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The Gauss–Bonnet Theorem for Coherent Tangent Bundles over Surfaces with Boundary and Its Applications

Wojciech Domitrz[1] · **Michał Zwierzyński**[1]

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Abstract

In Saji et al. (J Math 62:259–280, 2008; Ann Math 169:491–529, 2009; J Geom Anal 222):383–409, 2012) the Gauss–Bonnet formulas for coherent tangent bundles over compact-oriented surfaces (without boundary) were proved. We establish the Gauss–Bonnet theorem for coherent tangent bundles over compact-oriented surfaces with boundary. We apply this theorem to investigate global properties of maps between surfaces with boundary. As a corollary of our results, we obtain a special version of Fukuda–Ishikawa's theorem. We also study geometry of the affine-extended wave fronts for planar-closed non-singular hedgehogs (rosettes). In particular, we find a link between the total geodesic curvature on the boundary and the total singular curvature of the affine-extended wave front, which leads to a relation of integrals of functions of the width of a rosette.

Keywords Coherent tangent bundle · Wave front · Gauss–Bonnet formula

Mathematics Subject Classification Primary: 57R45 · Secondary: 53A05

1 Introduction

The local and global geometry of fronts and coherent tangent bundles, which are natural generalizations of fronts, has been recently very carefully studied in [19,29,30,35–38]. In particular in [35,36] the results of Kossowski [20,21] and Langevin et al. [24] were

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✉ Michał Zwierzyński
zwierzynskim@mini.pw.edu.pl

Wojciech Domitrz
domitrz@mini.pw.edu.pl

1 Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warszawa, Poland

generalized to the following Gauss–Bonnet-type formulas for the singular coherent tangent bundle $\mathcal{E}$ over a compact surface M whose set of singular points Σ admits at most peaks:

$$2\pi\chi(M) = \int_M K \mathrm{d}A + 2\int_\Sigma \kappa_s \mathrm{d}\tau, \tag{1.1}$$

$$\frac{1}{2\pi}\int_M K \mathrm{d}\hat{A} = \chi(M^+) - \chi(M^-) + \#P^+ - \#P^-. \tag{1.2}$$

In the above formulas K is the Gaussian curvature, κ_s is the singular curvature, $\mathrm{d}\tau$ is the arc length measure on Σ, $\mathrm{d}\hat{A}$ (respectively $\mathrm{d}A$) is the signed (respectively unsigned) area form, M^+ (respectively M^-) is the set of regular points in M, where $\mathrm{d}\hat{A} = \mathrm{d}A$ (respectively $\mathrm{d}\hat{A} = -\mathrm{d}A$), P^+ (respectively P^-) is the set of positive (respectively negative) peaks (see [35] and Sect. 2 for details). Saji et al. also found several interesting applications of the above formulas (see especially [37]).

The classical Gauss–Bonnet theorem was formulated for compact-oriented surfaces with boundary. Therefore, it is natural to find the analogous Gauss–Bonnet formulas for coherent tangent bundles over compact-oriented surfaces with boundary (see Theorem 2.20). Coherent tangent bundles over compact oriented surfaces with boundary also appear in many problems. In this paper, we apply the Gauss–Bonnet formulas to study smooth maps between compact-oriented surfaces with boundary and affine-extended wave fronts of the planar non-singular hedgehogs (rosettes). As a result, we obtain a new proof of a special version of Fukuda–Ishikawa's theorem [12] and we find a link between the total geodesic curvature on the boundary and the total singular curvature of the affine-extended wave front of a rosette. This leads to a relation between the integrals of the function of the width of the rosette, in particular of the width of an oval (see Theorem 5.24 and Conjecture 5.28).

In Sect. 2, we briefly sketch the theory of coherent tangent bundles and state the Gauss–Bonnet theorem for coherent tangent bundles over compact-oriented surfaces with boundary (Theorem 2.20), which is the main result of this paper. The proof of Theorem 2.20 is presented in Sect. 3. We apply this theorem to study the global properties of maps between compact-oriented surfaces with boundary in Sect. 4. The last section contains the results on the geometry of the affine-extended wave fronts of rosettes.

2 The Gauss–Bonnet Theorem

In this section, we formulate the Gauss–Bonnet-type theorem for coherent tangent bundles over compact-oriented surfaces with boundary. The proof of this theorem is presented in the next section. Coherent tangent bundles are intrinsic formulation of wave fronts. The theory of coherent tangent bundles were introduced and developed in [35–37]. We recall basic definitions and facts of this theory (for details see [35,37]).

Definition 2.1 Let M be a 2-dimensional compact-oriented surface (possibly with boundary). A *coherent tangent bundle* over M is a 5-tuple $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \psi)$, where

$\mathcal{E}$ is an orientable vector bundle over M of rank 2, $\langle\cdot,\cdot\rangle$ is a metric, D is a metric connection on $(\mathcal{E}, \langle\cdot,\cdot\rangle)$ and ψ is a bundle homomorphism

$$\psi : TM \to \mathcal{E},$$

such that for any smooth vector fields X, Y on M

$$D_X\psi(Y) - D_Y\psi(X) = \psi([X, Y]). \tag{2.1}$$

The pull-back metric $\mathrm{d}s^2 := \psi^* \langle\cdot,\cdot\rangle$ is called the *first fundamental form* on M. Let $\mathcal{E}_p$ denote the fiber of $\mathcal{E}$ at a point $p \in M$. If $\psi_p := \psi|_{T_pM} : T_pM \to \mathcal{E}_p$ is not a bijection at a point $p \in M$, then p is called a *singular point*. Let Σ denote the set of singular points on M. If a point $p \in M$ is not a singular point, then p is called a *regular point*. Let us notice that the first fundamental form on M is positive definite at regular points and it is not positive definite at singular points.

Let $\mu \in \mathrm{Sec}(\mathcal{E}^* \wedge \mathcal{E}^*)$ be a smooth non-vanishing skew-symmetric bilinear section such that for any orthonormal frame $\{e_1, e_2\}$ on $\mathcal{E}$ $\mu(e_1, e_2) = \pm 1$. The existence of such μ is a consequence of the assumption that $\mathcal{E}$ is orientable. A *co-orientation* of the coherent tangent bundle is a choice of μ. An orthonormal frame $\{e_1, e_2\}$ such that $\mu(e_1, e_2) = 1$ (respectively $\mu(e_1, e_2) = -1$) is called *positive* (respectively *negative*) with respect to the co-orientation μ.

From now on, we fix a co-orientation μ on the coherent tangent bundle.

Definition 2.2 Let $(U; u, v)$ be a positively oriented local coordinate system on M. Then $\mathrm{d}\hat{A} := \psi^*\mu = \lambda_\psi du \wedge dv$ (respectively $dA := |\lambda_\psi| du \wedge dv$) is called the *signed area form* (respectively the *unsigned area form*), where

$$\lambda_\psi := \mu(\psi_u, \psi_v), \psi_u := \psi\left(\frac{\partial}{\partial u}\right), \psi_v := \psi\left(\frac{\partial}{\partial v}\right).$$

The function λ_ψ is called the *signed area density function* on U.

The set of singular points on U is expressed as

$$\Sigma \cap U := \left\{p \in U : \lambda_\psi(p) = 0\right\}.$$

Let us notice that the signed and unsigned area forms, $\mathrm{d}\hat{A}$ and dA, give globally defined 2-forms on M and they are independent of the choice of positively oriented local coordinate system (u, v). Let us define

$$M^+ := \left\{p \in M \setminus \Sigma \,\middle|\, \mathrm{d}\hat{A}_p = \mathrm{d}A_p\right\}, \quad M^- := \left\{p \in M \setminus \Sigma \,\middle|\, \mathrm{d}\hat{A}_p = -\mathrm{d}A_p\right\}.$$

We say that a singular point $p \in \Sigma$ is *non-degenerate* if $\mathrm{d}\lambda_\psi$ does not vanish at p. Let p be a non-degenerate singular point. There exists a neighborhood U of p such that the set $\Sigma \cap U$ is a regular curve, which is called the *singular curve*. The *singular direction* is the tangential direction of the singular curve. Since p is non-degenerate, the rank of ψ_p is 1. The *null direction* is the direction of the kernel of ψ_p. Let $\eta(t)$ be the smooth (non-vanishing) vector field along the singular curve $\sigma(t)$ which gives the null direction.

Let $\wedge$ be the exterior product on TM.

Definition 2.3 Let $p \in M$ be a non-degenerate singular point and let $\sigma(t)$ be a singular curve such that $\sigma(0) = p$. The point p is called an A_2*-point* (or an *intrinsic cuspidal edge*) if the null direction at p (i.e. $\eta(0)$) is transversal to the singular direction at p (i.e. $\dot\sigma(0) := \frac{\mathrm{d}\sigma}{dt}\big|_{t=0}$). The point p is called an A_3*-point* (or an *intrinsic swallowtail*) if the point p is not an A_2*-point* and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot\sigma(t) \wedge \eta(t))|_{t=0} \neq 0.$$

Definition 2.4 Let p be a singular point $p \in M$ which is not an A_2-point. The point p is called a *peak* if there exists a coordinate neighborhood $(U; u, v)$ of p such that:

(i) if $q \in (\Sigma \cap U) \setminus \{p\}$ then q is an A_2-point;
(ii) the rank of the linear map $\psi_p : T_pM \to \mathcal{E}_p$ at p is equal to 1;
(iii) the set $\Sigma \cap U$ consists of finitely many C^1-regular curves emanating from p.

A peak is a non-degenerate if it is a non-degenerate singular point.

From now on, we assume that the set of singular points Σ *admits at most peaks*, i.e. Σ consists of A_2-points and peaks.

Furthermore, let us fix a Riemannian metric g on M. Since the first fundamental form $\mathrm{d}s^2$ degenerates on Σ, there exists a (1, 1)-tensor field I on M such that

$$\mathrm{d}s^2(X, Y) = g(IX, Y),$$

for smooth vector fields X, Y on M. We fix a singular point $p \in \Sigma$. Since Σ admits at most peaks, the point p is an A_2-point or a peak. Let $\lambda_1(p), \lambda_2(p)$ be the eigenvalues of $I_p := I\big|_{T_pM} : T_pM \to T_pM$. Since the kernel of ψ_p is one-dimensional, the only one of $\lambda_1(p), \lambda_2(p)$ vanishes. Let us assume that $\lambda_1(p) = 0$. Then $\lambda_2(p) > 0$. Thus, there exists a neighborhood V of p such that for every point $q \in V$ the map I_q has two distinct eigenvalues $\lambda_1(q), \lambda_2(q)$, such that $0 \leqslant \lambda_1(q) < \lambda_2(q)$. Furthermore, there exists a coordinate neighborhood $(U; u, v)$ of p such that U is a subset of V and the u-curves (respectively v-curves) give the λ_1-eigendirections (respectively λ_2-eigendirections), because the eigenvectors of eigenvalues $\lambda_1(q), \lambda_2(q)$ depends smoothly on q. Such a local coordinate system $(U; u, v)$ is called a *g-coordinate system* at p.

Definition 2.5 Let $\gamma(t)$ $(0 \leq t < 1)$ be a C^1-regular curve on M such that $\gamma(0) = p$. The $\mathcal{E}$*-initial vector* of γ at p is the following limit

$$\Psi_\gamma := \lim_{t\to 0+} \frac{\psi(\dot\gamma(t))}{|\psi(\dot\gamma(t))|} \in \mathcal{E}_p \tag{2.2}$$

if it exists.

Remark 2.6 If p is a regular point of M then the $\mathcal{E}$-initial vector of γ at p is the unit tangent vector of γ at p with respect to the first fundamental form $\mathrm{d}s^2$.

Proposition 2.7 (Proposition 2.6 in [35]). *Let γ be a C^1-regular curve emanating from an A_2-point or a peak p such that $\dot\gamma(0)$ is a not a null vector or γ is a singular curve. Then, the $\mathcal{E}$-initial vector of γ at p exists.*

Since, we study coherent tangent bundles over surfaces with boundary, we also consider a curve γ on the boundary which is tangent to the null direction at a singular point p on the boundary. We prove that in this case the $\mathcal{E}$-initial vector of γ at p exists if the singular direction is transversal to the boundary at p.

