part in one direction. The following lemma exactly explains such situation.

Lemma 4.1 *Let r be a local defining function for M near p on some holomorphic coordinate (z). Then the following four conditions are equivalent:*

(i) *r is not convenient (i.e. $\rho_1(M, p; (z)) = \infty$);*
(ii) *There exists $k \in \{1, \ldots, n\}$ such that $\mathcal{N}_+(r)$ does not intersect ξ_k-axis;*
(iii) *There exists $k \in \{1, \ldots, n\}$ such that the Taylor series of r does not contain any term consisting of z_k and $\bar{z}_k$ only;*
(iv) *There exists $k \in \{1, \ldots, n\}$ such that $(r \circ \gamma_k)(t) = O(t^N)$ for every $N \in \mathbb{N}$, where $\gamma_k = (0, \ldots, \overset{(k)}{t}, \ldots, 0) \in \Gamma^{\mathrm{reg}}$.*

The following lemma expresses a property of flat hypersurfaces by using the language of Newton polyhedra.

Lemma 4.2 *The following two conditions are equivalent:*

(i) *There exists a standard coordinate for M at p such that F in (3.1) is flat at the origin;*
(ii) *There exists a holomorphic coordinate (z) at p such that $p = 0$ and the Newton polyhedron of a defining function r for M on (z) takes the form: $\mathcal{N}_+(r) = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}_+^n : \xi_n \geq 1\}$.*

Since the above two lemmas are easy, their proofs are omitted.

4.1 Proof of Proposition 1.1

$(3) \iff (5)$ was stated in Lemma 4.1. $(6) \implies (7)$ is easy to see from Lemma 4.2. $(7) \implies (2)$ is shown in [9]. (Lemma 5 in [9] states $(7) \implies (1)$, but its proof actually implies the above stronger implication.) The other implications are obvious.

Remark 4.3 Let us consider the converse of each implication in Proposition 1.1. We give an example of the hypersurface showing that the converse does not hold in the implication with only one direction. All our examples of hypersurface M in $\mathbb{C}^{n+1}$ ($n \in \mathbb{N}$) are pseudoconvex and admit a good coordinate, i.e., M is locally expressed as in the model form: $2\mathrm{Re}(w) + F(z, \bar{z}) = 0$, as in (3.2). Thus, we only write a function $F(z, \bar{z})$ for each case. In the two-dimensional case, the converses of some implications are also true (see Sect. 5). In these cases, counterexamples must be constructed in $\mathbb{C}^n$ with $n \geq 3$.

- $((1) \not\Rightarrow (2))$ $F(z, \bar{z}) = |z_1^3 - z_2^2|^2 + |z_3|^2 + \cdots + |z_n|^2$ $(n \geq 2)$;
- $((2) \not\Rightarrow (3))$ $F(z, \bar{z}) = f(z_1, \bar{z}_1) + |z_2|^2 + \cdots + |z_n|^2$ $(n \in \mathbb{N})$;
- $((1) \not\Rightarrow (4))$ $F(z, \bar{z}) = f(z_1, \bar{z}_1) + |z_2|^2 + \cdots + |z_n|^2$ $(n \in \mathbb{N})$;
- $((4) \not\Rightarrow (3))$ $F(z, \bar{z}) = |z_1^3 - z_2^2|^2 + |z_3|^2 + \cdots + |z_n|^2$ $(n \geq 2)$;
- $((5) \not\Rightarrow (6))$ $F(z, \bar{z}) = |z_n|^2$ $(n \geq 2)$;
- $((2) \not\Rightarrow (7))$ $F(z, \bar{z}) = |z_n|^2$ $(n \geq 2)$;
- $((7) \not\Rightarrow (6))$ $F(z, \bar{z}) = f(z_n, \bar{z}_n)$ $(n \in \mathbb{N})$;
- $((6) \not\Rightarrow (8))$ $F(z, \bar{z}) = \exp(-|z_n|^{-2})$ $(n \in \mathbb{N})$.

Here $f \in C_0^\infty(\mathbb{C})$ is a subharmonic function admitting a divergent Taylor series at the origin. It was constructed in [9] (see Remark 5.8).

