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A Remark on Atomic Decompositions of Martingale Hardy's Spaces

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Abstract
We prove theorems and exhibit a counterexample concerning an atomic decomposition of martingale $\mathbb{H}^1$ with atoms satisfying simultaneous cancellation condition (3).

Keywords Hardy space · Atomic decomposition · Dyadic martingale

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1 Introduction

Let $f \in L^1(\mathbb{R})$. We say that $f \in \mathbb{H}^1$ (dyadic martingale Hardy space) if

$$Mf(x) = \sup_{I:x\in I} \frac{1}{|I|}\left|\int_I f\right| \in L^1, \qquad \|f\|_{\mathbb{H}^1} = \|Mf\|_{L^1}, \tag{1}$$

where the supremum is taken over all dyadic intervals I containing x. The basic property of Hardy spaces is the so called atomic decomposition. We say that a function a_I is a (dyadic) atom, or an $\mathbb{H}^1$ - atom, associated with a dyadic interval I if

- $\operatorname{supp} a_I \subset I$,
- $\|a_I\|_{L^\infty} \le \frac{1}{|I|}$,
- $\int a_I = 0$.

Dedicated to Guido Weiss on the occasion of his 90th Birthday.

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Any function $f \in \mathbb{H}^1$ admits a decomposition

$$f = \sum_I \lambda_I a_I, \qquad \text{in } \mathbb{H}^1, \tag{2}$$

where a_I are atoms and

$$\sum_I |\lambda_I| \le C\|f\|_{\mathbb{H}^1}.$$

The martingale Hardy spaces were first introduced in [3], with atomic decomposition implicitly appearing in [4], and the explicit proof appearing in [1]. More on the subject of atomic decompositions in the martingale setting can be found in [8]. Classical introduction to Hardy spaces can be found in [7]. Atomic decompositions in the classical case were developed in [2] and [5]. In this note we address the following question: Suppose we are given two weights w_1, w_2 on $\mathbb{R}$ and $f \in \mathbb{H}^1$ satisfying

$$1 \le w_i \le C, \qquad w_i \cdot f \in \mathbb{H}^1,\ i = 1, 2.$$

Can one obtain a decomposition (2) with atoms a_I satisfying simultaneously

$$\int a_I \cdot w_1 = \int a_I \cdot w_2 = 0? \tag{3}$$

One of the main results of this note is a negative answer to this natural question.

We also give a maximal function characterization of those $f \in \mathbb{H}^1$ which admit decomposition (2) with atoms a_I satisfying (3).

The Results

Given a weight function w we call $w\mathbb{H}^1$ the space $\{f : w \cdot f \in \mathbb{H}^1\}$ with norm $\|f\|_{w\mathbb{H}^1} = \|w \cdot f\|_{\mathbb{H}^1}$. A function a is called a $w\mathbb{H}^1$ atom if $w \cdot a$ is an atom. Since the weights w we consider are bounded and bounded away from 0, the only difference between an atom and a $w\mathbb{H}^1$ atom is in the cancellation condition.

Theorem 1 *There exist weights $1 \equiv w_1 \le w_2 \le c$ and a function f with $w_1 f \in \mathbb{H}^1$ and $w_2 f \in \mathbb{H}^1$, which does not have a joint atomic decomposition, that is it does not have a decomposition*

$$f = \sum \lambda_Q b_Q, \qquad \sum |\lambda_Q| \le C \max\left\{\|f\|_{w_1\mathbb{H}^1}, \|f\|_{w_2\mathbb{H}^1}\right\}, \tag{4}$$

with b_Q being both $w_1\mathbb{H}^1$ and $w_2\mathbb{H}^1$ atoms.

Remark Theorem is stated for $w_1 \equiv 1$. It is clear, that the same construction can be applied to the case of arbitrary $w_1 \sim 1$. That is for any weight $w_1 \sim 1$ there exists a w_2 satisfying $w_1 \le w_2 \le C$, such that the theorem holds.

