



Optimal Extensions of Conformal Mappings from the Unit Disk to Cardioid-Type Domains

Haiqing Xu¹

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Abstract

The conformal mapping $f(z) = (z + 1)^2$ from \mathbb{D} onto the standard cardioid has a homeomorphic extension of finite distortion to entire \mathbb{R}^2 . We study the optimal regularity of such extensions, in terms of the integrability degree of the distortion and of the derivatives, and these for the inverse. We generalize all outcomes to the case of conformal mappings from \mathbb{D} onto cardioid-type domains.

Keywords Extensions · Homeomorphisms of finite distortion · Inner cusp

Mathematics Subject Classification 30C20

1 Introduction

The standard cardioid domain

$$\Delta = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 < 0\} \quad (1.0.1)$$

is the image of the unit disk \mathbb{D} under the conformal mapping $g(z) = (z + 1)^2$. Since the origin is an inner-cusp point of $\partial\Delta$, the Ahlfors' three-point property fails, and hence $\partial\Delta$ is not a quasicircle. Therefore the preceding conformal mapping does not possess a quasiconformal extension to the entire plane. However, there is a homeomorphic extension $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by the Schoenflies theorem, see [10, Theorem 10.4]. Recall that homeomorphisms of finite distortion form a much larger class of homeomorphisms than quasiconformal mappings. A natural question arises: can we extend g as a homeomorphism of finite distortion? If we can, how good an extension can we find? Our first result gives a rather complete answer.

✉ Haiqing Xu
hqxu@mail.ustc.edu.cn

¹ Department of Mathematics and Statistics, University of Jyväskylä, PO BOX 35, 40014 Jyväskylä, Finland

Theorem 1.1 *Let \mathcal{F} be the collection of homeomorphisms $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of finite distortion such that $f(z) = (z + 1)^2$ for all $z \in \mathbb{D}$. Then $\mathcal{F} \neq \emptyset$. Moreover*

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F} \cap W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)\} = +\infty, \tag{1.0.2}$$

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}, K_f \in L_{loc}^q(\mathbb{R}^2)\} = 2, \tag{1.0.3}$$

$$\begin{aligned} \sup\{q \in (0, +\infty) : f \in \mathcal{F} \cap W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2) \text{ for a fixed } p > 1 \text{ and } K_f \in L_{loc}^q(\mathbb{R}^2)\} \\ = 1, \end{aligned} \tag{1.0.4}$$

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F}, f^{-1} \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)\} = \frac{5}{2} \tag{1.0.5}$$

and

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}, K_{f^{-1}} \in L_{loc}^q(\mathbb{R}^2)\} = 5. \tag{1.0.6}$$

The cardioid curve $\partial\Delta$ contains an inner-cusp point of asymptotic polynomial degree $3/2$. Motivated by this, we introduce a family of cardioid-type domains Δ_s with degree $s > 1$, see (2.3.2). Our second result is an analog of Theorem 1.1.

Theorem 1.2 *Let g be a conformal map from \mathbb{D} onto Δ_s , where Δ_s is defined in (2.3.2) and $s > 1$. Suppose that $\mathcal{F}_s(g)$ is the collection of homeomorphisms $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of finite distortion such that $f|_{\mathbb{D}} = g$. Then $\mathcal{F}_s(g) \neq \emptyset$. Moreover*

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F}_s(g) \cap W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)\} = +\infty, \tag{1.0.7}$$

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}_s(g), K_f \in L_{loc}^q(\mathbb{R}^2)\} = \max\left\{\frac{1}{s-1}, 1\right\}, \tag{1.0.8}$$

$$\begin{aligned} \sup\{q \in (0, +\infty) : f \in \mathcal{F}_s(g) \cap W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2) \text{ for a fixed } p > 1 \text{ and } K_f \in L_{loc}^q(\mathbb{R}^2)\} \\ = \max\left\{\frac{1}{s-1}, \frac{3p}{(2s-1)p+4-2s}\right\}, \end{aligned} \tag{1.0.9}$$

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F}_s(g), f^{-1} \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)\} = \frac{2(s+1)}{2s-1} \tag{1.0.10}$$

and

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}_s(g), K_{f^{-1}} \in L_{loc}^q(\mathbb{R}^2)\} = \frac{s+1}{s-1}. \tag{1.0.11}$$

Let us recall previous extension results. In [3,4], sufficient conditions on Ω are introduced to guarantee that a conformal mapping $g : \mathbb{D} \rightarrow \Omega$ has a homeomorphic extension of locally exponentially integrable distortion to the whole plane. Specially, when Ω is a Jordan domain with an outer-cusp point on its boundary, the authors from [8] established the optimal exponential regularity of distortion of homeomorphic extensions.

In Sect. 2, we recall some basic definitions and facts. We also introduce auxiliary mappings and domains. In Sect. 3, we give upper bounds for integrability degrees of potential extensions. Section 4 is devoted to the proof of Theorem 1.2. In Sect. 5 we prove Theorem 1.1.

2 Preliminaries

2.1 Notation

By $s \gg 1$ and $t \ll 1$ we mean that s is sufficiently large and t is sufficiently small, respectively. By $f \lesssim g$ we mean that there exists a constant $M > 0$ such that $f(x) \leq Mg(x)$ for every x . We write $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$ hold. By \mathcal{L}^2 (respectively \mathcal{L}^1) we mean the 2-dimensional (1-dimensional) Lebesgue measure. Furthermore we refer to the disk with center P and radius r by $B(P, r)$, and $S(P, r) = \partial B(P, r)$. For a set $E \subset \mathbb{R}^2$ we denote by \overline{E} the closure of E . If $A \in \mathbb{R}^{2 \times 2}$ is a matrix, $adj A$ is the adjoint matrix of A .

2.2 Basic Definitions and Facts

Definition 2.1 Let $\Omega \subset \mathbb{R}^2$ and $\Omega' \subset \mathbb{R}^2$ be domains. A homeomorphism $f : \Omega \rightarrow \Omega'$ is called K -quasiconformal if $f \in W_{loc}^{1,2}(\Omega, \Omega')$ and if there is a constant $K \geq 1$ such that

$$|Df(z)|^2 \leq K J_f(z)$$

holds for \mathcal{L}^2 -a.e. $z \in \Omega$.

Definition 2.2 Let $\Omega \subset \mathbb{R}^2$ be a domain. We say that a mapping $f : \Omega \rightarrow \mathbb{R}^2$ has finite distortion if $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^2)$, $J_f \in L_{loc}^1(\Omega)$ and

$$|Df(z)|^2 \leq K_f(z) J_f(z) \quad \mathcal{L}^2\text{-a.e. } z \in \Omega, \tag{2.2.1}$$

where

$$K_f(z) = \begin{cases} \frac{|Df(z)|^2}{J_f(z)} & \text{for all } z \in \{J_f > 0\}, \\ 1 & \text{for all } z \in \{J_f = 0\}. \end{cases}$$

Note that a necessary condition in Definition 2.2 is that $J_f(z) \geq 0$ for \mathcal{L}^2 -a.e. $z \in \Omega$. When $J_f(z) \leq 0$ for \mathcal{L}^2 -a.e. $z \in \Omega$, we also define mappings of finite distortion. Modification on (2.2.1) is that $|Df(z)|^2 \leq -K_f(z) J_f(z)$ for \mathcal{L}^2 -a.e. $z \in \Omega$ with

$$K_f(z) = \begin{cases} \frac{|Df(z)|^2}{-J_f(z)} & \text{for all } z \in \{J_f < 0\}, \\ 1 & \text{for all } z \in \{J_f = 0\}. \end{cases}$$

Analogous explanation is applied to Definition 2.1.

Definition 2.3 Given $A \subset \mathbb{R}^2$, a map $f : A \rightarrow \mathbb{R}^2$ is called an (l, L) -bi-Lipschitz mapping if $0 < l \leq L < \infty$ and

$$l|x - y| \leq |f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in A$.

If $\Omega \subset \mathbb{R}^2$ is a domain and $f : \Omega \rightarrow \mathbb{R}^2$ is an orientation-preserving bi-Lipschitz mapping, then f is quasiconformal.

Definition 2.4 Given a function φ defined on set $A \subset \mathbb{R}^2$, its modulus of continuity is defined as

$$\omega(\delta) \equiv \omega(\delta, \varphi, A) = \sup\{|\varphi(z_1) - \varphi(z_2)| : z_1, z_2 \in A, |z_1 - z_2| \leq \delta\}$$

for $\delta \geq 0$. Then φ is called Dini-continuous if

$$\int_0^\pi \frac{\omega(t)}{t} dt < \infty,$$

where the integration bound π can be replaced by any positive constant.

We say that a curve C is *Dini-smooth* if it has a parametrization $\alpha(t)$ for $t \in [0, 2\pi]$ so that $\alpha'(t) \neq 0$ for all $t \in [0, 2\pi]$ and α' is Dini-continuous.

Definition 2.5 Let $\Omega \subset \mathbb{R}^2$ be open and $f : \Omega \rightarrow \mathbb{R}^2$ be a mapping. We say that f satisfies the Lusin (N) condition if $\mathcal{L}^2(f(E)) = 0$ for any $E \subset \Omega$ with $\mathcal{L}^2(E) = 0$. Similarly, f satisfies the Lusin (N^{-1}) condition if $\mathcal{L}^2(f^{-1}(E)) = 0$ for any $E \subset f(\Omega)$ with $\mathcal{L}^2(E) = 0$.

Lemma 2.1 ([6, Theorem A.35]) *Let $\Omega \subset \mathbb{R}^2$ be open and $f \in W_{loc}^{1,1}(\Omega, \mathbb{R}^2)$. Suppose that η is a nonnegative Borel measurable function on \mathbb{R}^2 . Then*

$$\int_\Omega \eta(f(x))|J_f(x)| dx \leq \int_{f(\Omega)} \eta(y)N(f, \Omega, y) dy, \tag{2.2.2}$$

where the multiplicity function $N(f, \Omega, y)$ of f is defined as the number of preimages of y under f in Ω . Moreover (2.2.2) is an equality if we assume in addition that f satisfies the Lusin (N) condition.

Let $\Omega \subset \mathbb{R}^2$ be open. Via Lemma 2.1, we have that

$$\text{if } f \text{ is a } W_{loc}^{1,1}(\Omega, \mathbb{R}^2) \text{ homeomorphism, then } J_f \in L^1_{loc}(\Omega). \tag{2.2.3}$$

Lemma 2.2 ([6, Lemma A.28]) *Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism which belongs to $W_{loc}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$. Then f is differentiable \mathcal{L}^2 -a.e. on \mathbb{R}^2 .*

Lemma 2.2 and a simple computation show that

$$\max_{\theta \in [0, 2\pi]} |\partial_\theta f(z)| = K_f(z) \min_{\theta \in [0, 2\pi]} |\partial_\theta f(z)| \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2 \tag{2.2.4}$$

when $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism of finite distortion. Here $\partial_\theta f(z) = \cos(\theta)f_x(z) + \sin(\theta)f_y(z)$ for $\theta \in [0, 2\pi]$.

Lemma 2.3 ([5, Theorem 1.2], [6, Theorem 1.6]) *Let $\Omega \subset \mathbb{R}^2$ be a domain and $f : \Omega \rightarrow \mathbb{R}^2$ be a homeomorphism of finite distortion. Then $f^{-1} : f(\Omega) \rightarrow \Omega$ is also a homeomorphism of finite distortion. Moreover*

$$|Df^{-1}(y)|^2 \leq K_{f^{-1}}(y)J_{f^{-1}}(y) \quad \mathcal{L}^2\text{-a.e. } y \in f(\Omega). \tag{2.2.5}$$

Lemma 2.4 ([14, Theorem 2.1.11]) *Let all $\Omega \subset \mathbb{R}^2$, $\Omega_1 \subset \mathbb{R}^2$ and $\Omega_2 \subset \mathbb{R}^2$ be open, and $T \in Lip(\Omega_1, \Omega_2)$. Suppose that both $f \in W_{loc}^{1,p}(\Omega, \Omega_1)$ and $T \circ f \in L_{loc}^p(\Omega, \Omega_2)$ hold for some p with $1 \leq p \leq \infty$. Then $T \circ f \in W_{loc}^{1,p}(\Omega, \Omega_2)$ and*

$$D(T \circ f)(z) = DT(f(z))Df(z) \quad \mathcal{L}^2\text{-a.e. } z \in \Omega.$$

Definition 2.6 A rectifiable Jordan curve Γ in the plane is a chord-arc curve if there is a constant $C > 0$ such that

$$\ell_\Gamma(z_1, z_2) \leq C|z_1 - z_2|$$

for all $z_1, z_2 \in \Gamma$, where $\ell_\Gamma(z_1, z_2)$ is the length of the shorter arc of Γ joining z_1 and z_2 .

It is a well-known fact that a chord-arc curve is the image of the unit circle under a bi-Lipschitz mappings of the plane, see [7]. Thus chord-arc curves form a special class of quasicircles. The connections between chord-arc curves and quasiconformal theory can be found in [1,12].