It is shown in [16,22] (see also Corollary 6.4 in this paper) that the second example shows $(2) \not\Rightarrow (3)$ and $(1) \not\Rightarrow (4)$ in the two-dimensional case. The higher dimensional case can be easily shown. It is easy to check that the other hypersurfaces are counterexamples.

4.2 Proof of Theorem 1.3

It suffices to show the implication: $(1) \Longrightarrow (5)$.

Let (z) be an $\mathcal{N}$-canonical coordinate for M at p and let r be a local defining function for M near p on the coordinate (z). From Theorem 2.3, the condition (1) implies $\rho_1(M, p; (z)) = \infty$, which is equivalent to the condition (5) from Lemma 4.1.

5 Two Dimensional Case

In this section, we more precisely consider Question 1 in the Introduction in the case when a real hypersurface M is in $\mathbb{C}^2$. Let $p \in M$.

In the two dimensional case, many implications in Proposition 1.1 can be refined by equivalences.

Proposition 5.1 *Let M be a real hypersurface in $\mathbb{C}^2$ and let p lie in M. Among the above eight conditions for M at p in Proposition* 1.1, *the following implications hold:*
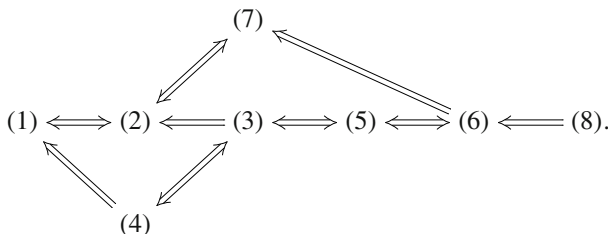
Proof $(5) \Longrightarrow (6)$ is obvious. It is known in [7] that (1), (2), (7) are equivalent. (4) $\Longrightarrow$ (3) will be shown in Lemma 5.2, below. □

Lemma 5.2 *If $\gamma \in \Gamma$ is tangent to $M \subset \mathbb{C}^2$ at p to infinite order, then $\gamma \in \Gamma^{\mathrm{reg}}$.*

Proof Let (z, w) be a standard coordinate for M at p such that M is expressed as in (3.1).

First, if F is flat at 0, then the curve γ, satisfying $(r \circ \gamma)(t) = O(t^N)$ for every $N \in \mathbb{N}$, essentially takes the form: $\gamma(t) = (t, 0)$. This curve is regular.

Next, let us consider the case when $\mathrm{ord}(F) = m$ with some $m \in \mathbb{N}$. In this case, the Newton polyhedron of r takes the form $\mathcal{N}_+(r) = \{\xi \in \mathbb{R}_+^2 : \xi_1/m + \xi_2 \geq 1\}$. Let $\kappa := \{\xi \in \mathbb{R}_+^2 : \xi_1/m + \xi_2 = 1\}$, which is the only compact facet of $\mathcal{N}_+(r)$. If r_κ were $\mathcal{N}$-nondegenerate, then Theorem 2.3 implies $\Delta_1(M, p) = m < \infty$, which is a contradiction. If r_κ is not $\mathcal{N}$-nondegenerate, then a desired curve must be written in the form $\gamma(t) = (t, ct^m + O(t^{m+1}))$ with $c \neq 0$ from the definition of the $\mathcal{N}$-nondegeneracy. This curve is also regular. $\qquad\square$

From Theorem 2.3, we can see the following.

Theorem 5.3 *Let $M \subset \mathbb{C}^2$. If there exists an $\mathcal{N}$-canonical coordinate for M at p, then the seven conditions (1)–(7) in Proposition 5.1 are equivalent.*

The following lemma is essentially the same as Theorem 2 in [16] (see also [17]).

Lemma 5.4 *Let $M \subset \mathbb{C}^2$. If $\Delta_1(M, p) = \infty$, then the Taylor series of F in (3.1) at the origin consists of only pure terms on every standard coordinate.*

Proof Let us assume the existence of a standard coordinate (z, w) on which the Taylor series of F contains a mixed term. Let m be the minimum order of the mixed terms of Taylor series of F and let $p_m(z, \bar{z})$ denote the sum of the mixed terms of order m.