Proof Consider the interval [0, 1] and arbitrary $n \in \mathbb{N}$. Let I_k be consecutive, adjacent intervals of length 2^{-k}

$$I_k = \Big[\frac{2^k-2}{2^k}, \frac{2^k-1}{2^k}\Big], \quad k = 1, 2, \dots, n.$$

Each of these intervals is an element of the standard dyadic grid. The left half of I_k is denoted by I_k^+ and the right half by I_k^-:

$$I_k^+ = \Big[\frac{2^k-2}{2^k}, \frac{2^k-3/2}{2^k}\Big], \qquad I_k^- = \Big[\frac{2^k-3/2}{2^k}, \frac{2^k-1}{2^k}\Big].$$

We define:

$$a_{I_k}(x) = \begin{cases} 2^k\Big(1+\frac{1}{n}\Big)^{k-1} & : x \in I_k^+, \\ -2^k\Big(1+\frac{1}{n}\Big)^{k-1} & : x \in I_k^-, \\ 0 & : x \notin I_k. \end{cases}$$

The functions a_{I_k} satisfy:

- $\operatorname{supp} a_{I_k} \subset I_k$,
- $\int a_{I_k} = 0$,
- $\|a_{I_k}\|_{L^\infty} = 2^k\big(1+\frac{1}{n}\big)^{k-1} = \dfrac{(1+\frac{1}{n})^{k-1}}{|I_k|} \le \dfrac{e}{|I_k|}$,

that is they are e - multiples of $\mathbb{H}^1$ atoms. We let

$$f_0 = \sum_{k=1}^n a_{I_k},$$

and thus

$$\|f_0\|_{\mathbb{H}^1} \le \sum_{k=1}^n \|a_{I_k}\|_{\mathbb{H}^1} \le C\,n.$$

Let us define the weight w_2

$$w_2(x) = \begin{cases} \big(1+\frac{1}{n}\big) & : x \in I_k^-, \quad k = 1, 2, \dots, \\ 1 & : x \notin \bigcup I_k^-. \end{cases}$$

We obtain the decomposition

$$w_2 \cdot f_0 = 2\mathbb{1}_{I_1^+} - 2^n \cdot \Big(1+\frac{1}{n}\Big)^n \mathbb{1}_{I_n^-} + \sum_{k=1}^{n-1} b_{J_k},$$

where

$$J_k = I_k^- \cup I_{k+1}^+,$$

and

$$b_{J_k}(x) = \begin{cases} -2^k \left(1+\frac{1}{n}\right)^k & : x \in I_k^-, \\ 2^{k+1} \left(1+\frac{1}{n}\right)^k & : x \in I_{k+1}^+, \\ 0 & : x \notin J_k. \end{cases}$$

Observe, that b_{J_k} are $3e/2$ multiples of $\mathbb{H}^1$ atoms.

- $\operatorname{supp} b_{J_k} = J_k$,
- $\int b_{J_k} = -\frac{1}{2}(1+\frac{1}{n})^k + \frac{1}{2}(1+\frac{1}{n})^k = 0$,
- $\|b_{J_k}\|_{L^\infty} = 2^{k+1}(1+\frac{1}{n})^k = \dfrac{3(1+\frac{1}{n})^k}{2|J_k|} \le \dfrac{3e}{2|J_k|}$.

Note that

$$J_k = I_k^- \cup I_{k+1}^+ = \left[1-\frac{3}{2^{k+1}}, 1-\frac{3}{2^{k+2}}\right] \subset \left[1-\frac{1}{2^{k-1}}, 1\right] = \tilde{J}_k,$$

where $\tilde{J}_k$ is an element of the standard dyadic grid, with length comparable to that of J_k

$$|J_k| = \frac{3}{2^{k+2}}, \qquad |\tilde{J}_k| = \frac{1}{2^{k-1}}.$$

Thus

$$\left\| \sum_{k=1}^{n-1} b_{J_k} \right\|_{\mathbb{H}^1} \le \sum_{k=1}^{n-1} \|b_{J_k}\|_{\mathbb{H}^1} \le Cn.$$