2.3 Definition of Cardioid-Type Domains

Let $s > 1$. We introduce a class of cardioid-type domains Δ_s whose boundaries contain internal polynomial cusps of order s , see Fig. 1. For technical reasons we do this in the following manner. Denote

$$\ell_1(s) = \{(u, v) \in \mathbb{R}^2 : u \in [-1, 0], v = (-u)^s\}$$

and

$$\ell_2(s) = \{(u, v) \in \mathbb{R}^2 : u \in [-1, 0], v = -(-u)^s\}.$$

Write $\ell_1(s)$ and $\ell_2(s)$ in the polar coordinate system as

$$\begin{aligned} \ell_1(s) &= \{Re^{i\Theta} : R = (-u)(1 + (-u)^{2(s-1)})^{\frac{1}{2}} \\ &\text{and } \Theta = \pi - \arctan((-u)^{s-1}) \text{ for } u \in [-1, 0]\} \end{aligned}$$

and

$$\ell_2(s) = \{Re^{i\Theta} : R = (-u)(1 + (-u)^{2(s-1)})^{\frac{1}{2}}$$

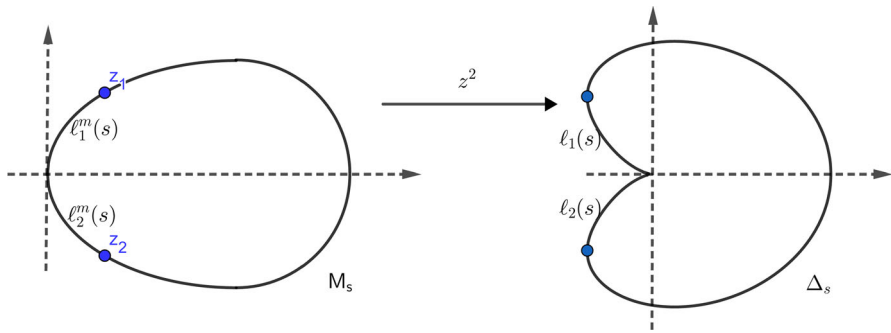


Fig. 1 M_s and Δ_s

and $\Theta = -\pi + \arctan((-u)^{s-1})$ for $u \in [-1, 0]$.

Take the branch of complex-valued function $z = w^{1/2}$ with $1^{1/2} = 1$. Denote by $\ell_1^m(s)$ and $\ell_2^m(s)$ the images of $\ell_1(s)$ and $\ell_2(s)$ under the preceding $z = w^{1/2}$, respectively. Then we can write $\ell_1^m(s)$ and $\ell_2^m(s)$ in the polar coordinate system as

$$\begin{aligned} \ell_1^m(s) &= \{re^{i\theta} : r = \sqrt{-u}(1 + (-u)^{2(s-1)})^{\frac{1}{4}} \\ &\text{and } \theta = \frac{\pi - \arctan((-u)^{s-1})}{2} \text{ for } u \in [-1, 0]\} \end{aligned} \tag{2.3.1}$$

and

$$\begin{aligned} \ell_2^m(s) &= \{re^{i\theta} : r = \sqrt{-u}(1 + (-u)^{2(s-1)})^{\frac{1}{4}} \\ &\text{and } \theta = \frac{-\pi + \arctan((-u)^{s-1})}{2} \text{ for } u \in [-1, 0]\}. \end{aligned}$$

Denote by z_1 and z_2 the end points of $\ell_1^m(s) \cup \ell_2^m(s)$. Notice that there is a unique circle sharing both the tangent of $\ell_1^m(s)$ at z_1 and the one of $\ell_2^m(s)$ at z_2 . This circle is divided into two arcs by z_1 and z_2 . Concatenating $\ell_1^m(s) \cup \ell_2^m(s)$ with the arc located on the right-hand side of the line through z_1 and z_2 , we then obtain a Jordan curve $\ell^m(s)$. Denote by $\ell(s)$ the image of $\ell^m(s)$ under z^2 . Let

$$M_s \text{ and } \Delta_s \text{ be the interior domains of } \ell^m(s) \text{ and } \ell(s), \text{ respectively.} \tag{2.3.2}$$

Then Δ_s is the desired cardioid-type domain with degree s . Moreover $\ell^m(s)$, $\ell(s)$, M_s and Δ_s are symmetric with respect to the real axis.

By the Riemann mapping theorem, there is a conformal mapping from $\mathbb{D} \cap \mathbb{R}_+^2$ onto $M_s \cap \mathbb{R}_+^2$ such that $\mathbb{D} \cap \mathbb{R}$ is mapped onto $M_s \cap \mathbb{R}$. It follows from the Schwarz reflection principle that there is a conformal mapping

$$g_s : \mathbb{D} \rightarrow M_s. \tag{2.3.3}$$

such that $g_s(\bar{z}) = \overline{g_s(z)}$ for all $z \in \mathbb{D}$. Moreover by the Osgood–Carathéodory theorem g_s has a homeomorphic extension from $\overline{\mathbb{D}}$ onto $\overline{M_s}$, still denoted g_s .

Lemma 2.5 *Let M_s and g_s be as in (2.3.2) and (2.3.3) with $s > 1$. Then g_s is a bi-Lipschitz mapping on \mathbb{D} .*

Proof If ∂M_s were a Dini-smooth Jordan curve, from [11, Theorem 3.3.5] it would follow that g'_s is continuous on \mathbb{D} and $g'_s(z) \neq 0$ for all $z \in \mathbb{D}$. Since M_s is convex, the mean value theorem would then yield that g_s is a bi-Lipschitz map from \mathbb{D} onto $\overline{M_s}$.

In order to prove that ∂M_s is a Dini-smooth Jordan curve, we first analyze ∂M_s in a neighborhood of the origin. For any point in ℓ^m_1 with Euclidean coordinate (x, y) , we have

$$x = r \cos \theta \text{ and } y = r \sin \theta. \tag{2.3.4}$$

where both r and θ share the expression in (2.3.1). We then obtain that

$$r \approx \sqrt{-u}, \theta \approx \frac{\pi}{2}, \frac{\partial r}{\partial u} \approx \frac{-1}{\sqrt{-u}} \text{ and } \frac{\partial \theta}{\partial u} \approx (-u)^{s-2} \tag{2.3.5}$$

whenever $|u| \ll 1$. Therefore from (2.3.4) and (2.3.5), it follows that

$$x \approx (-u)^{s-\frac{1}{2}}, y \approx (-u)^{\frac{1}{2}}, \frac{\partial x}{\partial u} \approx -(-u)^{s-\frac{3}{2}} \text{ and } \frac{\partial y}{\partial u} \approx -(-u)^{-\frac{1}{2}}.$$

Together with symmetry of ∂M_s , we conclude that $\frac{\partial x}{\partial y} \approx |y|^{2(s-1)}$ whenever $|y| \ll 1$. Next, notice that the part of ∂M_s away from the origin is piecewise smooth. By parametrizing ∂M_s as $\alpha(y) = (x(y), y)$, we then obtain that the modulus of continuity of α' satisfies

$$\omega(\delta, \alpha', \partial M_s) \leq \max\{\delta^{2(s-1)}, \delta\} \quad \forall \delta \ll 1.$$

Consequently α' is Dini-continuous. Therefore ∂M_s is a Dini-smooth Jordan curve. \square

Remark 2.1 Since $g_s : \mathbb{S}^1 \rightarrow \partial M_s$ is a bi-Lipschitz map by Lemma 2.5, via [13, Theorem A] there is a bi-Lipschitz mapping $g^c_s : \mathbb{D}^c \rightarrow M^c_s$ such that $g^c_s|_{\mathbb{S}^1} = g_s$. Let

$$G_s(z) = \begin{cases} g_s(z) & \forall z \in \mathbb{D}, \\ g^c_s(z) & \forall z \in \mathbb{D}^c. \end{cases} \tag{2.3.6}$$

Then G_s is an orientation-preserving bi-Lipschitz mapping.

Lemma 2.6 *Let $h_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a homeomorphism of finite distortion, and $h_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an (l, L) -bi-Lipschitz, orientation-preserving mapping. Then $h_1 \circ h_2$ is a homeomorphism of finite distortion.*

Proof Since h_2 is an orientation-preserving bi-Lipschitz mapping, we have that h_2 is quasiconformal. From [2, Corollary 3.7.6] it then follows that

$$h_2 \text{ satisfies Lusin } (N) \text{ and } (N^{-1}) \text{ condition,} \tag{2.3.7}$$

$$J_{h_2} > 0 \quad \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{2.3.8}$$

By Lemma 2.2 we have

$$\text{both } h_1 \text{ and } h_2 \text{ are differentiable } \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{2.3.9}$$

From (2.3.9) and (2.3.7) it therefore follows that $h_1 \circ h_2$ is differentiable \mathcal{L}^2 -a.e. on \mathbb{R}^2 , and

$$D(h_1 \circ h_2)(z) = Dh_1(h_2(z))Dh_2(z) \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{2.3.10}$$

From (2.3.10) and the distortion inequalities for h_1 and h_2 it follows that

$$\begin{aligned} |D(h_1 \circ h_2)(z)|^2 &\leq |Dh_1(h_2(z))|^2 |Dh_2(z)|^2 \leq K_{h_1}(h_2(z))K_{h_2}(z)J_{h_1}(h_2(z))J_{h_2}(z) \\ &= K_{h_1}(h_2(z))K_{h_2}(z)J_{h_1 \circ h_2}(z) \end{aligned} \tag{2.3.11}$$

for \mathcal{L}^2 -a.e. $z \in \mathbb{R}^2$.

To prove that $h_1 \circ h_2$ is a homeomorphism of finite distortion, via (2.2.3) and (2.3.11) it is sufficient to prove that $h_1 \circ h_2 \in W_{loc}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$. Since h_2 is an (l, L) -bi-Lipschitz orientation-preserving mapping, by (2.3.9) and (2.2.4) we then have that

$$l \leq |Dh_2(z)| \leq L \text{ and } 1 \leq K_{h_2}(z) \leq \frac{L}{l} \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{2.3.12}$$

From (2.3.8), (2.3.12), and (2.2.1) it then follows that

$$\frac{l^3}{L} \leq J_{h_2}(z) \leq L^2 \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{2.3.13}$$

By (2.3.10), (2.3.12), (2.3.13), and Lemma 2.1, we therefore have

$$\begin{aligned} \int_M |D(h_1 \circ h_2)(z)| \, dz &\leq \int_M |Dh_1(h_2(z))| \frac{|Dh_2(z)|}{J_{h_2}(z)} J_{h_2}(z) \, dz \\ &\approx \int_M |Dh_1(h_2(z))| J_{h_2}(z) \, dz \\ &= \int_{h_2(M)} |Dh_1(w)| \, dw < \infty \end{aligned}$$

for any compact set $M \subset \mathbb{R}^2$, where the last inequality is from $h_1 \in W_{loc}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$. □

3 Bounds for Integrability Degrees

For a given $s > 1$, let M_s as in (2.3.2). Define

$$\begin{aligned} \mathcal{E}_s &= \{f : f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is a homeomorphism of finite distortion} \\ &\text{and } f(z) = z^2 \text{ for all } z \in \overline{M_s}\}. \end{aligned} \tag{3.0.1}$$

Lemma 3.1 *Let \mathcal{E}_s be as in (3.0.1) with $s > 1$, and $f \in \mathcal{E}_s$. Suppose that $f^{-1} \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p \geq 1$. Then necessarily $p < 2(s + 1)/(2s - 1)$.*

Proof Given $x \in (-1, 0)$, denote by I_x the line segment connecting the points $(x, |x|^s)$ and $(x, -|x|^s)$. Since $f^{-1} \in W_{loc}^{1,p}$ for some $p \geq 1$, by the ACL-property of Sobolev functions it follows that

$$\text{osc}_{I_x} f^{-1} \leq \int_{I_x} |Df^{-1}(x, y)| \, dy \tag{3.0.2}$$

holds for \mathcal{L}^1 -a.e. $x \in (-1, 0)$. Applying Jensen’s inequality to (3.0.2), we have

$$\frac{(\text{osc}_{I_x} f^{-1})^p}{(-x)^{s(p-1)}} \leq \int_{I_x} |Df^{-1}(x, y)|^p \, dy. \tag{3.0.3}$$

Since $f(z) = z^2$ for all $z \in \partial M_s$, we have

$$(-x)^{1/2} \lesssim \text{osc}_{I_x} f^{-1} \quad \forall x \in (-1, 0). \tag{3.0.4}$$

Combining (3.0.3) with (3.0.4), we hence obtain

$$(-x)^{\frac{p}{2}-s(p-1)} \lesssim \int_{I_x} |Df^{-1}(x, y)|^p \, dy \quad \mathcal{L}^1\text{-a.e. } x \in (-1, 0). \tag{3.0.5}$$

Integrating (3.0.5) with respect to $x \in (-1, 0)$ therefore implies

$$\int_{-1}^0 (-x)^{\frac{p}{2}-s(p-1)} \, dx \lesssim \int_{B(0, \sqrt{2})} |Df^{-1}(x, y)|^p \, dx \, dy. \tag{3.0.6}$$

Since $f^{-1} \in W_{loc}^{1,p}$, from (3.0.6) we necessarily obtain $\frac{p}{2} - s(p - 1) > -1$, which is equivalent to $p < 2(s + 1)/(2s - 1)$. □

Our next proof borrows some ideas from [9, Theorem 1].