It is easy to construct a standard coordinate (z, w^*) on which M can be expressed by the equation:

$$2\mathrm{Re}(w^*) + p_m(z, \bar{z}) + Q(z, \bar{z}) + R_1^* \cdot \mathrm{Im}(w^*) + R_2^* = 0,$$

where $Q \in C_0^\infty(\mathbb{C})$ with $\mathrm{ord}(Q) \geq m + 1$ and R_1^*, R_2^* have the same properties as those of R_1, R_2 in (3.1), respectively. Since $2\mathrm{Re}(w^*) + p_m(z)$ is $\mathcal{N}$-nondegenerate, the above (z, w^*) is $\mathcal{N}$- canonical for M at p. It follows from Theorem 2.3 that $\Delta_1(M, p) = m < \infty$, which is a contradiction. $\qquad\square$

From the above lemma, when $\Delta_1(M, p) = \infty$, the Taylor series of F can be expressed as $2 \sum_{j=2}^\infty \mathrm{Re}(c_j z^j)$ where $c_j \in \mathbb{C}$. When the hypersurface M and $p \in M$ are fixed, the sequence of complex numbers $\{c_j\}_{j \in \mathbb{N}}$ is determined by the chosen standard coordinates (z, w) only. For a given coordinate (z, w), we define the formal power series:

$$\mathcal{S}(z) = \sum_{j=2}^\infty c_j z^j \quad (z \in \mathbb{C}). \tag{5.1}$$

If M is real analytic, then the series $\mathcal{S}(z)$ converges near the origin. But, its converse is not always true. From Cauchy-Hadamard's formula, positivity of the convergence radius of the power series $\mathcal{S}(z)$ is equivalent to the condition $\limsup_{j \to \infty} |c_j|^{1/j} < \infty$.

Hereafter in this section, we only consider the case when the hypersurface admits a good coordinate at p. That is to say, M can be expressed as in the model form

$$r(z, w, \bar{z}, \bar{w}) = 2\mathrm{Re}(w) + F(z, \bar{z}) = 0, \qquad (5.2)$$

where F is the same as that in (3.1).

The following theorem gives equivalence conditions for Question 1 in the Introduction in the two-dimensional model case.

Theorem 5.5 *Let M be a real hypersurface in $\mathbb{C}^2$ admitting a good coordinate at p as in (5.2). If $\Delta_1(M, p) = \infty$, then the following three conditions are equivalent.*

(i) *There exists a $\gamma \in \Gamma^{\mathrm{reg}}$ tangent to M at p to infinite order;*
(ii) *There exists a good coordinate (z, w) for M at p on which F is flat;*
(iii) *There exists a good coordinate (z, w) for M at p such that the convergence radius of the power series $\mathcal{S}(z)$ in (5.1) is positive.*

Proof (ii) $\Longrightarrow$ (i) and (ii) $\Longrightarrow$ (iii) are obvious.

First, let us show (iii) $\Longrightarrow$ (ii). From (iii), the power series $\mathcal{S}(z)$ can be regarded as a holomorphic function defined on some open neighborhood of $z = 0$. Putting $w^* = w - \mathcal{S}(z)$, we can express the hypersurface M on the good coordinate (z, w^*) by the equation $2\mathrm{Re}(w^*) + F^*(z, \bar{z}) = 0$, where $F^*(z, \bar{z}) = F(z, \bar{z}) - 2\mathrm{Re}(\mathcal{S}(z))$. Since $F(z) - 2\mathrm{Re}(\mathcal{S}(z))$ is flat at $z = 0$, the existence of coordinate in (ii) is shown.

Next, let us show (i) $\Longrightarrow$ (iii). We may assume that the regular holomorphic curve in (i) can be expressed as $\gamma(t) = (t, -h(t))$ on a good coordinates (z, w) satisfying that $(r \circ \gamma)(t) = O(t^N)$ for every $N \in \mathbb{N}$. Here h is holomorphic near the origin and satisfies $h(0) = 0$. Let $\sum_{j=1}^{\infty} a_j t^j$ be the Taylor series of h at $t = 0$, which converges on an open neighborhood of $t = 0$. Now, after the first finite sum of the Taylor series of h and F are substituted into the equations $(r \circ \gamma)(t) = -2\mathrm{Re}(h(t)) + F(t, \bar{t}) = 0$, we have

$$\mathrm{Re}\left(\sum_{j=1}^{N} a_j t^j\right) - \sum_{j=2}^{N} \mathrm{Re}(c_j t^j) = O(t^{N+1}),$$

for every $N \in \mathbb{N}$. From the above equality, we can see that $c_j = a_j$ for every $j \in \mathbb{N}$. This means that $\mathcal{S}(t)$ converges and $\mathcal{S}(t) = h(t)$ on an open neighborhood of $t = 0$. □