Proposition 2.8 *Let $(\mathcal{E}, \langle\cdot,\cdot\rangle, D, \psi)$ be a coherent tangent bundle over an compact oriented surface M with boundary. Let p be an A_2-point in the boundary ∂M. If the boundary ∂M is transversal to Σ at p and $\gamma : (-\varepsilon, \varepsilon) \to \partial M$ is a C^2-regular curve such that $\gamma(0) = p$, $\gamma\big((-\varepsilon,\varepsilon)\big) \cap \Sigma = \{p\}$ and $\dot\gamma(0)$ is the null vector at p, then the $\mathcal{E}$-initial vector Ψ_γ of γ at p exists, $D_{\frac{d}{dt}}\left(\psi\big(\dot\gamma(t)\big)\right)\big|_{t=0} \neq 0$, and*

$$\Psi_\gamma = \frac{D_{\frac{d}{dt}}\left(\psi\big(\dot\gamma(t)\big)\right)\big|_{t=0}}{\left|D_{\frac{d}{dt}}\left(\psi\big(\dot\gamma(t)\big)\right)\big|_{t=0}\right|} \in \mathcal{E}_p. \tag{2.3}$$

Proof Let $\sigma : [0,\varepsilon) \to \Sigma$ be a singular curve such that $\sigma(0) = p$. Let $(U; u, v)$ be a g-coordinate system at p i.e. the null direction at $\sigma(t)$ is spanned by $\frac{\partial}{\partial u}$. Since $\lambda_\psi(\sigma(t)) = 0$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_\psi\big(\sigma(t)\big)\right)\big|_{t=0} = \mathrm{d}\lambda_\psi\big|_p \cdot \dot\sigma(0) = 0. \tag{2.4}$$

Let us notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_\psi\big(\gamma(t)\big)\right)\big|_{t=0} = d\lambda_\psi\big|_p \cdot \dot\gamma(0) \neq 0 \tag{2.5}$$

since the vectors $\dot\sigma(0)$ and $\dot\gamma(0)$ span the space T_pM and $\mathrm{d}\lambda_\psi\big|_p \neq 0$.

On the other hand, since $\lambda_\psi\big(\gamma(t)\big) = \mu\left(\psi_u\big(\gamma(t)\big), \psi_v\big(\gamma(t)\big)\right)$ and $\psi_u\big(\gamma(0)\big) = 0$, we get the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_\psi\big(\gamma(t)\big)\right)\big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\,\mu\left(\psi_u\big(\gamma(t)\big), \psi_v\big(\gamma(t)\big)\right)\big|_{t=0}\\ &= \mu\left(D_{\frac{\mathrm{d}}{\mathrm{d}t}}\left(\psi_u\big(\gamma(t)\big)\Big|_{t=0}\right), \psi_v\big(\gamma(0)\big)\right)\end{aligned} \tag{2.6}$$

By (2.5) and (2.6) we get that $D_{\frac{\mathrm{d}}{\mathrm{d}t}}\left(\psi_u\big(\gamma(t)\big)\Big|_{t=0}\right), \psi_v\big(\gamma(0)\big)$ are linearly independent.

The vector field $\dot\gamma$ can be written in the following form $\dot\gamma(t) = \dot u(t)\frac{\partial}{\partial u} + \dot v(t)\frac{\partial}{\partial v}$, where $u(t) = t(a + h(t))$, $v(t) = t^2 g(t)$, $a \neq 0$ and h, g are some functions such that

$h(0) = 0$. Similarly, since $\psi_u\big(\gamma(0)\big) = 0$ and $D_{\frac{\mathrm{d}}{\mathrm{d}t}}\ \big(\psi_u\big(\gamma(t)\big)\big)\big|_{t=0} \neq 0$ we can write $\psi_u\big(\gamma(t)\big) = t\xi(t)$, where $\xi(t) \in \mathcal{E}_{\gamma(t)}$ and $\xi(0) \neq 0$.

Now, we will prove the formula (2.3).

$$
\begin{aligned}
\lim_{t\to 0^+} \frac{\psi\big(\dot\gamma(t)\big)}{\big|\psi\big(\dot\gamma(t)\big)\big|} &= \lim_{t\to 0^+} \frac{\dot u(t)\psi_u\big(\gamma(t)\big) + \dot v(t)\psi_v\big(\gamma(t)\big)}{\big|\dot u(t)\psi_u\big(\gamma(t)\big) + \dot v(t)\psi_v\big(\gamma(t)\big)\big|} \\
&= \lim_{t\to 0^+} \frac{t\left(\big(a + h(t) + t\dot h(t)\big)\xi(t) + \big(2g(t) + t\dot g(t)\big)\psi_v\big(\gamma(t)\big)\right)}{t\left|\big(a + h(t) + t\dot h(t)\big)\xi(t) + \big(2g(t) + t\dot g(t)\big)\psi_v\big(\gamma(t)\big)\right|} \\
&= \frac{a\xi(0) + 2g(0)\psi_v\big(\gamma(0)\big)}{\big|a\xi(0) + 2g(0)\psi_v\big(\gamma(0)\big)\big|},
\end{aligned}
$$

where the expression $a\xi(0) + 2g(0)\psi_v\big(\gamma(0)\big)$ is non-zero since the vectors $\xi(0) = D_{\frac{\mathrm{d}}{\mathrm{d}t}}\ \big(\psi_u\big(\gamma(t)\big)\big)\big|_{t=0}$ and $\psi_v\big(\gamma(0)\big)$ are linearly independent.

Since $D_{\frac{\mathrm{d}}{\mathrm{d}t}}\ \big(\psi\big(\dot\gamma(t)\big)\big)\big|_{t=0} = a\xi(0) + 2g(0)\psi_v\big(\gamma(0)\big)$, the equality (2.3) holds. □

Proposition 2.9 *Under the assumptions of Proposition 2.8, if* $\overline{\gamma}(t) := \gamma(-t)$*, then*

$$
\Psi_\gamma = \Psi_{\overline{\gamma}}. \tag{2.7}
$$

Proof Since $\overline{\gamma}(t) = \gamma(-t)$, we get that $\dot{\overline{\gamma}}(t) = -\dot\gamma(-t)$ and in particular $\dot{\overline{\gamma}}(0) = -\dot\gamma(0)$. Since

$$
D_{\frac{\mathrm{d}}{\mathrm{d}t}}\ \big(\psi\ \big(\dot{\overline{\gamma}}(t)\big)\big) = D_{\frac{\mathrm{d}}{\mathrm{d}t}}\ (-\psi\ (\dot\gamma(-t))) = -D_{\frac{\mathrm{d}}{\mathrm{d}t}}\ (\psi\ (\dot\gamma(-t))) = D_{-\frac{\mathrm{d}}{\mathrm{d}t}}\ (\psi\ (\dot\gamma(-t)))\,,
$$

the equality (2.7) holds. □

Definition 2.10 Let γ_1 and γ_2 be two C^1-regular curves emanating from p such that $\mathcal{E}$-initial vectors of γ_1 and γ_2 at p exist. Then the angle

$$
\arccos(\langle \Psi_{\gamma_1}, \Psi_{\gamma_2} \rangle) \in [0, \pi]
$$

is called the *angle between the initial vectors* of γ_1 and γ_2 at p.

We generalize the definition of singular sectors from [35] to the case of coherent tangent bundles over surfaces with boundary.

Let U be a (sufficiently small) neighborhood of a singular point p. Let σ_1 and σ_2 be curves in U starting at p such that both are singular curves or one of them is a singular curve and the other one is in ∂M. A domain Ω is called a *singular sector* at p if it satisfies the following conditions

(i) the boundary of $\Omega \cap U$ consists of σ_1, σ_2 and the boundary of U.
(ii) $\Omega \cap \Sigma = \emptyset$.

If the peak $p \in M \setminus \partial M$ is an isolated singular point than the domain $U \setminus \{p\}$ is a singular sector at p, where U is a neighborhood of p such that $U \cap \Sigma = \{p\}$. We assume that singular direction is transversal to the boundary of M. Therefore, there are no isolated singular points on the boundary.

We define the interior angle of a singular sector. If p is in ∂M, then the *interior angle* of a singular sector at p is the angle of the initial vectors of σ_1 and σ_2 at p.

While the interior angle of a singular sector may take value greater than π if $p \in M \setminus \partial M$, we can choose γ_j for $j = 0, \ldots, n$ inside the singular sector in a such way that the angel between $\Psi_{\gamma_{j-1}}$ and Ψ_{γ_j} is not greater than π.

Let Ω be a singular sector at the peak p. Then, there exists a positive integer n and C^1-regular curves starting at p $\gamma_0 = \sigma_0, \gamma_1, \cdots, \gamma_n = \sigma_1$ satisfying the assumptions of Proposition 2.7 and the following conditions:

(i) if $i \neq j$ then $\gamma_i \cap \gamma_j = \emptyset$ in Ω,
(ii) for each $j = 1, \ldots, n$ there exists a sector domain $\omega_j \subset \Omega$ such that ω_j is bounded by γ_{j-1} and γ_j and $\omega_j \cap \gamma_i = \emptyset$ for $i \neq j-1, j$,
(iii) if $n \geqslant 2$ the vectors $\dot{\gamma}_{j-1}(0), \dot{\gamma}_j(0))$ are linearly independent and form a positively oriented frame for $j = 1, \ldots, n$.

If the peak p is an isolated singular point then there exist curves $\gamma_0, \gamma_1, \gamma_2$ satisfying the above assumptions and conditions (i)–(iii). We also put $\gamma_3 = \gamma_0$.

The *interior angle of the singular sector* Ω is

$$\sum_{j=1}^{n} \arccos\left(\langle \Psi_{\gamma_{j-1}}, \Psi_{\gamma_j} \rangle\right).$$

If Ω is a singular sector at a singular point p then Ω is contained in M^+ or M^-. The singular sector Ω is called *positive* (respectively *negative*) if $\Omega \subset M^+$ (respectively $\Omega \subset M^-$).

Definition 2.11 Let p be a singular point. Then, $\alpha_+(p)$ (respectively $\alpha_-(p)$) is the sum of all interior angles of positive (respectively negative) singular sectors at p.

Proposition 2.12 (Theorem A in [35]) *Let $p \in M \setminus \partial M$ be a peak. The sum $\alpha_+(p)$ of all interior angles of positive singular sectors at p and the sum $\alpha_-(p)$ of all interior angles of negative singular sectors at p satisfy*

$$\alpha_+(p) + \alpha_-(p) = 2\pi,$$
$$\alpha_+(p) - \alpha_-(p) \in \Big\{ -2\pi, 0, 2\pi \Big\}.$$

Theorem 2.13 *Let $p \in \partial M$ be a singular point. We assume that the singular direction is transversal to the boundary ∂M at p.*

If the null direction is transversal to the boundary ∂M at p, then

$$\alpha_+(p) + \alpha_-(p) = \pi,$$
$$\alpha_+(p) - \alpha_-(p) \in \Big\{ -\pi, \pi \Big\}.$$

If the null direction is tangent to the boundary ∂M at p, then

$$\alpha_+(p) = \alpha_-(p).$$

Proof The first part of this theorem follows from Proposition 2.15 in [35]. By Proposition 2.9, we get the second part. □

Definition 2.14 A peak p in $M \setminus \partial M$ is called *positive* (*null*, *negative*, respectively) if $\alpha_+(p) - \alpha_-(p) > 0$ ($\alpha_+(p) - \alpha_-(p) = 0$, $\alpha_+(p) - \alpha_-(p) < 0$, respectively).

Definition 2.15 A singular point p in ∂M is called *positive* (*null*, *negative*, respectively) if $\alpha_+(p) - \alpha_-(p) > 0$ ($\alpha_+(p) - \alpha_-(p) = 0$, $\alpha_+(p) - \alpha_-(p) < 0$, respectively).

Remark 2.16 It is easy to see that a peak p in ∂M is not null if ∂M is transversal to the singular direction at p and an A_2-singular point p in ∂M is null if the null vector at p is tangent to ∂M.

Definition 2.17 Let p be a peak in ∂M. We say that p is in the *positive boundary* (respectively in the *negative boundary*) if there exists a neighborhood U in M of p such that $(U \setminus \{p\}) \cap \partial M \subset M^+$ (respectively $(U \setminus \{p\}) \cap \partial M \subset M^-$).