Lemma 3.2 *Let \mathcal{E}_s be as in (3.0.1) with $s > 1$. Let $f \in \mathcal{E}_s$ and suppose that $K_{f^{-1}} \in L_{loc}^q(\mathbb{R}^2)$ for a given $q \geq 1$. Then $q < (s + 1)/(s - 1)$.*

Proof For a given $t \ll 1$, we denote

$$E_t = \{(x, y) \in \mathbb{R}^2 : x \in (-t^2, -(\frac{t}{2})^2) \text{ and } y = -|x|^s\}$$

and

$$F_t = \{(x, y) \in \mathbb{R}^2 : x \in (-t^2, -(\frac{t}{2})^2) \text{ and } y = |x|^s\}.$$

Let $\tilde{E}_t = f^{-1}(E_t)$ and $\tilde{F}_t = f^{-1}(F_t)$. Set

$$L_t^1 = \min\{|z| : z \in \tilde{F}_t\}, L_t^2 = \max\{|z| : z \in \tilde{F}_t\},$$

$$L_t = \text{dist}(\tilde{E}_t, \tilde{F}_t), L_0 = \max\{|f^{-1}(z)| : \text{Re}z = -1, \text{Im}z \in [-1, 1]\}.$$

Since $f(z) = z^2$ for all $z \in \partial M_s$, we have $L_t^1 \approx t/2$, $L_t^2 \approx t$ and $L_t \approx t$ whenever $t \ll 1$. Given $w \in A_t := \{w \in \mathbb{R}^2 : L_t^1 \leq |w| \leq L_t^2\}$, set $\rho(w) = L_t^2/(L_t|w|)$. Define

$$v(z) = \begin{cases} 1 & \text{for all } z \in B(0, L_0) \setminus A_t, \\ \inf_{\gamma_z} \int_{\gamma_z} \rho \, ds & \text{for all } z \in A_t, \end{cases} \tag{3.0.7}$$

where the infimum is taken over all curves $\gamma_z \subset A_t$ joining z and \tilde{E}_t . From (3.0.7) it follows that for any $z_1, z_2 \in A_t$ and any curve $\gamma_{z_1z_2} \subset A_t$ connecting z_1 and z_2 we have

$$|v(z_1) - v(z_2)| \leq \int_{\gamma_{z_1z_2}} \rho \, ds. \tag{3.0.8}$$

Therefore v is a Lipschitz function on A_t . By Rademacher’s theorem, v is differentiable \mathcal{L}^2 -a.e. on A_t . Hence (3.0.8) together with the continuity of ρ gives

$$|Dv(z)| \leq \rho(z) \quad \mathcal{L}^2\text{-a.e. } z \in A_t. \tag{3.0.9}$$

Integrating (3.0.9) over $\tilde{Q}_t = A_t \setminus M_s$ then yields

$$\int_{\tilde{Q}_t} |Dv|^2 \leq \int_{\tilde{Q}_t} \rho^2 \approx \int_{L_t^1}^{L_t^2} \frac{1}{r} \, dr \approx \log 2. \tag{3.0.10}$$

By Lemma 2.3 we have $f^{-1} \in W_{\text{loc}}^{1,1}$. Let $u = v \circ f^{-1}$. From Lemma 2.4 we then have $u \in W_{\text{loc}}^{1,1}(f(B(0, L_0)))$ and

$$|Du(z)| \leq |Dv(f^{-1}(z))| |Df^{-1}(z)| \quad \mathcal{L}^2\text{-a.e. in } f(A_t). \tag{3.0.11}$$

By (3.0.7), $v(z) = 0$ for all $z \in \tilde{E}_t$. Hence $u(z) = 0$ for all $z \in E_t$. Whenever $z \in \tilde{F}_t$, we have $\mathcal{L}^1(\gamma_z) \geq L_t$ for any curve $\gamma_z \subset A_t$ joining z and \tilde{E}_t . Therefore $v(z) \geq 1$ for all $z \in \tilde{F}_t$. Hence $u(z) \geq 1$ for all $z \in F_t$. By the ACL-property of Sobolev functions and Hölder’s inequality, we therefore have that

$$1 \leq \int_{-x^s}^{x^s} |Du(x, y)| dy \leq \left(\int_{-x^s}^{x^s} |Du(x, y)|^p dy \right)^{\frac{1}{p}} (2x^s)^{\frac{p-1}{p}} \tag{3.0.12}$$

for any $p > 1$ and \mathcal{L}^1 -a.e. $x \in [-t^2, -(t/2)^2]$. Define

$$R_t = \{(x, y) \in \mathbb{R}^2 : x \in (-t^2, -(t/2)^2), y \in (-|x|^s, |x|^s)\}.$$

Fubini’s theorem and (3.0.12) then give

$$\begin{aligned} \int_{R_t} |Du(x, y)|^p dx dy &= \int_{-t^2}^{-(t/2)^2} \int_{-x^s}^{x^s} |Du(x, y)|^p dy dx \\ &\gtrsim \int_{-t^2}^{-(t/2)^2} x^{s(1-p)} dx \approx t^{2(1+s(1-p))}. \end{aligned} \tag{3.0.13}$$

Set $Q_t = f(\tilde{Q}_t)$. Then for any $z \in R_t \setminus Q_t$ there is an open disk $B_z \subset R_t \setminus Q_t$ such that $z \in B_z$ and $u|_{B_z} \equiv 1$. Therefore

$$\int_{Q_t} |Du|^p \geq \int_{Q_t \cap R_t} |Du|^p = \int_{R_t} |Du|^p. \tag{3.0.14}$$

Combining (3.0.13) with (3.0.14) gives that

$$t^{2(1+s(1-p))} \lesssim \int_{Q_t} |Du|^p \tag{3.0.15}$$

for all $p \geq 1$.

For any $p \in (0, 2)$, by (3.0.11), (2.2.5), and Hölder’s inequality, we have

$$\begin{aligned} \int_{Q_t} |Du|^p &\leq \int_{Q_t} |Dv \circ f^{-1}|^p |Df^{-1}|^p \\ &\leq \int_{Q_t} |Dv \circ f^{-1}|^p J_{f^{-1}}^{\frac{p}{2}} K_{f^{-1}}^{\frac{p}{2}} \\ &\leq \left(\int_{Q_t} |Dv \circ f^{-1}|^2 J_{f^{-1}} \right)^{\frac{p}{2}} \left(\int_{Q_t} K_{f^{-1}}^{\frac{p}{2-p}} \right)^{\frac{2-p}{2}} \\ &\leq \left(\int_{\tilde{Q}_t} |Dv|^2 \right)^{\frac{p}{2}} \left(\int_{Q_t} K_{f^{-1}}^{\frac{p}{2-p}} \right)^{\frac{2-p}{2}} \end{aligned} \tag{3.0.16}$$

where the last inequality comes from Lemma 2.1. Let $q = p/(2 - p)$. Via (3.0.10) and (3.0.15), we conclude from (3.0.16) that

$$t^{2(1+q+s(1-q))} \lesssim \int_{Q_t} K_{f^{-1}}^q \tag{3.0.17}$$

for all $q \geq 1$. We now consider the set Q_t for $t = 2^{-j}$ with $j \geq j_0$ for a fixed large j_0 . Since

$$\sum_{j=j_0}^{\infty} \chi_{Q_{2^{-j}}}(x) \leq 2\chi_{\mathbb{D}}(x) \quad \forall x \in \mathbb{R}^2,$$

by (3.0.17) we have that

$$\sum_{j=j_0}^{+\infty} 2^{j2(s(q-1)-q-1)} \lesssim \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_{f^{-1}}^q \leq 2 \int_{\mathbb{D}} K_{f^{-1}}^q. \tag{3.0.18}$$

The series in (3.0.18) diverges when $q \geq \frac{s+1}{s-1}$ and hence $K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)$ can only hold when $q < (s + 1)/(s - 1)$. □

We continue with properties of our homeomorphism f . The following lemma is a version of [4, Theorem 4.4].

Lemma 3.3 *Let \mathcal{E}_s be as in (3.0.1) with $s > 1$. If $f \in \mathcal{E}_s$ and $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$, then $q < \max\{1, 1/(s - 1)\}$.*

Proof Denote

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-1, 0), x_2 \in (-|x_1|^s, |x_1|^s)\}.$$

For a given $t \ll 1$, set

$$\begin{aligned} \Omega_t^1 &= \{(x_1, x_2) \in \Omega : x_1 \in (-1, -t^2)\}, \\ \tilde{Q}_t &= \{(x_1, x_2) \in \Omega : x_1 \in [-t^2, -(\frac{t}{2})^2]\} \text{ and } \Omega_t^2 = \Omega \setminus (\Omega_t^1 \cup \tilde{Q}_t). \end{aligned}$$

Define

$$v(x_1, x_2) = \begin{cases} 1 & \forall (x_1, x_2) \in \Omega_t^1, \\ 1 - \left(\int_{-t^2}^{-x_1} \frac{dx}{(-x)^s} \right)^{-1} \int_{-t^2}^{x_1} \frac{dx}{(-x)^s} & \forall (x_1, x_2) \in \tilde{Q}_t, \\ 0 & \forall (x_1, x_2) \in \Omega_t^2. \end{cases} \tag{3.0.19}$$

Then v is a Lipschitz function on Ω . Let $u = v \circ f$. By Lemma 2.4, we have $u \in W^{1,1}_{loc}(f^{-1}(\Omega))$ and

$$Du(z) = Dv(f(z))Df(z) \quad \mathcal{L}^2\text{-a.e. } z \in f^{-1}(\Omega). \tag{3.0.20}$$

Let $P_1 = f^{-1}((-t^2, t^{2s}))$, $P_2 = f^{-1}((-t/2)^2, (t/2)^{2s})$, and O be the origin. Denote by L_t^1 and L_t^2 the length of line segment $P_1 P_2$ and of $P_1 O$, respectively. Then $L_t^1 < L_t^2$. Since $f(z) = z^2$ for all $z \in \partial M_s$, we have

$$L_t^1 \approx \frac{t}{2} \text{ and } L_t^2 \approx t \quad \text{whenever } t \ll 1. \tag{3.0.21}$$

Let $\hat{S}(P_1, r) = S(P_1, r) \cap f^{-1}(\Omega)$. From the ACL-property of Sobolev functions and Hölder’s inequality, we have that

$$\text{osc}_{\hat{S}(P_1, r)} u \leq \int_{\hat{S}(P_1, r)} |Du| \, ds \leq (2\pi r)^{\frac{p-1}{p}} \left(\int_{\hat{S}(P_1, r)} |Du|^p \, ds \right)^{\frac{1}{p}} \tag{3.0.22}$$

for any $p > 1$ and \mathcal{L}^1 -a.e. $r \in (L_t^1, L_t^2)$. Since $\text{osc}_{\hat{S}(P_1, r)} u = 1$ for all $r \in (L_t^1, L_t^2)$, we conclude from (3.0.22) that

$$\int_{\hat{S}(P_1, r)} |Du|^p \, ds \gtrsim r^{1-p} \quad \mathcal{L}^1\text{-a.e. } r \in (L_t^1, L_t^2). \tag{3.0.23}$$

Let $A_t = f^{-1}(\Omega) \cap B(P_1, L_t^2) \setminus \overline{B(P_1, L_t^1)}$. By Fubini’s theorem and (3.0.21), we deduce from (3.0.23) that

$$\int_{A_t} |Du|^p = \int_{L_t^1}^{L_t^2} \int_{\hat{S}(P_1, r)} |Du|^p \, ds \, dr \gtrsim \int_{L_t^1}^{L_t^2} r^{1-p} \, dr \approx t^{2-p}. \tag{3.0.24}$$

Let $Q_t = f^{-1}(\tilde{Q}_t)$. From (3.0.19), we have $|Du(z)| = 0$ for all $z \in A_t \setminus Q_t$. We hence conclude from (3.0.24) that

$$\int_{Q_t} |Du|^p \geq \int_{Q_t \cap A_t} |Du|^p = \int_{A_t} |Du|^p \gtrsim t^{2-p} \tag{3.0.25}$$

for any $p \geq 1$.

From (3.0.20), (2.2.1), and Hölder’s inequality, it follows that for any $p \in (0, 2)$

$$\begin{aligned} \int_{Q_t} |Du|^p &\leq \int_{Q_t} |Dv \circ f|^p |Df|^p \leq \int_{Q_t} |Dv \circ f|^p J_f^{\frac{p}{2}} K_f^{\frac{p}{2}} \\ &\leq \left(\int_{Q_t} |Dv \circ f|^2 J_f \right)^{\frac{p}{2}} \left(\int_{Q_t} K_f^{\frac{2-p}{2}} \right)^{\frac{2-p}{2}} \\ &\leq \left(\int_{\tilde{Q}_t} |Dv|^2 \right)^{\frac{p}{2}} \left(\int_{Q_t} K_f^{\frac{2-p}{2}} \right)^{\frac{2-p}{2}}, \end{aligned} \tag{3.0.26}$$

where the last inequality is from Lemma 2.1. From (3.0.19), we have that

$$\begin{aligned} \int_{\tilde{Q}_t} |Dv(x_1, x_2)|^2 \, dx_1 \, dx_2 &= \left(\int_{-t^2}^{-(t/2)^2} \frac{dx}{(-x)^s} \right)^{-2} \int_{-t^2}^{-(t/2)^2} \int_{-|x_1|^s}^{|x_1|^s} \frac{1}{(-x_1)^{2s}} \, dx_2 \, dx_1 \\ &\approx \left(\int_{-t^2}^{-(t/2)^2} \frac{dx}{(-x)^s} \right)^{-1} \approx t^{2(s-1)}. \end{aligned} \tag{3.0.27}$$

Let $q = p/(2 - p)$. Then $q \in [1, +\infty)$ whenever $p \in [1, 2)$. Combining (3.0.27), (3.0.25) with (3.0.26) yields

$$t^{2+2(1-s)q} \lesssim \int_{Q_t} K_f^q \tag{3.0.28}$$

for all $q \geq 1$. We now consider the set Q_t for $t = 2^{-j}$ with $j \geq j_0$ for a fixed large j_0 . Analogously to (3.0.18), it follows from (3.0.28) that

$$\sum_{j=j_0}^{+\infty} 2^{2j((s-1)q-1)} \lesssim \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_f^q \leq 2 \int_{B(0,1)} K_f^q. \tag{3.0.29}$$

Whenever $s \geq 2$, the sum in (3.0.29) diverges if $q \geq 1$. Whenever $s \in (1, 2)$, the sum in (3.0.29) also diverges if $q \geq 1/(s - 1)$. Hence $K_f \in L^q_{loc}(\mathbb{R}^2)$ is possible only when $q < \max\{1, 1/(s - 1)\}$. \square

In Lemma 3.3, we obtained an estimate for those q for which $K_f \in L^q_{loc}$. We continue with the additional assumption that $f \in W^{1,p}_{loc}$ for some $p > 1$.