Remark 5.6 The condition (iii) in Theorem 5.5 is weaker than the condition: M is real analytic near p. Indeed, as mentioned in the Introduction, the real analyticity of M implies the existence of desired curves in (i) in the theorem [7,11,19]. The hypersurface M defined by $2\mathrm{Re}(w) + e^{-1/|z|^2} = 0$ with $p = (0, 0)$ satisfies the conditions (ii) and (iii), but M is not real analytic at p.

In [3,16,22], smooth real hypersurfaces in $\mathbb{C}^2$ admitting no curve γ_∞ in (1.2) are constructed. It follows from Theorem 5.5 that there are many such hypersurfaces.

Corollary 5.7 *Let M be a real hypersurface defined by the equation (5.2). If $\Delta_1(M, p) = \infty$, then the following two conditions are equivalent.*

(i) *There is no $\gamma \in \Gamma$ tangent to M at 0 to infinite order;*

(ii) *The convergence radius of the power series $\mathcal{S}(z)$ equals zero.*

Remark 5.8 From (ii) in the above corollary, it is quite easy to construct smooth hypersurfaces satisfying the condition (i). Let $\{c_j\}_{j \in \mathbb{N}}$ be a sequence of complex numbers such that the convergence radius of $\sum_{j=2}^{\infty} c_j z^j$ is zero. By using a classical lemma of E. Borel (cf. [21], Theorem 1.5.4, or [14], Theorem 1.2.6), for the formal power series of $(x, y) \in \mathbb{R}^2$ ($z = x + iy$):

$$\sum_{j=2}^{\infty} \mathrm{Re}(c_j z^j) = \sum_{(j,k)\in\mathbb{Z}_+^2} C_{jk} x^j y^k \text{ with } C_{jk} \in \mathbb{R}, \qquad (5.3)$$

there exists a real-valued C^{∞} function f defined near the origin in $\mathbb{R}^2$ whose Taylor series at the origin is (5.3). Then the real hypersurface defined by $2\mathrm{Re}(w) + f(z, \bar{z}) = 0$ satisfies the condition (i). We remark that the above hypersurface cannot always be uniquely determined from the sequence $\{c_j\}_{j \in \mathbb{N}}$.

In [16,22], similar examples of hypersurfaces have been found. Since their constructions do not use a lemma of E. Borel directly, they look much more elaborate.

Furthermore, it is shown in [9] that when $\{c_j\}_{j \in \mathbb{N}}$ is an increasing sequence of positive real numbers, the above function f can be selected to be a subharmonic function on $\mathbb{C}$. Therefore, there are many pseudoconvex real hypersurfaces satisfying the condition (i) in the corollary.

6 Higher Dimensional Case

In this section, we generalize results given in the previous section to the higher dimensional case. Let M be a real hypersurface in $\mathbb{C}^{n+1}$ ($n \geq 1$) and let p lie in M. In this section, we always consider the case when M is of Bloom–Graham infinite type at p. First, let us recall the exact definition of the Bloom–Graham type in [3]. We remark that the following definition is an equivalence condition for their original type. This equivalence is also shown in [3].

Definition 6.1 Let $\mathcal{X}$ be a set of n-dimensional complex submanifolds containing p. We say that the Bloom–Graham type of M is m ($< \infty$), if there is a $X \in \mathcal{X}$ tangent to M at p to order m but no $X \in \mathcal{X}$ tangent to a higher order. Otherwise, we say that the Bloom–Graham type of M is infinity at p (see the condition (7) in Proposition 1.1).

In the case of Bloom–Graham infinity type, a similar property to Lemma 5.4 can be seen.

Lemma 6.2 *If the Bloom–Graham type of M at p is infinity, then the Taylor series of F in (3.1) at the origin consists of only pure terms on every standard coordinate (see Sect. 3).*

Proof Let us assume the existence of a standard coordinate (z, w) on which the Taylor series of F contains a mixed term. Let m be the minimum order of the mixed terms of Taylor series of F.