Let $\sigma(t)$ ($t \in (a; b)$) be a C^2-regular curve on M. We assume that if $\sigma(t) \in \Sigma$ then $\dot{\sigma}(t)$ is transversal to the null direction at $\sigma(t)$. Then, the image $\psi(\dot{\sigma}(t))$ does not vanish. Thus, we take a parameter τ of σ such that

$$\left\langle \psi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)\right), \psi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)\right)\right\rangle \equiv 1.$$

Definition 2.18 Let $n(\tau)$ be a section of $\mathcal{E}$ along $\sigma(\tau)$ such that $\{\psi(\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)), n(\tau)\}$ is a positive orthonormal frame. Then

$$\hat{\kappa}_g(\tau) := \left\langle D_{\frac{\mathrm{d}}{\mathrm{d}\tau}}\psi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)\right), n(\tau)\right\rangle = \mu\left(\psi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)\right), D_{\frac{\mathrm{d}}{\mathrm{d}\tau}}\psi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)\right)\right)$$

is called the *$\mathcal{E}$-geodesic curvature* of σ, which gives the geodesic curvature of the curve σ with respect to the orientation of $\mathcal{E}$.

We assume that the curve σ is a singular curve consisting of A_2-points. Take a null vector field $\eta(\tau)$ along $\sigma(\tau)$ such that $\{\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau), \eta(\tau)\}$ is a positively oriented field along $\sigma(\tau)$ for each τ. Then, the *singular curvature function* is defined by

$$\kappa_s(\tau) := \mathrm{sgn}(\mathrm{d}\lambda_\psi(\eta(\tau))) \cdot \hat{\kappa}_g(\tau),$$

where $\mathrm{sgn}(\mathrm{d}\lambda_\psi(\eta(\tau)))$ denotes the sign of the function $\mathrm{d}\lambda_\psi(\eta)$ at τ. In a general parameterization of $\sigma = \sigma(t)$, the singular curvature function is defined as follows

$$\kappa_s(t) = \mathrm{sgn}\left(\mathrm{d}\lambda_\psi(\eta(t))\right) \cdot \frac{\mu\left(\psi(\dot{\sigma}(t)), D_{\frac{\mathrm{d}}{\mathrm{d}t}}\psi(\dot{\sigma}(t))\right)}{|\psi(\dot{\sigma}(t))|^3},$$

where $^{\boldsymbol{\cdot}} := \frac{\mathrm{d}}{\mathrm{d}t}$, $|\xi| := \sqrt{\langle \xi, \xi \rangle}$.

By Proposition 1.7 in [35] the singular curvature function does not depend on the orientation of M, the orientation on $\mathcal{E}$, nor the parameter t of the singular curve $\sigma(t)$.

By Proposition 2.11 in [35] the singular curvature measure $\kappa_s \mathrm{d}\tau$ is bounded on any singular curve, where $\mathrm{d}\tau$ is the arclength measure of this curve with respect to the first fundamental form $\mathrm{d}s^2$. Now, we prove the following proposition concerning the geodesic curvature measure on the boundary of M.

Proposition 2.19 *Let $\gamma : [0, \varepsilon) \to \partial M$ be a C^2-regular curve such that $\Sigma \cap \gamma([0, \varepsilon)) = \{\gamma(0)\}$ is an A_2-point and the vector $\dot{\gamma}(0)$ is the null vector at $\gamma(0)$. Then, the geodesic curvature measure $\hat{\kappa}_g d\tau$ is continuous on $[0, \varepsilon)$, where $d\tau$ is the arclength measure with respect to the first fundamental form ds^2.*

Proof The point $\gamma(0) \in \partial M$ is a null A_2-point. By Proposition 2.8 we can write that $\Psi(\dot{\gamma}(t)) = t\zeta(t)$ for $t \in [0, \tilde{\varepsilon})$ for sufficiently small $\tilde{\varepsilon} \leq \varepsilon$, where $\zeta(t) \in \mathcal{E}_{\gamma(t)}$ and $\zeta(0) = D_{\frac{\mathrm{d}}{\mathrm{d}t}} \psi(\dot{\gamma}(t))\,|_{t=0} \neq 0$. The geodesic curvature in a general parameterization has the following form

$$\hat{\kappa}_g(t) = \frac{\mu\left(\psi(\dot{\gamma}(t)),\, D_{\frac{\mathrm{d}}{\mathrm{d}t}} \psi(\dot{\gamma}(t))\right)}{|\psi(\dot{\gamma}(t))|^3}.$$

Thus, the geodesic curvature measure

$$\hat{\kappa}_g(\tau)\mathrm{d}\tau = \hat{\kappa}_g(t)\,|\psi(\dot{\gamma}(t))|\,\mathrm{d}t = \frac{\mu\left(\zeta(t),\, D_{\frac{\mathrm{d}}{\mathrm{d}t}} \zeta(t)\right)}{|\zeta(t)|^2}\mathrm{d}t$$

is bounded and continuous on $[0, \tilde{\varepsilon})$. It implies that the geodesic curvature measure is continuous on $[0, \varepsilon)$ since $\Sigma \cap \gamma([0, \varepsilon)) = \{\gamma(0)\}$. ◻

Let $U \subset M$ be a domain and let $\{e_1, e_2\}$ be a positive orthonormal frame field on $\mathcal{E}$ defined on U. Since D is a metric connection, there exists a unique 1-form ω on U such that

$$D_X e_1 = -\omega(X)e_2, \quad D_X e_2 = \omega(X)e_1,$$

for any smooth vector field X on U. The form ω is called the *connection form* with respect to the frame $\{e_1, e_2\}$. It is easy to check that $\mathrm{d}\omega$ does not depend on the choice of a frame $\{e_1, e_2\}$ and gives a globally defined 2-form on M. Since D is a metric connection and it satisfies (2.1) we have

$$\mathrm{d}\omega = K\mathrm{d}\hat{A} = \begin{cases} K\mathrm{d}A & \text{on } M_+, \\ -K\mathrm{d}A & \text{on } M_-, \end{cases}$$

where K is the Gaussian curvature of the first fundamental form $\mathrm{d}s^2$ (see [35,36]).

The next theorem is a generalization of the Gauss–Bonnet theorem for coherent tangent bundles over smooth compact-oriented surfaces with boundary.

Theorem 2.20 (The Gauss–Bonnet type formulas) *Let $\mathcal{E}$ be a coherent tangent bundle on a smooth compact-oriented surface M with boundary whose set of singular points Σ admits at most peaks. If the set of singular points Σ is transversal to the boundary ∂M, then*

$$
\begin{aligned}
2\pi\chi(M) = {} & \int_M K \mathrm{d}A + 2\int_\Sigma \kappa_s \mathrm{d}\tau \\
& + \int_{\partial M\cap M^+} \hat{\kappa}_g \mathrm{d}\tau - \int_{\partial M\cap M^-} \hat{\kappa}_g \mathrm{d}\tau \\
& - \sum_{p\in null(\Sigma\cap\partial M)} (2\alpha_+(p) - \pi),
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
\int_M K \mathrm{d}\hat{A} + \int_{\partial M} \hat{\kappa}_g \mathrm{d}\tau = {} & 2\pi\left(\chi(M^+) - \chi(M^-)\right) + 2\pi\left(\#P^+ - \#P^-\right) \\
& + \pi\left(\#(\Sigma\cap\partial M)^+ - \#(\Sigma\cap\partial M)^-\right) \\
& + \pi\left(\#P_{\partial M^+} - \#P_{\partial M^-}\right),
\end{aligned} \tag{2.9}
$$

where $d\tau$ is the arc length measure, P^+ (respectively P^-) is the set of positive (respectively negative) peaks in $M\setminus\partial M$, $(\Sigma\cap\partial M)^+$ (respectively $(\Sigma\cap\partial M)^-$, $null(\Sigma\cap\partial M)$) is the set of positive (respectively negative, null) singular points in $\Sigma\cap\partial M$, $P_{\partial M^+}$ (respectively $P_{\partial M^-}$) is the set of peaks in the positive (respectively negative) boundary.

3 The Proof of Theorem 2.20

We use the method presented in the proof of Theorem B in [35]. First, we formulate the local Gauss–Bonnet theorem for admissible triangles.

Definition 3.1 A curve $\sigma(t)$ ($t\in[a,b]$) is *admissible on the surface with boundary* if it satisfies one of the following conditions:

(1) σ is a C^2-regular curve such that $\sigma((a,b))$ does not contain a peak, and the tangent vector $\dot\sigma(t)$ ($t\in[a,b]$) is transversal to the singular direction, the null direction if $\sigma(t)\in\Sigma$ and $\dot\sigma(t)$ is transversal to the boundary if $\sigma(t)\in\partial M$.
(2) σ is a C^1-regular curve such that the set $\sigma([a,b])$ is contained in the set of singular points Σ and the set $\sigma((a,b))$ does not contain a peak.
(3) σ is C^2-regular curve such that the set $\sigma([a,b])$ is contained in the boundary ∂M, the set $\sigma((a,b))$ does not contain a singular point and the tangent vector $\dot\sigma(t)$ ($t\in\{a,b\}$) is transversal to the singular direction if $\sigma(t)\in\Sigma$.

Remark 3.2 A curve $\sigma(t)$ is admissible in the sense of Definition 2.12 in [35] if it satisfies conditions (1) or (2) in Definition 3.1. For the purpose of this paper we add (3) in Definition 3.1 and the transversality of the admissible curve to the boundary in (1).

Let U be a domain in M.

Definition 3.3 (*See Definition* 3.1 *in* [35]) Let $\overline{T} \subset U$ be the closure of a simply connected domain T which is bounded by three admissible arcs γ_1, γ_2, γ_3. Let A, B and C be the distinct three boundary points of T which are intersections of these three arcs. Then $\overline{T}$ is called an *admissible triangle on the surface with boundary* if it satisfies the following conditions:

(1) $\overline{T}$ admits at most one peak on $\{A, B, C\}$.
(2) the three interior angles at A, B and C with respect to the metric g are all less than π.
(3) if γ_j for $j = 1, 2, 3$ is not a singular curve, it is C^2-regular, namely it is a restriction of a certain open C^2-regular arc.

We write $\Delta ABC := \overline{T}$ and we denote by

$$\overline{BC} := \gamma_1,\ \overline{CA} := \gamma_2,\ \overline{AB} := \gamma_3$$

the regular arcs whose boundary points are $\{B, C\}$, $\{C, A\}$, $\{A, B\}$, respectively.

We give the orientation of $\partial\Delta ABC$ compatible with respect to the orientation of M. We denote by $\angle A$, $\angle B$, $\angle C$ the interior angles (with respect to the first fundamental form ds^2) of the piecewise smooth boundary of ΔABC at A, B and C, respectively if A, B and C are regular points.

If $A \in M \setminus \partial M$ is a singular point and $(U; u, v)$ is a g-coordinate system at A, then we set (see Proposition 2.15 in [35])

$$\angle A := \begin{cases} \pi & \text{if the } u\text{-curve passing through } A \text{ separates } \overline{AB} \text{ and } \overline{AC}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma(t)$ be an admissible curve. We define a *geometric curvature* $\tilde{\kappa}_g(t)$ in the following way:

$$\tilde{\kappa}_g(t) = \begin{cases} \hat{\kappa}_g(t) \text{ if } \sigma(t) \in M^+, \\ -\hat{\kappa}_g(t) \text{ if } \sigma(t) \in M^-, \\ \kappa_s(t) \text{ it } \sigma(t) \in \Sigma, \end{cases}$$

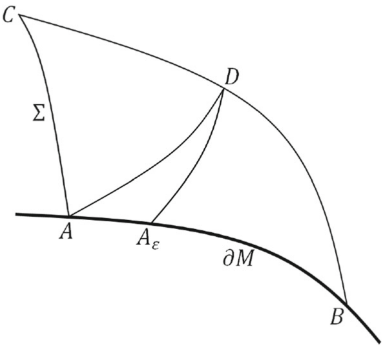
where $\hat{\kappa}_g$ is the geodesic curvature with respect to the orientation of M and κ_s is the singular curvature.

Proposition 3.4 *Let ΔABC be an admissible triangle on the surface with boundary such that A is an A_2-point, $\overline{AB} \subset \partial M$ and $\Delta ABC \setminus \overline{AC}$ lies in M^+ or in M^-. Suppose that the boundary ∂M is transversal to Σ at A and let $T_A\partial M$ be a null direction at A. Then*

$$\angle A + \angle B + \angle C - \pi = \int_{\partial\Delta ABC} \tilde{\kappa}_g d\tau + \int_{\Delta ABC} K dA. \tag{3.1}$$

Proof Without loss of generality, let us assume that $\Delta ABC \setminus \overline{AC}$ lies in M^+. If the arc $\overline{AC} \subset \Sigma$ or the interior angle $\angle BAC$ with respect to the metric g is greater than

Fig. 1 A decomposition of the triangle ABC into admissible triangles



$\frac{\pi}{2}$, we decompose the triangle ΔABC into admissible triangles ΔABD and ΔADC such that the interior angle $\angle BAD$ with respect to the metric g is in the interval $(0, \frac{\pi}{2})$ and the arc $\overline{AD}$ is transversal to the arc $\overline{BC}$ at D, see Fig. 1. The formula (3.1) for ΔADC follows from Theorem 3.3 in [35], so it is enough to prove the formula (3.1) for the triangle ΔABD.