Lemma 3.4 *Let \mathcal{E}_s be as in (3.0.1) with $s > 2$. If $f \in \mathcal{E}_s$, $f \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \in (0, 1)$, then $q < 3p/((2s - 1)p + 4 - 2s)$.*

Proof Let f be a homeomorphism with the above properties. By [5, Theorem 4.1] we have $f^{-1} \in W^{1,r}_{loc}(\mathbb{R}^2)$ where

$$r = \frac{(q + 1)p - 2q}{p - q}.$$

Moreover

$$r < \frac{2(s + 1)}{2s - 1} \Leftrightarrow q < \frac{3p}{(2s - 1)p + 4 - 2s}.$$

Hence the claim follows from Lemma 3.1. \square

Remark 3.1 Notice that in the proof of Lemma 3.3 we only care about the property of f in a small neighborhood of the origin. Let $t \ll 1$. By modifying $\partial M_s \cap B(0, t)$, we may generalize Lemma 3.3. For example, we modify $\partial M_{3/2} \cap B(0, t)$ such that its image under $f(z) = z^2$ is

$$\{(x, y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = c|x|^3\}$$

where c is a positive constant. If $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$, by the analogous arguments as for Lemma 3.3 we have $q < 2$. Similarly, one may extend Lemmas 3.1, 3.2 and 3.4 to the above setting.

Lemma 3.5 *Let Δ_s be as in (2.3.2) with $s > 1$. Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism of finite distortion such that f maps \mathbb{D} conformally onto Δ_s . We have that*

- (1) *if $f^{-1} \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p \geq 1$ then $p < 2(s + 1)/(2s - 1)$,*
- (2) *if $K_{f^{-1}} \in L_{loc}^q(\mathbb{R}^2)$ for some $q \geq 1$ then $q < (s + 1)/(s - 1)$,*
- (3) *if $K_f \in L_{loc}^q(\mathbb{R}^2)$ for some $q \geq 1$ then $q < \max\{1, 1/(s - 1)\}$,*
- (4) *if $s > 2$, $f \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_f \in L_{loc}^q$ for some $q \in (0, 1)$, then $q < 3p/((2s - 1)p + 4 - 2s)$.*

Proof Let g_s be as in (2.3.3), and $h_s = z^2 \circ g_s$. Since $h_s : \mathbb{D} \rightarrow \Delta_s$ is conformal, there is a Möbius transformation

$$m_s(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} \quad \text{where } \theta \in [0, 2\pi] \text{ and } |a| < 1$$

such that $f(z) = h_s \circ m_s(z)$ for all $z \in \mathbb{D}$. Since $m_s : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a bi-Lipschitz mapping, by [13, Theorem A] there is a bi-Lipschitz mapping $m_s^c : \mathbb{D}^c \rightarrow \Delta_s^c$ such that $m_s^c|_{\mathbb{S}^1} = m_s$. Define

$$\mathfrak{M}_s(z) = \begin{cases} m_s(z) & z \in \overline{\mathbb{D}}, \\ m_s^c(z) & z \in \mathbb{D}^c. \end{cases} \tag{3.0.30}$$

Then $\mathfrak{M}_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a bi-Lipschitz, orientation-preserving mapping. Let G_s be as in (2.3.6). Define

$$E = f \circ \mathfrak{M}_s^{-1} \circ G_s^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

Lemma 2.6 implies that $E \in \mathcal{E}_s$, where \mathcal{E}_s is from (3.0.1). From Lemmas 2.2 and 2.3, it follows that

$$\text{both } f^{-1} \text{ and } E^{-1} \text{ are differentiable } \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{3.0.31}$$

Since

$$\begin{aligned} \frac{|f^{-1}(z_1) - f^{-1}(z_2)|}{|z_1 - z_2|} &= \frac{|E^{-1}(z_1) - E^{-1}(z_2)|}{|z_1 - z_2|} \frac{|(G_s^{-1}(E^{-1}(z_1)) - (G_s^{-1}(E^{-1}(z_2))))|}{|E^{-1}(z_1) - E^{-1}(z_2)|} \times \\ &\quad \times \frac{|\mathfrak{M}_s^{-1}(G_s^{-1} \circ E^{-1}(z_1)) - \mathfrak{M}_s^{-1}(G_s^{-1} \circ E^{-1}(z_2))|}{|G_s^{-1} \circ E^{-1}(z_1) - G_s^{-1} \circ E^{-1}(z_2)|} \end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}^2$ with $z_1 \neq z_2$, by (3.0.31) and the bi-Lipschitz properties of G_s^{-1} and \mathfrak{M}_s^{-1} we have that

$$|Df^{-1}(z)| \approx |DE^{-1}(z)|, \tag{3.0.32}$$

$$\begin{aligned} \max_{\theta \in [0, 2\pi]} |\partial_\theta f^{-1}(z)| &\approx \max_{\theta \in [0, 2\pi]} |\partial_\theta E^{-1}(z)|, \\ \min_{\theta \in [0, 2\pi]} |\partial_\theta f^{-1}(z)| &\approx \min_{\theta \in [0, 2\pi]} |\partial_\theta E^{-1}(z)| \end{aligned} \tag{3.0.33}$$

for \mathcal{L}^2 -a.e. $z \in \mathbb{R}^2$. If $f^{-1} \in W_{loc}^{1,p}$ for some $p \geq 1$, Lemma 3.2 together with (3.0.34) gives $p < 2(s + 1)/(2s - 1)$. By (3.0.33) and (2.2.4) we have that

$$K_{f^{-1}}(z) \approx K_{E^{-1}}(z) \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{3.0.34}$$

If $K_{f^{-1}} \in L_{loc}^q(\mathbb{R}^2)$ for some $q \geq 1$, combining (3.0.32) and Lemma 3.1 then yields $q < (s + 1)/(s - 1)$.

By Lemma 2.2 and 2.6, we have that

$$\text{both } f \text{ and } E \text{ are differentiable } \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{3.0.35}$$

From [2, Corollary 3.7.6], $G_s \circ \mathfrak{M}_s$ satisfies Lusin (N) and (N^{-1}) conditions. Since

$$\begin{aligned} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} &= \frac{|E(G_s \circ \mathfrak{M}_s(z_1)) - E(G_s \circ \mathfrak{M}_s(z_2))|}{|G_s \circ \mathfrak{M}_s(z_1) - G_s \circ \mathfrak{M}_s(z_2)|} \frac{|G_s(\mathfrak{M}_s(z_1)) - G_s(\mathfrak{M}_s(z_2))|}{|\mathfrak{M}_s(z_1) - \mathfrak{M}_s(z_2)|} \times \\ &\times \frac{|\mathfrak{M}_s(z_1) - \mathfrak{M}_s(z_2)|}{|z_1 - z_2|} \end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}^2$ with $z_1 \neq z_2$, from (3.0.35) and the bi-Lipschitz properties of G_s and \mathfrak{M}_s we have that

$$|Df(z)| \approx |DE(G_s \circ \mathfrak{M}_s(z))|, \tag{3.0.36}$$

$$\max_{\theta \in [0, 2\pi]} |\partial_\theta f(z)| \approx \max_{\theta \in [0, 2\pi]} |\partial_\theta E(G_s \circ \mathfrak{M}_s(z))|, \tag{3.0.37}$$

$$\min_{\theta \in [0, 2\pi]} |\partial_\theta f(z)| \approx \min_{\theta \in [0, 2\pi]} |\partial_\theta E(G_s \circ \mathfrak{M}_s(z))| \tag{3.0.38}$$

for \mathcal{L}^2 -a.e. $z \in \mathbb{R}^2$. By (2.2.4), (3.0.37), and (3.0.38), we have that

$$K_f(z) \approx K_E(G_s \circ \mathfrak{M}_s(z)) \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{3.0.39}$$

Via the same reasons as for (2.3.13), we have that

$$J_{G_s \circ \mathfrak{M}_s}(z) \approx 1 \quad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{3.0.40}$$

By (3.0.40) and Lemma 2.1, we derive from (3.0.39) that

$$\begin{aligned} \int_A K_f^q(z) \, dz &= \int_A K_E^q(G_s \circ \mathfrak{M}_s(z)) \frac{J_{G_s \circ \mathfrak{M}_s}(z)}{J_{G_s \circ \mathfrak{M}_s}(z)} \, dz \\ &\approx \int_A K_E^q(G_s \circ \mathfrak{M}_s(z)) J_{G_s \circ \mathfrak{M}_s}(z) \, dz = \int_{G_s \circ \mathfrak{M}_s(A)} K_E^q(w) \, dw \end{aligned} \tag{3.0.41}$$

for any $q \geq 0$ and any compact set $A \subset \mathbb{R}^2$. By (3.0.36) and Lemma 2.1, we obtain that

$$\begin{aligned} \int_A |Df(z)|^p &= \int_A |DE(G_s \circ \mathfrak{M}_s(z))|^p \frac{J_{G_s \circ \mathfrak{M}_s}(z)}{J_{G_s \circ \mathfrak{M}_s}(z)} \, dz \\ &\approx \int_A |DE(G_s \circ \mathfrak{M}_s(z))|^p J_{G_s \circ \mathfrak{M}_s}(z) \, dz = \int_{G_s \circ \mathfrak{M}_s(A)} |DE|^p(w) \, dw \end{aligned} \tag{3.0.42}$$

for any $p \geq 0$. If $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$, Lemma 3.3 together with (3.0.41) gives that $q < \max\{1, 1/(s - 1)\}$. If $f \in W^{1,p}_{loc}$ and $K_f \in L^q_{loc}$ for some $p > 1$ and some $q \in (0, 1)$, combining Lemma 3.4 with (3.0.42) then implies $q < 3p/((2s - 1)p + 4 - 2s)$. □

Under a more general assumption that f in Lemma 3.5 is K -quasiconformal from \mathbb{D} onto Δ_s , authors from [4, Theorem 4.4] showed a result analogous to Lemma 3.5 (3).

4 Proof of Theorem 1.2

4.1 Prove that the Class $\mathcal{F}_s(g)$ from Theorem 1.2 is Nonempty

Proof Let g be as in Theorem 1.2. The beginning of proof for Lemma 3.4 shows that

$$g = z^2 \circ g_s \circ m_s,$$

where $m_s : \mathbb{D} \rightarrow \mathbb{D}$ is a Möbius transformation and $g_s : \mathbb{D} \rightarrow M_s$ from (2.3.3) is a conformal mapping. Recall that m_s (or g_s) has a bi-Lipschitz extension $\mathfrak{M}_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (or $G_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$) as in (3.0.30) (or (2.3.6)). Via Lemma 2.6, it suffices to prove that $z^2 : M_s \rightarrow \Delta_s$ has a homeomorphic extension $E : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of finite distortion. Then

$$f := E \circ G_s \circ \mathfrak{M}_s \in \mathcal{F}_s(g). \tag{4.1.1}$$

We divide the construction of E into two steps.

Step 1: we construct E_1 in a neighborhood of the cusp point, see Fig. 2. To be precise, we define f_1, \dots, f_4 and let E_1 be the sum of compositions of f_1, \dots, f_4 .

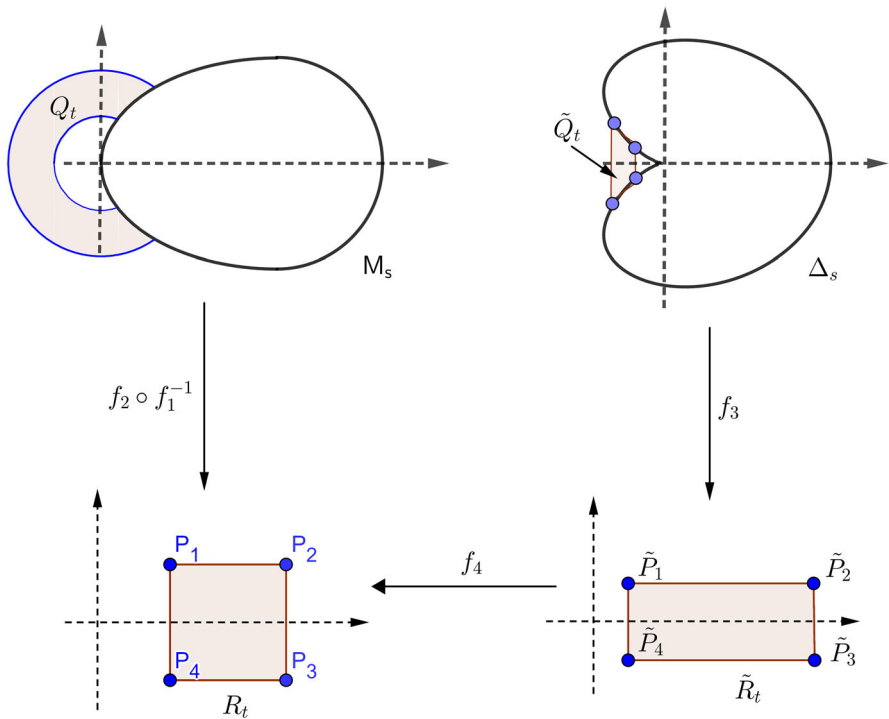


Fig. 2 The construction $f_3^{-1} \circ f_4^{-1} \circ f_2 \circ f_1^{-1} : Q_t \rightarrow \tilde{Q}_t$