To account for the remaining parts of $w_2 \cdot f_0$ we extend f_0 to $[1, 2]$ and modify w_2 there. Let

$$\alpha = \int f_0 w_2 = \frac{1}{2} - \frac{1}{2}\left(1+\frac{1}{n}\right)^n, \qquad -1 \le \alpha \le -\frac{1}{2}.$$

For $x \in [1, 2]$ we let

$$w_2(x) = \begin{cases} 1 & : x \in [1, \frac{3}{2}], \\ \left(1+\frac{1}{n}\right) & : x \in (\frac{3}{2}, 2], \end{cases}$$

and

$$f(x) = \begin{cases} f_0(x) & : x \in [0, 1], \\ 2\alpha n & : x \in [1, \frac{3}{2}], \\ -2\alpha n & : x \in (\frac{3}{2}, 2], \\ 0 & : x \notin [0, 2]. \end{cases}$$

Then

$$\|f\|_{\mathbb{H}^1} \leq \|f_0\|_{\mathbb{H}^1} + 2|\alpha|n \leq Cn.$$

Moreover $w_2 \cdot f$ decomposes as

$$w_2 \cdot f = \sum_{k=1}^{n-1} b_{J_k} + A,$$

where

$$A = 2\mathbb{1}_{I_1^+} - 2^n\left(1 + \tfrac{1}{n}\right)^n \mathbb{1}_{I_n^-} + 2\alpha n \mathbb{1}_{[1,3/2]} - 2\alpha n\left(1 + \tfrac{1}{n}\right)\mathbb{1}_{(3/2,2]}.$$

It is a straightforward argument to show that $\|A\|_{\mathbb{H}^1} \leq Cn$. Similar straightforward computation shows the same estimate holds for the classical, non-martingale Hardy's space. We will comment on this later. Thus

$$\|w_2 \cdot f\|_{\mathbb{H}^1} \leq \left\| \sum_{k=1}^{n-1} b_{J_k} \right\|_{\mathbb{H}^1} + \|A\|_{\mathbb{H}^1} \leq Cn.$$

We have just shown

$$\|f\|_{w_1\mathbb{H}^1} \leq Cn, \qquad \|f\|_{w_2\mathbb{H}^1} \leq Cn.$$

Consider the function

$$\begin{aligned} g(x) &= n(w_1(x) - w_2(x)) \cdot f(x) \\ &= \begin{cases} 2^k(1 + \frac{1}{n})^{k-1} & : x \in I_k^-, \quad k = 1, 2, \ldots, n \\ 2\alpha n & : x \in (\frac{3}{2}, 2], \\ 0 & : \text{ otherwise.} \end{cases} \end{aligned}$$

We will show, that

$$\|g\|_{\mathbb{H}^1} = \|Mg\|_{L^1} \geq C\, n^2.$$

Let us consider $x \in I_k^-$. Immediate dyadic parents of I_k^- are

$$I_k, \quad [1-2^{-i}, 1],\ i = 0, 1, \dots, k-1.$$

We compute the average of g over $I = [1-2^{-(k-1)}, 1]$.

$$\begin{aligned}\frac{1}{|I|}\int_I g(y)\,dy &= 2^{k-1}\sum_{i=k}^{n}\int_{I_i^-} 2^i\Big(1+\frac{1}{n}\Big)^{i-1}dy\\ &= 2^{k-1}\sum_{i=k}^{n} 2^{-(i+1)}2^i\Big(1+\frac{1}{n}\Big)^{i-1}\\ &= 2^{k-2}\sum_{i=k-1}^{n-1}\Big(1+\frac{1}{n}\Big)^{i}\\ &= 2^{k-2}n\left(\Big(1+\frac{1}{n}\Big)^{n}-\Big(1+\frac{1}{n}\Big)^{k-1}\right).\end{aligned}$$

Thus

$$Mg(x) \ge 2^{k-2}n\left(\Big(1+\frac{1}{n}\Big)^{n}-\Big(1+\frac{1}{n}\Big)^{k-1}\right), \qquad x \in I_k^-.$$

Integrating Mg we obtain

$$\|Mg\|_{L^1} \ge \sum_{k=1}^{n}|I_k^-|2^{k-2}n\left(\Big(1+\frac{1}{n}\Big)^{n}-\Big(1+\frac{1}{n}\Big)^{k-1}\right) = \frac{n^2}{8}.$$