It is easy to construct a new standard coordinate (z, w^*) on which M can be expressed by the equation

$$2\mathrm{Re}(w^*) + P_m(z, \bar{z}) + Q(z, \bar{z}) + R_1^* \cdot \mathrm{Im}(w^*) + R_2^* = 0,$$

where P_m is a non-zero mixed homogenous polynomial of degree m without pure terms, $Q \in C_0^\infty(\mathbb{C}^n)$ with $\mathrm{ord}(Q) \geq m + 1$ and R_1^*, R_2^* have the same properties as those of R_1, R_2 in (3.1), respectively. It follows from the definition that the Bloom–Graham type of M at p equals $m < \infty$, which is a contradiction. $\square$

From Lemma 6.2, the Taylor series of F at the origin in $\mathbb{C}^n$ can be expressed as $2\sum_{\alpha \in \mathbb{Z}_+^n} \mathrm{Re}(c_\alpha z^\alpha)$ where $c_\alpha \in \mathbb{C}$. When the hypersurface M and $p \in M$ are fixed, the sequence $\{c_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is determined by the chosen standard coordinates (z, w). For a given coordinate (z, w), we define the formal power series

$$\mathcal{S}(z) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha \quad (z \in \mathbb{C}^n). \tag{6.1}$$

Hereafter in this section, we only consider the case when the hypersurface admits a good coordinate at p. That is to say, M can be expressed as in the model form

$$r(z, w, \bar{z}, \bar{w}) = 2\mathrm{Re}(w) + F(z, \bar{z}) = 0, \tag{6.2}$$

where F is the same as in (3.1).

Let $\hat{\Gamma}$ be the set of (germs of) nonconstant holomorphic curves $\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_n) :$ $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$. Let $\sum_{k=1}^\infty a_{jk} t^k$ be the Taylor series of $\hat{\gamma}_j$ for $j = 1, \ldots, n$. After these Taylor series are substituted into $\mathcal{S}(z_1, \ldots, z_n)$ with $z_j = \hat{\gamma}_j(t)$, a formal computation gives the formal power series, denoted by $(\mathcal{S} \circ \hat{\gamma})(t)$, in the following.

$$(\mathcal{S} \circ \hat{\gamma})(t) = \mathcal{S}(\hat{\gamma}_1(t), \ldots, \hat{\gamma}_n(t)) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha \prod_{j=1}^n \left(\sum_{k=1}^\infty a_{jk} t^k\right)^{\alpha_j}$$

$$=: \sum_{j=2}^\infty c_j t^j. \tag{6.3}$$

The relationship between F and $\mathcal{S}$ implies that

$$(F \circ \hat{\gamma})(t) = 2\sum_{j=2}^N \mathrm{Re}(c_j t^j) + O(t^{N+1}), \tag{6.4}$$

for every $N \in \mathbb{N}$.

The following theorem is a natural generalization of Theorem 5.6. Let Γ denote the set of nonconstant holomorphic mappings $\gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^{n+1}, p)$.

Theorem 6.3 *Let M be a real hypersurface in $\mathbb{C}^{n+1}$ $(n \geq 1)$ admitting a good coordinate at p as in (6.2). If the Bloom–Graham type of M at p is infinity, then the following three conditions are equivalent.*

(i) *There exists a $\gamma \in \Gamma$ tangent to M at p to infinite order;*

(ii) *There exists a good coordinate (z, w) for M at p on which $(F \circ \hat{\gamma})(t)$ is flat at $t = 0$ for some $\hat{\gamma} \in \hat{\Gamma}$;*

(iii) *There exists a good coordinate (z, w) for M at p such that the formal power series $(\mathcal{S} \circ \hat{\gamma})(t)$ in (6.3) converges on an open neighborhood of $t = 0$ for some $\hat{\gamma} \in \hat{\Gamma}$, where $\mathcal{S}$ is as in (6.1).*

Proof First, let us show (ii) $\Longrightarrow$ (i). Let $\gamma(t) = (\hat{\gamma}(t), 0) \in \Gamma$, where $\hat{\gamma}$ is as in (ii), and let r be a defining function for M as in (6.2). Then we have $(r \circ \gamma)(t) = (F \circ \hat{\gamma})(t) = O(t^N)$ for every $N \in \mathbb{N}$, which implies (i).