Aim 1: to define f_1 and f_2 . Fix $s > 1$, and define

$$\eta(x) = \sqrt{x}(1 + x^{2(s-1)})^{\frac{1}{4}} \quad \text{for all } x > 0. \tag{4.1.2}$$

Then

$$\eta'(x) = \frac{(1 + x^{2(s-1)})^{\frac{1}{4}}}{2\sqrt{x}} \left(1 + \frac{(s-1)x^{2s-2}}{1 + x^{2(s-1)}} \right). \tag{4.1.3}$$

For a given $t \ll 1$, let

$$L_t^1 = \eta((t/2)^2), \quad L_t^2 = \eta(t^2), \quad \text{and } \sigma_t = L_t^2 - L_t^1. \tag{4.1.4}$$

Then $L_t^1 \approx t/2$, $L_t^2 \approx t$ and $\sigma_t \approx t/2$ whenever $t \ll 1$. Set

$$Q_t = \overline{B(0, L_t^2)} \setminus (B(0, L_t^1) \cup M_s), \quad \text{and } f_1(x, y) = xe^{iy} \quad \forall x \geq 0 \text{ and } y \in [0, 2\pi]. \tag{4.1.5}$$

Let $\ell(r)$ be the length of $f_1^{-1}(Q_t) \cap \{(x, y) \in \mathbb{R}^2 : x = r\}$. Define

$$f_2(r, \theta) = \left(r, \frac{\sigma_t}{\ell(r)}(\pi - \theta) \right) \quad \forall (r, \theta) \in f_1^{-1}(Q_t). \tag{4.1.6}$$

Since ∂M_s is mapped onto $\partial \Delta_s$ by z^2 , we have that

$$\ell(r) = \pi + \arctan \tau^{2(s-1)} \text{ and } r = \eta(\tau^2) \tag{4.1.7}$$

for all $\tau \in (t/2, t)$. Then $\ell(r) \approx \pi$ and $r \approx \tau$ whenever $\tau \ll 1$. From (4.1.3), it follows that $\frac{\partial r}{\partial \tau} \approx 1$. Together with $\frac{\partial \ell}{\partial \tau} \approx \tau^{2s-3}$, we have that

$$\frac{\partial \ell(r)}{\partial r} \approx r^{2s-3} \quad \text{for all } r \ll 1. \tag{4.1.8}$$

Denote $R_t = f_2 \circ f_1^{-1}(Q_t)$. Then $R_t = [L_t^1, L_t^2] \times [-\sigma_t/2, \sigma_t/2]$. Combining (4.1.5) with (4.1.6) implies

$$f_1 \circ f_2^{-1}(x, y) = \left(-x \cos \frac{\ell(x)y}{\sigma_t}, x \sin \frac{\ell(x)y}{\sigma_t} \right) \quad \forall (x, y) \in R_t.$$

Therefore

$$Df_1 \circ f_2^{-1}(x, y) = \begin{bmatrix} -\cos \frac{\ell(x)y}{\sigma_t} + \frac{xy\ell'(x)}{\sigma_t} \sin \frac{\ell(x)y}{\sigma_t} & \frac{x\ell(x)}{\sigma_t} \sin \frac{\ell(x)y}{\sigma_t} \\ \sin \frac{\ell(x)y}{\sigma_t} + \frac{xy\ell'(x)}{\sigma_t} \cos \frac{\ell(x)y}{\sigma_t} & \frac{x\ell(x)}{\sigma_t} \cos \frac{\ell(x)y}{\sigma_t} \end{bmatrix}. \tag{4.1.9}$$

By (4.1.4), (4.1.7), and (4.1.8), we deduce from (4.1.9) that

$$|Df_1 \circ f_2^{-1}(x, y)| \lesssim 1 \text{ and } J_{f_1 \circ f_2^{-1}}(x, y) = -\frac{x\ell(x)}{\sigma} \approx -1 \tag{4.1.10}$$

for all $t \ll 1$ and each $(x, y) \in R_t$. Since $K_{f_1 \circ f_2^{-1}} \geq 1$, from (4.1.10) we have

$$K_{f_1 \circ f_2^{-1}} \approx 1. \tag{4.1.11}$$

By (4.1.10) again we have that

$$|Df_2 \circ f_1^{-1}| = \frac{|adj Df_1 \circ f_2^{-1}|}{|J_{f_1 \circ f_2^{-1}}|} \approx |Df_1 \circ f_2^{-1}| \lesssim 1 \text{ and } J_{f_2 \circ f_1^{-1}} = \frac{1}{J_{f_1 \circ f_2^{-1}}} \approx -1. \tag{4.1.12}$$

Analogously to (4.1.11), we have that

$$K_{f_2 \circ f_1^{-1}}(x, y) \approx 1 \quad \forall t \ll 1 \text{ and } \forall (x, y) \in Q_t. \tag{4.1.13}$$

Aim 2: to define $f_3 : \tilde{Q}_t \rightarrow \tilde{R}_t$. Let

$$\tilde{Q}_t = \{(x, y) \in \mathbb{R}^2 : x \in [-t^2, -(t/2)^2], |y| \leq |x|^s\}.$$

Define

$$f_3(u, v) = \left(-u, \frac{t^{2s}}{(-u)^s} v \right) \quad \forall (u, v) \in \tilde{Q}_t.$$

Then f_3 is diffeomorphic and

$$Df_3(u, v) = \begin{bmatrix} -1 & 0 \\ \frac{st^{2s}}{(-u)^{s+1}} v & \frac{t^{2s}}{(-u)^s} \end{bmatrix}. \tag{4.1.14}$$

From (4.1.14) we have that

$$|Df_3| \lesssim 1 \text{ and } J_{f_3} \approx -1 \quad \forall (u, v) \in \tilde{Q}_t. \tag{4.1.15}$$

Analogously to (4.1.11), we have that

$$K_{f_3}(u, v) \approx 1 \quad \forall t \ll 1 \text{ and } \forall (u, v) \in \tilde{Q}_t. \tag{4.1.16}$$

Let $\tilde{R}_t = f_3(\tilde{Q}_t)$. Then $\tilde{R}_t = [(t/2)^2, t^2] \times [-t^{2s}, t^{2s}]$. The same reasons as for (4.1.12) and (4.1.13) imply that

$$|Df_3^{-1}(x, y)| \lesssim 1, \quad J_{f_3^{-1}}(x, y) \approx -1 \text{ and } K_{f_3^{-1}}(x, y) \approx 1 \tag{4.1.17}$$

for all $t \ll 1$ and $(x, y) \in \tilde{R}_t$.

Aim 3: to define $f_4 : \tilde{R}_t \rightarrow R_t$. Denote by P_1, P_2, P_3, P_4 and $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ the four vertices of \tilde{R}_t and R_t , respectively. Then

$$P_1 = \left(L_t^1, \frac{\sigma_t}{2} \right), \quad P_2 = \left(L_t^2, \frac{\sigma_t}{2} \right), \quad P_3 = \left(L_t^2, -\frac{\sigma_t}{2} \right), \quad P_4 = \left(L_t^1, -\frac{\sigma_t}{2} \right)$$

and

$$\tilde{P}_1 = \left((t/2)^2, t^{2s} \right), \quad \tilde{P}_2 = \left(t^2, t^{2s} \right), \quad \tilde{P}_3 = \left(t^2, -t^{2s} \right), \quad \tilde{P}_4 = \left((t/2)^2, -t^{2s} \right).$$

Since ∂M_s is mapped onto $\partial \Delta_s$ by z^2 , the line segment $\tilde{P}_1 \tilde{P}_2$ is mapped onto $P_1 P_2$ by

$$(u, t^{2s}) \mapsto \left(\eta(u), \frac{\sigma_t}{2} \right) \quad \forall u \in [(t/2)^2, t^2],$$

and the line segment $\tilde{P}_4 \tilde{P}_3$ is mapped onto $P_4 P_3$ by

$$(u, -t^{2s}) \mapsto \left(\eta(u), -\frac{\sigma_t}{2} \right) \quad \forall u \in [(t/2)^2, t^2].$$

Define

$$f_4(u, v) = \left(\eta(u), \frac{\sigma_t}{2t^{2s}}v \right) \quad \forall (u, v) \in \tilde{R}_t. \tag{4.1.18}$$

Then f_4 is a diffeomorphism from \tilde{R}_t onto R_t and

$$Df_4(u, v) = \begin{bmatrix} \eta'(u) & 0 \\ 0 & \frac{\sigma_t}{2t^{2s}} \end{bmatrix}. \tag{4.1.19}$$

By (4.1.3) and (4.1.4) we have that $\eta'(u) \approx t^{-1}$ and $\frac{\sigma_t}{2t^{2s}} \approx t^{1-2s}$ whenever $t \ll 1$ and $(u, v) \in \tilde{R}_t$. It follows from (4.1.19) that

$$|Df_4(u, v)| \approx t^{1-2s} \text{ and } J_{f_4}(u, v) \approx t^{-2s} \tag{4.1.20}$$

for all $t \ll 1$ and all $(u, v) \in \tilde{R}_t$. Then

$$K_{f_4}(u, v) = \frac{|Df_4(u, v)|^2}{J_{f_4}(u, v)} \approx t^{2-2s} \quad \forall t \ll 1 \text{ and } (u, v) \in \tilde{R}_t. \tag{4.1.21}$$

The same reasons as for (4.1.12) and (4.1.13) imply that

$$|Df_4^{-1}(x, y)| \approx t, \quad J_{f_4^{-1}}(x, y) \approx t^{2s} \text{ and } K_{f_4^{-1}}(x, y) \approx t^{2-2s} \tag{4.1.22}$$

for all $t \ll 1$ and all $(x, y) \in R_t$.

Aim 4: to define E_1 . Set

$$F_t = f_3^{-1} \circ f_4^{-1} \circ f_2 \circ f_1^{-1}.$$

Then F_t is a diffeomorphism from Q_t onto \tilde{Q}_t . Therefore

$$DF_t(z) = Df_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z)) Df_4^{-1}(f_2 \circ f_1^{-1}(z)) D(f_2 \circ f_1^{-1})(z)$$

for all $z \in Q_t$. From (4.1.17), (4.1.22), and (4.1.12) it then follows that

$$\begin{aligned} \int_{Q_t} |DF_t|^p \, dz &\leq \int_{Q_t} \left| Df_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1}) \right|^p \left| Df_4^{-1}(f_2 \circ f_1^{-1}) \right|^p \left| Df_2 \circ f_1^{-1} \right|^p \, dz \\ &\lesssim t^p \mathcal{L}^2(Q_t) \approx t^{2+p} \end{aligned} \tag{4.1.23}$$

for all $p \geq 0$. For a fixed large j_0 , we now consider the set Q_t with $t = 2^{-j}$ for all $j \geq j_0$. Define

$$E_1 = \sum_{j=j_0}^{+\infty} F_{2^{-j}} \chi_{Q_{2^{-j}}}. \tag{4.1.24}$$

Denote $\Omega_1 = \cup_{j=j_0}^{+\infty} Q_{2^{-j}}$ and $\tilde{\Omega}_1 = \cup_{j=j_0}^{+\infty} \tilde{Q}_{2^{-j}}$. Then E_1 is a homeomorphism from Ω_1 onto $\tilde{\Omega}_1$, and satisfies (2.2.1) for E_1 on \mathcal{L}^2 -a.e. Ω_1 . In order to prove that E_1 has finite distortion on Ω_1 , via (2.2.3) it thus suffices to prove that $E_1 \in W_{loc}^{1,1}(\Omega_1, \tilde{\Omega}_1)$. Actually, from (4.1.23) we have that

$$\int_{\Omega_1} |DE_1|^p = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} |DF_{2^{-j}}(z)|^p dz \lesssim \sum_{j=j_0}^{+\infty} 2^{-j(2+p)} < \infty \tag{4.1.25}$$

for all $p \geq 1$.

Step 2: we construct E_2 on the domain away from the cusp point. Denote

$$\Omega_2 = M_s^c \setminus \Omega_1 \text{ and } \tilde{\Omega}_2 = \Delta_s^c \setminus \tilde{\Omega}_1.$$

Notice that both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$ are piecewise smooth Jordan curves with nonzero angles at the two corners. Therefore both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$ are chord-arc curves. By [7] there are bi-Lipschitz mappings

$$H_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ and } H_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \tag{4.1.26}$$

such that $H_1(\mathbb{S}^1) = \partial\Omega_2$ and $H_2(\mathbb{S}^1) = \partial\tilde{\Omega}_2$. Define

$$h(z) = \begin{cases} E_1(z) & \forall z \in \partial\Omega_2 \cap \partial\tilde{\Omega}_1, \\ z^2 & \forall z \in \partial\Omega_2 \cap \partial M_s. \end{cases}$$

Then h is a bi-Lipschitz mapping in terms of the arc lengths. By the chord-arc properties of both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$, we have that h is also a bi-Lipschitz mapping with respect to the Euclidean distances. Taking (4.1.26) into account, we conclude that $H_2^{-1} \circ h \circ H_1 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a bi-Lipschitz mapping. By [13, Theorem A] there is then a bi-Lipschitz mapping

$$H : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \tag{4.1.27}$$

such that $H|_{\mathbb{S}^1} = H_2^{-1} \circ h \circ H_1$. Define

$$E_2 = H_2 \circ H \circ H_1^{-1}. \tag{4.1.28}$$

By (4.1.26) and (4.1.27), we have that E_2 is a bi-Lipschitz extension of h . Furthermore since $\deg_{M_s}(h, w) = 1$, we obtain that E_2 is orientation-preserving. Hence E_2 is a quasiconformal mapping. The same reasons as for (2.3.12) and (2.3.13) imply

$$|DE_2(z)|, K_{E_2}(z), \text{ and } J_{E_2}(z) \text{ are bounded from both above and below} \tag{4.1.29}$$

for \mathcal{L}^2 -a.e. $z \in \mathbb{R}^2$, and

$$\left| DE_2^{-1}(w) \right|, K_{E_2}^{-1}(w) \text{ and } J_{E_2}^{-1}(w) \text{ are bounded from both above and below} \tag{4.1.30}$$

for \mathcal{L}^2 -a.e. $w \in \mathbb{R}^2$.

