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On the Spectrum of Differential Operators Under Riemannian Coverings

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Abstract

For a Riemannian covering $p\colon M_2 \to M_1$, we compare the spectrum of an essentially self-adjoint differential operator D_1 on a bundle $E_1 \to M_1$ with the spectrum of its lift D_2 on $p^*E_1 \to M_2$. We prove that if the covering is infinite sheeted and amenable, then the spectrum of D_1 is contained in the essential spectrum of any self-adjoint extension of D_2. We show that if the deck transformations group of the covering is infinite and D_2 is essentially self-adjoint (or symmetric and bounded from below), then D_2 (or the Friedrichs extension of D_2) does not have eigenvalues of finite multiplicity and in particular, its spectrum is essential. Moreover, we prove that if M_1 is closed, then p is amenable if and only if it preserves the bottom of the spectrum of some/any Schrödinger operator, extending a result due to Brooks.

Keywords Spectrum of differential operator · Amenable covering · Bottom of spectrum · Schrödinger operator

Mathematics Subject Classification 58J50 · 35P15 · 53C99

1 Introduction

A basic problem in Geometric Analysis is the investigation of relations between the geometry of a manifold and the spectrum of a differential operator on it. In this direction, it is natural to study the behavior of the spectrum under maps between Riemannian manifolds, which respect the geometry of the manifolds to some extent. In this paper, we deal with this problem for Riemannian coverings.

Let $p\colon M_2 \to M_1$ be a Riemannian covering of connected manifolds with (possibly empty) smooth boundary. A Schrödinger operator S_1 on M_1 is an operator of the form $S_1 = \Delta + V$, where Δ is the (non-negative definite) Laplacian and $V\colon M_1 \to \mathbb{R}$ is

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smooth and bounded from below. For such an operator S_1 on M_1, its lift on M_2 is the operator $S_2 = \Delta + V \circ p$. The first results involving possibly infinite-sheeted coverings and establishing connections between properties of the covering and the (Dirichlet) spectra of S_1 and S_2 are related to the change of the bottom (that is, the minimum) of the spectrum and were proved by Brooks [6,7]. He showed that if the underlying manifold is complete, of finite topological type, without boundary and the covering is normal and amenable, then the bottom of the spectrum of the Laplacian is preserved. Bérard and Castillon [4] extended this result by showing that if the covering is amenable and the underlying manifold is complete with finitely generated fundamental group and without boundary, then the bottom of the spectrum of any Schrödinger operator is preserved. Recently, it was proved in [2] that the bottom of the spectrum of a Schrödinger operator is preserved under amenable coverings, without any topological or geometric assumptions.

In this paper, we prove a global result about this problem in a more general context. Instead of comparing the bottoms of the spectra, we prove inclusion of spectra under some reasonable assumptions. Moreover, our context allows us to impose various boundary conditions on Schrödinger operators (for instance, Dirichlet, Neumann, mixed, and Robin), while the former results involve only Dirichlet conditions. Furthermore, our theorems are applicable to a broad class of differential operators, including Schrödinger operators with magnetic potential (that is, first-order term), Dirac operators, higher-order Laplacian, and Laplace-type operators on vector bundles. It is worth to point out that the Hodge Laplacian is a special case of the latter ones, as well as the Jacobi (stability) operator of a minimal submanifold. Furthermore, in this context, we may consider the Laplacian on weighted manifolds (or Laplacian with density).

In order to simplify the statements of our results, we need to set up some notation. Consider a Riemannian or Hermitian vector bundle $E_1 \to M_1$ endowed with a (not necessarily metric) connection ∇. Let D_1 be a (not necessarily elliptic) differential operator of arbitrary order on E_1. We consider the pullback bundle $E_2 := p^* E_1 \to M_2$ endowed with the corresponding metric and connection, and the lift D_2 of D_1.

If M_1 has empty boundary, we consider the space of compactly supported smooth sections of E_i as the domain of D_i, $i = 1, 2$. If M_1 has non-empty boundary, as the domain of D_1 we consider the space of compactly supported smooth sections η of E_1 satisfying a number of boundary conditions of the form

$$\sum_{j=0}^{k} a_j \nabla_n^{(j)} \eta = 0 \text{ on } \partial M_1,$$

where n is the inward pointing normal to ∂M_1 and a_j's are functions defined on ∂M_1. For example, in this context, we may impose boundary conditions of the form $\eta = \nabla_n \eta = \cdots = \nabla_n^{(k)} \eta = 0$ on ∂M_1, for some $k \in \mathbb{N}$. As the domain of D_2 we consider the space of compactly supported, smooth sections of E_2 that satisfy analogous boundary conditions to the ones imposed on the domain of D_1.

Let μ_1 be a measure expressed via a positive smooth density in terms of the volume element of M_1, that is, $d\mu_1 = h d\,\mathrm{Vol}$. Let μ_2 be the corresponding measure on M_2, that is, $d\mu_2 = (h \circ p) d\,\mathrm{Vol}$. We consider the operators D_i restricted to the above domains as densely defined operators in $L^2(E_i, \mu_i)$, $i = 1, 2$.

For sake of simplicity, we present here special versions of our main results involving self-adjoint operators. The results are stated for infinite-sheeted coverings, since this is the interesting case of amenable coverings. However, we also prove the analogous results for finite-sheeted coverings. Our first result provides inclusion of the spectrum $\sigma(\overline{D}_1)$ of the closure of D_1, as long as it is self-adjoint, in the essential spectrum $\sigma_{\text{ess}}(D_2')$ of any self-adjoint extension D_2' of D_2.

Theorem 1.1 *Assume that D_1 is essentially self-adjoint and let D_2' be a self-adjoint extension of D_2. If the covering is infinite sheeted and amenable, then the spectra of the operators satisfy $\sigma(\overline{D}_1) \subset \sigma_{\text{ess}}(D_2')$.*

Recall that a Schrödinger operator on a complete manifold is essentially self-adjoint on the space of compactly supported smooth functions vanishing on the boundary (if it is non-empty). Therefore, in the context of Schrödinger operators, it follows that if the underlying manifold is complete and the covering is infinite sheeted and amenable, then the spectrum of S_1 is contained in the essential spectrum of S_2.

An important case where the above theorem cannot be applied is that of Schrödinger operators on non-complete Riemannian manifolds. A Schrödinger operator on such a manifold does not have a unique self-adjoint extension, when restricted to the above domain, and we are interested in the spectrum of its Friedrichs extension. According to [2], if the covering is amenable, then the bottoms of the spectra of S_1 and S_2 coincide. The amenability is used only to establish $\lambda_0(S_2) \leq \lambda_0(S_1)$, since the inverse inequality holds for any covering, where λ_0 stands for the bottom of the spectrum. This motivates us to establish the following theorem, which compares the bottom $\lambda_0(D_1^{(F)})$ of the spectrum of the Friedrichs extension of D_1 with the bottom $\lambda_0^{\text{ess}}(D_2^{(F)})$ of the essential spectrum of the Friedrichs extension of D_2, when the operators are symmetric and bounded from below.

Theorem 1.2 *Assume that D_i is symmetric and bounded from below, and denote by $D_i^{(F)}$ its Friedrichs extension, $i = 1, 2$. If the covering is infinite sheeted and amenable, then $\lambda_0^{\text{ess}}(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$.*

In particular, for Schrödinger operators, it follows that if the covering is infinite sheeted and amenable, then the bottom of the spectrum of S_1 is equal to the bottom of the essential spectrum of S_2, without any topological or geometric assumptions.

The above results involve amenable coverings. However, the deck transformations group of a (possibly non-amenable) covering provides information about the group of isometries of the covering space. This motivates us to work in a more general context than Riemannian coverings and prove that under some symmetry assumptions, an essentially self-adjoint differential operator does not have eigenvalues of finite multiplicity and in particular, its spectrum is essential. Moreover, we show the analogous result for the Friedrichs extension of a symmetric and bounded from below differential operator. In the context of Riemannian coverings, we obtain the following immediate consequences.

Corollary 1.3 *Assume that D_2 is essentially self-adjoint. If the deck transformations group of the covering is infinite, then $\overline{D}_2$ does not have eigenvalues of finite multiplicity and in particular, $\sigma(\overline{D}_2) = \sigma_{\text{ess}}(\overline{D}_2)$.*

Corollary 1.4 *Assume that D_2 is symmetric and bounded from below, and denote by $D_2^{(F)}$ its Friedrichs extension. If the deck transformations group of the covering is infinite, then $D_2^{(F)}$ does not have eigenvalues of finite multyplicity and in particular, $\sigma(D_2^{(F)}) = \sigma_{\text{ess}}(D_2^{(F)})$.*

For Schrödinger operators, it follows that if the deck transformations group of the covering is infinite, then the spectrum of S_2 is essential, without any assumptions on the manifolds.

All the above results provide information about the spectra from properties of the covering (amenability or infinite deck transformations group). In the converse direction, Brooks [6] proved that if a normal Riemannian covering of a closed manifold (that is, compact without boundary) preserves the bottom of the spectrum of the Laplacian, then the covering is amenable. In this paper, we extend this result to Schrödinger operators and to not necessarily normal coverings. Recall that local isometries between complete Riemannian manifolds are (not necessarily normal) Riemannian coverings. In the following theorem, we denote by $h^{\text{ess}}(M)$ the supremum of the Cheeger's constants over complements of compact and smoothly bounded domains of M.

Theorem 1.5 *Let $p\colon M_2 \to M_1$ be a Riemannian covering with M_1 closed. Then the following are equivalent:*

(i) *p is infinite sheeted and amenable,*
(ii) *$\sigma(S_1) \subset \sigma_{\text{ess}}(S_2)$ for some/any Schrödinger operator S_1 on M_1 and its lift S_2,*
(iii) *$\lambda_0(S_1) = \lambda_0^{\text{ess}}(S_2)$ for some/any Schrödinger operator S_1 on M_1 and its lift S_2,*
(iv) *$h^{\text{ess}}(M_2) = 0$.*

It is worth to point out that Brooks proved his theorem in a quite complicated way, relying heavily on geometric measure theory. Our proof of the above theorem is significantly simpler and avoids the use of geometric measure theory. Moreover, this result yields that the assumption of amenability is natural in Theorems 1.1 and 1.2. Indeed, if we restrict ourselves to Schrödinger operators and coverings of closed manifolds, amenability is actually equivalent to the conclusions of these theorems.