Second, let us show (iii) $\Longrightarrow$ (ii). Let $I := \{j : \hat{\gamma}_j \not\equiv 0\}$. Define the map $T_I : \mathbb{C}^n \to \mathbb{C}^n$ by $(w_1, \ldots, w_n) = T_I(z_1, \ldots, z_n)$ where $w_j = z_j$ if $j \in I$ and $w_j = 0$ otherwise. It follows from Abel's lemma (cf. [12]) that the convergence of $(\mathcal{S} \circ \hat{\gamma})(t)$ for some $t \neq 0$ implies that the power series $(\mathcal{S} \circ T_I)(z)$ converges on an open neighborhood of $z = 0$, which means that $\mathcal{S} \circ T_I$ can be regarded as a holomorphic function there. Letting $w^* = w - (\mathcal{S} \circ T_I)(z)$, we can express the hypersurface M on the good coordinate (z, w^*) by the equation: $2\text{Re}(w^*) + F^*(z, \bar{z}) = 0$, where $F^*(z, \bar{z}) = F(z, \bar{z}) - 2\text{Re}((\mathcal{S} \circ T_I)(z))$. By using the equality $\mathcal{S} \circ T_I \circ \hat{\gamma} = \mathcal{S} \circ \hat{\gamma}$, $(F^* \circ \hat{\gamma})(t) = (F \circ \hat{\gamma})(t) - 2\text{Re}(\mathcal{S} \circ \hat{\gamma})(t)$ is flat at $t = 0$, which implies (ii).

Third, let us show (i) $\Longrightarrow$ (iii). We may assume that a holomorphic curve in (i) can be expressed as $\gamma(t) = (\hat{\gamma}(t), -h(t))$, where $\hat{\gamma} \in \hat{\Gamma}$ and $h \in \mathcal{O}_0(\mathbb{C})$ with $h(0) = 0$, on a good coordinates (z, w). Note that (i) is equivalent to the condition $(r \circ \gamma)(t) = O(t^N)$ for every $N \in \mathbb{N}$. Let $\sum_{j=1}^{\infty} a_j t^j$ be the Taylor series of h at $t = 0$, which converges on an open neighborhood of $t = 0$. Now, after the first finite sum of the Taylor series of h and $F \circ \hat{\gamma}$ in (6.4) are substituted into the equations $(r \circ \gamma)(t) = -2\text{Re}(h(t)) + (F \circ \hat{\gamma})(t) = O(t^N)$ for every $N \in \mathbb{N}$, we have

$$\text{Re}\left(\sum_{j=1}^{N} a_j t^j\right) - \sum_{j=2}^{N} \text{Re}(c_j t^j) = O(t^{N+1}),$$

for every $N \in \mathbb{N}$. From the above equality, we can see that $c_j = a_j$ for every $j \in \mathbb{N}$. This means that $(\mathcal{S} \circ \hat{\gamma})(t)$ converges and $(\mathcal{S} \circ \hat{\gamma})(t) = h(t)$ on an open neighborhood of $t = 0$. $\qquad\square$

In [9], smooth pseudoconvex real hypersurfaces in $\mathbb{C}^{n+1}$ of Bloom–Graham infinite type admitting no curve γ_∞ in (1.2) are constructed. It follows from Theorem 6.3 that many such hypersurfaces can be easily constructed.

Corollary 6.4 *Let M be a real hypersurface defined by the equation (6.2). If the Bloom–Graham type of M at 0 is infinity, then the following three conditions are equivalent.*

(i) *There is no $\gamma \in \Gamma$ tangent to M at 0 to infinite order;*

(ii) *For all $\hat{\gamma} \in \hat{\Gamma}$, the formal power series $(\mathcal{S} \circ \hat{\gamma})(t)$ does not converge at any point on a delated open neighborhood of $t = 0$;*

(iii) *The formal power series $\mathcal{S}(z)$ does not converge at any point on a delated open neighborhood of $z = 0$.*

Proof We remark that (ii) $\Longrightarrow$ (iii) can be shown by using Abel's lemma (cf. the proof of (iii) $\Longrightarrow$ (ii) in Theorem 6.3). The other implications can be directly obtained from Theorem 6.3. □

Remark 6.5 From (iii) in the above corollary, it is easy to construct smooth pseudo-convex hypersurfaces of Bloom–Graham infinite type satisfying the condition (i). One of simple examples of hypersurfaces is given by the equation $\text{Re}(w) + f_1(z_1, \bar{z}_1) + \cdots + f_n(z_n, \bar{z}_n) = 0$, where f_j ($j = 1, \ldots, n$) are subharmonic functions constructed in [9] (see also Remark 5.8). The example constructed in [9] takes the same form, but this example needs some additional conditions for each f_j.