Via (4.1.24) and (4.1.28), we set

$$E(x, y) = \begin{cases} E_1(x, y) & \text{for all } (x, y) \in \Omega_1, \\ E_2(x, y) & \text{for all } (x, y) \in \Omega_2, \\ (x^2 - y^2, 2xy) & \text{for all } (x, y) \in \overline{M}_s. \end{cases} \tag{4.1.31}$$

By the properties of E_1 and E_2 , we conclude that $E \in \mathcal{E}_s$. □

4.2 Proof of (1.0.7), (1.0.10), and (1.0.11) in Theorem 1.2

Proof of (1.0.7) Let g be as in Theorem 1.2. It suffices to check that there is $f \in \mathcal{F}_s(g)$ satisfying that $f \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. Let f be as in (4.1.1) and E be as in (4.1.31). By (4.1.25), (4.1.29), and the fact that $E(z) = z^2$ for all $z \in M_s$, we obtain that $E \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. By (3.0.42) $f \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. □

Proof of (1.0.10) Let g be as in Theorem 1.2. By Lemma 3.5 (1) it suffices to construct a $f \in \mathcal{F}_s(g)$ satisfying that $f^{-1} \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < 2(s + 1)/(2s - 1)$. Let f be as in (4.1.1) and E be as in (4.1.31). Via (3.0.32) it suffices to check that $E^{-1} \in W_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < 2(s + 1)/(2s - 1)$.

By (4.1.15), (4.1.20), and (4.1.10), we have that

$$\left| DF_{2^{-j}}^{-1}(w) \right| \leq \left| Df_1 \circ f_2^{-1}(f_4 \circ f_3(w)) \right| \left| Df_4(f_3(w)) \right| \left| Df_3(w) \right| \lesssim 2^{j(2s-1)}$$

for all $j \geq j_0$ and \mathcal{L}^2 -a.e. $w \in \tilde{Q}_{2^{-j}}$. Together with $\mathcal{L}^2(\tilde{Q}_{2^{-j}}) \approx 2^{-2j(s+1)}$, we hence obtain that

$$\int_{\tilde{\Omega}_1} \left| DE_1^{-1} \right|^p = \sum_{j=j_0}^{+\infty} \int_{\tilde{Q}_{2^{-j}}} \left| DF_{2^{-j}}^{-1} \right|^p \lesssim \sum_{j=j_0}^{+\infty} 2^{-j(2(s+1)+p(1-2s))} < \infty \tag{4.2.1}$$

for all $p < 2(s + 1)/(2s - 1)$. Since

$$\left| DE^{-1}(u, v) \right| \lesssim (u^2 + v^2)^{-1/4} \quad \forall (u, v) \in \Delta_s, \tag{4.2.2}$$

by a change of variables we have that

$$\int_{\Delta_s} \left| DE^{-1}(w) \right|^p dw \lesssim \int_0^{2\pi} \int_0^1 r^{1-\frac{p}{2}} dr d\theta \approx \int_0^1 r^{1-\frac{p}{2}} dr < \infty \tag{4.2.3}$$

for all $p < 2(s + 1)/(2s - 1)$. By (4.1.30), (4.2.1), and (4.2.3), we conclude that $E^{-1} \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < 2(s + 1)/(2s - 1)$. \square

Proof of (1.0.11) Let E be as in (4.1.31). Analogously to the proof of (1.0.10), it suffices to check that $K_{E^{-1}} \in L_{loc}^q(\mathbb{R}^2)$ for all $q < (s + 1)/(s - 1)$. Note that Lemma 3.5 (2) and (3.0.34) play game now. From (4.1.11), (4.1.21), and (4.1.16), we have that

$$K_{F_{2^{-j}}^{-1}}(w) = K_{f_1 \circ f_2^{-1}}(f_4 \circ f_3(w))K_{f_4}(f_3(w))K_{f_3}(w) \approx 2^{j(2s-2)}$$

for all $j \geq j_0$ and \mathcal{L}^2 -a.e. $w \in \tilde{Q}_{2^{-j}}$. Together with $\mathcal{L}^2(\tilde{Q}_{2^{-j}}) \approx 2^{-j2(s+1)}$, we then obtain that

$$\int_{\tilde{\Omega}_1} K_{E^{-1}}^q = \sum_{j=j_0}^{+\infty} \int_{\tilde{Q}_{2^{-j}}} K_{F_{2^{-j}}^{-1}}^q \lesssim \sum_{j=j_0}^{+\infty} 2^{2j[(s-1)q-(s+1)]} < \infty \tag{4.2.4}$$

for all $q < (s + 1)/(s - 1)$. By (4.1.30), (4.2.4), and the fact that E is conformal on M_s , we conclude that $K_{E^{-1}} \in L_{loc}^q(\mathbb{R}^2)$ for all $q < (s + 1)/(s - 1)$. \square

4.3 Proof of (1.0.8) in Theorem 1.2

Proof Analogously to the proof of (1.0.10) in Sect. 4.2, via Lemma 3.5 (3) and (3.0.41) it suffices to construct a $E \in \mathcal{E}_s$ satisfying that $K_E \in L_{loc}^q$ for all $q < \max\{1, 1/(s-1)\}$. The construction is divided into two cases.

Case 1: $s \in (1, 2)$. Let E be as in (4.1.31). From (4.1.17), (4.1.22), and (4.1.13), it follows that

$$K_{F_{2^{-j}}}(z) = K_{f_3^{-1}}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z))K_{f_4^{-1}}(f_2 \circ f_1^{-1}(z))K_{f_2 \circ f_1^{-1}}(z) \approx 2^{2j(s-1)}$$

for all $j \geq j_0$ and \mathcal{L}^2 -a.e. $z \in Q_{2^{-j}}$. Together with $\mathcal{L}^2(Q_{2^{-j}}) \approx 2^{-2j}$ we then have that

$$\int_{\Omega_1} K_E^q = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_{F_{2^{-j}}}^q \approx \sum_{j=j_0}^{+\infty} 2^{2j(q(s-1)-1)} < \infty \tag{4.3.1}$$

for all $q < 1/(s - 1)$. By (4.3.1), (4.1.29), and the fact that E is conformal on M_s , we conclude that $K_E \in L_{loc}^q(\mathbb{R}^2)$ for all $q < 1/(s - 1)$. Therefore we have proved (1.0.8) whenever $s \in (1, 2)$.

Case 2: $s \in [2, \infty)$. Except for redefining $f_4^{-1} : R_t \rightarrow \tilde{R}_t$ as in (4.1.18), we follow all processes in Sect. 4.1 to define a new E , see Fig. 3. To redefine f_4^{-1} , we should define mappings A, B, C .

We begin with notation. Let α_t and β_t be the length of sides of \tilde{R}_t , and γ_t be the length of a side of R_t . Whenever $t \ll 1$, we have that

$$\alpha_t = t^2 - (t/2)^2 \approx t^2, \beta_t = 2t^{2s} \text{ and } \gamma_t = \eta(t^2) - \eta((t/2)^2) \approx t. \tag{4.3.2}$$

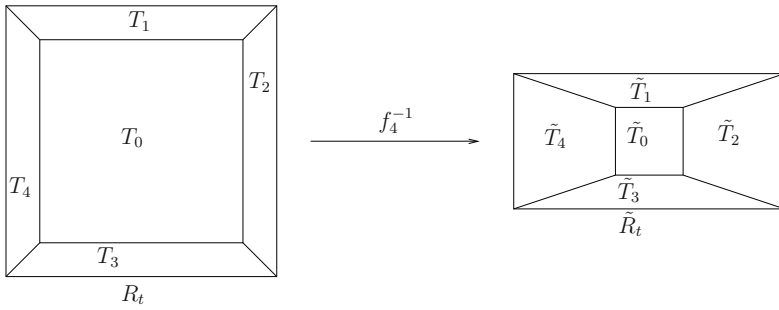


Fig. 3 The redefined $f_4^{-1} : R_t \rightarrow \tilde{R}_t$

Let $\tilde{T}_0 = \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3\tilde{Q}_4$ be the concentric square of \tilde{R}_t with side length $\beta_t/2$. Set

$$\delta_t = \exp(-t^{-1}) \quad \text{for } t > 0 \tag{4.3.3}$$

and let $T_0 = Q_1Q_2Q_3Q_4$ be the concentric square of R_t with side length $\gamma_t(1 - 2\delta_t)$. We divide $R_t \setminus T_0$ into four isosceles trapezoids $T_1, T_2, T_3,$ and T_4 . Similarly, we obtain isosceles trapezoids $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4$ from $\tilde{R}_t \setminus \tilde{T}_0$, see Fig. 3.

Aim 1: define $A : T_1 \rightarrow \tilde{T}_1$. Set

$$A_2(x, y) = \frac{\beta_t}{4\delta_t\gamma_t} \left(y - \gamma_t \left(\frac{1}{2} - \delta_t \right) \right) + \frac{\beta_t}{4} \quad \forall (x, y) \in T_1. \tag{4.3.4}$$

For a given $(x, y) \in T_1$, let $(x_p, y) = P_1Q_1 \cap \{(X, Y) \in \mathbb{R}^2 : Y = y\}$, $(\tilde{x}_p, A_2) = \tilde{P}_1\tilde{Q}_1 \cap \{(X, Y) \in \mathbb{R}^2 : Y = A_2(x, y)\}$, $\ell(y)$ be the length of $T_1 \cap \{(X, Y) : Y = y\}$, and $\tilde{\ell}(y)$ be the length of $\tilde{T}_1 \cap \{(X, Y) : Y = A_2\}$. Denote $(P_1)_1$ by the first coordinate of P_1 . Then

$$x_p = -y + \frac{\gamma_t}{2} + (P_1)_1 \text{ and } \tilde{x}_p = \frac{2\alpha_t - \beta_t}{\beta_t} \left(\frac{\beta_t}{2} - A_2 \right) + (\tilde{P}_1)_1, \tag{4.3.5}$$

$$\ell(y) = 2y \approx \gamma_t \text{ and } \tilde{\ell}(y) = \frac{4\alpha_t - 2\beta_t}{\beta_t} A_2(x, y) + \beta_t - \alpha_t \geq \frac{\beta_t}{2}. \tag{4.3.6}$$

Let $u = \frac{\gamma_t}{\ell(y)}(x - x_p) + (P_1)_1$ for $(x, y) \in T_1$, and η be as in (4.1.2). Define

$$A_1(x, y) = \frac{\tilde{\ell}(y)}{\alpha_t} \left(\eta^{-1}(u) - (\tilde{P}_1)_1 \right) + \tilde{x}_p \quad \forall (x, y) \in T_1. \tag{4.3.7}$$

By (4.3.7) and (4.3.4), we have that

$$A = (A_1, A_2) \tag{4.3.8}$$

is a diffeomorphism from T_1 onto \tilde{T}_1 . We next give some estimates for A . By (4.3.2) we have that

$$\frac{\partial A_2(x, y)}{\partial y} = \frac{\beta_t}{4\delta_t \gamma_t} \approx \frac{t^{2s-1}}{\delta_t} \quad \forall (x, y) \in T_1. \tag{4.3.9}$$

From (4.1.3), (4.3.6), and (4.3.2) it follows that

$$\frac{\partial A_1(x, y)}{\partial x} = \frac{\tilde{\ell}(y)}{\alpha_t} (\eta^{-1})'(u) \frac{\partial u}{\partial x} \approx \frac{\tilde{\ell}(y)}{t} \quad \forall (x, y) \in T_1. \tag{4.3.10}$$

Moreover, by (4.3.5) and (4.3.6) we have that

$$\frac{\partial x_p}{\partial y} = -1, \quad \frac{\partial \tilde{x}_p}{\partial y} = \frac{\beta_t - 2\alpha_t}{\beta_t} \frac{\partial A_2}{\partial y}, \quad \frac{\partial \ell(y)}{\partial y} = 2 \text{ and } \frac{\partial \tilde{\ell}(y)}{\partial y} = \frac{4\alpha_t - 2\beta_t}{\beta_t} \frac{\partial A_2}{\partial y}. \tag{4.3.11}$$

It follows from (4.3.11) that

$$\begin{aligned} \frac{\partial A_1}{\partial y} &= \frac{\partial \tilde{x}_p}{\partial y} + \frac{\partial \tilde{\ell}(y)}{\alpha_t \partial y} \left(\eta^{-1}(u) - (\tilde{P}_1)_1 \right) + \frac{\tilde{\ell}(y)}{\alpha_t} (\eta^{-1})'(u) \frac{\partial u}{\partial y} \\ &= \frac{2\alpha_t - \beta_t}{\beta_t} \frac{\partial A_2}{\partial y} \left[-1 + \frac{2}{\alpha_t} (\eta^{-1}(u) - (\tilde{P}_1)_1) \right] \\ &\quad + \frac{\gamma_t \tilde{\ell}(y)}{\alpha_t \ell(y)} (\eta^{-1})'(u) \left[1 - \frac{2}{\ell(y)} (x - x_p) \right]. \end{aligned} \tag{4.3.12}$$

Notice that $0 \leq \eta^{-1}(u) - (\tilde{P}_1)_1 \leq \alpha_t$ and $0 \leq x - x_p \leq \ell(y)$ for all $(x, y) \in T_1$. Therefore (4.3.12) together with (4.3.2) and (4.3.9) implies

$$\left| \frac{\partial A_1(x, y)}{\partial y} \right| \lesssim \frac{2\alpha_t - \beta_t}{\beta_t} \frac{\partial A_2(x, y)}{\partial y} \approx \frac{t}{\delta_t} \quad \forall (x, y) \in T_1. \tag{4.3.13}$$

We conclude from (4.3.9), (4.3.10), and (4.3.13) that

$$|DA(x, y)| \lesssim \max \left\{ \left| \frac{\partial A_1}{\partial x} \right|, \left| \frac{\partial A_1}{\partial y} \right|, \left| \frac{\partial A_2}{\partial x} \right|, \left| \frac{\partial A_2}{\partial y} \right| \right\} \lesssim \frac{t}{\delta_t} \tag{4.3.14}$$

and

$$J_A(x, y) = \frac{\partial A_1}{\partial x} \frac{\partial A_2}{\partial y} \approx \frac{t^{2s-2} \tilde{\ell}(y)}{\delta_t} \tag{4.3.15}$$

for all $t \ll 1$ and all $(x, y) \in T_1$. Moreover by (4.3.14), (4.3.15), and (4.3.6) we have that

$$K_A(x, y) = \frac{|DA(x, y)|^2}{J_A(x, y)} \lesssim \frac{t^{4-2s}}{\delta_t \tilde{\ell}(y)} \lesssim \frac{t^{4(1-s)}}{\delta_t} \tag{4.3.16}$$

holds for all $t \ll 1$ and all $(x, y) \in T_1$.