Furthermore, Brooks [7], and more recently, Roblin and Tapie [22] proved that under some more general (but still quite restrictive) assumptions, if the bottom of the spectrum of the Laplacian is preserved, then the covering is amenable. In particular, these assumptions imply that the bottom of the spectrum of the Laplacian on M_1 is not in the essential spectrum. Moreover, Brooks [7] provided examples demonstrating that without these conditions, the bottom of the spectrum of the Laplacian may be preserved even if the covering is non-amenable. This suggests that under some assumptions on the geometry and the spectrum of the Laplacian on M_1, the bottom of the spectrum is preserved under a weaker assumption than amenability of the covering. In this direction, as an application of Theorem 1.1, we prove the following result.

Corollary 1.6 *Let $p\colon M_2 \to M_1$ be a Riemannian covering with M_1 complete. Let S_1 be a Schrödinger operator on M_1 with $\lambda_0(S_1) \in \sigma_{\text{ess}}(S_1)$, and S_2 its lift on M_2. If there exists a compact $K \subset M_1$, such that the image of the fundamental group of any connected component of $M_1 \smallsetminus K$ in $\pi_1(M_1)$ is amenable, then $\lambda_0(S_1) = \lambda_0(S_2)$.*

The paper is organized as follows: In Sect. 2, we give some preliminaries. In Sects. 3 and 4, we present the construction which is used in order to prove Theorem 1.2 and a more general result (Theorem 4.1) than Theorem 1.1. The proofs are given in Sect. 4, where we also present the analogous results for finite-sheeted coverings. In Sect. 5, we study manifolds with high symmetry and establish Corollaries 1.3 and 1.4. In Sect. 6, we present an alternative proof of Brooks' Theorem [6], extending it to not necessarily normal Riemannian coverings. In Sect. 7, we introduce the notion of renormalized Schrödinger operators, which is used to prove Theorem 1.5. Moreover, in this section we establish Corollary 1.6 and we present a simple example demonstrating that the behavior of the bottom of the spectrum of the connection Laplacian under a covering depends on the corresponding metric connection. Therefore, a main point in our results is the independence from the vector bundles, the connections, and the differential operators.

2 Preliminaries

We first recall some basic facts from functional analysis. For more details, see [17]. Let $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a closed (linear) operator on a separable Hilbert space $\mathcal{H}$ over a field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The *spectrum* of A is given by

$$\sigma(A) := \{\lambda \in \mathbb{F} : (A - \lambda)\colon \mathcal{D}(A) \to \mathcal{H} \text{ not bijective}\}.$$

The *essential spectrum* of A is defined as

$$\sigma_{\text{ess}}(A) := \{\lambda \in \mathbb{F} : (A - \lambda)\colon \mathcal{D}(A) \to \mathcal{H} \text{ not Fredholm}\}.$$

Recall that an operator is called *Fredholm* if its kernel is finite-dimensional and its range is closed and of finite codimension. The *discrete spectrum* of A is the complement of the essential spectrum in the spectrum of A, that is, $\sigma_d(A) := \sigma(A) \setminus \sigma_{\text{ess}}(A)$.

The *approximate point spectrum* of A, denoted by $\sigma_{\text{ap}}(A)$, is defined as the set of all $\lambda \in \mathbb{F}$, such that there exists $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}(A)$ with $\|v_k\| = 1$ and $(A - \lambda)v_k \to 0$ in $\mathcal{H}$. For $\lambda \in \mathbb{F}$, a *Weyl sequence* for A and λ is a sequence $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}(A)$, such that $\|v_k\| = 1$, $v_k \rightharpoonup 0$ and $(A - \lambda)v_k \to 0$ in $\mathcal{H}$, where "$\rightharpoonup$" denotes the weak convergence in $\mathcal{H}$. The *Weyl spectrum* of A, denoted by $\sigma_W(A)$, is the set of all $\lambda \in \mathbb{F}$, such that there exists a Weyl sequence for A and λ.

The following proposition is the characterization of the spectrum of a self-adjoint operator as the set of approximate eigenvalues and the well-known Weyl's criterion for the essential spectrum.

Proposition 2.1 *If A is self-adjoint, then $\sigma_{\text{ap}}(A) = \sigma(A)$, $\sigma_W(A) = \sigma_{\text{ess}}(A)$ and $\sigma_d(A)$ consists of isolated eigenvalues of A of finite multiplicity. In particular, $\sigma_{\text{ess}}(A)$ consists of eigenvalues of A of infinite multiplicity and accumulation points of $\sigma(A)$.*

Since we are interested in closures of operators, we need the following elementary lemma, characterizing the approximate point spectrum and the Weyl spectrum of the closure in terms of the initial operator.

Lemma 2.2 *Assume that A is the closure of an operator $B\colon \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ and consider $\lambda \in \mathbb{F}$. Then:*

(i) $\lambda \in \sigma_{\mathrm{ap}}(A)$ *if and only if there exists $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}(B)$, such that $\|v_k\| = 1$ and $(B-\lambda)v_k \to 0$ in $\mathcal{H}$,*
(ii) $\lambda \in \sigma_W(A)$ *if and only if there exists $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}(B)$, such that $\|v_k\| = 1$, $v_k \rightharpoonup 0$ and $(B-\lambda)v_k \to 0$ in $\mathcal{H}$.*

For an operator $B\colon \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ and $v \in \mathcal{D}(B) \smallsetminus \{0\}$, the *Rayleigh quotient* of v with respect to B is defined as

$$\mathcal{R}_B(v) := \frac{\langle Bv, v\rangle}{\|v\|^2}.$$

If B is symmetric, then $\mathcal{R}_B(v) \in \mathbb{R}$, for any $v \in \mathcal{D}(B) \smallsetminus \{0\}$, and B is *bounded from below* if the infimum of $\mathcal{R}_B(v)$, with $v \in \mathcal{D}(B) \smallsetminus \{0\}$, is finite. In this case, this infimum is called the *lower bound* of B.

The spectrum of a self-adjoint operator A is contained in $\mathbb{R}$ and the *bottom* (that is, the minimum) of the spectrum and the bottom of the essential spectrum of A are denoted by $\lambda_0(A)$ and $\lambda_0^{\mathrm{ess}}(A)$, respectively. The following characterization of the bottom of the spectrum is due to Rayleigh.

Proposition 2.3 *If $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint, then*

$$\lambda_0(A) = \inf_{v\in\mathcal{D}(A)\smallsetminus\{0\}} \mathcal{R}_A(v).$$

If, in addition, A is the closure of an operator $B\colon \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$, then the bottom of the spectrum of A is given by

$$\lambda_0(A) = \inf_{v\in\mathcal{D}(B)\smallsetminus\{0\}} \mathcal{R}_B(v).$$

Throughout the paper, manifolds are connected, with possibly empty, smooth, and not necessarily connected boundary, unless otherwise stated. Let $p\colon M_2 \to M_1$ be a Riemannian covering of m-dimensional manifolds, $E_1 \to M_1$ a Riemannian or Hermitian vector bundle of rank ℓ, and $D_1\colon \Gamma(E_1) \to \Gamma(E_1)$ a differential operator of order d. Consider the pullback bundle $E_2 := p^*E_1$ on M_2, $y \in M_2$ and set $x := p(y)$. Let U_2 be an open neighborhood of y, such that the restriction $p|_{U_2}$ is an isometry onto its image U_1. The *lift* $D_2\colon \Gamma(E_2) \to \Gamma(E_2)$ of D_1 is the differential operator defined by

$$D_2\eta(z) := (p|_{U_2})^*(D_1((p|_{U_2}^{-1})^*\eta)(p(z))),$$

for any $\eta \in \Gamma(E_2)$ and $z \in U_2$. Without loss of generality, we may assume that U_1 is contained in a coordinate neighborhood and there exists a local trivialization $E_1|_{U_1} \to U_1 \times \mathbb{F}^\ell$. With respect to this coordinate system and trivialization, D_1 is expressed as

$$D_1 = \sum_{|\alpha| \le d} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{1}$$

where A^α are smooth maps defined on U_1, with values $\ell \times \ell$ matrices with entries in $\mathbb{F}$. Then, with respect to the lifted coordinate system and the corresponding trivialization $E_2|_{U_2} \to U_2 \times \mathbb{F}^\ell$, D_2 has the local expression

$$D_2 = \sum_{|\alpha| \le d} (A^\alpha \circ p) \frac{\partial^{|\alpha|}}{\partial y^\alpha}.$$

Let M be a Riemannian manifold and $E \to M$ a Riemannian or Hermitian vector bundle endowed with a connection ∇. Assume that M has non-empty boundary, denote by n the inward pointing normal to ∂M, and extend n locally as the velocity of unit speed geodesics normal to the boundary. For $\eta \in \Gamma_c(E)$ and $k \in \mathbb{N}$, consider the following sections defined in a neighborhood of the boundary

$$\nabla_n^{(k)} \eta := \nabla_n (\nabla_n^{(k-1)} \eta), \text{ where } \nabla_n^{(0)} \eta := \eta.$$

Similarly, for $f \in C_c^\infty(M)$ and $k \in \mathbb{N}$, consider the following functions defined in a neighborhood of the boundary

$$n^{(k)}(f) := n(n^{(k-1)} f), \text{ where } n^{(0)} f := f.$$

Lemma 2.4 *Let M be a Riemannian manifold, $E \to M$ a Riemannian or Hermitian vector bundle endowed with a connection ∇ and $D \colon \Gamma(E) \to \Gamma(E)$ a differential operator. If M has empty boundary, set $\mathcal{D}(D) := \Gamma_c(E)$. If M has non-empty boundary, consider $\nu \in \mathbb{N}$, and for $l = 1, \dots, \nu$, let $k_l \in \mathbb{N}$ and $a_{j,l}$ be real or complex-valued functions (depending on whether E is Riemannian or Hermitian) defined on ∂M, $j = 0, \dots, k_l$. Let n be the inward pointing normal to ∂M and consider*

$$\mathcal{D}(D) := \{\eta \in \Gamma_c(E) : \sum_{j=0}^{k_l} a_{j,l} \nabla_n^{(j)} \eta = 0 \text{ on } \partial M,\ l = 1, \dots, \nu\}.$$

Let μ be a measure on M expressed via a positive smooth density with respect to the volume element of M; that is, there exists a positive $h \in C^\infty(M)$, such that $d\mu = h d\,\mathrm{Vol}$. Then the operator $D \colon \mathcal{D}(D) \subset L^2(E, \mu) \to L^2(E, \mu)$ is closable.