We can take the arc $\overline{AD}$ and rotate it around D with respect to the canonical metric $\mathrm{d}u^2 + \mathrm{d}v^2$ on the uv-plane. Then, we obtain a smooth one-parameter family of C^2-regular arcs starting at D. Since the interior angle $\angle BAD$ is in $(0, \frac{\pi}{2})$ and $\overline{BD}$, $\overline{AD}$ are transversal at D, restricting the image of this family to the triangle ΔABD, we obtain a family of C^2-regular curves

$$\gamma_\varepsilon : [0, 1] \to \Delta ABD,$$

where $\varepsilon \in [0, 1]$ and:

(i) γ_0 parameterizes $\overline{AD}$ and $\gamma_0(0) = A$, $\gamma_0(1) = D$,
(ii) $\gamma_\varepsilon(1) = D$ for all $\varepsilon \in [0, 1]$,
(iii) the correspondence $\sigma : \varepsilon \mapsto \gamma_\varepsilon(0)$ gives a subarc of $\overline{AB}$. We set $A_\varepsilon = \gamma_\varepsilon(0)$, where $A_0 = A$.

Since $\Delta A_\varepsilon BD$ for $\varepsilon > 0$ is an admissible triangle, then by Theorem 3.3 in [35] we get that

$$\angle A_\varepsilon + \angle B + \angle A_\varepsilon DB - \pi = \int_{\partial\Delta A_\varepsilon BD} \tilde{\kappa}_g \mathrm{d}\tau + \int_{\Delta A_\varepsilon BD} K \mathrm{d}A.$$

Since ΔABD is admissible and $\tilde{\kappa}_g$ is bounded on both $\overline{AB}$ and $\overline{AD}$, by taking the limit as $\varepsilon \to 0^+$, we have that

$$\lim_{\varepsilon\to 0^+} \angle A_\varepsilon + \angle B + \angle D - \pi = \int_{\partial\Delta ABD} \tilde{\kappa}_g \mathrm{d}\tau + \int_{\Delta ABD} K \mathrm{d}A.$$

By Proposition 2.8 we have

$$\lim_{\varepsilon\to 0^+} \angle A_\varepsilon = \lim_{\varepsilon\to 0^+} \arccos \frac{\left\langle \psi\left(\frac{d\gamma_\varepsilon(t)}{\mathrm{d}t}\right)\big|_{t=0}, \psi\left(\frac{\mathrm{d}\sigma(\varepsilon)}{\mathrm{d}\varepsilon}\right)\right\rangle}{\left|\psi\left(\frac{d\gamma_\varepsilon(t)}{\mathrm{d}t}\right)\big|_{t=0}\right| \cdot \left|\psi\left(\frac{\mathrm{d}\sigma(\varepsilon)}{\mathrm{d}\varepsilon}\right)\right|} = \arccos\left\langle \Psi_{\gamma_0}, \Psi_\sigma\right\rangle. \tag{3.2}$$

This completes the proof. □

Remark 3.5 By Theorem 3.3 in [35] and Proposition 3.4 the equality (3.1) holds for any admissible triangle on a surface with boundary.

Let $\overline{X}$, X°, ∂X, respectively, denote the closure of a subset X of M, the interior of X and the boundary of X, respectively.

Let us triangulate M by admissible triangles such that each point in the set $P \cup (\Sigma \cap \partial M) =: P^\star$ is a vertex, where P is the set all peaks in M°. Let T, E and V, respectively, denote the set of all triangles, the set of all edges and the set all of vertices in the given triangulation, respectively.

Lemma 3.6 *The following relation holds:*

$$\begin{aligned}\#\left\{v \in V \mid v \in (M^+)^\circ\right\} = \chi(\overline{M}^+) &+ \frac{1}{2}\#\left\{\Delta \in T \mid \Delta \subset \overline{M}^+\right\}\\ &+ \frac{1}{2}\#\left\{e \in E \mid e \subset \partial M^+\right\}\\ &- \#\left\{v \in V \mid v \in \partial M^+ \setminus P^\star\right\} - \#P^\star.\end{aligned}$$

Proof By the definition of Euler's characteristic we get that

$$\#\left\{v \in V \mid v \in \overline{M}^+\right\} = \chi(\overline{M}^+) - \#\left\{\Delta \in T \mid \Delta \subset \overline{M}^+\right\} + \#\left\{e \in E \mid e \subset \overline{M}^+\right\}. \tag{3.3}$$

Furthermore, it is easy to verify that

$$\#\left\{e \in E \mid e \subset \overline{M}^+\right\} = \frac{3}{2}\#\left\{\Delta \in F \mid \Delta \subset \overline{M}^+\right\} + \frac{1}{2}\#\left\{e \in E \mid e \subset \partial M^+\right\} \tag{3.4}$$

and

$$\begin{aligned}\#\left\{v \in V \mid v \in (M^+)^\circ\right\} = \#&\left\{v \in V \mid v \in \overline{M}^+\right\}\\ &- \#\left\{v \in V \mid v \in \partial M^+ \setminus P^\star\right\} - \#\left\{p \in V \mid p \in P^\star\right\}.\end{aligned} \tag{3.5}$$

Combining together (3.3), (3.4) and (3.5) we end the proof. □

Let us define the sum $\sum_{\Delta ABC\in T,\Delta\subset\overline{M}^+}(\angle A+\angle B+\angle C-\pi)$ by S_+.
Then,

$$\begin{aligned}S_+ &= 2\pi\#\left\{v\in V \mid v\in (M^+)^\circ\right\}+\pi\#\left\{v\in V\mid v\in\partial M^+\setminus P^\star\right\}\\&\quad+\sum_{p\in P^\star}\alpha_+(p)-\pi\#\left\{\Delta\in T\mid \Delta\subset\overline{M}^+\right\}.\end{aligned}$$

By Lemma 3.6 we get that

$$\begin{aligned}S_+ &= 2\pi\chi(\overline{M}^+)+\pi\#\left\{e\in E\mid e\in\partial M^+\right\}-\pi\#\left\{v\in V\mid v\in\partial M^+\setminus P^\star\right\}\\&\quad-2\pi\#P^\star+\sum_{p\in P^\star}\alpha_+(p)\\&=2\pi\chi(\overline{M}^+)+\frac{\pi}{2}\sum_{v\in V,v\in\partial M^+}\deg_{\partial M^+}(v)-\pi\#\left\{v\in V\mid v\in\partial M^+\right\}\\&\quad-\pi\#P^\star+\sum_{p\in P^\star}\alpha_+(p),\end{aligned}$$

where $\deg_X(v)=\#\{e\in E\mid e\subset X, v\in e\}$, where X is a subset of M. Since ∂M^+ is an Eulerian graph, the number $\deg_{\partial M^+}(v)$ is even and let us write that $m_+(v):=\frac{1}{2}\deg_{\partial M^+}(v)$. Furthermore, if $v\in(V\cap\partial M^+)\setminus P^\star$ then $\deg_{\partial M^+}(v)=2$ and we get the relation

$$\frac{1}{2}\sum_{v\in V\cap\partial M^+}\deg_{\partial M^+}(v)-\#\left\{v\in V\mid v\in\partial M^+\right\}=\sum_{p\in P^\star}(m_+(p)-1).$$

Hence we get the following:

$$S_+=2\pi\chi(\overline{M}^+)+\sum_{p\in P^\star}(\alpha_+(p)+\pi m_+(p))-2\pi\#P^\star. \tag{3.6}$$

Similarly we get that

$$S_-=2\pi\chi(\overline{M}^-)+\sum_{p\in P^\star}(\alpha_-(p)+\pi m_-(p))-2\pi\#P^\star, \tag{3.7}$$

where $S_-=\sum_{\Delta ABC\in T,\Delta\subset\overline{M}^-}(\angle A+\angle B+\angle C-\pi)$ and $m_-(v):=\frac{1}{2}\deg_{\partial M^-}(v)$.
It is easy to see that

$$m_+(p)=m_-(p)\text{ for }p\in P^\star\setminus\partial M, \tag{3.8}$$
$$m_+(p)+m_-(p)=\deg_\Sigma(p)\text{ for }p\in P^\star\setminus\partial M, \tag{3.9}$$
$$m_+(p)+m_-(p)=\deg_{\Sigma\cup\partial M}(p)-1\text{ for }p\in\Sigma\cap\partial M. \tag{3.10}$$

Furthermore if $p \in \Sigma \cap \partial M$, then

$$m_+(p) - m_-(p) = \begin{cases} 1 & \text{if } p \text{ is a peak in the positive boundary,} \\ -1 & \text{if } p \text{ is a peak in the negative boundary,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Lemma 3.7 *The Euler characteristic of Σ is equal to*

$$\chi(\Sigma) = \#P^\star - \frac{1}{2}\sum_{p\in P^\star}(m_+(p) + m_-(p)) + \frac{1}{2}\#(\Sigma \cap \partial M).$$

Proof We know that

$$\begin{aligned}\chi(\Sigma) &= \#\{v \in V \mid v \in \Sigma\} - \#\{e \in E \mid e \subset \Sigma\} \\ &= \#\{v \in V \mid v \in \Sigma\} - \frac{1}{2}\sum_{v\in V\cap\Sigma}\deg_\Sigma v.\end{aligned}$$

If $p \in P \setminus \partial M$ then $\deg_\Sigma(p) = \deg_{\Sigma\cup\partial M}(p)$ and if $p \in \Sigma \cap \partial M$ then $\deg_\Sigma(p) = \deg_{\Sigma\cup\partial M}(p) - 2$. By (3.9) and (3.10) we get that

$$\begin{aligned}\chi(\Sigma) &= \#P^\star - \frac{1}{2}\sum_{p\in P\setminus\partial M}(m_+(p) + m_-(p)) - \frac{1}{2}\sum_{p\in\Sigma\cap\partial M}(m_+(p) + m_-(p) - 1) \\ &= \#P^\star - \frac{1}{2}\sum_{p\in P^\star}(m_+(p) + m_-(p)) + \frac{1}{2}\#(\Sigma \cap \partial M).\end{aligned}$$

□

Lemma 3.8 *The following equality holds:*

$$S_+ + S_- = 2\pi\chi(M) + \sum_{p\in null(\Sigma\cap\partial M)}(2\alpha_+(p) - \pi).$$

Proof Since $\chi(\overline{M}^+) + \chi(\overline{M}^-) = \chi(M) + \chi(\Sigma)$, by (3.6), (3.7), Lemma 3.7 and Theorem 2.13 we get that:

$$\begin{aligned}S_+ + S_- &= 2\pi\chi(M) + 2\pi\chi(\Sigma) + \sum_{p\in P^\star}(\alpha_+(p) + \alpha_-(p)) \\ &\quad + \pi\sum_{p\in P^\star}(m_+(p) + m_-(p)) - 4\pi\#P^\star \\ &= 2\pi\chi(M) + \pi\#(\Sigma\cap\partial M) + \sum_{p\in P^\star}(\alpha_+(p) + \alpha_-(p)) - 2\pi\#P^\star \\ &= 2\pi\chi(M) + \pi\#(\Sigma\cap\partial M) + \sum_{p\in(\Sigma\cap\partial M)^+\cup(\Sigma\cap\partial M)^-}(\alpha_+(p) + \alpha_-(p))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p\in P\setminus\partial M}(\alpha_+(p)+\alpha_-(p))+\sum_{p\in\mathrm{null}(\Sigma\cap\partial M)}(\alpha_+(p)+\alpha_-(p))\\
&-2\pi\#(\Sigma\cap\partial M)-2\pi\#(P\setminus\partial M)\\
&=2\pi\chi(M)+\sum_{p\in(\Sigma\cap\partial M)^+\cup(\Sigma\cap\partial M)^-}\pi+\sum_{p\in P\setminus\partial M}2\pi\\
&+\sum_{p\in\mathrm{null}(\Sigma\cap\partial M)}2\alpha_+(p)-\pi\#(\Sigma\cap\partial M)-2\pi\#(P\setminus\partial M)\\
&=2\pi\chi(M)+\sum_{p\in\mathrm{null}(\Sigma\cap\partial M)}(2\alpha_+(p)-\pi).
\end{aligned}
$$

□

Lemma 3.9 *The following equality holds:*

$$
\begin{aligned}
S_+-S_-=2\pi\left(\chi(M^+)-\chi(M^-)\right)+2\pi\left(\#P^+-\#P^-\right)\\
+\pi\left(\#(\Sigma\cap\partial M)^+-\#(\Sigma\cap\partial M)^-\right)+\pi\left(\#P_{\partial M^+}-\#P_{\partial M^-}\right),
\end{aligned}
$$

where P^+ (respectively P^-) is the set of positive (respectively negative) peaks in $M\setminus\partial M$, $(\Sigma\cap\partial M)^+$ (respectively $(\Sigma\cap\partial M)^-$) is the set of positive (respectively negative) singular points in $\Sigma\cap\partial M$, $P_{\partial M^+}$ (respectively $P_{\partial M^-}$) is the set of peaks in the positive (respectively negative) boundary.