Aim 2: define $B : T_2 \rightarrow \tilde{T}_2$. Denote by P_c and \tilde{P}_c be the center of R_t and \tilde{R}_t , respectively. Given $(x, y) \in T_2$, we define

$$\begin{aligned} B_1(x, y) &= \frac{2\alpha_t - \beta_t}{4\delta_t\gamma_t} \left(x - (P_c)_1 - \frac{\gamma_t}{2}\right) + (\tilde{P}_c)_1 + \frac{\alpha_t}{2}, \quad B_2(x, y) \\ &= y \frac{a(x - (P_c)_1) + b}{c(x - (P_c)_1) + d}, \end{aligned}$$

where a, b, c, d satisfy

$$\begin{aligned} a\gamma_t \left(\frac{1}{2} - \delta_t\right) + b &= \frac{\beta_t}{4}, \quad a\frac{\gamma_t}{2} + b = \frac{\beta_t}{2}, \quad c\gamma_t \left(\frac{1}{2} - \delta_t\right) + d \\ &= \gamma_t \left(\frac{1}{2} - \delta_t\right), \quad c\frac{\gamma_t}{2} + d = \frac{\gamma_t}{2}. \end{aligned} \tag{4.3.17}$$

Then

$$B = (B_1, B_2) \tag{4.3.18}$$

is a diffeomorphism from T_2 onto \tilde{T}_2 . By (4.3.2) we have that

$$\frac{\partial B_1(x, y)}{\partial x} = \frac{2\alpha_t - \beta_t}{4\delta_t\gamma_t} \approx \frac{t}{\delta_t} \quad \forall (x, y) \in T_2. \tag{4.3.19}$$

Moreover, from (4.3.17) and (4.3.2) we have that

$$\frac{\partial B_2(x, y)}{\partial y} = \frac{a(x - (P_c)_1) + b}{c(x - (P_c)_1) + d} \approx \frac{\beta_t}{\gamma_t} \approx t^{2s-1} \tag{4.3.20}$$

and

$$\left| \frac{\partial B_2(x, y)}{\partial x} \right| = \frac{|y(ad - bc)|}{[c(x - (P_c)_1) + d]^2} \lesssim \frac{\gamma_t b}{\gamma_t^2} \approx t^{2s-1} \tag{4.3.21}$$

for all $(x, y) \in T_2$. We then conclude from (4.3.19), (4.3.20) and (4.3.21) that

$$|DB(x, y)| \lesssim \max \left\{ \left| \frac{\partial B_1}{\partial x} \right|, \left| \frac{\partial B_1}{\partial y} \right|, \left| \frac{\partial B_2}{\partial x} \right|, \left| \frac{\partial B_2}{\partial y} \right| \right\} \lesssim \frac{t}{\delta_t} \tag{4.3.22}$$

and

$$J_B(x, y) = \frac{\partial B_1}{\partial x} \frac{\partial B_2}{\partial y} \approx \frac{t^{2s}}{\delta_t}. \tag{4.3.23}$$

for all $t \ll 1$ and all $(x, y) \in T_2$. Moreover by (4.3.22) and (4.3.23) we have that

$$K_B(x, y) = \frac{|DB(x, y)|^2}{J_B(x, y)} \lesssim \frac{t^{2(1-s)}}{\delta_t} \tag{4.3.24}$$

for all $t \ll 1$ and all $(x, y) \in T_2$.

Aim 3: define $C : T_0 \rightarrow \tilde{T}_0$. By (4.3.8) and (4.3.18) we have that Q_1Q_2 is mapped onto $\tilde{Q}_1\tilde{Q}_2$ by $A_1(\cdot, \gamma_t(1/2 - \delta_t))$, and Q_2Q_3 is mapped onto $\tilde{Q}_2\tilde{Q}_3$ by $B_2((P_c)_1 + \gamma_t(1/2 - \delta_t), \cdot)$. For a given $(x, y) \in T_0$, define

$$C(x, y) = \left(A_1\left(x, \gamma_t\left(\frac{1}{2} - \delta_t\right)\right), B_2\left((P_c)_1 + \gamma_t\left(\frac{1}{2} - \delta_t\right), y\right) \right). \tag{4.3.25}$$

Then $C : T_0 \rightarrow \tilde{T}_0$ is diffeomorphic. By (4.3.10) and (4.3.20), we have that

$$\frac{\partial}{\partial x} A_1\left(x, \gamma_t(1/2 - \delta_t)\right) \approx t^{2s-1}, \quad \frac{\partial}{\partial y} B_2\left((P_c)_1 + \gamma_t(1/2 - \delta_t), y\right) \approx t^{2s-1}$$

for all $(x, y) \in T_0$. Therefore

$$|DC(x, y)| \lesssim t^{2s-1} \text{ and } K_C(x, y) \approx 1 \tag{4.3.26}$$

for all $t \ll 1$ and all $(x, y) \in T_0$.

Aim 4: redefine f_4^{-1} and E . Via (4.3.8), (4.3.18), and (4.3.25), we set $f_4^{-1} : R_t \rightarrow \tilde{R}_t$ in (4.1.18) as

$$f_4^{-1}(x, y) = \begin{cases} A(x, y) & \forall (x, y) \in T_1, \\ B(x, y) & \forall (x, y) \in T_2, \\ (A_1(x, -y), -A_2(x, -y)), & \forall (x, y) \in T_3, \\ (2(\tilde{P}_c)_1 - B_1(2(P_c)_1 - x, y), B_2(2(P_c)_1 - x, y)) & \forall (x, y) \in T_4, \\ C(x, y) & \forall (x, y) \in T_0. \end{cases} \tag{4.3.27}$$

Like in Sect. 4.1, by taking a fixed $j_0 \gg 1$ we then define $F_{2-j} : Q_{2-j} \rightarrow \tilde{Q}_{2-j}$ for all $j \geq j_0$, $E_1 : \Omega_1 \rightarrow \tilde{\Omega}_1$, $E_2 : \Omega_2 \rightarrow \tilde{\Omega}_2$, and $E : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. It is not difficult to see that the new-defined E is a homeomorphism such that $E(z) = z^2$ for all $z \in \overline{M_s}$ and satisfies (2.2.1) for E on \mathcal{L}^2 -a.e. \mathbb{R}^2 . To show that $E \in \mathcal{E}_s$, via (2.2.3) it is then

enough to prove that $E \in W_{\text{loc}}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$. By (4.1.12), (4.1.17), (4.3.14), (4.3.22), and (4.3.26), we have that

$$\begin{aligned}
 DF_{2^{-j}}(z) &= Df_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z))Df_4^{-1}(f_2 \circ f_1^{-1}(z))D(f_2 \circ f_1^{-1})(z) \\
 &\lesssim \begin{cases} \frac{2^{-j}}{\delta_{2^{-j}}} & \mathcal{L}^2\text{-a.e. } z \in f_1 \circ f_2^{-1}(\cup_{k=1}^4 T_k), \\ 2^{j(1-2s)} & \mathcal{L}^2\text{-a.e. } z \in f_1 \circ f_2^{-1}(T_0), \end{cases} \tag{4.3.28}
 \end{aligned}$$

for all $j \geq j_0$. Notice that

$$\mathcal{L}^2(T_0) = (\gamma_{2^{-j}}(1 - 2\delta_{2^{-j}}))^2 \approx 2^{-2j}, \quad \mathcal{L}^2(T_k) = \delta_{2^{-j}}\gamma_{2^{-j}}^2(1 - \delta_{2^{-j}}) \approx \delta_{2^{-j}}2^{-2j}$$

for all $k = 1, 2, 3, 4$ and all $j \geq j_0$. It hence follows from (4.1.10) that

$$\mathcal{L}^2(f_1 \circ f_2^{-1}(T_0)) \approx 2^{-2j}, \quad \mathcal{L}^2(f_1 \circ f_2^{-1}(T_k)) \approx \delta_{2^{-j}}2^{-2j} \quad \text{for all } k = 1, 2, 3, 4. \tag{4.3.29}$$

By (4.3.28) and (4.3.29) we then have that

$$\int_{Q_{2^{-j}}} |DF_{2^{-j}}| = \sum_{k=0}^4 \int_{f_1 \circ f_2^{-1}(T_k)} |DF_{2^{-j}}| \lesssim 2^{-3j} + 2^{-j(2s+1)} \lesssim 2^{-3j} \quad \forall j \geq j_0.$$

Therefore

$$\int_{\Omega_1} |DE_1| = \sum_{j=j_0}^{\infty} \int_{Q_{2^{-j}}} |DF_{2^{-j}}| \lesssim \sum_{j=j_0}^{\infty} 2^{-3j} < \infty. \tag{4.3.30}$$

By (4.1.29), (4.3.30), and the fact that $E(z) = z^2$ for all $z \in M_s$, we have that $E \in W_{\text{loc}}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$.

We next show $K_E \in L_{\text{loc}}^q(\mathbb{R}^2)$ for all $q < 1$. By (4.1.13), (4.1.17), (4.3.16), (4.3.24), and (4.3.26), we have that

$$K_{F_{2^{-j}}}(z) \lesssim \begin{cases} \frac{2^{4j(s-1)}}{\delta_{2^{-j}}} & \forall z \in f_1 \circ f_2^{-1}(T_1 \cup T_3), \\ \frac{2^{2j(s-1)}}{\delta_{2^{-j}}} & \forall z \in f_1 \circ f_2^{-1}(T_2 \cup T_4), \\ 1 & \forall z \in f_1 \circ f_2^{-1}(T_0). \end{cases} \tag{4.3.31}$$

for all $j \geq j_0$. For any $q \geq 0$, via (4.3.29) and (4.3.31) we obtain that

$$\int_{Q_{2^{-j}}} K_{F_{2^{-j}}}^q = \sum_{k=0}^4 \int_{f_1 \circ f_2^{-1}(T_k)} K_{F_{2^{-j}}}^q \lesssim \delta_{2^{-j}}^{1-q} 2^{j(4q(s-1)-2)} (1 + 2^{2qj(1-s)}) + 2^{-2j}$$

for all $j \geq j_0$. Therefore

$$\begin{aligned} \int_{\Omega_1} K_E^q &= \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_{F_{2^{-j}}}^q \\ &\lesssim \sum_{j=j_0}^{+\infty} \exp((q-1)2^j) 2^{j(4q(s-1)-2)} (1+2^{j2q(1-s)}) + \sum_{j=j_0}^{+\infty} 2^{-2j} < +\infty \end{aligned} \tag{4.3.32}$$

for all $q \in (0, 1)$ and each $s > 1$. By (4.1.29), (4.3.32), and the fact that E is conformal on M_s , we conclude that $K_E \in L_{loc}^q(\mathbb{R}^2)$ for all $q \in (0, 1)$. \square

4.4 Proof of (1.0.9) in Theorem 1.2

Proof Analogously to the proof of (1.0.10) in Sect. 4.2, via Lemma 3.5 (4), (3.0.41), and (3.0.42) it suffices to construct $E \in \mathcal{E}_s$ satisfying that $E \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_E \in L_{loc}^q$ for all $q < \max\{1/(s-1), M(p, s)\}$. Here we denote $M(p, s) = 3p/((2s-1)p+4-2s)$ with $p > 1$. The construction is divided into two cases.

Case 1: $s \in (1, 2)$. Let E be as in (4.1.31). Then $E \in \mathcal{E}_s$. By (4.1.25), (4.1.29), and the fact that $E(z) = z^2$ for all $z \in M_s$, we obtain that $E \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. From (4.1.17), (4.1.22), and (4.1.13), it follows that

$$K_{F_{2^{-j}}}(z) = K_{f_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z))} K_{f_4^{-1}(f_2 \circ f_1^{-1}(z))} K_{f_2 \circ f_1^{-1}(z)} \approx 2^{(2s-2)j}$$

for all $j \geq j_0$ and \mathcal{L}^2 -a.e. $z \in Q_{2^{-j}}$. Together with $\mathcal{L}^2(Q_{2^{-j}}) \approx 2^{-2j}$, we then obtain

$$\int_{\Omega_1} K_E^q = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_{F_{2^{-j}}}^q \approx \sum_{j=j_0}^{+\infty} 2^{-j2(1+q(1-s))} < \infty \tag{4.4.1}$$

for all $q < 1/(s-1)$. By (4.4.1), (4.1.29), and the fact that E is conformal on M_s , we have that $K_E \in L_{loc}^q(\mathbb{R}^2)$ for all $q < 1/(s-1)$.