Proof Consider the formal adjoint D^{ad} of D in $L^2(E)$, defined by

$$\langle D\eta, \theta \rangle_{L^2(E)} = \langle \eta, D^{\mathrm{ad}} \theta \rangle_{L^2(E)},$$

for all $\eta \in \mathcal{D}(D)$ and $\theta \in \Gamma_{cc}(E)$, where $\Gamma_{cc}(E)$ is the space of smooth sections, compactly supported in the interior of M. Evidently, for $\eta \in \mathcal{D}(D)$ and $\theta \in \Gamma_{cc}(E)$, we have

$$\langle D\eta, \theta\rangle_{L^2(E,\mu)} = \langle \eta, D'\theta\rangle_{L^2(E,\mu)}, \text{ where } D'\theta := \frac{1}{h}D^{\mathrm{ad}}(h\theta).$$

It is clear that the operator $D' \colon \Gamma_{cc}(E) \subset L^2(E,\mu) \to L^2(E,\mu)$ is densely defined and its adjoint satisfies $D \subset (D')^*$. Since the adjoint is closed, it follows that D is closable. □

A *Schrödinger operator* on a possibly non-connected Riemannian manifold M is an operator of the form $S := \Delta + V$, where Δ is the Laplacian and $V \colon M \to \mathbb{R}$ is smooth and bounded from below. If M is complete and without boundary, then S is essentially self-adjoint on $C_c^\infty(M)$, that is, the closure of $S \colon C_c^\infty(M) \subset L^2(M) \to L^2(M)$ is self-adjoint. If M is complete with non-empty boundary, then S is essentially self-adjoint on $\{f \in C_c^\infty(M) : f = 0 \text{ on } \partial M\}$. If M is non-complete, then S restricted to the above domain does not have a unique self-adjoint extension, and we are interested in the Friedrichs extension of S. By abuse of notation, the spectrum and the essential spectrum of the above- described self-adjoint operator are denoted by $\sigma(S)$ and $\sigma_{\mathrm{ess}}(S)$, respectively, and their bottoms by $\lambda_0(S)$ and $\lambda_0^{\mathrm{ess}}(S)$, respectively. These sets and quantities for the Laplacian on M are denoted by $\sigma(M)$, $\sigma_{\mathrm{ess}}(M)$ and $\lambda_0(M)$, $\lambda_0^{\mathrm{ess}}(M)$, respectively.

Let $p \colon M_2 \to M_1$ be a Riemannian covering of complete manifolds without boundary. For $x \in M_1$ and $y \in p^{-1}(x)$, the *fundamental domain* of p centered at y is defined by

$$D_y := \{z \in M_2 : d(z, y) \le d(z, y') \text{ for all } y' \in p^{-1}(x)\}.$$

Some basic properties of these fundamental domains are presented in [2]. It is clear that D_y is closed and M_2 is the union of D_y, with $y \in p^{-1}(x)$. It is worth to point out that the intersection of different fundamental domains is of measure zero. Moreover, ∂D_y and the cut locus $\mathrm{Cut}(x)$ of x are of measure zero and $p \colon D_y \smallsetminus \partial D_y \to M_1 \smallsetminus C_0$ is an isometry, where C_0 is a subset of $\mathrm{Cut}(x)$. The following two lemmas are proved in [2]. The lemma after these is proved similarly to Lemma 2.6. In these lemmas and in the sequel, we denote open and closed balls by B and C, respectively.

Lemma 2.5 *If $K \subset B(x,r)$, then $p^{-1}(K) \cap D_y \subset B(y,r)$. In particular, if K is compact, then $p^{-1}(K) \cap D_y$ is compact.*

Lemma 2.6 *For any $r > 0$, there exists $N(r) \in \mathbb{N}$, such that any $z \in M_2$ is contained in at most $N(r)$ of the balls $C(y,r)$, with $y \in p^{-1}(x)$.*

Lemma 2.7 *Consider the universal coverings $p_i \colon \tilde{M} \to M_i$, $i = 1, 2$. For any r, $r_0 > 0$, there exists $\tilde{N}(r, r_0) \in \mathbb{N}$, such that*

$$\#\{w \in p_2^{-1}(z) : B(w, r_0) \cap C(u, r) \neq \emptyset\} \le \tilde{N}(r, r_0),$$

for all $u \in p_1^{-1}(x)$ and $z \in M_2$.

It is worth to point out that the quantities $N(r)$ and $\tilde{N}(r, r_0)$ in the above lemmas depend on the choice of $x \in M_1$.

Finally, we recall the notions of amenable right action and amenable covering. For more details on amenable left actions, which are completely analogous to right actions, see [4, Sect. 2]. A right action of a countable group Γ on a countable set X is called *amenable* if there exists a Γ-invariant mean on $L^\infty(X)$. The following characterization is due to Følner.

Proposition 2.8 *The right action of a countable group Γ on a non-empty, countable set X is amenable if and only if for any finite $G \subset \Gamma$ and $\varepsilon > 0$, there exists a non-empty, finite $F \subset X$, such that*

$$\#(F \setminus Fg) < \varepsilon \#(F),$$

for all $g \in G$. Such a set F is called a Følner set *for G and ε.*

A countable group Γ is called *amenable* if the right action of Γ on itself is amenable. In this case, the right action of Γ on any countable set X is amenable. Moreover, it is clear that any right action on a non-empty, finite set is amenable.

A Riemannian covering $p \colon M_2 \to M_1$ is called *amenable* if the right action of $\pi_1(M_1)$ on $\pi_1(M_2)\backslash\pi_1(M_1)$ (that is, the set of right cosets of $\pi_1(M_2)$ in $\pi_1(M_1)$, when considered as deck transformations groups of the universal coverings) is amenable. Clearly, a normal covering is amenable if and only if its deck transformations group is amenable. Furthermore, finite-sheeted coverings are amenable.

The following criteria for amenability of groups are immediate consequences of the definition and Proposition 2.8.

Corollary 2.9 *Any finitely generated group of subexponential growth is amenable.*

Corollary 2.10 *A countable group Γ is amenable if and only if any finitely generated subgroup of Γ is amenable*

Corollary 2.11 *Any countable solvable group is amenable.*

Proof From Corollaries 2.9 and 2.10, it follows that any countable abelian group is amenable. From the definition, it is clear that an extension of an amenable group by an amenable group is also amenable. □

3 Coverings of Manifolds with Boundary

The aim of this section is to show the following proposition, according to which, any Riemannian covering of manifolds with boundary can be "extended" to a Riemannian covering of manifolds without boundary.

Proposition 3.1 *Let M be a Riemannian manifold with non-empty boundary. Then there exists a Riemannian manifold N of the same dimension, without boundary and an isometric embedding $i \colon M \to N$, such that, after identifying M with $i(M)$, any Riemannian covering $p \colon M' \to M$ can be extended to a Riemannian covering $p \colon N' \to N$.*

In order to prove this proposition, we need to establish some auxiliary lemmas.

Lemma 3.2 *Let M be a Riemannian manifold with non-empty boundary. Then there exists a Riemannian manifold N of the same dimension, without boundary, an isometric embedding $i : M \to N$, and a strong deformation retraction of N onto $i(M)$.*

Proof Consider the space $\partial M \times [0, +\infty)$ and the map $\Psi : \partial M \to \partial M \times [0, +\infty)$, defined by $\Psi(x) := (x, 0)$. Then $N := M \cup_\Psi (\partial M \times [0, +\infty))$ is a smooth manifold and there exists a smooth embedding $i : M \to N$. Therefore, M can be identified with $i(M)$. Since M is connected, so is N, and there exists a strong deformation retraction of N onto M, obtained by considering $F_t(x, r) := (x, (1 - t)r)$ in the glued ends $\partial M \times [0, +\infty)$.

It remains to extend the Riemannian metric of M to a Riemannian metric of N. Any $x \in \partial M$ has an open neighborhood U_x in N, such that there exists a smooth frame field $\{e_1, \dots, e_m\}$ in U_x, where m is the dimension of the manifolds. Let $g_{jk} := \langle e_j, e_k\rangle$, $1 \leq j, k \leq m$, be the components of the Riemannian metric of M. Since they are smooth up to the boundary of M, they can be extended smoothly to a neighborhood of x. After passing to a smaller neighborhood of x if needed, we may assume that g_{jk}'s are smooth in U_x and their matrix is symmetric and positive definite at any point of U_x. Hence, they express a Riemannian metric in U_x.

Clearly, ∂M can be covered with such neighborhoods U_x. Consider the interior of M as an open subset of N endowed with its Riemannian metric and $N \smallsetminus M$ with an arbitrary Riemannian metric. Combining these Riemannian metrics via a partition of unity subordinate to this open cover of N, gives rise to a Riemannian metric of N, which is an extension of the Riemannian metric of M. □

Lemma 3.3 *Let M be a Riemannian manifold with non-empty boundary. Consider N as in the previous lemma and identify M with $i(M)$. Let $q : \tilde{N} \to N$ be the universal covering of N. Then the restriction $q : q^{-1}(M) \to M$ is the universal covering of M.*

Proof Since there exists a strong deformation retraction of N onto M, every loop in N can be homotoped to a loop in M. This implies that for any $x \in M$ and $y_1, y_2 \in q^{-1}(x)$, there exists a path in $q^{-1}(M)$ from y_1 to y_2. Since M is connected, it follows that so is $q^{-1}(M)$ and the restriction $q : q^{-1}(M) \to M$ is a covering of (connected) manifolds.

Let $r_M : N \to M$ be a retraction. Then the map $r_M \circ q : \tilde{N} \to M$ is continuous and $r_M \circ q = q$ in $q^{-1}(M)$. From the Lifting Theorem, it follows that $r_M \circ q$ has a continuous lift $\tilde{r}_M : \tilde{N} \to q^{-1}(M)$, with $\tilde{r}_M(y_0) = y_0$, for some $y_0 \in q^{-1}(M)$. Since $\tilde{r}_M|_{q^{-1}(M)}$ is a deck transformation of the covering $q : q^{-1}(M) \to M$, it follows that $\tilde{r}_M : \tilde{N} \to q^{-1}(M)$ is a retraction. Since $\tilde{N}$ is simply connected, this yields that so is $q^{-1}(M)$. □

Proof of Proposition 3.1: Consider N and $q : \tilde{N} \to N$ as in the above lemmas, identify M with $i(M)$ and set $\tilde{M} := q^{-1}(M)$. Denote by Γ_N and Γ_M the deck transformations groups of $q : \tilde{N} \to N$ and $q : \tilde{M} \to M$, respectively. It is clear that for $g \in \Gamma_N$, we have $g|_{\tilde{M}} \in \Gamma_M$, and any $\gamma \in \Gamma_M$ has a unique extension $\gamma' \in \Gamma_N$. For any Riemannian covering $p : M' \to M$, there exists a subgroup $\Gamma \subset \Gamma_M$, such that $M' = \tilde{M}/\Gamma$. For $\Gamma' := \{\gamma' \in \Gamma_N : \gamma \in \Gamma\}$ and $N' := \tilde{N}/\Gamma'$, the inclusion $\tilde{M} \hookrightarrow \tilde{N}$ descends to an isometric embedding $M' \to N'$, which completes the proof. □

4 Spectrum of Operators Under Amenable Coverings

Throughout this section, we work in the following context, which is briefly described in the Introduction.