In fact, we can show a stronger estimate, namely $\|g\|_{\mathbb{H}^1} \ge Cn^2$, where the norm is in the classical Hardy's space. We will comment on that in a remark below. To see this stronger estimate, let us fix a test function

$$\mathbb{1}_{[-1,1]} \le \Phi \le \mathbb{1}_{[-3/2,3/2]}.$$

Then

$$\begin{aligned}\Phi_t * g(x) &= \Big(\Phi_t * \sum_{k=1}^{n}2^k\Big(1+\frac{1}{n}\Big)^{k-1}\mathbb{1}_{I_k^-}\Big)(x) + \Big(\Phi_t * (2\alpha n\mathbb{1}_{(3/2,2]})\Big)(x)\\ &= \Big(\Phi_t * F\Big)(x) + \Big(\Phi_t * G\Big)(x),\\ &= \frac{1}{t}\int_{x-\frac{3}{2}t}^{x+\frac{3}{2}t}\Phi\Big(\frac{x-y}{t}\Big)F(y)\,dy + \frac{1}{t}\int_{x-\frac{3}{2}t}^{x+\frac{3}{2}t}\Phi\Big(\frac{x-y}{t}\Big)G(y)\,dy\end{aligned}$$

where

$$\operatorname{supp} F = \bigcup_{k=n/2}^{n} I_k^- \subset [1 - 3 \cdot 2^{-n/2-1}, 1], \quad \operatorname{supp} G = (3/2, 2].$$

Now, take $x \in [0, 1 - 3 \cdot 2^{-n/2}]$ ($n \geq 4$), and $t = 1 - x$. Then the second integral vanishes due to disjoint supports. Thus

$$\Phi_t * g(x) = \frac{1}{t} \int_{x-\frac{3}{2}t}^{x+\frac{3}{2}t} \Phi\Big(\frac{x-y}{t}\Big) F(y)\, dy \geq \frac{1}{t} \int_{x-t}^{x+t} F(y)\, dy$$

Since $t = 1 - x$, we have $x + t = 1$ and $x - t = 2x - 1 \leq 1 - 3 \cdot 2^{-n/2-1}$, so the integration interval covers the entire support of F. Thus, in the chosen range of x

$$\begin{aligned}
\big|Mg(x)\big| &\geq \Big|\Phi_{1-x} * g(x)\Big| \\
&\geq \frac{1}{1-x} \int_{2x-1}^{1} F(y)\, dy \\
&= \frac{1}{1-x} \sum_{k=n/2}^{n} \Big(1 + \frac{1}{n}\Big)^{k-1} \cdot \frac{1}{2} \\
&\geq \frac{1}{1-x} \cdot \frac{n}{4}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|Mg\|_{L^1} &\geq \int_0^{1-3\cdot 2^{-n/2}} |Mg(x)|\, dx \\
&\geq \frac{n}{4} \int_0^{1-3\cdot 2^{-n/2}} \frac{dx}{1-x} \\
&= \frac{n}{4} \cdot \Big(\frac{n}{2} \log 2 - \log 3\Big) \\
&\geq Cn^2.
\end{aligned}$$

We continue with the proof of the theorem. Suppose f does have a decomposition

$$f = \sum_Q \lambda_Q b_Q,$$

where b_Q are both $w_1\mathbb{H}^1$ and $w_2\mathbb{H}^1$ atoms, as in the statement of the theorem. Then, by estimates above

$$\sum_Q |\lambda_Q| \leq Cn.$$

This would imply

$$\begin{aligned}\|g\|_{\mathbb{H}^1} &= \left\| n(w_1 - w_2) \sum_Q \lambda_Q \cdot b_Q \right\|_{\mathbb{H}^1} \\ &\leq \sum_Q |\lambda_Q| \cdot \|n(w_1 - w_2) b_Q\|_{\mathbb{H}^1} \\ &\leq Cn,\end{aligned}$$

since each $n(w_1 - w_2)b_Q$ is an $\mathbb{H}^1$ atom:

- supp $n(w_1 - w_2)b_Q \subset Q$,
- $\int n(w_1 - w_2)b_Q = 0$,
- $\|n(w_1 - w_2)b_Q\|_{L^\infty} \leq \|b_Q\|_{L^\infty} \leq \frac{1}{|Q|}$.