Case 2: $s \in [2, \infty)$. Redefining δ_t in (4.3.3) as

$$\delta_t = t^{\frac{p+2}{p-1}} \log^{\frac{p}{p-1}}(t^{-1}).$$

We follow the methods in Sect. 4.3 to define a new f_4^{-1} . Set $j_0 \gg 1$. There are then new $F_{2^{-j}} : Q_{2^{-j}} \rightarrow \tilde{Q}_{2^{-j}}$ for all $j \geq j_0$, $E_1 : \Omega_1 \rightarrow \tilde{\Omega}_1$, $E_2 : \Omega_2 \rightarrow \tilde{\Omega}_2$, and $E : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. It is not difficult to see that the new E is homeomorphic and satisfies (2.2.1) for E on \mathcal{L}^2 -a.e. \mathbb{R}^2 . To show that E satisfies all requirements, it is enough to check that $E \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ and $K_E \in L_{loc}^q(\mathbb{R}^2)$ for all $q \in (0, M(p, s))$.

From (4.1.12), (4.1.17), (4.3.14), (4.3.22), and (4.3.26), we have that

$$|DF_{2^{-j}}(z)| \lesssim \begin{cases} \frac{2^{-j}}{\delta_{2^{-j}}} & \forall z \in f_1 \circ f_2^{-1}(\cup_{k=1}^4 T_k), \\ 2^{j(1-2s)} & \forall z \in f_1 \circ f_2^{-1}(T_0), \end{cases} \tag{4.4.2}$$

for all $j \geq j_0$. It follows from (4.4.2) and (4.3.29) that

$$\int_{Q_{2^{-j}}} |DF_{2^{-j}}|^p = \sum_{k=0}^4 \int_{f_1 \circ f_2^{-1}(T_k)} |DF_{2^{-j}}|^p \lesssim \delta_{2^{-j}}^{1-p} 2^{-j(2+p)} + 2^{j(p(1-2s)-2)}.$$

Therefore

$$\int_{\Omega_1} |DE|^p = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} |DF_{2^{-j}}|^p \lesssim \sum_{j=j_0}^{+\infty} \frac{1}{j^p} + \sum_{j=j_0}^{+\infty} 2^{-j(p(2s-1)+2)} < \infty. \tag{4.4.3}$$

By (4.4.3), (4.1.29), and the fact that $E(z) = z^2$ for all $z \in M_s$, we conclude that $E \in W_{loc}^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$. By (4.1.12), (4.1.13), Lemma 2.1, and (4.1.17), we have

$$\begin{aligned} \int_{f_1 \circ f_2^{-1}(T_1)} K_{F_{2^{-j}}}^q &\approx \int_{f_1 \circ f_2^{-1}(T_1)} K_{f_3^{-1}}^q(f_4^{-1} \circ f_2 \circ f_1^{-1}) K_{f_4^{-1}}^q(f_2 \circ f_1^{-1}) K_{f_2 \circ f_1^{-1}}^q |J_{f_2 \circ f_1^{-1}}| \\ &\leq \int_{T_1} K_{f_3^{-1}}^q(f_4^{-1}) K_{f_4^{-1}}^q \\ &\lesssim \int_{T_1} K_{f_4^{-1}}^q \end{aligned} \tag{4.4.4}$$

for all $q \geq 0$ and all $j \geq j_0$. Notice $\tilde{\ell}(\gamma_{2^{-j}}/2) = \alpha_{2^{-j}}$ and $\tilde{\ell}(\gamma_{2^{-j}}(\frac{1}{2} - \delta_{2^{-j}})) = \beta_{2^{-j}}/2$ for all $j \geq 1$. By Fubini's theorem, (4.3.16), (4.3.6), and (4.3.2), we then have

$$\begin{aligned} \int_{T_1} K_{f_4^{-1}}^q &\lesssim \int_{\gamma_{2^{-j}}(\frac{1}{2} - \delta_{2^{-j}})}^{\frac{\gamma_{2^{-j}}}{2}} \int_{x_p}^{x_p + \ell(y)} \left(\frac{2^{j(2s-4)}}{\delta_{2^{-j}} \tilde{\ell}(y)} \right)^q dx dy \\ &\approx \frac{2^{jq(2s-4)} \gamma_{2^{-j}}}{\delta_{2^{-j}}^q} \int_{\gamma_{2^{-j}}(\frac{1}{2} - \delta_{2^{-j}})}^{\frac{\gamma_{2^{-j}}}{2}} \frac{1}{\tilde{\ell}^q(y)} dy \\ &= \frac{2^{jq(2s-4)} \gamma_{2^{-j}}}{(1-q)\delta_{2^{-j}}^q} \frac{2\delta_{2^{-j}} \gamma_{2^{-j}}}{2\alpha_{2^{-j}} - \beta_{2^{-j}}} \left(\tilde{\ell}^{1-q}(\frac{\gamma_{2^{-j}}}{2}) - \tilde{\ell}^{1-q}(\gamma_{2^{-j}}(\frac{1}{2} - \delta_{2^{-j}})) \right) \\ &\lesssim \frac{\delta_{2^{-j}}^{1-q} 2^{-2j[1+q(1-s)]}}{1 - M(p, s)} \end{aligned} \tag{4.4.5}$$

for any fixed $q \in (0, M(p, s))$. Combining (4.4.4) with (4.4.5) implies that

$$\int_{f_1 \circ f_2^{-1}(T_1)} K_{F_{2^{-j}}}^q \lesssim \delta_{2^{-j}}^{1-q} 2^{-2j[1+q(1-s)]} \quad \forall j \geq j_0. \tag{4.4.6}$$

By symmetry of f_4^{-1} between T_1 and T_3 , it follows from (4.4.6) that

$$\int_{f_1 \circ f_2^{-1}(T_3)} K_{F_{2^{-j}}}^q = \int_{f_1 \circ f_2^{-1}(T_1)} K_{F_{2^{-j}}}^q \lesssim \delta_{2^{-j}}^{1-q} 2^{-2j|1+q(1-s)|} \tag{4.4.7}$$

for all $j \geq j_0$. By (4.3.31) and (4.3.29), we have that

$$\int_{f_1 \circ f_2^{-1}(T_0)} K_{F_{2^{-j}}}^q \lesssim 2^{-2j} \tag{4.4.8}$$

and

$$\int_{f_1 \circ f_2^{-1}(T_2 \cup T_4)} K_{F_{2^{-j}}}^q \lesssim \delta_{2^{-j}} 2^{-2j} \left(\frac{2^{2j(s-1)}}{\delta_{2^{-j}}} \right)^q = \delta_{2^{-j}}^{1-q} 2^{2j[q(s-1)-1]} \tag{4.4.9}$$

for all $j \geq j_0$. From (4.4.6), (4.4.7), (4.4.8), and (4.4.9), we conclude that

$$\begin{aligned} \int_{\Omega_1} K_E^q &= \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_{F_{2^{-j}}}^q = \sum_{j=j_0}^{+\infty} \sum_{k=0}^4 \int_{f_1 \circ f_2^{-1}(T_k)} K_{F_{2^{-j}}}^q \\ &\lesssim \sum_{j=j_0}^{+\infty} 2^{-2j} + 2^{-j} \left(\frac{\delta_{2^{-j}}^{(p+2)(1-q)} + 2[1+q(1-s)]}{\delta_{2^{-j}}} \right) \log \frac{\delta_{2^{-j}}^{p(1-q)}}{\delta_{2^{-j}}} (2^j). \end{aligned} \tag{4.4.10}$$

Note that

$$\frac{(p+2)(1-q)}{p-1} + 2[1+q(1-s)] > 0 \Leftrightarrow q < M(p, s).$$

It from (4.4.10) follows that $\int_{\Omega_1} K_E^q < \infty$ for all $q \in (0, M(p, s))$. Together with (4.1.29) and the fact that E is conformal on M_s , we conclude that $K_E \in L_{loc}^q(\mathbb{R}^2)$ for all $q \in (0, M(p, s))$. □

5 Proof of Theorem 1.1

Proof Let Δ be as in (1.0.1). The representation of $\partial\Delta$ in Cartesian coordinates is

$$(x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 = 0.$$

Hence we can parametrize $\partial\Delta$ in a neighborhood of the origin as

$$\tilde{\Gamma}_0 = \{(x, y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = d(x)\},$$

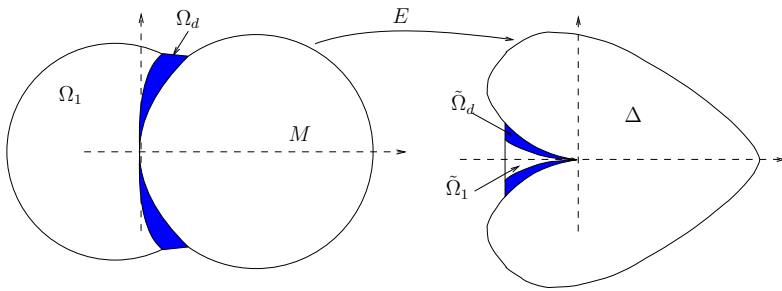


Fig. 4 The existence of an extension

where $j_0 \gg 1$ and $d(x) = \frac{-x^3(4-x)}{2-x^2+2x+\sqrt{1+2x}}$. Since $d(x) \approx |x|^3$ for all $|x| \ll 1$, there are $c_1 > 0, c_2 > 0$ such that

$$-c_1x^3 \leq d(x) \leq -c_2x^3 \quad \forall x \in [-2^{-j_0}, 0].$$

Denote

$$\begin{aligned} \tilde{\Gamma}_1 &= \{(x, y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = -c_1x^3\}, \\ \tilde{\Gamma}_2 &= \{(x, y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = -c_2x^3\}, \\ \tilde{\Gamma}_3 &= \{(x, y) \in \mathbb{R}^2 : x = -2^{-j_0}, y^2 \in [c_1(2^{-j_0})^3, d(-2^{-j_0})]\}, \\ \tilde{\Gamma}_4 &= \{(x, y) \in \mathbb{R}^2 : x = -2^{-j_0}, y^2 \in [d(-2^{-j_0}), c_2(2^{-j_0})^3]\}. \end{aligned}$$

Let $\tilde{\Omega}_u$ and $\tilde{\Omega}_d$ be the domains bounded by $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_4$ and $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_3$, respectively. Denote by Ω_u, Ω_d and Γ_k for $k = 0, \dots, 4$ the images of $\tilde{\Omega}_u, \tilde{\Omega}_d$ and $\tilde{\Gamma}_k$ under the branch of complex-valued function $z^{1/2}$ with $1^{1/2} = 1$, respectively.

We first prove the existence of an extension, see Fig. 4.

Let $r = (2^{-2j_0} + c_12^{-3j_0})^{1/4}$. Denote

$$\begin{aligned} M &= \{(x + 1, y) \in \mathbb{R}^2 : (x, y) \in \mathbb{D}\}, \\ \Omega_1 &= \overline{B(0, r)} \setminus (M \cup \Omega_d), \quad \Omega_2 = \mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_d \cup M), \\ \tilde{\Omega}_1 &= \{(x, y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 \leq c_1|x|^3\} \text{ and } \tilde{\Omega}_2 = \mathbb{R}^2 \setminus (\tilde{\Omega}_1 \cup \tilde{\Omega}_d \cup \Delta). \end{aligned}$$

Analogously to the arguments in Sect. 4.1, we define $E_1 : \Omega_1 \rightarrow \tilde{\Omega}_1$ and $E_2 : \Omega_2 \rightarrow \tilde{\Omega}_2$. Here $\eta(x) = \sqrt{x}(1 + c_1x)^{1/4}$ and $s = 3/2$. Define

$$E(x, y) = \begin{cases} E_1(x, y) & \forall (x, y) \in \Omega_1, \\ E_2(x, y) & \forall (x, y) \in \Omega_2, \\ (x^2 - y^2, 2xy) & \forall (x, y) \in M \cup \Omega_d, \end{cases} \tag{5.0.1}$$

and $f_0(x, y) = E(x + 1, y)$. By the analogous arguments as in Sect. 4.1, we have that $f_0 \in \mathcal{F}$.

We next prove (1.0.3). Suppose $f \in \mathcal{F}$. Then $\hat{f}(u, v) = f(u - 1, v)$ is a homeomorphism of finite distortion on \mathbb{R}^2 and $\hat{f}(M \setminus \Omega_u) = \Delta \setminus \tilde{\Omega}_u$. By Remark 3.1, we have that if $K_{\hat{f}} \in L_{\text{loc}}^q(\mathbb{R}^2)$ then $q < 2$. Therefore if $K_f \in L_{\text{loc}}^q(\mathbb{R}^2)$ then $q < 2$. In order to prove (1.0.3), it then suffices to construct a mapping $f_0 \in \mathcal{F}$ such that $K_{f_0} \in L_{\text{loc}}^q(\mathbb{R}^2)$ for all $q < 2$. Let E be as in (5.0.1) and $f_0(x, y) = E(x + 1, y)$. Then $f_0 \in \mathcal{F}$. The same arguments as for the case $s \in (1, 2)$ in Sect. 4.3 show that $K_E \in L_{\text{loc}}^q(\mathbb{R}^2)$ for all $q < 2$. Therefore $K_{f_0} \in L_{\text{loc}}^q(\mathbb{R}^2)$ for all $q < 2$.

The strategies to prove (1.0.2), (1.0.4), (1.0.5), and (1.0.6) are same as the one to prove (1.0.3). We leave the details to the interested reader. \square

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