Let $p\colon M_2 \to M_1$ be a Riemannian covering, $E_1 \to M_1$ a Riemannian or Hermitian vector bundle endowed with a connection ∇, and $D_1\colon \Gamma(E_1) \to \Gamma(E_1)$ a differential operator on E_1. Let $E_2 \to M_2$ be the pullback bundle, endowed with the corresponding metric and connection ∇, and $D_2\colon \Gamma(E_2) \to \Gamma(E_2)$ the lift of D_1. If M_1 has empty boundary, we consider the space of compactly supported smooth sections of E_i as the domain of D_i, that is, $\mathcal{D}(D_i) := \Gamma_c(E_i)$, $i = 1, 2$.

If M_1 has non-empty boundary, consider $\nu \in \mathbb{N}$, and for $l = 1, \dots, \nu$, let $k_l \in \mathbb{N}$ and $a^{(1)}_{j,l}$ be real or complex-valued functions (depending on whether E_1 is Riemannian or Hermitian) defined on ∂M_1, $j = 0, \dots, k_l$. It is worth to point out that we do not impose any assumptions on the functions $a^{(1)}_{j,l}$. Let n_i be the inward pointing normal to ∂M_i, set $a^{(2)}_{j,l} := a^{(1)}_{j,l} \circ p$, and consider

$$\mathcal{D}(D_i) := \{\eta \in \Gamma_c(E_i) : \sum_{j=0}^{k_l} a^{(i)}_{j,l} \nabla^{(j)}_{n_i} \eta = 0 \text{ on } \partial M_i,\ l = 1, \dots, \nu\},\ i = 1, 2.$$

Let μ_1 be a measure on M_1 expressed via a positive smooth density with respect to the volume element of M_1; that is, there exists a positive $h \in C^\infty(M_1)$, such that $d\mu_1 = hd\,\mathrm{Vol}$. Let μ_2 be the corresponding measure on M_2, i.e., $d\mu_2 = (h \circ p)d\,\mathrm{Vol}$. We consider the operator D_i as a densely defined operator

$$D_i\colon \mathcal{D}(D_i) \subset L^2(E_i, \mu_i) \to L^2(E_i, \mu_i), \tag{2}$$

$i = 1, 2$. When we refer to closability, symmetry, or essential self-adjointness of D_i, we consider the operator as in (2), $i = 1, 2$. From Lemma 2.4, the operator D_i is closable and we denote by $\overline{D}_i$ its closure, $i = 1, 2$.

Our aim in this section is to prove Theorem 1.2 and the following more general version of Theorem 1.1.

Theorem 4.1 *Let D_2' be a closed extension of D_2. If the covering is infinite sheeted and amenable, then $\sigma_{\mathrm{ap}}(\overline{D}_1) \subset \sigma_W(D_2')$.*

For convenience of the reader, we briefly describe the outline of the proof of this theorem. Given $\eta \in \mathcal{D}(D_1)$ with $\|\eta\|_{L^2(E_1,\mu_1)} = 1$, $\lambda \in \mathbb{F}$, and $\varepsilon > 0$, we want to prove that there exists $\zeta \in \mathcal{D}(D_2)$, such that

$$\frac{\|(D_2 - \lambda)\zeta\|_{L^2(E_2,\mu_2)}}{\|\zeta\|_{L^2(E_2,\mu_2)}} \leq \|(D_1 - \lambda)\eta\|_{L^2(E_1,\mu_1)} + \varepsilon. \tag{3}$$

First consider the case where the manifolds are complete without boundary. Then there exists $r > 0$, such that $\operatorname{supp}\eta \subset B(x, r)$, for some $x \in M_1$. For $y \in p^{-1}(x)$, we consider a function $\varphi_y \in C^\infty_c(M_2)$ centered at y, whose profile is essentially independent from y. For a finite subset P of $p^{-1}(x)$, we consider the test section $\chi\theta \in \mathcal{D}(D_2)$, where θ is the lift of η and $\chi = \sum_{y \in P} \varphi_y$. For such a section we establish pointwise estimates of the form $\|(D_2 - \lambda)(\chi\theta)\| \leq C$ in M_2, where C is a constant independent from P.

Consider $p^{-1}(x)$ as a discrete graph, where two points are connected if their distance is less than $2r+2$. For a point $y \in P \setminus \partial P$, it follows that $\chi\theta = \theta$ in $B(y, r)$. Moreover, $\chi\theta$ is supported in the union of the balls $B(y, r)$, with y in P or y connected to some point in ∂P. From Lemma 2.6, there are at most $N(2r + 2)\#(\partial P)$ many $y \in p^{-1}(x)$ that are connected to some point in ∂P.

Since the covering is amenable, it follows that there exist finite subsets P of $p^{-1}(x)$ with arbitrarily small isoperimetric ratio. Hence, the corresponding sections $\chi\theta$ coincide with θ in a relatively large part of their supports, while in the rest of their supports they satisfy the aforementioned estimates, which are independent from P. Therefore, the corresponding test sections $\chi\theta$ satisfy (3). Moreover, since p is infinite sheeted, given a compact $K \subset M_2$, we may choose a finite $P \subset p^{-1}(x)$, so that $\chi\theta$ satisfies (3) and $\operatorname{supp}(\chi\theta)$ does not intersect K. This completes the proof of the theorem in case the manifolds are complete without boundary.

If the manifolds are non-complete without boundary, then we consider conformal Riemannian metrics that make the manifolds complete, and exploit the method described above.

If the manifolds have non-empty boundary, then we extend the given Riemannian covering to a Riemannian covering of manifolds without boundary, according to Proposition 3.1. Then we consider conformal Riemannian metrics that make the manifolds complete and exploit a slight variation of the above method. In this case, it is important to require that this new Riemannian metric on M_1 coincides with the original Riemannian metric in a compact neighborhood of $\operatorname{supp} \eta$, so that this construction respects the imposed boundary conditions; that is, if $\eta \in \mathcal{D}(D_1)$, then $\chi\theta \in \mathcal{D}(D_2)$, for any finite subset P of $p^{-1}(x)$.

4.1 Partition of Unity

In this subsection, we construct a special partition of unity, which is used in the sequel to obtain cut-off functions on M_2.

Let K_0 be a compact subset of M_1. Consider the universal coverings $p_i : \tilde{M} \to M_i$ and denote by Γ_i the deck transformations group of p_i, $i = 1, 2$. If M_1 has empty boundary, consider a Riemannian metric $\mathfrak{h}_1$, conformal to the original metric $\mathfrak{g}_1$, such that $(M_1, \mathfrak{h}_1)$ is complete. Such a metric exists according to [21].

If M_1 has non-empty boundary, let n_i be the inward pointing normal to ∂M_i, $i = 1, 2$, and $\tilde{n}$ the inward pointing normal to $\partial \tilde{M}$. Consider a Riemannian manifold $(N_1, \mathfrak{g}_1)$ containing M_1, as in Proposition 3.1, and a Riemannian metric $\mathfrak{h}_1$, conformal to the original metric $\mathfrak{g}_1$, such that $(N_1, \mathfrak{h}_1)$ is complete. Since K_0 is compact, we may assume that $\mathfrak{h}_1$ coincides with $\mathfrak{g}_1$ in a compact neighborhood of K_0. From Proposition 3.1, it follows that the Riemannian covering $p \colon M_2 \to M_1$ can be extended to a Riemannian covering $p \colon N_2 \to N_1$. Moreover, according to Lemma 3.3, $\tilde{M}$ can be identified with a domain of the simply connected covering space $\tilde{N}$ of N_1.

From now on, geodesics are considered with respect to $\mathfrak{h}_1$ and its lifts. If M_1 has empty boundary, distances are considered with respect to $\mathfrak{h}_1$ or its lifts. In this case, we denote the open (respectively, closed) ball of radius r around a point z by $B(z,r)$ (respectively, $C(z,r)$). If M_1 has non-empty boundary, the distance between two points is considered in $(N_1, \mathfrak{h}_1)$ or its corresponding covering space. In this case, $B(z,r)$ and $C(z,r)$ stand for the corresponding balls in M_1, M_2, or $\tilde{M}$. For example, for $u \in \tilde{M}$ and $r > 0$, we have

$$B(u,r) = \{z \in \tilde{M} : d(z,u) < r\},$$

where $d(\cdot,\cdot)$ is the distance function of $\tilde{N}$ induced by the lift of $\mathfrak{h}_1$.

Fix $x \in M_1$, $u \in p_1^{-1}(x)$, and $r > 0$, such that $K_0 \subset B(x,r)$. If M_1 has non-empty boundary, consider r large enough, so that $B(u,r) \cap \partial\tilde{M} \neq \emptyset$.