Proof It is a consequence of (3.6), (3.7), Lemma 3.7 and Theorem 2.13 and the fact that $\chi(\overline{M}^+)-\chi(\overline{M}^-)=\chi(M^+)-\chi(M^-)$. □

Since the integration of the geometric curvature on curves which are not included in $\Sigma\cup\partial M$ are canceled by opposite integrations and the singular curvature does not depend on the orientation of the singular curve, by Proposition 3.4 and Theorem 3.3 in [35] we get that

$$
S_\pm=\int_{M^\pm}K\mathrm{d}A+\int_{\partial M^\pm}\tilde{\kappa}_g\mathrm{d}\tau=\int_{M^\pm}K\mathrm{d}A+\int_\Sigma\kappa_s\mathrm{d}\tau\pm\int_{\partial M\cap M^\pm}\hat{\kappa}_g\mathrm{d}\tau.
$$

Hence

$$
S_++S_-=\int_M K\mathrm{d}A+2\int_\Sigma\kappa_s\mathrm{d}\tau+\int_{\partial M\cap M^+}\hat{\kappa}_g\mathrm{d}\tau-\int_{\partial M\cap M^-}\hat{\kappa}_g\mathrm{d}\tau, \tag{3.12}
$$

$$
S_+-S_-=\int_M K\mathrm{d}\hat{A}+\int_{\partial M}\hat{\kappa}_g\mathrm{d}\tau. \tag{3.13}
$$

By Lemma 3.8, Lemma 3.9, (3.12) and (3.13) we complete the proof of Theorem 2.20.

4 Applications of the Gauss–Bonnet Formulas to Maps

As a corollary of Theorem 2.20 we get a special version of Fukuda–Ishikawa's theorem (Theorem 1.1 in [12], see also [22]), which is the generalization of Quine's formula (Theorem 1 in [33]) for surfaces with boundary (see also Proposition 3.6 in [37]). We assume that the set of singular points of a map is transversal to the boundary of a surface.

Proposition 4.1 *Let M and N both be compact oriented connected surfaces with boundary. Let $f : M \to N$ be a C^∞-smooth map such that $f(\partial M) \subset \partial N$ and $f^{-1}(\partial N) = \partial M$ and whose set of singular points consists of folds and cusps. If the set of singular points of f is transversal to ∂M then the topological degree of f satisfies*

$$deg(f)\chi(N) = \chi(M_f^+) - \chi(M_f^-) + S_f^+ - S_f^-, \tag{4.1}$$

where M_f^+ (respectively M_f^-) is the set of regular points at which f preserves (respectively reverses) the orientation, S_f^+ (respectively S_f^-) is the number of positive cusps (respectively the number of negative cusps).

Proof Let h be a Riemannian metric on N and let D be the Levi–Civita connection on (N, h). Then, the tuple (f^*TN, h, D, df) is a coherent tangent bundle on M (see [37]). Since $f(\partial M) \subset \partial N$ and the set of singular points of f is transversal to ∂M, there are no cusps in ∂M and all folds in ∂M are null singular points. Therefore, by Theorem 2.20 we get that:

$$\int_M K \mathrm{d}\hat{A} + \int_{\partial M} \hat{\kappa}_g \mathrm{d}\tau = 2\pi \left(\chi(M_f^+) - \chi(M_f^-)\right) + 2\pi \left(S_f^+ - S_f^-\right). \tag{4.2}$$

The following identity holds

$$\int_M K \mathrm{d}\hat{A} = \int_M f^*\Omega_{12},$$

where Ω_{12} is a curvature 2-form.

Furthermore, it is well known that $\int_M f^*\Omega_{12} = \deg(f) \int_N \Omega_{12}$ (see for instance Remark 1 in [11] page 111). On the other hand, we have $\int_N \Omega_{12} = \int_N K_N dA$, where K_N is the Gaussian curvature of N. By the Gauss–Bonnet theorem for N we get $\int_N K_N dA = 2\pi\chi(N) - \int_{\partial N} \kappa_g \mathrm{d}\tau$, where κ_g is the geodesic curvature of ∂N in N. Thus,

$$\int_M K \mathrm{d}\hat{A} = \deg(f) \left(2\pi\chi(N) - \int_{\partial N} \kappa_g \mathrm{d}\tau\right). \tag{4.3}$$

Since $f(\partial M) \subset \partial N$ and $f^{-1}(\partial N) = \partial M$ and $\langle \cdot, \cdot \rangle_p = h_{f(p)}(\cdot, \cdot)$ for p in M, we obtain that

$$\int_{\partial M} \hat{\kappa}_g \mathrm{d}\tau = \deg\left(f\big|_{\partial M}\right) \int_{\partial N} \kappa_g \mathrm{d}\tau. \tag{4.4}$$

By Theorem 13.2.1 [11, p. 105] we get $\deg(f) = \deg\left(f\big|_{\partial M}\right)$.
By (4.2)–(4.4) we obtain the formula (4.1). □

We can also get easily the generalization of Proposition 3.7 in [37] by the Gauss–Bonnet formulas.

Proposition 4.2 *Let (N, h) be an oriented Riemannian 2-manifold, let M be a compact oriented 2-manifold with boundary. Let $f : M \to N$ be a C^∞-smooth map whose set of singular points consists of folds and cusps and is transversal to ∂M. Then the total singular curvature $\int_\Sigma \kappa_s \mathrm{d}\tau$ with respect to the length element $\mathrm{d}\tau$ (with respect to h) on the set of singular points Σ is bounded, and satisfies the following identity*

$$\begin{aligned} 2\pi\chi(M) = &\int_M \left(\widetilde{K} \circ f\right) \left|f^* \mathrm{d}A_h\right| + 2\int_\Sigma \kappa_s \mathrm{d}\tau \\ &+ \int_{\partial M \cap M_f^+} \hat{\kappa}_g \mathrm{d}\tau - \int_{\partial M \cap M_f^-} \hat{\kappa}_g \mathrm{d}\tau - \sum_{p \in null(\Sigma \cap \partial M)} (2\alpha_+(p) - \pi). \end{aligned}$$

where M_f^+ (respectively M_f^-) is the set of regular points at which f preserves (respectively reverses) the orientation, $\widetilde{K}$ is the Gaussian curvature function on (N, h), $\hat{\kappa}_g$ is a geodesic curvature, $|f^* \mathrm{d}A_h|$ is the pull-back of the Riemannian measure of (N, h) and

$$\alpha_+(p) = arccos\left(h\left(\frac{D_{\frac{\mathrm{d}}{dt}}\left(\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t)\right)}{\left|D_{\frac{\mathrm{d}}{dt}}\left(\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t)\right)\right|}, \frac{\mathrm{d}}{d\tau}(f \circ \sigma)(\tau)\right)\right),$$

where D is the Levi–Civita connection on N, γ is a C^2-regular parameterization of the boundary ∂M in the neighborhood of p and σ is a parameterization of Σ in the neighborhood of p.

5 Geometry of the Affine-Extended Wave Front

In this section, we apply Theorem 2.20 to an affine-extended wave front of a planar non-singular hedgehog. Fronts are examples of coherent tangent bundles (see [35]).

Planar hedgehogs are curves which can be parameterized using their Gauss map. A hedgehog can be also viewed as the Minkowski difference of convex bodies (see [23, 25–28]). The non-singular hedgehogs are also known as the rosettes (see [2,31,44]).

The singularities and the geometry of affine λ-equidistants were very widely studied in many papers [1,7–9,16,18,34,41]. The envelope of affine diameters (the centre symmetry set) was studied in [5,13–15,17].

Let $\mathcal{C}$ be a smooth parameterized curve on the affine plane $\mathbb{R}^2$, i.e. the image of the C^∞-smooth map from an interval to $\mathbb{R}^2$. We say that a smooth curve is *closed* if it is the image of a C^∞-smooth map from S^1 to $\mathbb{R}^2$. A smooth curve is *regular* if its velocity does not vanish. A closed regular curve is called an m-rosette if its signed curvature is positive and its rotation number is m. A *convex* curve is a 1-rosette.

Definition 5.1 A pair of points $a, b \in \mathcal{C}$ ($a \neq b$) is called a *parallel pair* if the tangent lines to $\mathcal{C}$ at a and b are parallel.

Definition 5.2 An *affine λ-equidistant* is the following set:

$$E_\lambda(\mathcal{C}) = \left\{ \lambda a + (1-\lambda)b \;\middle|\; a, b \text{ is a parallel pair of } \mathcal{C} \right\}.$$

The set $E_{\frac{1}{2}}(\mathcal{C})$ will be called the *Wigner caustic* of $\mathcal{C}$.

A *chord* passing through a parallel pair $a, b \in C$ is the following set:

$$\left\{ \lambda a + (1-\lambda)b \;\middle|\; \lambda \in [0,1] \right\}.$$

Definition 5.3 The *centre symmetry set* of $\mathcal{C}$, which we will denote as $CSS(\mathcal{C})$, is the envelope of all chords passing through parallel pairs of $\mathcal{C}$.

If $\mathcal{C}$ is a generic convex curve, then the Wigner caustic of $\mathcal{C}$, $E_\lambda(\mathcal{C})$, for a generic λ, and $CSS(\mathcal{C})$ are smooth closed curves with at most cusp singularities [1,13,15,17], the number of cusps of the Wigner caustic and the centre symmetry set of $\mathcal{C}$ are odd and not smaller than 3 [1,13], the number of cusps of $CSS(\mathcal{C})$ is not smaller than the number of cusps of $E_{\frac{1}{2}}(\mathcal{C})$ [5] and the number of cusps of $E_\lambda(\mathcal{C})$ is even for a generic $\lambda \neq \frac{1}{2}$ [10]. Moreover, cusp singularities of all $E_\lambda(\mathcal{C})$ are lying on smooth parts of $CSS(\mathcal{C})$ [15]. In addition, if $\mathcal{C}$ is a convex curve, then the Wigner caustic is contained in a closure of the region bounded by the centre symmetry set ([3], see Fig. 2). The Wigner caustic also appears in one of the two constructions of bi-dimensional improper affine spheres. This construction can be generalized to higher even dimensions [4]. The oriented area of the Wigner caustic improves the classical planar isoperimetric inequality and gives the relation between the area and the perimeter of smooth convex bodies of constant width [42–44]. Recently, the properties of the middle hedgehog, which is a generalization of the Wigner caustic in the case of non-smooth convex bodies, were studied in [39,40].

Definition 5.4 *The extended affine space* is the space $\mathbb{R}^3_e = \mathbb{R} \times \mathbb{R}^2$ with coordinate $\lambda \in \mathbb{R}$ (called the affine time) on the first factor and a projection on the second factor denoted by $\pi : \mathbb{R}^3_e \ni (\lambda, x) \mapsto x \in \mathbb{R}^2$.