Thus, since the constants are independent of n, we have obtained a contradiction

$$n^2 \lesssim \|g\|_{\mathbb{H}^1} \lesssim n.$$

We call just constructed function f_n, and the weight w^n. Both are localized on [0, 2]. It is now routine to appropriately $\mathbb{H}^1$-scale and shift thus constructed f_n's, together with w^n's (both operations necessarily dyadic), so they are all localized within [0, 1], with disjoint supports. The sum of $n^{-\frac{3}{2}} f_n$'s over a dyadic n's, together with weight w, being the sum of w^n's is the required example for which the condition (4) cannot hold. This completes the proof. □

Remark Observe, that the above theorem is also valid in the case of classical Hardy's space.

We point to another possible construction of the weight w from Theorem 1, very much in the spirit of tweaks known from the theory of Cauchy Integral. Let $I \subset [0, 1]$ be dyadic. We let w^n be given by (5) below (modified weight from the above theorem), and denote by w^n_I this weight re-scaled and translated to I. Suppose $\{n_k\}$ is a sequence of naturals increasing to infinity sufficiently fast. We construct a sequence of weights ω^k.

(i) We put $\omega^1 = w^{n_1}_{[0,1]}$.
(ii) Assume $\omega^1, \ldots, \omega^k$ have already been constructed. Let $I_{k,j},\ j = 1, \ldots, l_k$ be the maximal dyadic intervals on which $\omega^k = c_{k,j}$ is constant. Then, for $x \in I_{k,j}$ we put $\omega^{k+1}(x) = c_{k,j} w^{n_{k+1}}_{I_{k,j}}(x)$. Observe that by construction $\bigcup_j I_{k,j} = [0, 1]$
(iii) We put $w(x) = \lim_{k\to\infty} \omega^k(x)$.

The weight obviously satisfies requirements of the Theorem 1 (together with the function f, which consists of parts constructed in the proof, but summed differently). We also point out (we leave the proof to the reader), that it satisfies the condition (6) below.

We will now prove a maximal function characterization of those functions on [0, 1] that do admit atomic decomposition with atoms satisfying double cancellation condition. From now on we fix n and the weights $w_1 \equiv 1$ and $w_2 = w$ on [0, 1] constructed in the proof of Theorem 1. Let us recall

$$w(x) = \begin{cases} \left(1 + \frac{1}{n}\right) & : x \in I_k^-, \quad k = 1, 2, \dots, \\ 1 & : x \notin \bigcup I_k^-, \end{cases} \tag{5}$$

(we do no restrict k to be $\leq n$, thus $\operatorname{supp} w = [0, 1]$). We will state a quantitative version of our result for these weights. The argument clearly extends to any pair w_1, w_2 satisfying condition (6) below, with $w = w_2 w_1^{-1}$. Typical examples of such weights are those defined by lacunary Fourier series or positive polynomials. See Corollary following Theorem 2.

Let us recall that we are working in the setting of the standard dyadic martingale on $\mathbb{R}$. Our aim is to define a maximal function which would characterize an atomic space with atoms simultaneously orthogonal to both 1 and w. We put

$$\Delta_I(x) = \mathbb{1}_I(x)(\beta_I - w(x))$$

with the constant β_I chosen so that $\int \Delta_I = 0$. Obviously,

$$\frac{\mathbb{1}_I(x)}{|I|^{1/2}}, \qquad \frac{\Delta_I(x)}{\|\Delta_I\|_{L^2(I)}}$$

are orthonormal functions in $L^2(I)$, obtained by Gramm-Schmidt orthogonalization of weights $\mathbb{1}_I$ and $\mathbb{1}_I \cdot w$ on I. We define the following maximal operator

$$\mathcal{M} f(x) = \sup_{\substack{I \subset [0,1]-\text{dyad.} \\ \alpha \in \mathbb{R}}} \frac{\mathbb{1}_I(x)}{|I| \cdot \|\alpha - w\|_{L^\infty(I)}} \left| \int_I f \cdot (\alpha - w) \right|.$$