Lemma 4.2 *There exists a non-negative $\psi_u \in C_c^\infty(\tilde{M})$, with* $\operatorname{supp}\psi_u \subset C(u, r+1)$ *and $\psi_u = 1$ in $C(u, r+1/2)$. Moreover, if M_1 has non-empty boundary, ψ_u can be chosen such that $\tilde{n}^{(i)}\psi_u = 0$ on $\partial\tilde{M} \cap p_1^{-1}(K_0)$, for any $i \in \mathbb{N}$.*

Proof It is clear that there exists a non-negative $\psi_u' \in C_c^\infty(\tilde{M})$ with $\operatorname{supp}\psi_u' \subset C(u, r+1)$ and $\psi_u' = 1$ in $C(u, r+1/2)$. If M_1 has empty boundary, this is the desired function. Otherwise, let $K := \partial\tilde{M} \cap C(u, r+2)$ and denote by n the inward pointing normal to $\partial\tilde{M}$ with respect to the lift of $\mathfrak{h}_1$. Since K is compact, there exists $\varepsilon > 0$, with $\varepsilon < 1/8$, such that the map $\Phi\colon K \times [0, 2\varepsilon) \to \tilde{M}$, defined by $\Phi(z,t) := \exp_z(tn)$ is a diffeomorphism onto its image K_ε. Let $K_1 := \partial\tilde{M} \cap C(u, r + 1/2 + 2\varepsilon)$ and $K_2 := \partial\tilde{M} \cap C(u, r+1-2\varepsilon)$. Clearly, there exists a non-negative $\tau \in C_c^\infty(\partial\tilde{M})$, with $\operatorname{supp}\tau \subset K_2$ and $\tau = 1$ in K_1. Extend it to τ' in K_ε by $\tau'(\Phi(z,t)) := \tau(z)$, for all $(z,t) \in K \times [0, 2\varepsilon)$. Consider a smooth $f\colon \mathbb{R} \to \mathbb{R}$, with $0 \le f \le 1$, $f(t) = 1$ for $t \le \varepsilon$, and $f(t) = 0$ for $t \ge 3\varepsilon/2$, and the function v defined in K_ε by $v(\Phi(z,t)) = f(t)$, for all $(z,t) \in K \times [0, 2\varepsilon)$. Extend v by zero outside K_ε and set

$$\psi_u := v\tau' + (1-v)\psi_u'.$$

Since $\operatorname{supp}(v\tau') \subset C(u, r+1)$, $\operatorname{supp}\psi_u' \subset C(u, r+1)$, it follows that $\operatorname{supp}\psi_u \subset C(u, r+1)$. Since $\varepsilon < 1/8$, the points where v is not smooth are not in $C(u, r+1)$, which yields that $\psi_u \in C_c^\infty(\tilde{M})$. Since $\psi_u' = 1$ in $C(u, r+1/2)$ and $\tau' = 1$ in $C(u, r+1/2) \cap K_\varepsilon$, it follows that $\psi_u = 1$ in $C(u, r+1/2)$. In $\Phi(K \times [0, \varepsilon))$, which is a neighborhood of $\operatorname{supp}\psi_u \cap \partial\tilde{M}$, we have $\psi_u = \tau'$. In particular, in a neighborhood of the boundary, ψ_u is constant along geodesics (with respect to the lift of $\mathfrak{h}_1$) that are normal to the boundary. This yields that $n^{(i)}\psi_u = 0$ on $\partial\tilde{M}$, for any $i \in \mathbb{N}$. Since $\mathfrak{h}_1$ coincides with $\mathfrak{g}_1$ in a compact neighborhood of K_0, it follows that $\tilde{n}^{(i)}\psi_u = 0$ on $\partial\tilde{M} \cap p_1^{-1}(K_0)$, for any $i \in \mathbb{N}$. □

Let ψ_u be a function as in the above lemma and for any $y \in p^{-1}(x)$, consider $u(y) \in p_2^{-1}(y)$ and $g(y) \in \Gamma_1$, such that $u(y) = g(y)u$. Consider the functions $\psi_{u(y)} := \psi_u \circ g(y)^{-1}$ in $\tilde{M}$ and ψ_y in M_2 defined by

$$\psi_y(z) := \sum_{w \in p_2^{-1}(z)} \psi_{u(y)}(w). \tag{4}$$

It is clear that $\psi_y \in C_c^\infty(M_2)$, $\operatorname{supp} \psi_y \subset C(y, r+1)$, and $\psi_y \geq 1$ in $C(y, r+1/2)$, for any $y \in p^{-1}(x)$. Moreover, if M_1 has non-empty boundary, then $n_2^{(i)} \psi_y = 0$ on $\partial M_2 \cap p^{-1}(K_0)$, for any $y \in p^{-1}(x)$ and $i \in \mathbb{N}$. From Lemma 2.6, there exists $N(r+2) \in \mathbb{N}$, such that for any $z \in M_2$, the ball $B(z, 1)$ intersects at most $N(r+2)$ of the supports of ψ_y, with $y \in p^{-1}(x)$. Therefore, $\sum_{y \in p^{-1}(x)} \psi_y$ is locally a finite sum and hence, well defined and smooth.

If M_1 is compact, we choose r large enough, so that $\sum_{y \in p^{-1}(x)} \psi_y \geq 1$ in M_2. In this case, set $\psi_1 := 0$ in M_2. If M_1 is non-compact, consider $f_1 \in C_c^\infty(M_1)$ with $0 \leq f_1 \leq 1$, $f_1 = 1$ in $C(x, r)$, $\operatorname{supp} f_1 \subset B(x, r+1/2)$, and let ψ_1 be the lift of $1 - f_1$ on M_2. Then $\psi_1 \in C^\infty(M_2)$, $\psi_1 \geq 0$ in M_2 and $\psi_1 = 0$ in $C(y, r)$, for all $y \in p^{-1}(x)$. Evidently, $\psi_1 + \sum_{y \in p^{-1}(x)} \psi_y \geq 1$ in M_2.

Consider the smooth partition of unity consisting of the functions

$$\varphi_1 := \frac{\psi_1}{\psi_1 + \sum_{y' \in p^{-1}(x)} \psi_{y'}} \text{ and } \varphi_y := \frac{\psi_y}{\psi_1 + \sum_{y' \in p^{-1}(x)} \psi_{y'}}, \tag{5}$$

with $y \in p^{-1}(x)$.

Remark 4.3 It is evident that $\operatorname{supp} \varphi_1 = \operatorname{supp} \psi_1$, and for any $y \in p^{-1}(x)$, we have $\operatorname{supp} \varphi_y = \operatorname{supp} \psi_y$, $\sum_{y' \in p^{-1}(x)} \varphi_{y'} = 1$ in $C(y, r)$, and $\varphi_y > 0$ in $C(y, r+1/2)$. Since $K_0 \subset B(x, r)$, it follows that $\psi_1 = 0$ in a neighborhood of $p^{-1}(K_0)$. If M_1 has non-empty boundary, then $n_2^{(i)} \psi_y = 0$ on $\partial M_2 \cap p^{-1}(K_0)$, for any $y \in p^{-1}(x)$ and $i \in \mathbb{N}$. This yields that $n_2^{(i)} \varphi_y = 0$ on $\partial M_2 \cap p^{-1}(K_0)$, for all $y \in p^{-1}(x)$ and $i \in \mathbb{N}$.

Let $\eta \in \mathcal{D}(D_1)$ and $\theta \in \Gamma(E_2)$ be the lift of η. Fix $x \in M_1$, $u \in p_1^{-1}(x)$, and $r > 0$, such that $K_0 := \operatorname{supp} \eta \subset B(x, r)$. If M_1 has non-empty boundary, we choose r large enough, so that $B(u, r) \cap \partial \tilde{M} \neq \emptyset$. Consider a partition of unity associated with K_0, x, u and r as in (5) and for a finite $P \subset p^{-1}(x)$, set $\chi := \sum_{y \in P} \varphi_y$.

Remark 4.4 Since P is finite, it follows that $\chi \in C_c^\infty(M_2)$ and $\chi\theta \in \Gamma_c(E_2)$. If M_1 has empty boundary, this yields that $\chi\theta \in \mathcal{D}(D_2)$. If M_1 has non-empty boundary, from Remark 4.3, we have that $n_2^{(i)} \varphi_y = 0$ on $\partial M_2 \cap \operatorname{supp} \theta$, for any $y \in p^{-1}(x)$ and $i \in \mathbb{N}$. In particular, if η satisfies a boundary condition of the form

$$\sum_{j=0}^{k} a_j \nabla_{n_1}^{(j)} \eta = 0 \text{ on } \partial M_1,$$

then for $\chi\theta$ we have

$$\sum_{j=0}^{k} (a_j \circ p) \nabla_{n_2}^{(j)} (\chi\theta) = \sum_{y \in P} \sum_{j=0}^{k} \varphi_y (a_j \circ p) \nabla_{n_2}^{(j)} \theta = 0 \text{ on } \partial M_2.$$

Hence, $\chi\theta$ satisfies analogous boundary conditions to η. Since $\eta \in \mathcal{D}(D_1)$, it follows that $\chi\theta \in \mathcal{D}(D_2)$.

Proposition 4.5 *There exists a constant C, independent from P, such that for any $z \in M_2$, we have $\|D_2(\chi\theta)(z)\| \leq C$.*

It is worth to point out that the constant in this proposition, as well as the estimates in the sequel, depend on various choices we made in this construction. For instance, they depend on the conformal Riemannian metric $\mathfrak{h}_1$, on r and on the choice of ψ_u. The main point of this proposition is that there is no dependence on P.

Proof Consider $\delta > 0$, such that for any $x' \in C(x, r+1)$, the ball $B(x', 2\delta)$ is evenly covered and contained in a coordinate neighborhood, and $E_1|_{B(x',2\delta)}$ is trivial. Let $x_1, \dots, x_k \in C(x, r+1)$, such that the balls $B(x_i, \delta)$, with $1 \leq i \leq k$, cover $C(x, r+1)$. In any ball $B(x_i, 2\delta)$, D_1 has a local expression of the form (1), with A^α smooth. This yields that in $B(x_i, \delta)$, D_1 is expressed in the form (1), with A^α smooth and bounded. For any such ball, we fix a coordinate system (which can be extended to the corresponding ball of radius 2δ) and a trivialization. Since $C(x, r+1)$ is covered by finitely many such balls, it follows that there exists $C_1 > 0$, such that in any of these balls, we have $\|A^\alpha\| \leq C_1$, for all multi-indices α of absolute value less or equal to the order d of D_1.

Since η is smooth and compactly supported in $B(x, r)$, there exists $C_2 > 0$, such that in any of these balls, denoting by $(\eta^{(1)}, \dots, \eta^{(\ell)})$ the local expression of η, we have that

$$\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (\eta^{(1)}, \dots, \eta^{(\ell)}) \right\| \leq C_2,$$

for all multi-indices α of absolute value less or equal to d, that is, we have *uniform estimates up to order d for η (with respect to this system of trivializations).* We lift these balls and the corresponding coordinate systems and trivializations to M_2 and $\tilde{M}$. Similarly, if $\psi_1 \neq 0$, we obtain uniform estimates up to order d for f_1, which yield uniform estimates up to order d for ψ_1 (with respect to the lifted system on M_2).