Roughly speaking, when the flatness of hypersurfaces is stronger, it becomes easier to find the curve tangent to M to higher order. Thus, the following question seems to be more difficult: does there exist a smooth pseudoconvex real hypersurface in $\mathbb{C}^{n+1}$ ($n \geq 2$) of the Bloom–Graham *finite type* that admits no $\gamma \in \Gamma$ tangent to M to infinite order? The following theorem gives an affirmative answer.

Theorem 6.6 *Let* $n \geq 2$. *There exists a smooth pseudoconvex real hypersurface* M *in* $\mathbb{C}^{n+1}$ *with* $\Delta_1(M, p) = \infty$ *and* $\Delta_1^{\text{reg}}(M, p) < \infty$ *(in particular,* M *is of Bloom–Graham finite type at* p*) that admits no* $\gamma \in \Gamma$ *tangent to* M *at* p *to infinite order.*

Proof Notice that since a desired real hypersurface M satisfies that $\Delta_1^{\text{reg}}(M, p) < \Delta_1(M, p)$, there is no $\mathcal{N}$-canonical coordinate for M from Theorem 2.3.

First, let us construct a desired real hypersurface M in $\mathbb{C}^3$. Let $\{c_j\}_{j \in \mathbb{N}}$ be an increasing sequence of positive real numbers such that the power series $\sum_{j=8}^{\infty} c_j z^j$ does not converge away from the origin. Let f be a real-valued smooth subharmonic function defined near the origin in $\mathbb{C}$ whose Taylor series is $2 \sum_{j=8}^{\infty} \text{Re}(c_j z^j)$ (see Remark 5.8). Let us consider the smooth function F defined near the origin in $\mathbb{C}^2$ as

$$F(z, \bar{z}) = \left| z_1^3 - z_2^2 \right|^2 + f(z_1, \bar{z}_1). \tag{6.5}$$

Let M be a smooth real hypersurface in $\mathbb{C}^3$ defined by $r(z, w) = 2\text{Re}(w) + F(z, \bar{z}) = 0$. The pseudoconvexity of M is obvious.

Now, let us consider $\Delta_1(M, 0)$ and $\Delta_1^{\text{reg}}(M, 0)$ by using the Newton polyhedron of $\mathcal{N}_+(r)$. It is easy to see

$$\mathcal{N}_+(r) := \{\xi \in \mathbb{R}_+^3 : \xi_1/6 + \xi_2/4 + \xi_3 \geq 1\}.$$

Let $\kappa_0 := \{(\xi_1, \xi_2, 0) \in \mathcal{N}_+(r) : \xi_1/6 + \xi_2/4 = 1\}$, which is a compact face of $\mathcal{N}_+(r)$. Then F_{κ_0} is not $\mathcal{N}$-nondegenerate, while F_κ's are $\mathcal{N}$-nondegenerate for the other compact faces κ. Therefore, Theorems 7.3 and 8.4 in [15] implies that if $\gamma \in \Gamma$ does not take the form

$$\gamma(t) = (t^2, \pm t^3 + o(t^3), o(t^{12})), \tag{6.6}$$

then we have

$$1 \leq \frac{\text{ord}(r \circ \gamma)}{\text{ord}(\gamma)} \leq 6.$$

In particular, every $\gamma \in \Gamma^{\text{reg}}$ does not take the form (6.6), so it is easy to see $\Delta_1^{\text{reg}}(M, 0) = 6$. More precisely, we consider the case when γ takes the form (6.6).

Lemma 6.7 *Let N be an arbitrary integer with $N \geq 10$. Then the following two conditions are equivalent.*

(i) $\text{ord}(r \circ \gamma) \geq 2N + 2$;
(ii) $\gamma(t) = (t^2, \pm t^3 + O(t^{N+1}), -\sum_{j=8}^{N} c_j t^{2j} + O(t^{2N+2}))$.