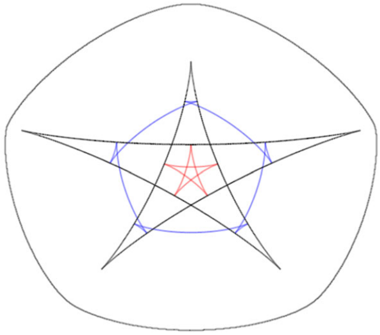


Fig. 2 An oval $\mathcal{C}$ and $E_{\frac{1}{2}}(\mathcal{C})$, $E_{\frac{2}{5}}(\mathcal{C})$, $CSS(\mathcal{C})$. The support function of $\mathcal{C}$ is equal to $p(\theta) = 31 + 2\cos 2\theta + \sin 5\theta$

Definition 5.5 Let R_m be an m-rosette. *The affine extended wave front* of R_m is the following set:

$$\mathbb{E}(R_m) = \bigcup_{\lambda\in[0,1]} \{\lambda\} \times E_\lambda(R_m) \subset \mathbb{R}^3_e.$$

$\mathbb{E}(R_m)$ is the union of all $E_\lambda(R_m)$ for $\lambda \in [0, 1]$, each embedded into its own slice of the extended affine space.

Note that, when R_m is a circle on the plane, then $\mathbb{E}(R_m)$ is the double cone, which is a smooth manifold with the nonsingular projection π everywhere, but at its singular point, which projects to the center of the circle (the center of symmetry).

We will study the geometry of $\mathbb{E}(R_m)$ through the support function of R_m [2,44]. Take a point O as the origin of our frame. Let θ be the oriented angle from the positive x_1-axis. Let $p(\theta)$ be the oriented perpendicular distance from O to the tangent line at a point on R_m and let this ray and x_1-axis form an angle θ. The function p is a single valued periodic function of θ with period $2m\pi$ and the parameterization of R_m in terms of θ and $p(\theta)$ is as follows

$$[0, 2m\pi) \ni \theta \mapsto \gamma(\theta) = \big(p(\theta)\cos\theta - p'(\theta)\sin\theta,\, p(\theta)\sin\theta + p'(\theta)\cos\theta\big) \in \mathbb{R}^2. \tag{5.1}$$

Then, the radius of curvature ρ of R_m is in the following form

$$\rho(\theta) = \frac{\mathrm{d}s}{\mathrm{d}\theta} = p(\theta) + p''(\theta) > 0, \tag{5.2}$$

or equivalently, the curvature κ of R_m is given by

$$\kappa(\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{p(\theta) + p''(\theta)} > 0. \tag{5.3}$$

In Fig. 3 we illustrate (with different opacities) the surface $\mathbb{E}(R_1)$, where R_1 is an oval represented by the support function $p(\theta) = 11 - \frac{1}{2}\cos 2\theta + \sin 3\theta$. We also

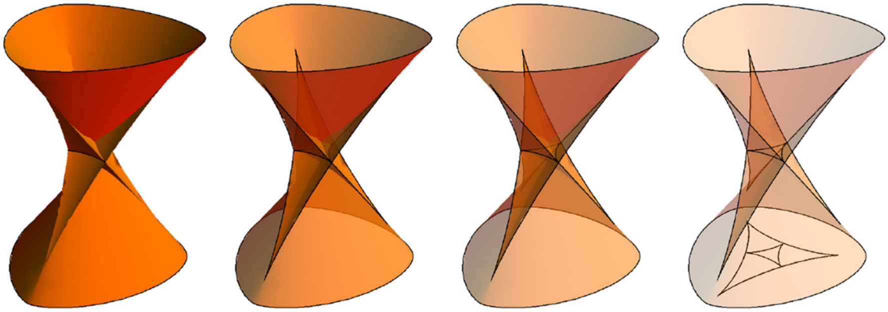


Fig. 3 The affine extended wave front of an oval

present the following curves: $\{0\} \times R_1$, $\{1\} \times R_1$, $\{\frac{1}{2}\} \times E_{\frac{1}{2}}(R_1)$, $\{0\} \times E_{\frac{1}{2}}(R_1)$ and $\{0\} \times CSS(R_1)$.

Let Σ be a set of singular points of $\mathbb{E}$. It is well known that $\pi(\Sigma(\mathbb{E}(R_1))) = CSS(R_1)$ and the map $\Sigma(\mathbb{E}(R_1)) \ni p \mapsto \pi(p) \in CSS(R_1)$ is the double covering of $CSS(R_1)$.

Let $\mathbb{E}_k(R_m)$ for $k = 1, \ldots, m$ be a branch of $\mathbb{E}(R_m)$ which has the following parameterization

$$f_k(\lambda, \theta) = (\lambda, \lambda\gamma(\theta) + (1-\lambda)\gamma(\theta + k\pi)). \tag{5.4}$$

We use the following notation:

$$(f_k)_\lambda := \frac{\partial}{\partial\lambda} f_k(\lambda, \theta), \quad (f_k)_\theta := \frac{\partial}{\partial\theta} f_k(\lambda, \theta). \tag{5.5}$$

In Figs. 4 and 5 we illustrate (with different opacities) the branches $\mathbb{E}_1(R_2)$ and $\mathbb{E}_2(R_2)$, respectively, where R_2 is a 2-rosette represented by the support function $p(\theta) = 11 + \sin\frac{\theta}{2} - 7\cos\frac{3\theta}{2} - \frac{1}{2}\sin 2\theta$.

Directly by Definition 5.5 we get the following proposition.

Proposition 5.6 *Every branch of* $\mathbb{E}(R_m)$ *is a ruled surface.*

It is well known that the Gaussian curvature of a ruled surface at a non-singular point is non-positive. By direct calculation we get the following proposition.

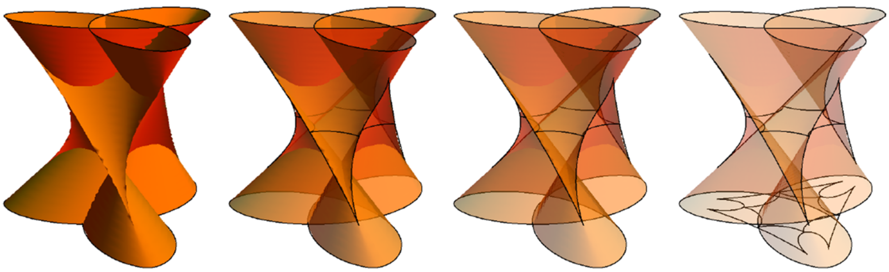


Fig. 4 A singular branch of the affine extended wave front of a 2-rosette

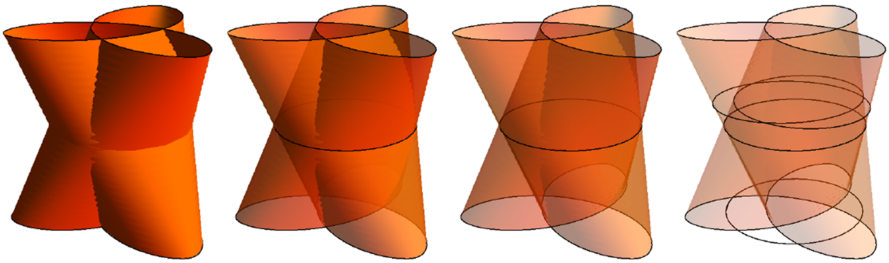


Fig. 5 A non-singular branch of the affine extended wave front of a 2-rosette

Proposition 5.7 *Let R_m be an m-rosette.*

(i) *A point (λ, θ) is a singular point of $\mathbb{E}_k(R_m)$ if and only if*

$$\frac{\kappa(\theta)}{\kappa(\theta + k\pi)} = (-1)^{k+1}\frac{\lambda}{1-\lambda}. \tag{5.6}$$

(ii) *A singular point (λ_0, θ_0) is a cuspidal edge if and only if*

$$\left(\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right)'\Big|_{(\lambda_0,\theta_0)} \neq 0. \tag{5.7}$$

(iii) *A singular point (λ_0, θ_0) is a swallowtail if and only if*

$$\left(\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right)'\Big|_{(\lambda_0,\theta_0)} = 0 \text{ and } \left(\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right)''\Big|_{(\lambda_0,\theta_0)} \neq 0. \tag{5.8}$$

Proof We use (5.4) as the parameterization of $\mathbb{E}_k(R_m)$. Let us notice that f_k is singular if and only if $(f_k)_\lambda \times (f_k)_\theta = 0$. This condition is equivalent to (5.6). By Fact 1.5 in [36] we get (5.7) and (5.8). □

Remark 5.8 By Theorem 3.3 in [15] there exists an open and dense subset of the space of rosettes such that the affine extended wave front $\mathbb{E}(R_m)$ has only A_2 and A_3 singularities (cuspidal edges and swallowtails) for any rosette R_m in this subset. Thus, by Proposition 5.7 a rosette R_m is called *generic* if there do not exist θ and $k \in \{1, \cdots, m\}$ such that

$$\left(\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right)' = \left(\frac{\kappa(\theta + k\pi)}{\kappa(\theta)}\right)'' = 0. \tag{5.9}$$

By direct calculation we get the following proposition (see also Definition 2.2).

Proposition 5.9 *If R_m is generic then every singular point of $\mathbb{E}(R_m)$ is non-degenerate.*

Remark 5.10 In [6,10,44] we study in details the geometry of affine λ-equidistants of rosettes. We show among other things that there exist m branches of $E_{\frac{1}{2}}(R_m)$ and

$2m-1$ branches of $E_\lambda(R_m)$ for $\lambda \neq 0, \frac{1}{2}, 1$. Let $E_{\frac{1}{2},k}(R_m)$ for $k = 1, 2, \ldots, m$ denote different branches of $E_{\frac{1}{2}}(R_m)$ and let $E_{\lambda,k}(R_m)$ for $k = 1, 2, \ldots, 2m-1$ denote different branches of $E_\lambda(R_m)$ for $\lambda \neq 0, \frac{1}{2}, 1$. Then, the support function of $E_{\frac{1}{2},k}(R_m)$ for $k = 1, \ldots, m$ is in the form (5.10), the support function of $E_{\lambda,k}(R_m)$ for $k = 1, 2, \ldots, m$ (respectively $k = m+1, m+2, \ldots, 2m-1$) in the form (5.11) (respectively in the form (5.12)), where

$$p_{\frac{1}{2},k}(\theta) = \frac{1}{2}\big(p(\theta) + (-1)^k p(\theta + k\pi)\big), \tag{5.10}$$

$$p_{\lambda,k}(\theta) = \lambda p(\theta) + (-1)^k (1-\lambda) p(\theta + k\pi), \tag{5.11}$$

$$p_{\lambda,k}(\theta) = (1-\lambda) p(\theta) + (-1)^k \lambda p(\theta + (k-m)\pi). \tag{5.12}$$

Let $\gamma_{\lambda,k}$ denote the parameterization of $E_{\lambda,k}$ in terms of the support function accordingly to (5.10), (5.11) and (5.12), respectively. Furthermore each branch of $E_\lambda(R_m)$, except $E_{\frac{1}{2},m}(R_m)$, has the rotation number equal to m. The rotation number of $E_{\frac{1}{2},m}(R_m)$ is equal to $\frac{m}{2}$. If R_m is a generic m-rosette then for $\lambda \in (0,1) - \{\frac{1}{2}\}$ only branches $E_{\lambda,k}(R_m)$ for $k = 1, 3, \ldots, 2\lceil \frac{1}{2}m \rceil - 1, m+1, m+3, \ldots, m + 2\lceil \frac{1}{2}m \rceil - 1$ can admit cusp singularities and branches $E_{\frac{1}{2},k}(R_m)$ for $k = 1, 3, \ldots, 2\lceil \frac{1}{2}m \rceil - 1$ has cusp singularities. By [13] we known that if a, b is parallel pair of R_m and R_m is parameterized at a and b in different directions and $\kappa(a), \kappa(b)$ denote the signed curvatures of R_m at a and b, respectively, then the point $\frac{a\kappa_1 + b\kappa_2}{\kappa_1 + \kappa_2}$, which is lying on the line between a and b, belongs to $CSS(R_m)$.

Corollary 5.11 *Let R_m be a generic m-rosette. Then, $CSS(R_m)$ which is created from singular points of $E_\lambda(R_m)$ for $\lambda \in [0,1]$ consists of exactly $2\lceil \frac{1}{2}m \rceil - 1$ branches.*

Proof It is a consequence of Remark 5.10. □

Let $CSS_k(R_m)$ for $k = 1, 3, \ldots, 2\lceil \frac{1}{2}m \rceil - 1$ denote a branch of $CSS(R_m)$. Then, the parameterization of $CSS_k(R_m)$ is in the following form

$$\gamma_{CSS_k(R_m)}(\theta) = \frac{\kappa(\theta)}{\kappa(\theta) + \kappa(\theta + k\pi)} \gamma(\theta) + \frac{\kappa(\theta + k\pi)}{\kappa(\theta) + \kappa(\theta + k\pi)} \gamma(\theta + k\pi), \tag{5.13}$$

where if $k < m$ then $\theta \in [0, 2m\pi]$ and if $k = m$ then $\theta \in [0, m\pi]$.