Since ψ_u is smooth and compactly supported in $C(u, r+1)$, which intersects finitely many balls of the lifted system on $\tilde{M}$, there exist uniform estimates up to order d for ψ_u. Since $\psi_{u(y)}$ is a composition of ψ_u with an element of Γ_1, we obtain the same uniform estimates up to order d for $\psi_{u(y)}$, for all $u(y)$. Recall the definition of ψ_y in (4). Consider a ball $B(z', \delta)$ of the lifted system on M_2, which intersects supp ψ_y, and the corresponding coordinate system. It is clear that for any $w \in p_2^{-1}(z')$, the lifted system on $\tilde{M}$ contains the ball $B(w, \delta)$ and the corresponding coordinate system. From Lemma 2.7, there exists $\tilde{N}(r+1, \delta) \in \mathbb{N}$, independent from y and z', such that at most $\tilde{N}(r+1, \delta)$ such balls intersect the support of $\psi_{u(y)}$. Since we have uniform estimates up to order d for $\psi_{u(y)}$, which are independent from $y \in p^{-1}(x)$, we obtain the same uniform estimates up to order d for ψ_y, for all $y \in p^{-1}(x)$. From Lemma 2.6, it follows that at most $N(r+1+\delta)$ of the supports of ψ_y, with $y \in p^{-1}(x)$, intersect the open ball $B(z, \delta)$, for any $z \in M_2$. This yields that there exist uniform estimates up to order d for $\psi_1 + \sum_{y \in p^{-1}(x)} \psi_y$.

Recall the definition of φ_y in (5). Since the denominator is greater or equal to 1 and we have uniform estimates (independent from y) up to order d for the numerator and the denominator, we obtain the same uniform estimates up to order d for φ_y, for all $y \in p^{-1}(x)$. From Lemma 2.6, at most $N(r+1+\delta)$ of the supports of φ_y, with $y \in p^{-1}(x)$, intersect the ball $B(z,\delta)$, for any $z \in M_2$. Therefore, we obtain uniform estimates up to order d for χ, which are independent from P.

Clearly, for $z \in \operatorname{supp}(\chi\theta)$, we have that $z \in B(y,r)$, for some $y \in p^{-1}(x)$, and in particular, z is contained in a ball of the system. With respect to the corresponding coordinate system and trivialization, denoting by $(\theta^{(1)},\dots,\theta^{(\ell)})$ the local expression of θ, we have

$$\begin{aligned}\|D_2(\chi\theta)(z)\| &= \left\| \sum_{|\alpha|\le d} (A^\alpha \circ p)(z) \frac{\partial^{|\alpha|}}{\partial y^\alpha}(\chi(\theta^{(1)},\dots,\theta^{(\ell)}))(z) \right\| \\ &\le \sum_{|\alpha|\le d} C_1 \left\| \frac{\partial^{|\alpha|}}{\partial y^\alpha}(\chi(\theta^{(1)},\dots,\theta^{(\ell)}))(z) \right\| \\ &\le C_1 C_2 C_3 C(d,\ell),\end{aligned}$$

where C_3 is the uniform bound up to order d for χ (which is independent from P) and $C(d,\ell)$ is a constant depending only on d and ℓ. □

Corollary 4.6 *There exists a constant C', independent from P, such that for any point $z \in M_2$, we have $|\langle D_2(\chi\theta)(z), (\chi\theta)(z)\rangle| \le C'$.*

Proof Follows immediately from Proposition 4.5. □

4.2 Amenable Coverings

In this subsection, we continue to work in the setting of the previous subsection; that is, we consider the Riemannian covering $p\colon M_2 \to M_1$ and a fixed compact subset K_0 of M_1. Consider the universal coverings $p_i\colon \tilde{M} \to M_i$ and denote by Γ_i the deck transformations group of p_i, $i = 1, 2$.

If M_1 has empty boundary, we consider a Riemannian metric $\mathfrak{h}_1$ conformal to the original metric $\mathfrak{g}_1$, such that $(M_1, \mathfrak{h}_1)$ is complete. Distances are considered with respect to $\mathfrak{h}_1$ or its lift $\mathfrak{h}_2$ on M_2. Similarly, the distance on $\tilde{M}$ is considered with respect to the lift of $\mathfrak{h}_1$. For $x \in M_1$ and $y \in p^{-1}(x)$, we denote by D_y the fundamental domain of the Riemannian covering $p\colon (M_2,\mathfrak{h}_2) \to (M_1,\mathfrak{h}_1)$ centered at y.

If M_1 has non-empty boundary, we extend the Riemannian covering $p\colon M_2 \to M_1$ to a Riemannian covering $p\colon N_2 \to N_1$, according to Proposition 3.1. We consider a Riemannian metric $\mathfrak{h}_1$ on N_1 conformal to original Riemannian metric $\mathfrak{g}_1$, that coincides with $\mathfrak{g}_1$ in a compact neighborhood of K_0, such that $(N_1,\mathfrak{h}_1)$ is complete. From Lemma 3.3, $\tilde{M}$ can be identified with a domain of the simply connected covering space $\tilde{N}$ of N_1. Denote by $\mathfrak{h}_2$ and $\tilde{\mathfrak{h}}$ the lift of $\mathfrak{h}_1$ on N_2 and $\tilde{N}$, respectively. As distance function on M_1, M_2, and $\tilde{M}$, we consider the restriction of the distance function of $(N_1,\mathfrak{h}_1)$, $(N_2,\mathfrak{h}_2)$, and $(\tilde{N},\tilde{\mathfrak{h}})$, respectively. For $x \in M_1$ and $y \in p^{-1}(x)$, we denote by D_y the part of the fundamental domain of $p\colon (N_2, \mathfrak{h}_2) \to (N_1, \mathfrak{h}_1)$ centered at y that lies in M_2; that is,

$$D_y = \{z \in M_2 : d(z, y) \le d(z, y') \text{ for any } y' \in p^{-1}(x)\},$$

where $d(\cdot, \cdot)$ is the distance function of N_2 induced by $\mathfrak{h}_2$.

Fix $x \in M_1$ and $u \in p_1^{-1}(x)$. It is quite convenient to identify $\Gamma_2\backslash\Gamma_1$ with $p^{-1}(x)$, that is, $\Gamma_2\gamma$ is identified with $p_2(\gamma u)$, and study induced right action of Γ_1 on $p^{-1}(x)$. Clearly, if $y = p_2(\gamma u)$, for some $\gamma \in \Gamma_1$, then $y \cdot g = p_2(\gamma g u)$, for any $g \in \Gamma_1$. It is worth to point out that p is amenable if and only if this right action of Γ_1 on $p^{-1}(x)$ is amenable.

For $r > 0$, consider the finite set

$$G_r := \{g \in \Gamma_1 : d(u, gu) < r\}$$

and the subgroup $\langle G_r \rangle$ of Γ_1 generated by G_r. We are interested in the right action of $\langle G_r \rangle$ on $p^{-1}(x)$. The next remark is a simple description of the orbits of this action.

Remark 4.7 Let $y \in p^{-1}(x)$ and $g \in G_r$. Then there exists $\gamma \in \Gamma_1$, with $y = p_2(\gamma u)$ and $y \cdot g = p_2(\gamma g u)$. Clearly, we have

$$d(y, y \cdot g) = d(p_2(\gamma u), p_2(\gamma g u)) \le d(\gamma u, \gamma g u) = d(u, gu) < r.$$

Conversely, let $y_1, y_2 \in p^{-1}(x)$ with $d(y_1, y_2) < r$. Then there exist $\gamma_1, \gamma_2 \in \Gamma_1$, such that $y_i = p_2(\gamma_i u)$, for $i = 1, 2$, and there exists $\sigma \in \Gamma_2$, such that

$$d(\sigma\gamma_1 u, \gamma_2 u) = d(p_2(\gamma_1 u), p_2(\gamma_2 u)) = d(y_1, y_2) < r.$$

This yields that $\gamma_1^{-1}\sigma^{-1}\gamma_2 =: g \in G_r$. It follows that $\Gamma_2\gamma_2 = \Gamma_2\gamma_1 g$, i.e., $y_2 = y_1 \cdot g$.

Hence, two points $z_1, z_2 \in p^{-1}(x)$ are in the same orbit of the action of $\langle G_r \rangle$ on $p^{-1}(x)$ if and only if there exist $k \in \mathbb{N}$ and $y_1, \ldots, y_k \in p^{-1}(x)$, such that $y_1 = z_1$, $y_k = z_2$, and $d(y_i, y_{i+1}) < r$, for $i = 1, \ldots, k-1$.

Lemma 4.8 *If $p\colon M_2 \to M_1$ is infinite sheeted, then there exists $R > 0$, such that one of the following holds:*

(i) *either for any $r \ge R$, the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has only infinite orbits,*
(ii) *or for any $r \ge R$, the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has infinitely many finite orbits.*

Proof Assume to the contrary that the statement does not hold. Then there exists $r_0 > 0$, such that the action of $\langle G_{r_0} \rangle$ on $p^{-1}(x)$ has only finitely many finite orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$, for some $k \in \mathbb{N}$. Since p is infinite sheeted, there exists also an infinite orbit $\mathcal{O}$. Since the action of Γ_1 on $p^{-1}(x)$ is transitive, for $y_i \in \mathcal{O}_i$, there exists $g_i \in \Gamma_1$, such that $y_i \cdot g_i \in \mathcal{O}$, for $i = 1, \ldots, k$. Then there exists $R > 0$, such that $G_{r_0} \cup \{g_1, \ldots, g_k\} \subset G_R$ and the action of $\langle G_R \rangle$ on $p^{-1}(x)$ has only infinite orbits. It is clear that so does the action of $\langle G_r \rangle$ on $p^{-1}(x)$, for any $r \ge R$, which is a contradiction. $\square$

Let $r > 0$, such that $K_0 \subset B(x, r)$. If M_1 has non-empty boundary, consider r large enough, so that $B(u, r) \cap \partial \tilde{M} \neq \emptyset$. If p is infinite sheeted, we choose $r \geq R$, where R is the constant from Lemma 4.8. Consider a partition of unity consisting of the functions φ_1 and φ_y, with $y \in p^{-1}(x)$, associated with K_0, x, u, and r as in (5). For a finite $P \subset p^{-1}(x)$, let $\chi := \sum_{y \in P} \varphi_y$ and consider the sets

$$\begin{aligned} Q_+ &:= \{y \in p^{-1}(x) : \chi = 1 \text{ in } B(y, r)\} \\ Q_- &:= \{y \in p^{-1}(x) : 0 < \chi(z) < 1 \text{ for some } z \in B(y, r)\}, \\ Q &:= Q_+ \cup Q_- = \{y \in p^{-1}(x) : \chi(z) \neq 0 \text{ for some } z \in B(y, r)\}. \end{aligned} \tag{6}$$

Clearly, $\chi = 0$ in $B(y, r)$, for any $y \in p^{-1}(x) \smallsetminus Q$. Since χ is compactly supported, it follows that Q is finite. The proof of the following lemma is essentially presented in [2], but since we are in a different situation here, we repeat it.