It is immediate that $\mathcal{M} f \leq C M_{HL} f$, where $M_{HL} f$ is the standard Hardy-Littlewood maximal function on [0, 1]. We will use the following

Lemma *Let $I \subset [0, 1]$. Then, for some constant C independent on n, we have*

$$\left| \frac{\Delta_I(x)}{\|\Delta_I\|_{L^2}^2} \int_I f \cdot \Delta_I \right| \leq C \mathcal{M} f(x)$$

Proof The lemma follows immediately from the following condition satisfied by the weight $\alpha - w$: there is the constant C independent of α and n such that for any dyadic interval I we have

$$|I| \cdot \|\alpha - w\|_{L^\infty(I)}^2 \leq C \|\alpha - w\|_{L^2}^2 \tag{6}$$

To see this, suppose $I \subset [0, 1]$ is a dyadic interval. Since $\bigcup_{k \geq 1} I_k = [0, 1)$ one of the following cases has to hold.

(i) There exists a k such that $I \subset I_k$ properly. Then $I \subset I_k^-$ or $I \subset I_k^+$ and $\alpha - w$ is constant on I.

(ii) There exists a k such that $I = I_k$. Then $\alpha - w$ is constant on both I_k^-, I_k^+.

(iii) There exists a k such that $I_k \subset I$ properly. We denote by k_0 minimal such k. Let $J^\#$ denote the immediate dyadic parent of J. Then $I_{k_0}^\# = [\frac{2^{k_0}-2}{2^{k_0}}, 1] \subset I$. If $I_{k_0}^\# \subset I$ properly, than $I_{k_0-1} \subset (I_{k_0}^\#)^\# \subset I$, contradicting the definition of k_0. Hence $I = I_{k_0}^\#$, $I_{k_0} \subset I$, $2|I_{k_0}| = |I|$ and $\alpha - w$ takes exactly two values on I. We note that both values are taken on $I_{k_0}^-, I_{k_0}^+$.

To summarize, I either is contained within some I_k, or contains a number of I_k's in their entirety. In either case the function $\alpha - w$ on I is constant, or assumes exactly 2 values, spread over sets of equal measure. In any case, the norm equivalence condition (6) is immediate. □

The following two theorems have motivated the construction of the counterexample in Theorem 1. We recall that we work with the weight w constructed for a fixed n.

Theorem 2 *If* $\mathcal{M}f \in L^1([0,1])$, supp $f \subset [0,1]$, $\int f = \int wf = 0$, *then* f *admits decomposition*

$$f = \sum_{I-dyad.} \lambda_I \, a_I,$$

where, for some constant C *independent on* n

$$\sum_I |\lambda_I| \le C \big\|\mathcal{M}f\big\|_{L^1},$$

and a_I *are atoms, satisfying double cancellation condition*

$$\int a_I = \int a_I \, w = 0.$$

Proof We note that the argument we use in the proof is standard, the only difference is in the cancellation statements involved. We begin with a definition of an auxiliary maximal operator $\mathcal{M}^*$, playing the role of the classical grand maximal operator.

$$\mathcal{M}^* f(x) = \sup_{\substack{I \subset [0,1], I\text{-dyad.} \\ \alpha \in \mathbb{R}}} \frac{\mathbb{1}_{10I}(x)}{|I| \cdot \|\alpha - w\|_{L^\infty(I)}} \left| \int_I f \cdot (\alpha - w) \right|.$$

It is an easy consequence of the definition of $\mathcal{M}$ and $\mathcal{M}^*$ that

$$\big|\{x : \mathcal{M}^* f(x) > \lambda\}\big| \le 10 \big|\{x : \mathcal{M}f(x) > \lambda\}\big|.$$