Lemma 5.12 *Let $\mathcal{C}$ be a closed smooth curve with at most cusp singularities and let the rotation number of $\mathcal{C}$ be m. If m is an integer, then the number of cusp singularities is even. If m is the form $\frac{1}{2}d$, where d is an odd integer, then the number of cusp singularities is odd.*

Proof A continuous normal vector field to the germ of a curve with the cusp singularity is directed outside the cusp on the one of two connected regular components and is directed inside the cusp on the other component as it is shown in Fig. 6. If m is an integer, then the number of cusps of $\mathcal{C}$ is even, otherwise is odd. □

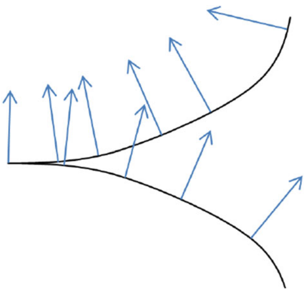


Fig. 6 A continuous normal vector field to the germ of a curve with the cusp singularity

Proposition 5.13 *Let R_m be a generic m-rosette. If $k = m$ and m is an odd number, then the number of cusp singularities of $CSS_k(R_m)$ is odd and not smaller than the number of cusp singularities of $E_{\frac{1}{2},k}(R_m)$, otherwise the number of cusp singularities of $CSS_k(R_m)$ is even and not smaller than the number of cusp singularities of $E_{\frac{1}{2},k}(R_m)$, which is even and positive.*

Proof The parity of the number of cusp singularities of $CSS_k(R_m)$ is a consequence of (5.13) and Lemma 5.12.

Let m be even and $k \leqslant m$ or m be odd and $k < m$. By Theorem 2.9 in [44] we know that $E_{\frac{1}{2},k}(R_m)$ has at least 2 cusp singularities. Because the cusp in $E_{\frac{1}{2}}$ appears when $\frac{\kappa(a)}{\kappa(b)} = 1$ and cusp in CSS appears when $\left(\frac{\kappa(a)}{\kappa(b)}\right)' = 0$ [5,13], where a, b is a parallel pair and $'$ is used to denote the derivative with respect to the parameter along the corresponding segment of a curve. Therefore, by Roll's theorem we get that the number of cusp singularities of $CSS_k(R_m)$ is not smaller than the number of cusp singularities of $E_{\frac{1}{2},k}(R_m)$. The same arguments works when m is odd and $k = m$. $\square$

Corollary 5.14 *Let R_m be an m-rosette. Then, the number of branches of $\mathbb{E}(R_m)$ is equal to m and a branch $\mathbb{E}_k(R_m)$ is singular if and only if k is odd.*

In Figs. 4 and 5 we present two branches of $\mathbb{E}(R_2)$: $\mathbb{E}_1(R_2)$ and $\mathbb{E}_2(R_2)$, respectively.

Proposition 5.15 *Let R_m be an m-rosette and let p be a non-singular point of $\mathbb{E}_k(R_m)$. Then, the Gaussian curvature of $\mathbb{E}_k(R_m)$ at p is equal to* 0.

Proof The surface is parameterized by (5.4).

At a non-singular point (λ, θ) the Gaussian curvature K of $\mathbb{E}_k$ is equal to

$$K_k(\lambda,\theta) = \frac{\det\left((f_k)_{\lambda\lambda}, (f_k)_\lambda, (f_k)_\theta\right)\cdot\det\left((f_k)_{\theta\theta}, (f_k)_\lambda, (f_k)_\theta\right) - \det^2\left((f_k)_{\lambda\theta}, (f_k)_\lambda, (f_k)_\theta\right)}{\left(|(f_k)_\lambda|^2|(f_k)_\theta|^2 - ((f_k)_\lambda\cdot(f_k)_\theta)^2\right)^2}. \tag{5.14}$$

Since $(f_k)_{\lambda\lambda} = 0$ and vectors $(f_k)_\theta$ and $(f_k)_{\lambda\theta}$ are linearly dependent, the Gaussian curvature K_k at a non-singular point of $\mathbb{E}_k$ is equal to zero. $\square$

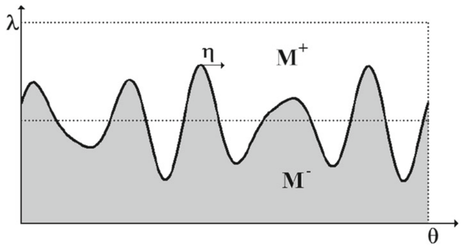


Fig. 7 A singular curve with a null vector η

Definition 5.16 Let R_m be an m-rosette. Let $k \in \{1, 2, \ldots, m\}$. Then, the *k-width* of R_m for an oriented angle θ is the following

$$w_k(\theta) = p(\theta) - (-1)^k p(\theta + k\pi). \tag{5.15}$$

Remark 5.17 Let R_m be a generic m-rosette and $k \leqslant m$ be an odd number. From now on we set

$$\begin{aligned} &M := [0, 1] \times S^1, \\ &M \ni (\lambda, \theta) \mapsto f_k(\lambda, \theta) \in \mathbb{E}_k(R_m) \subset \mathbb{R}^3, \\ &M \ni (\lambda, \theta) \mapsto \nu_k(\lambda, \theta) := \frac{(w_k(\theta), \mathbb{n}(\theta))}{\sqrt{1 + w_k^2(\theta)}} \in S^2. \end{aligned}$$

The map (f_k, ν_k) is a front. Then, the coherent tangent bundle $\mathcal{E}^{f_k}$ over M has the following fiber at $p \in M$

$$\mathcal{E}_p^{f_k} := \left\{ X \in T_{f_k(p)}\mathbb{R}^3 \mid \langle X, \nu_k(p) \rangle = 0 \right\}.$$

The set of singular points Σ_k is parameterized by $(\lambda_k(\theta), \theta)$, where $\lambda_k(\theta) = \frac{\kappa(\theta)}{\kappa(\theta)+\kappa(\theta+k\pi)}$. Let us notice that

$$M^- = \left\{ (\lambda, \theta) \in M \;\middle|\; \lambda < \lambda_k(\theta) \right\}, M^+ = \left\{ (\lambda, \theta) \in M \;\middle|\; \lambda > \lambda_k(\theta) \right\}.$$

Furthermore, if the function $\lambda_k(\theta)$ has a local minimum, then the point $(\lambda_k(\theta), \theta)$ is a negative peak and if $\lambda_k(\theta)$ has a local maximum, then this point is a positive peak. See Fig. 7.

Proposition 5.18 *Let R_m be a generic m-rosette. Let k be an odd number and let $\lambda \in [0, 1]$. Then, the $\mathcal{E}^{f_k}$-geodesic curvature of a curve $\{\lambda\} \times S^1$ in M at a non-singular point is equal to*

$$\hat{\kappa}_{k,g}(\theta) := \frac{w_k(\theta)}{|\lambda\rho(\theta) - (1-\lambda)\rho(\theta + k\pi)|\sqrt{1 + w_k^2(\theta)}}. \tag{5.16}$$

Proof Let $s_k(\lambda,\theta) := \lambda\rho(\theta) - (1-\lambda)\rho(\theta+k\pi)$. Then (5.16) follows from the formula

$$\hat{\kappa}_{k,g}(\theta) = \frac{\det(\gamma'_{k,\lambda}(\theta), \gamma''_{k,\lambda}(\theta), \nu_k(\lambda,\theta))}{|\gamma'_{k,\lambda}(\theta)|^3}.$$

□

Proposition 5.19 *Let R_m be a generic m-rosette. Let k be an odd number. Then the singular curvature on a cuspidal edge at a point $\left(\frac{\kappa(\theta)}{\kappa(\theta)+\kappa(\theta+k\pi)}, \theta\right)$ is equal to*

$$\kappa_{k,s}(\theta) = \kappa_{CSS_k}(\theta) \cdot \frac{\sqrt{1+w_k^2(\theta)}}{w_k(\theta)} \cdot \left(\frac{w_k^2(\theta)+w_k'^2(\theta)}{1+w_k^2(\theta)+w_k'^2(\theta)}\right)^{\frac{3}{2}}, \tag{5.17}$$

where $\kappa_{CSS_k}(\theta)$ is a the curvature of $CSS_k(R_m)$, which is given by the following formula:

$$\kappa_{CSS_k}(\theta) = \frac{-\big(\kappa(\theta)+\kappa(\theta+k\pi)\big)^3}{\kappa(\theta)\kappa(\theta+k\pi)\big|\kappa'(\theta+k\pi)\kappa(\theta) - \kappa'(\theta)\kappa(\theta+k\pi)\big|} \cdot \frac{w_k(\theta)}{\big(w_k^2(\theta)+w_k'^2(\theta)\big)^{\frac{3}{2}}}. \tag{5.18}$$

Proof It is a direct consequence of the formula of the singular curvature and the formula of the curvature of the centre symmetry set (see Lemma 2.6 in [10]). □

By Theorem 1.6 in [36] we know that the singular curvature does not depend on the orientation of the parameter θ, the orientation of M, the choice of ν, nor the orientation of the singular curve. The sign of the singular curvature have a geometric interpretation, if the singular curvature is positive (respectively negative) then the cuspidal edge is positively (respectively negatively) curved. See Fig. 8.

We find a formula which gives us the relation between the total singular curvature on set of singular points and the total geodesic curvature on the boundary of M. The integrals in (5.19)–(5.22) can be seen as integrals on $f_k(\Sigma_k)$ and $f_k(\{\lambda\}\times S^1) = \{\lambda\}\times E_{k,\lambda}(R_m)$ since the arclength measure, the singular curvature and $\mathcal{E}^{f_k}$-geodesic curvature are defined with respect to the first fundamental form ds^2 which is the pullback of metric $\langle\cdot,\cdot\rangle$ on $\mathbb{E}_k(R_m) \subset \mathbb{R}^3$.

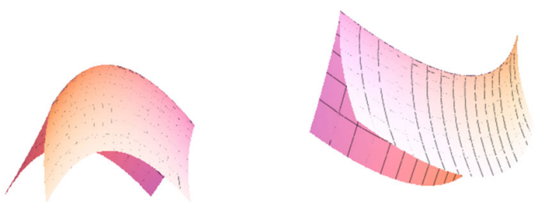


Fig. 8 Examples of positively (on the left) and negatively (on the right) curved cuspidal edges

Theorem 5.20 *Let k be an odd number. Let R_m be a generic m-rosette. Then*

$$\int_{\Sigma_k} \kappa_{k,s} d\tau + \int_{\{1\}\times S^1} \hat{\kappa}_{k,g} d\tau = 0, \tag{5.19}$$

where $d\tau$ denote the arc length measure and the orientation of $\{1\} \times S^1$ is compatible with the orientation of M.

Proof By Remark 5.17 we get that $(f_k, \nu_k) : M \to \mathbb{R}^3 \times S^2$ is a front. The boundary of M does not intersect the set of singular points Σ. By genericity of R_m this front satisfies the assumptions of Theorem 2.20. Since $\lambda_k(\theta) + \lambda_k(\theta + k\pi) = 1$, we get that M^+ and M^- are symmetric. Hence $\chi(M^+) = \chi(M^-)$ and $\#P^- = \#P^+$.