Lemma 4.9 *If p is amenable, then for any $\varepsilon > 0$, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that*

$$\frac{\#(Q_-)}{\#(Q_+)} < \varepsilon.$$

Proof From Proposition 2.8, since p is amenable, for any $\delta > 0$, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that

$$\#(P \smallsetminus Pg) < \delta \#(P),$$

for all $g \in G_{2r+2}$. From Remark 4.3, we have that $\operatorname{supp} \varphi_{y_0} \subset C(y_0, r+1)$, $\varphi_{y_0} > 0$ in $B(y_0, r + 1/2)$, and $\sum_{y \in p^{-1}(x)} \varphi_y = 1$ in $B(y_0, r)$, for any $y_0 \in p^{-1}(x)$. Clearly, P is contained in Q, which implies that $\#(P) \leq \#(Q)$.

For $y \in Q_-$, there exists $z \in B(y, r)$, such that $0 < \chi(z) < 1$. Therefore, there exist $y_1 \in P$ and $y_2 \in p^{-1}(x) \smallsetminus P$, such that $\varphi_{y_i}(z) > 0$, which yields that $d(y_i, z) < r+1$, for $i = 1, 2$. It follows that $d(y_1, y_2) < 2r+2$ and from Remark 4.7, there exists $g \in G_{2r+2}$, such that $y_1 = y_2 \cdot g$. In particular, $y_1 \in P \smallsetminus Pg$. Since $d(y, y_1) < 2r+1$, from Lemma 2.6, for a fixed y_1, there exist at most $N(2r+1)$ such y. Since $y_1 \in P \smallsetminus Pg$, for some $g \in G_{2r+2}$, there exist at most $\delta \#(P) \#(G_{2r+2})$ such y_1. Hence, it follows that

$$\#(Q_-) \leq \delta \#(P) \#(G_{2r+2}) N(2r+1) \leq \delta \#(Q) \#(G_{2r+2}) N(2r+1).$$

Since Q is the disjoint union of Q_+ and Q_-, for $\delta \#(G_{2r+2}) N(2r+1) < 1$, we have

$$\frac{\#(Q_-)}{\#(Q_+)} \leq \frac{\delta \#(G_{2r+2}) N(2r+1)}{1 - \delta \#(G_{2r+2}) N(2r+1)}.$$

This completes the proof, since $\delta > 0$ is arbitrarily small. $\square$

Proposition 4.10 *If $p\colon M_2 \to M_1$ is infinite sheeted and amenable, then for any $\varepsilon > 0$ and $K \subset M_2$ compact, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that* $\operatorname{supp} \chi$ *does not intersect K and*

$$\frac{\#(Q_-)}{\#(Q_+)} < \varepsilon.$$

Proof First assume that the second statement of Lemma 4.8 holds. Then the action of $\langle G_{2r+2} \rangle$ on $p^{-1}(x)$ has infinitely many finite orbits $\mathcal{O}_n$, with $n \in \mathbb{N}$. Clearly, for $P := \mathcal{O}_n$, we have that Q_- is empty. Indeed, if there exists $y_0 \in Q_-$, then there exist $z \in B(y_0, r)$, $y_1 \in P$, and $y_2 \in p^{-1}(x) \smallsetminus P$, such that $\varphi_{y_i}(z) > 0$, $i = 1, 2$. It follows that $d(z, y_i) < r + 1$, $i = 1, 2$, which yields that $d(y_1, y_2) < 2r + 2$. From Remark 4.7, there exists $g \in G_{2r+2}$, such that $y_2 = y_1 \cdot g$, which is a contradiction, since P is an orbit of the action of $\langle G_{2r+2} \rangle$ on $p^{-1}(x)$.

For a compact $K \subset M_2$, the set $P_K := p^{-1}(x) \cap B(K, r + 2)$ is finite and in particular, intersects only finitely many orbits $\mathcal{O}_n$. Let P be an orbit that does not intersect P_K. Since $\operatorname{supp} \varphi_y \subset C(y, r + 1)$, for any $y \in p^{-1}(x)$, it is clear that for such P, the support of χ does not intersect K.

Assume now that the first statement of Lemma 4.8 holds, that is, the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has only infinite orbits. For a compact subset K of M_2, consider the finite set $P_K := p^{-1}(x) \cap B(K, r + 2)$. From Lemma 4.9, for any $\varepsilon > 0$, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that

$$\frac{\#(Q_-)}{\#(Q_+)} < \delta := \frac{\varepsilon}{1 + (1 + \varepsilon)N(2r + 1)\#(P_K)},$$

where $N(2r + 1)$ is the constant from Lemma 2.6.

Since the action of $\langle G_r \rangle$ on $p^{-1}(x)$ has only infinite orbits, it follows that Q_- is non-empty. Indeed, since P is non-empty and this action has only infinite orbits, there exists an infinite orbit $\mathcal{O}$ and $z_1 \in P \cap \mathcal{O}$. Since P is finite, there exists $z_2 \in \mathcal{O} \smallsetminus P$, and from Remark 4.7, there exist $k \in \mathbb{N}$ and $y_1, \dots, y_k \in p^{-1}(x)$, with $y_1 = z_1$, $y_k = z_2$, and $d(y_i, y_{i+1}) < r$, for $i = 1, \dots, k - 1$. Since $y_1 \in P$ and $y_k \notin P$, there exists $1 \leq j < k$, such that $y_j \in P$ and $y_{j+1} \notin P$. Since $d(y_j, y_{j+1}) < r$, it follows that $0 < \chi(y_{j+1}) < 1$ and in particular, $y_j \in Q_-$.

Evidently, Q_+ is contained in P. Since Q_- is non-empty, it is clear that

$$\frac{1}{\delta} \leq \#(Q_+) \leq \#(P),$$

which yields that $\#(P) > \#(P_K)$, from the choice of δ. In particular, the finite set $P' := P \smallsetminus P_K$ is non-empty. Consider the function χ' and the sets Q'_+, Q'_-, and Q' corresponding to P' as in (6). Clearly, the support of χ' does not intersect K, since $\operatorname{supp} \varphi_y \subset C(y, r + 1)$, for any $y \in p^{-1}(x)$.

From Lemma 2.6, it follows that for any $y_0 \in p^{-1}(x)$, the support of φ_{y_0} intersects at most $N(2r+1)$ open balls $B(y,r)$, with $y \in p^{-1}(x)$. Hence, we have that

$$\#(Q'_-) \leq \#(Q_-) + N(2r+1)\#(P_K),$$
$$\#(Q'_+) \geq \#(Q_+) - N(2r+1)\#(P_K).$$

Therefore, we obtain

$$\frac{\#(Q'_-)}{\#(Q'_+)} \leq \frac{\#(Q_-) + N(2r+1)\#(P_K)}{\#(Q_+) - N(2r+1)\#(P_K)} < \varepsilon,$$

from the choice of δ. □

Remark 4.11 After endowing M_1 or N_1 with $\mathfrak{h}_1$ (depending on whether M_1 has empty boundary or not) and the covering space with its lift $\mathfrak{h}_2$, we have that $p\colon D_y \to M_1$ is an isometry up to sets of measure zero, for any $y \in p^{-1}(x)$. Thus, for $f \in C_c(M_1)$, we have

$$\int_{D_y} (f \circ p) d\mathrm{Vol}_{\mathfrak{h}_2} = \int_{M_1} f d\mathrm{Vol}_{\mathfrak{h}_1}, \tag{7}$$

where $\mathrm{Vol}_{\mathfrak{h}_i}$ (respectively, $\mathrm{Vol}_{\mathfrak{g}_i}$) is the volume element of M_i induced by $\mathfrak{h}_i$ (respectively, $\mathfrak{g}_i$), $i = 1, 2$. Since $\mathfrak{g}_1$ and $\mathfrak{h}_1$ are conformal, it is clear that there exists a positive $\varphi_v \in C^\infty(M_1)$, such that

$$d\mathrm{Vol}_{\mathfrak{g}_1} = \varphi_v d\mathrm{Vol}_{\mathfrak{h}_1} \text{ and } d\mathrm{Vol}_{\mathfrak{g}_2} = (\varphi_v \circ p) d\mathrm{Vol}_{\mathfrak{h}_2}.$$

In particular, for any $f \in C_c(M_1)$ and $y \in p^{-1}(x)$, from (7), we obtain that

$$\begin{aligned} \int_{D_y} (f \circ p) d\mu_2 &= \int_{D_y} (f \circ p)(h \circ p)(\varphi_v \circ p) d\mathrm{Vol}_{\mathfrak{h}_2} = \int_{M_1} f h \varphi_v d\mathrm{Vol}_{\mathfrak{h}_1} \\ &= \int_{M_1} f d\mu_1. \end{aligned}$$

Similarly, for a compact $K \subset M_1$, we have $\mu_1(K) = \mu_2(p^{-1}(K) \cap D_y)$, for any $y \in p^{-1}(x)$.

Proposition 4.12 *Let $p\colon M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering. Let $\eta \in \mathcal{D}(D_1)$ with $\|\eta\|_{L^2(E_1,\mu_1)} = 1$ and $\lambda \in \mathbb{F}$. Then for any $\varepsilon > 0$ and $K \subset M_2$ compact, there exists $\zeta \in \mathcal{D}(D_2)$, with $\|\zeta\|_{L^2(E_2,\mu_2)} = 1$, $\operatorname{supp}\zeta \cap K = \emptyset$, $\operatorname{supp}\zeta \subset p^{-1}(\operatorname{supp}\eta)$, and $\|(D_2 - \lambda)\zeta\|_{L^2(E_2,\mu_2)} \leq \|(D_1 - \lambda)\eta\|_{L^2(E_1,\mu_1)} + \varepsilon$.*

Proof Let $K_0 := \operatorname{supp}\eta$. If M_1 has non-empty boundary, extend the Riemannian covering $p\colon M_2 \to M_1$ according to Proposition 3.1. Consider conformal Riemannian metrics and distance functions as described in the beginning of this subsection.