If $x \in \{x : \mathcal{M}^* f(x) > \lambda\}$ than $x \in 10I$ for some dyadic interval I such that $I \subset \{x : \mathcal{M}f(x) > \lambda\}$. Now, if we write

$$\{x : \mathcal{M}f(x) > \lambda\} = \bigcup \tilde{I}$$

where $\tilde{I}$ are maximal dyadic. Thus each $I \subset \tilde{I}$ for some $\tilde{I}$ and consequently

$$\left|\{x : \mathcal{M}^* f(x) > \lambda\}\right| \le 10\left|\{x : \mathcal{M} f(x) > \lambda\}\right|.$$

The immediate corollary is

$$\|\mathcal{M}^* f\|_{L^1} \le C\|\mathcal{M} f\|_{L^1}.$$

For the dyadic interval I we denote by $P_I(f)$ the orthonormal projection of f onto the space spanned by the weights $\mathbb{1}_I$ and $\mathbb{1}_I \cdot w$

$$P_I(f)(x) = \frac{\mathbb{1}_I(x)}{|I|}\int_I f(y)\,dy + \frac{\Delta_I(x)}{\|\Delta_I\|_{L^2}^2}\int_I f(y)\Delta_I(y)\,dy. \tag{7}$$

We observe that

$$\langle P_I(f), 1\rangle = \langle f \cdot \mathbb{1}_I, 1\rangle, \quad \text{and} \quad \langle P_I(f), w\rangle = \langle f \mathbb{1}_I, w\rangle,$$

directly by the definition (7). Denote by A_s the set $\{\mathcal{M}^* f(x) > 2^s\}$ and let $A_1 = \bigcup_{r_1} I_{r_1}$ be the Whitney decomposition. Since by the construction $A_{s+1} \subset A_s$ so we can choose the Whitney decomposition $A_2 = \bigcup_{r_1,r_2} I_{r_1,r_2}$ in such a way that $I_{r_1,r_2} \subset I_{r_1}$. We continue this process, obtaining of a tree of dyadic intervals $\{I_{r_1}, \cdots, I_{r_1,r_2,\cdots,r_s}, \cdots\}$. We write

$$f(x) = \sum_s \sum_{r_1,r_2,\cdots,r_s} \Bigg(\Big(\mathbb{1}_{I_{r_1,r_2,\cdots,r_s}}(x) - \sum_{r_{s+1}} \mathbb{1}_{I_{r_1,r_2,\cdots,r_s,r_{s+1}}}(x)\Big) \cdot f(x) - \\ - P_{I_{r_1,r_2,\cdots,r_s}}(f)(x) + \sum_{r_{s+1}} P_{I_{r_1,r_2,\cdots,r_s,r_{s+1}}}(f)(x)\Bigg)$$

Each component

$$b_{I_{r_1,r_2,\cdots,r_s}}(x) = \Big(\mathbb{1}_{I_{r_1,r_2,\cdots,r_s}}(x) - \sum_{r_{s+1}} \mathbb{1}_{I_{r_1,r_2,\cdots,r_s,r_{s+1}}}(x)\Big) \cdot f(x) - \\ - P_{I_{r_1,r_2,\cdots,r_s}}(f)(x) + \sum_{r_{s+1}} P_{I_{r_1,r_2,\cdots,r_s,r_{s+1}}}(f)(x)$$

satisfies

$$\begin{gathered} |b_{I_{r_1,r_2,\cdots,r_s}}(x)| \le C2^s \\ \sum_s \sum_{r_1,r_2,\cdots,r_s} 2^s |I_{r_1,r_2,\cdots,r_s}| \le C\|\mathcal{M}^* f\|_{L^1} \\ \langle b_{I_{r_1,r_2,\cdots,r_s}}, \mathbb{1}_I\rangle = 0, \qquad \langle b_{I_{r_1,r_2,\cdots,r_s}}, \Delta_I\rangle = 0. \end{gathered} \tag{8}$$

This means that

$$a_{I_{r_1,r_2,\cdots,r_s}}(x) = \frac{b_{I_{r_1,r_2,\cdots,r_s}}(x)}{2^s|I_{r_1,r_2,\cdots,r_s}|}$$

are double cancellation atoms and

$$f(x) = \sum \lambda_{I_{r_1,r_2,\cdots,r_s}} a_{I_{r_1,r_2,\cdots,r_s}}(x),$$

where

$$\sum |\lambda_{I_{r_1,r_2,\cdots,r_s}}| \le C\|\mathcal{M}^* f\|_{L^1}.$$

The theorem follows. □

We leave it to the reader to extend Theorem 2 to any pair w_1, w_2 having the property that $w = w_2 w_1^{-1}$ satisfies (6).