By Proposition 5.15 and Theorem 2.20 we get that

$$\int_{\{1\}\times S^1} \hat{\kappa}_{k,g} \mathrm{d}\tau = -\int_{\{0\}\times S^1} \hat{\kappa}_{k,g} \mathrm{d}\tau$$

and then we get (5.19). □

Theorem 5.21 *Let k be an odd number, R_m be a generic m-rosette and $\lambda \in [0, 1)$. If $E_{k,\lambda}(R_m)$ admits at most cusp singularities, then*

$$\int_{\{\lambda\}\times S^1} \hat{\kappa}_{k,g} \mathrm{d}\tau = -\int_{\{1\}\times S^1} \hat{\kappa}_{k,g} \mathrm{d}\tau, \tag{5.20}$$

$$\int_{(\{\frac{1}{2}\}\times S^1)\cap M^+} \hat{\kappa}_{k,g} \mathrm{d}\tau = \sum_{p\in C} \alpha_+(p) - \frac{1}{2}\pi \#C - \frac{1}{2}\int_{\{1\}\times S^1} \hat{\kappa}_{k,g} \mathrm{d}\tau, \tag{5.21}$$

$$\int_{(\{\frac{1}{2}\}\times S^1)\cap M^-} \hat{\kappa}_{k,g} \mathrm{d}\tau = -\sum_{p\in C} \alpha_+(p) + \frac{1}{2}\pi \#C - \frac{1}{2}\int_{\{1\}\times S^1} \hat{\kappa}_{k,g} \mathrm{d}\tau, \tag{5.22}$$

where the orientations of S^1 in the integrals on the left hand sides and the right-hand sides are opposite in the above formulas, $C = \Sigma_k \cap (\{\frac{1}{2}\} \times S^1)$, $d\tau$ is the arclength measure and

$$\alpha_+(p) := arccos\left(\sqrt{\frac{w_k^2(\theta) + w_k'^2(\theta)}{1 + w_k^2(\theta) + w_k'^2(\theta)}} \cos(\beta(\theta))\right), \tag{5.23}$$

where $p = (\frac{1}{2}, \theta)$ and $\beta(\theta)$ is the angle between the tangent vector to R_m at $\gamma(\theta)$ and the vector $\gamma(\theta + k\pi) - \gamma(\theta)$.

Proof Let $M_\lambda := [\lambda, 1] \times S^1$. By Remark 5.17 we get that $(f_k, \nu_k)\big|_{M_\lambda} : M_\lambda \to \mathbb{R}^3 \times S^2$ is a front. It is easy to see that $\chi(M_\lambda^+) = 0$ and $\chi(M_\lambda^-) = \#P^+ - \#P^-$ is the number of cusps of $\mathcal{E}^{f_k}\big|_{M_\lambda}$ (that is $\#\big(\Sigma_k \cap (\{\lambda\} \times S^1)\big)$). Since every point $p \in \Sigma_k \cap \partial M_\lambda$ is a null singular point, by Theorem 2.20 (see (2.9)) we get (5.20).

By the genericity of R_m the front $(f_k, \nu_k)\big|_{M_{\frac{1}{2}}}$ satisfies the assumptions of Theorem 2.20. Since $\int_{\Sigma_k} \kappa_s \mathrm{d}\tau = 2\int_{\Sigma_k \cap M_{\frac{1}{2}}} \kappa_s \mathrm{d}\tau$, we get (5.21) and (5.22).

The angle between initial vectors (see Definition 2.5) of the singular curve at p and of the boundary curve at p is $\alpha_+(p)$ (see Theorem 2.20). By Proposition 2.7 and Proposition 2.8 we get (5.23). ◻

Furthermore, directly by (2.9) we get the following proposition.

Proposition 5.22 *Let k be an odd number. Let R_m be a generic m-rosette. Let $\mathcal{C}^+$ (respectively $\mathcal{C}^-$) be a simple regular curve in M^+ (respectively M^-) which is smoothly homotopic to $\{1\}\times S^1$ (respectively $\{0\}\times S^1$). If the orientations of $\mathcal{C}^+$, $\mathcal{C}^-$ are opposite then*

$$\int_{\mathcal{C}^+} \kappa_{k,g} \mathrm{d}\tau + \int_{\mathcal{C}^-} \kappa_{k,g} \mathrm{d}\tau = 0,$$

where $\mathrm{d}\tau$ denote the arc length measure.

By Theorem 5.20 we can get the relation between integrals of the curvature of the centre symmetry set, the curvature of the rosette and the width of the rosette.

Corollary 5.23 *Let k be an odd number and let R_m be a generic m-rosette. Then*

$$\begin{aligned}&\int_{R_m} \kappa(\theta(s)) \cdot \frac{w_k(\theta(s))}{\sqrt{1+w_k^2(\theta(s))}} \mathrm{d}s \\ &\quad = \int_{CSS_k(R_m)} \kappa_{CSS_k(R_m)}(\theta(\ell)) \cdot \frac{\big(\rho(\theta(\ell)) + \rho(\theta(\ell)+k\pi)\big)\sqrt{1+w_k^2(\theta(\ell))}}{\big(1+w_k^2(\theta(\ell)) + w_k'^2(\theta(\ell))\big)^{\frac{3}{2}}} \mathrm{d}\ell,\end{aligned} \tag{5.24}$$

where s (respectively ℓ) is the arc length parameter on R_m (respectively on $CSS_m(R_m)$).

Theorem 5.24 *Let k be an odd number and let R_m be a generic m-rosette. Then*

$$\int_0^{2m\pi} \frac{w_k(\theta)}{\sqrt{1+w_k^2(\theta)}} \mathrm{d}\theta = \int_0^{2m\pi} \big(w_k(\theta) + w_k''(\theta)\big) \cdot \frac{\sqrt{1+w_k^2(\theta)}}{1+w_k^2(\theta)+w_k'^2(\theta)} \mathrm{d}\theta. \tag{5.25}$$

Proof The proof is a straightforward use of (5.16), (5.17) and the fact that $\rho(\theta)+\rho(\theta+k\pi) = w_k(\theta) + w_k''(\theta)$. ◻

Remark 5.25 Since $w_k(\theta) = \sinh(C_1\theta + C_2)$ for $C_1, C_2 \in \mathbb{R}$ is the general solution of

$$\frac{w_k(\theta)}{\sqrt{1+w_k^2(\theta)}} = \big(w_k(\theta) + w_k''(\theta)\big) \cdot \frac{\sqrt{1+w_k^2(\theta)}}{1+w_k^2(\theta)+w_k'^2(\theta)}, \tag{5.26}$$

the only periodic solution of (5.26) is a constant function. Therefore, the relation (5.25) is naively fulfilled only for rosettes of constant k-width.

Remark 5.26 The condition that w is C^2-smooth cannot be omitted. We can consider the function $w(\theta) = 1 + |x - \pi|^3$ and the interval $[0, 2\pi]$. One can check that relation (5.25) does not hold.

Remark 5.27 By (5.15) the odd coefficients of the Fourier series of a width of an oval vanish. Thus, a function $w(\theta) = 2 + \sin 3\theta$ is not a width of any oval but it satisfies the relation (5.25).

Conjecture 5.28 Let $w : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic C^2-smooth function. Then, w satisfies the relation

$$\int_0^{2\pi} \frac{w(\theta)}{\sqrt{1+w^2(\theta)}} \mathrm{d}\theta = \int_0^{2\pi} \big(w(\theta) + w''(\theta)\big) \cdot \frac{\sqrt{1+w^2(\theta)}}{1 + w^2(\theta) + w'^2(\theta)} \mathrm{d}\theta. \tag{5.27}$$

In [29,30] other invariants of cuspidal edges of fronts are introduced. Let $(f, \nu) : M \mapsto \mathbb{R}^3 \times S^2$ be a front. Let γ be a singular curve near an A_2-point (a cuspidal edge) and η be a null direction along γ such that the singular direction γ' and the null direction η form a positively oriented frame. We put $\hat{\gamma} = f \circ \gamma$, $f_\eta = df(\eta)$, $f_{\eta,\eta} = d(f_\eta)(\eta)$, $f_{\eta,\eta,\eta} = d(f_{\eta,\eta})(\eta)$. Then, *the limiting normal curvature along* γ is defined in the following way

$$\kappa_\nu(t) = \frac{\left\langle \hat{\gamma}''(t), \nu(\gamma(t)) \right\rangle}{|\hat{\gamma}'(t)|^2}. \tag{5.28}$$

The *cuspidal curvature along* γ is defined as follows:

$$\kappa_c(t) = \frac{|\hat{\gamma}(t)|^{\frac{3}{2}} \det\big(\hat{\gamma}(t), f_{\eta\eta}(\gamma(t)), f_{\eta\eta\eta}(\gamma(t))\big)}{\big|\hat{\gamma}(t) \times f_{\eta\eta}(\gamma(t))\big|^{\frac{5}{2}}}. \tag{5.29}$$

The *cusp-directional torsion* is defined by the formula

$$\kappa_t(t) = \frac{\det\big(\hat{\gamma}', f_{\eta\eta}(\gamma), (f_{\eta\eta}(\gamma))'\big)}{\big|\hat{\gamma}' \times f_{\eta\eta}(\gamma)\big|^2}(t) - \frac{\det\big(\hat{\gamma}', f_{\eta\eta}(\gamma), \hat{\gamma}''\big) \cdot \big\langle \hat{\gamma}', f_{\eta\eta}(\gamma)\big\rangle}{|\hat{\gamma}'|^2 |\hat{\gamma}' \times f_{\eta\eta}(\gamma)|^2}(t). \tag{5.30}$$

In [36], it was shown that a point p is a generic cuspidal edge if and only if $\kappa_\nu(p)$ does not vanish. The curvature κ_c is exactly the cuspidal curvature of the cusp of the plane curve obtained as the intersection of the surface by the plane H, where H is orthogonal to the tangential direction at a given cuspidal edge [30]. For the geometrical meaning of the cusp-directional torsion (5.30) see Proposition 5.2 in [29] and for global properties see Appendix A in [29]. By straightforward calculations we obtain the following lemma.

Lemma 5.29 *Let R_m be a generic m-rosette. Let k be an odd number. Then the normal curvature $\kappa_{k,\nu}$, the cuspidal curvature $\kappa_{k,c}$ and the cusp-directional torsion $\kappa_{k,t}$ of the cuspidal edge of $\mathbb{E}_k(R_m)$ at a point $\left(\frac{\kappa(\theta)}{\kappa(\theta)+\kappa(\theta+k\pi)}, \theta\right)$ are given by the following formulas*

$$\kappa_{k,\nu}(\theta) \equiv 0, \tag{5.31}$$

$$\kappa_{k,c}(\theta) = \frac{2\sqrt{\kappa(\theta)\kappa(\theta+k\pi)\big(\kappa(\theta)+\kappa(\theta+k\pi)\big)}}{\sqrt{\left|\left(\frac{\kappa(\theta+k\pi)}{\kappa(\theta)}\right)'\right|}} \cdot \frac{\left(1+w_k^2(\theta)+w_k'^2(\theta)\right)^{\frac{3}{4}}}{\left(1+w_k^2(\theta)\right)^{\frac{5}{4}}}, \tag{5.32}$$

$$\kappa_{k,t}(\theta) = -\frac{\big(\kappa(\theta)+\kappa(\theta+k\pi)\big)^2}{\kappa^2(\theta)\cdot\left(\frac{\kappa(\theta+k\pi)}{\kappa(\theta)}\right)'} \cdot \frac{1}{1+w_k^2(\theta)}. \tag{5.33}$$

Proposition 5.30 *Let R_m be a generic m-rosette. Let k be an odd number. Then*

(i) *cuspidal edges of $\mathbb{E}_k(R_m)$ are not generic,*
(ii) *the mean curvature of $\mathbb{E}_k(R_m)$ is not bounded,*
(iii) *the total torsion of the image of singular curve $\hat{\gamma}_k(\theta)$ for $\theta \in [0, 2k\pi]$ is equal to $2n\pi$ for some integer n, i.e.*

$$\int_{\gamma_k} \tau_k(s)\mathrm{d}s = 2n\pi, \tag{5.34}$$

where γ_k is the singular curve, τ_k is a torsion of $\hat{\gamma}_k$ and s is the arc length parameter of $\hat{\gamma}_k$.

Proof (i) It is a consequence of (5.31).
(ii) Since $\kappa_{k,c}(p) \neq 0$ for any cuspidal edge $p \in \Sigma$, then by Proposition 2.8 in [30] we get that the mean curvature of $CSS_k(R_m)$ is not bounded.
(iii) From Appendix A in [29] we know that in our case there is the following equality

$$\int_{\gamma_k} \kappa_{k,t}(s)\mathrm{d}s = \int_{\gamma_k} \tau_k(s)\mathrm{d}s - 2n\pi.$$

It is easy to see that $\int_{\gamma_k} \kappa_{k,t}(s)\mathrm{d}s = 0$. Hence (5.34) holds. □

Remark 5.31 For the geometrical meaning of the number n in Corollary 5.30(iii) see Appendix A in [29]. In [32], authors show that the total torsion of a closed line of curvature on a surface (i.e. a closed curve on a surface whose tangents are always in the direction of a principal curvature) is $l\pi$, where l is an integer. Furthermore, they show that if the total torsion of a closed curve is $l\pi$ for an integer l, then this curve can appear as a line of curvature on a surface and if l is even, then it can appear as a line of curvature on a surface of genus 1.

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