Proof By dividing the function r into a polynomial part and the remainder part, a defining function for M can be rewritten as follows.

$$r(z, w) = 2\text{Re}\left(w + \sum_{j=8}^{N} c_j z_1^j\right) + \left|z_1^3 - z_2^2\right|^2 + R_{N+1}(z_1, \bar{z}_1), \qquad (6.7)$$

where $R_{N+1} \in C_0^\infty(\mathbb{C})$ with $\text{ord}(R_{N+1}) \geq N + 1$.

First, we show (i) $\Longrightarrow$ (ii). It suffices to treat the curves of the form (6.6), which will be more exactly denoted by $\gamma(t) = (t^2, \pm t^3 + g(t), h(t))$ where $g \in \mathcal{O}_0(\mathbb{C})$ with $\text{ord}(g) > 3$ and $h \in \mathcal{O}_0(\mathbb{C})$ with $\text{ord}(h) > 12$. Substituting this $\gamma(t)$ into (6.7), we have

$$(r \circ \gamma)(t) = 2\text{Re}\left(h(t) + \sum_{j=8}^{N} c_j t^{2j}\right) + |g(t)|^2 + R_{N+1}(t^2, \overline{t^2}). \qquad (6.8)$$

Since $R_{N+1}(t^2, \overline{t^2}) = O(t^{2N+2})$ and the mixed terms and the pure terms cannot be canceled, g and h must satisfy that $g(t) = O(t^{N+1})$ and $h(t) + \sum_{j=8}^{N} c_j t^{2j} = O(t^{2N+2})$, which imply the condition (ii).

Next, we show (ii) $\Longrightarrow$ (i). Substituting the equation in (ii) into (6.7), we can easily see $\text{ord}(r \circ \gamma) \geq 2N + 2$. $\qquad \square$

It follows from the above lemma that $\Delta_1(M, p) = \infty$.

Now, let us assume that there exists a curve $\gamma_\infty \in \Gamma$ such that $\text{ord}(r \circ \gamma_\infty) \geq N$ for every $N \in \mathbb{N}$. Let $\gamma_\infty(t) =: (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. Since γ_∞ satisfies the condition (i) in Lemma 6.7, the condition (ii) implies that $\gamma_3(t) = -\sum_{j=8}^{N} c_j t^{2j} + O(t^{2N+2})$ for every $N \in \mathbb{N}$. But, $\sum_{j=8}^{\infty} c_j t^{2j}$ does not converge away from the origin, which is a contradiction to the holomorphy of γ_3. As a result, we see that there exists no $\gamma \in \Gamma$ tangent to M at the origin to infinite order.

In higher dimensional case $\mathbb{C}^{n+1}$ with $n \geq 3$, the following F is considered:

$$F(z, \bar{z}) = \left|z_1^3 - z_2^2\right|^2 + f(z_1, \bar{z}_1) + \sum_{j=3}^{n} |z_j|^2,$$

where f is the same as that in (6.5). It is easy to construct higher dimensional hyper-surfaces satisfying the properties in the theorem by using F in (6.5) in a similar fashion to the three-dimensional case. □

7 Open Problems

Theorems 5.5 and 6.3 only treat real hypersurfaces of the model form as in (3.2). The following problem is naturally raised.

Problem 7.1 Let $n \geq 1$ and let M be a general smooth (pseudoconvex) real hyper-surface in $\mathbb{C}^{n+1}$ and $p \in M$. Give equivalence conditions for the existence of $\gamma \in \Gamma$ tangent to M at p to infinite order, analogous to those in Theorems 5.5 and 6.3.

The second problem is concerned with Theorem 6.6.

Problem 7.2 Let $n \geq 2$ and let M be a smooth (pseudoconvex) real hypersurface in $\mathbb{C}^{n+1}$ with $\Delta_1^{\mathrm{reg}}(M, p) < \infty$ for $p \in M$. Give equivalence conditions for the existence of $\gamma \in \Gamma$ tangent to M at p to infinite order.

Theorem 6.6 only provides a simple example to the nonexistence of the desired $\gamma \in \Gamma$.

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