Corollary *Assume that w_1, w_2 are polynomials satisfying $1 \le w_1, w_2 \le C$ on $[0, 1]$ and that $f \in \mathbb{H}^1$, $\operatorname{supp} f \subset [0, 1]$ is simultaneously orthogonal to w_1 and w_2. Then f admits an atomic decomposition with atoms simultaneously orthogonal to w_1, w_2.*

Proof The proof is based on standard norm-comparison properties for polynomials . We need to check the assumption of the Theorem 2 for polynomials w_1 and w_2.

If $J \subset [0, 1]$ is an interval, $|J| = R$ and w is a polynomial of degree d (here $w = \alpha w_1 - w_2$), then

$$\sup_{x\in J} |w(x)| \approx \sum_{k=0}^{d} R^k |w^{(k)}(a)| \tag{9}$$

The implied constants in (9) do not depend on R and $a \in J$. Hence, for any $I \subset J$, we have $\sup_{x\in J} |w(x)| \ge CR \sup_{x\in I} |w^{(1)}(x)|$ again with the constant C independent of I and J. With this observation the estimate for $\mathcal{M}a$, a being a classical $\mathbb{H}^1$ atom, is an application of the standard cancellation argument. We note, that the proof of the corollary can be made independent of Theorem 2. We leave details to the reader. □

We present one more result.

Theorem 3 *Suppose $f \in w_1\mathbb{H}^1 \cap w_2\mathbb{H}^1$. Then there exist coefficients $\{\lambda_{r_1,r_2,\ldots,r_s} : r_i \in J_i - \text{finite},\ s \in \mathbb{N}\}$ and a tree of dyadic intervals $\{I_{r_1,r_2,\ldots,r_s} : r_i \in J_i - \text{finite},\ s \in \mathbb{N}\}$, with inclusions*

$$I_{r_1} \supset I_{r_1,r_2} \supset \cdots \supset I_{r_1,r_2,\ldots,r_s} \supset \ldots$$

such that f admits atomic decompositions

$$f = \sum_{\substack{r_i\in I_i, i=1,\ldots s\\ s\in\mathbb{N}}} \lambda_{r_1,r_2,\ldots,r_s} b^j_{r_1,r_2,\ldots,r_s}, \qquad j = 1, 2,$$

where the atoms are given by

$$b^j_{r_1,r_2,\dots,r_s}(x) = \frac{1}{\lambda_{r_1,r_2,\dots,r_s}}\Big(f\cdot \mathbb{1}_{I_{r_1,\dots,r_s}\setminus\bigcup_{r_{s+1}} I_{r_1,\dots,r_{s+1}}}(x) - \langle f\rangle_{I_{r_1,\dots,r_s},w_j} + \sum_{r_{s+1}} \mathbb{1}_{I_{r_1,\dots,r_{s+1}}}(x)\langle f\rangle_{I_{r_1,\dots,r_{s+1}},w_j}\Big),$$

and

$$\langle f\rangle_{I,w} = \frac{1}{|I|}\int_I f\cdot w.$$

We omit the proof which is similar to the classical case of dyadic atomic decomposition. The argument yielding Theorem 2 can be adapted here as well. The only change required is to consider level sets of $\mathcal{M}_1^*(w_1\cdot f)+\mathcal{M}_1^*(w_2\cdot f)$, for $\mathcal{M}_1^*$ being the dyadic grand maximal operator

$$\mathcal{M}^* f(x) = \sup_{\substack{I\subset[0,1],\, I\text{-dyad.}\\ \alpha\in\mathbb{R}}} \frac{\mathbb{1}_{10I}(x)}{|I|}\Big|\int_I f\Big|.$$

We leave details to the reader.

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