Let $p_1 : \tilde{M} \to M_1$ be the universal covering of M_1 and fix $x \in M_1$, $u \in p_1^{-1}(x)$, and $r \geq R$ (from Lemma 4.8), such that $K_0 \subset B(x,r)$. If M_1 has non-empty boundary, consider r large enough, so that $B(u,r) \cap \partial\tilde{M} \neq \emptyset$. Consider a partition of unity consisting of the functions φ_1 and φ_y, with $y \in p^{-1}(x)$, associated with K_0, x, u, and r as in (5), and let θ be the lift of η. From Remark 4.4, for any finite set $P' \subset p^{-1}(x)$ and $\chi' := \sum_{y \in P'} \varphi_y$, we have that $\chi'\theta \in \mathcal{D}(D_2)$. From Proposition 4.5, there exists $C > 0$, independent from P', such that $\|D_2(\chi'\theta)(z)\| \leq C$, for any $z \in M_2$. Hence, we obtain that

$$\max_{z \in M_2} \left\|(D_2 - \lambda)(\chi'\theta)(z)\right\| \leq C + |\lambda| \max_{w \in M_1} \|\eta(w)\| =: C_0.$$

From Proposition 4.10, there exists a non-empty, finite $P \subset p^{-1}(x)$, such that the support of $\chi := \sum_{y \in P} \varphi_y$ does not intersect K and

$$\frac{\#(Q_-)}{\#(Q_+)} < \frac{\varepsilon}{C_0^2 \mu_1(\operatorname{supp} \eta)},$$

where Q_+, Q_-, and Q are the sets corresponding to P as in (6).

Since $\chi\theta$ is in the domain of D_2, so is the normalized section $\zeta := (1/\|\chi\theta\|_{L^2(E_2,\mu_2)})\chi\theta$. Evidently, $\|\zeta\|_{L^2(E_2,\mu_2)} = 1$ and $\operatorname{supp}\zeta \subset p^{-1}(\operatorname{supp}\eta)$. From Lemma 2.5, we have that $\operatorname{supp}\zeta \cap D_y \subset B(y,r)$, for any $y \in p^{-1}(x)$, which yields that $\operatorname{supp}\zeta$ is contained in the union of the fundamental domains D_y, with $y \in Q$. Clearly, we have

$$\|\chi\theta\|^2_{L^2(E_2,\mu_2)} \geq \sum_{y \in Q_+} \int_{D_y} \|\chi\theta\|^2 \mathrm{d}\mu_2 = \sum_{y \in Q_+} \int_{D_y} \|\theta\|^2 \mathrm{d}\mu_2 = \#(Q_+),$$

from the definition of Q_+ and Remark 4.11. Therefore, we obtain that

$$\begin{aligned} \int_{M_2} \|(D_2 - \lambda)\zeta\|^2 \mathrm{d}\mu_2 \leq &\frac{1}{\#(Q_+)} \sum_{y \in Q_+} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 \mathrm{d}\mu_2 \\ &+ \frac{1}{\#(Q_+)} \sum_{y \in Q_-} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 \mathrm{d}\mu_2. \end{aligned}$$

For $y \in Q_+$, we have $\chi = 1$ in $B(y,r)$, which is a neighborhood of $\operatorname{supp}\theta \cap D_y$. This implies that

$$\begin{aligned} \frac{1}{\#(Q_+)} \sum_{y \in Q_+} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 \mathrm{d}\mu_2 &= \frac{1}{\#(Q_+)} \sum_{y \in Q_+} \int_{D_y} \|(D_2 - \lambda)\theta\|^2 \mathrm{d}\mu_2 \\ &= \int_{M_1} \|(D_1 - \lambda)\eta\|^2 \mathrm{d}\mu_1. \end{aligned}$$

Since $\|(D_2 - \lambda)(\chi\theta)(z)\| \leq C_0$, for any $z \in M_2$, it follows that

$$\frac{1}{\#(Q_+)} \sum_{y \in Q_-} \int_{D_y} \|(D_2 - \lambda)(\chi\theta)\|^2 \mathrm{d}\mu_2 \leq \frac{C_0^2}{\#(Q_+)} \sum_{y \in Q_-} \mu_2(\operatorname{supp}\theta \cap D_y)$$
$$= \frac{\#(Q_-)}{\#(Q_+)} C_0^2 \mu_1(\operatorname{supp}\eta) \leq \varepsilon.$$

Hence, $\|(D_2 - \lambda)\zeta\|^2_{L^2(E_2,\mu_2)} \leq \|(D_1 - \lambda)\eta\|^2_{L^2(E_1,\mu_1)} + \varepsilon.$ □

Proposition 4.13 *Let $p \colon M_2 \to M_1$ be an infinite sheeted, amenable Riemannian covering, and assume that the operators D_i are symmetric, $i = 1, 2$. Then for any section $\eta \in \mathcal{D}(D_1) \smallsetminus \{0\}$, $\varepsilon > 0$, and $K \subset M_2$ compact, there exists $\zeta \in \mathcal{D}(D_2) \smallsetminus \{0\}$, such that* $\operatorname{supp}\zeta \subset p^{-1}(\operatorname{supp}\eta)$, $\operatorname{supp}\zeta \cap K = \emptyset$, *and* $\mathcal{R}_{D_2}(\zeta) \leq \mathcal{R}_{D_1}(\eta) + \varepsilon$.

Proof The proof is similar to the proof of Proposition 4.12, using Corollary 4.6 instead of Proposition 4.5. □

Proof of Theorem 4.1: Consider $\lambda \in \sigma_{\mathrm{ap}}(\overline{D}_1)$. From Lemma 2.2, it follows that there exists $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(D_1)$, such that $\|\eta_k\|_{L^2(E_1,\mu_1)} = 1$ and $(D_1 - \lambda)\eta_k \to 0$ in $L^2(E_1, \mu_1)$. Consider an exhausting sequence $(K_k)_{k\in\mathbb{N}}$ of M_2. From Proposition 4.12, for any $k \in \mathbb{N}$, there exists $\zeta_k \in \mathcal{D}(D_2)$, with $\|\zeta_k\|_{L^2(E_2,\mu_2)} = 1$, such that $\|(D_2 - \lambda)\zeta_k\|_{L^2(E_2,\mu_2)} \leq \|(D_1 - \lambda)\eta_k\|_{L^2(E_1,\mu_1)} + 1/k$ and $\operatorname{supp}\zeta_k \cap K_k = \emptyset$. Therefore, $(D_2 - \lambda)\zeta_k \to 0$ in $L^2(E_2, \mu_2)$ and for any compact $K \subset M_2$, there exists $k_0 \in \mathbb{N}$, such that $\operatorname{supp}\zeta_k \cap K = \emptyset$, for all $k \geq k_0$. It follows that $(\zeta_k)_{k\in\mathbb{N}}$ is a Weyl sequence for D_2' and λ, and in particular, $\lambda \in \sigma_W(D_2')$. □

Proof of Theorem 1.1: Follows immediately from Theorem 4.1 and Proposition 2.1. □

Assume now that the operator $D_i \colon \mathcal{D}(D_i) \subset L^2(E_i, \mu_i) \to L^2(E_i, \mu_i)$ is symmetric and bounded from below, and let $D_i^{(F)}$ be its Friedrichs extension, $i = 1, 2$. For more details on the Friedrichs extension of a symmetric, bounded from below and densely defined linear operator on a Hilbert space, see [25]. It is well known that the Friedrichs extension of an operator preserves its lower bound. In particular, for $i = 1, 2$, we have

$$\lambda_0(D_i^{(F)}) = \inf_{\eta \in \mathcal{D}(D_i) \smallsetminus \{0\}} \mathcal{R}_{D_i}(\eta). \tag{8}$$

Recall the following proposition for the essential spectrum of a self-adjoint operator.

Proposition 4.14 ([13, Proposition 2.1]) *Let $A \colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a bounded from below, self-adjoint operator on a separable Hilbert space $\mathcal{H}$ over $\mathbb{R}$ or $\mathbb{C}$, and consider $\lambda \in \mathbb{R}$. Then the interval $(-\infty, \lambda]$ intersects the essential spectrum of A if and only if for any $\varepsilon > 0$, there exists an infinite-dimensional subspace $\mathcal{H}_\varepsilon \subset \mathcal{D}(A)$, such that $\mathcal{R}_A(v) < \lambda + \varepsilon$, for all $v \in \mathcal{H}_\varepsilon \smallsetminus \{0\}$.*

Proof of Theorem 1.2: From (8), it follows that there exists $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(D_1) \smallsetminus \{0\}$, such that $\mathcal{R}_{D_1}(\eta_k) \leq \lambda_0(D_1^{(F)}) + 1/k$, for any $k \in \mathbb{N}$. Proposition 4.13 yields that there exists $(\zeta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(D_2) \smallsetminus \{0\}$, such that $\mathcal{R}_{D_2}(\zeta_k) \leq \lambda_0(D_1^{(F)}) + 2/k$ and $\operatorname{supp}\zeta_k \cap \operatorname{supp}\zeta_{k'} = \emptyset$, for all $k, k' \in \mathbb{N}$, with $k \neq k'$. Evidently, for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that $\mathcal{R}_{D_2}(\zeta_k) < \lambda_0(D_1^{(F)})+\varepsilon$, for all $k \geq k_0$. Consider the subspace $\mathcal{H}_\varepsilon$ of $\mathcal{D}(D_2)$ spanned by $\{\zeta_k : k \geq k_0\}$. Since the sections ζ_k, with $k \in \mathbb{N}$, have disjoint supports, the space $\mathcal{H}_\varepsilon$ is infinite-dimensional. Clearly, any $\theta \in \mathcal{H}_\varepsilon$ is of the form $\theta := \sum_{i=k_0}^{k_0+k} m_i\zeta_i$, for some $k \in \mathbb{N}$ and $m_{k_0}, \dots, m_{k_0+k} \in \mathbb{F}$. Therefore, we have

$$\mathcal{R}_{D_2}(\theta) = \frac{\sum_{i=k_0}^{k_0+k} |m_i|^2 \langle D_2\zeta_i, \zeta_i\rangle_{L^2(E_2,\mu_2)}}{\sum_{i=k_0}^{k_0+k} |m_i|^2 \|\zeta_i\|^2_{L^2(E_2,\mu_2)}} \leq \max_{k_0\leq i\leq k_0+k} \mathcal{R}_{D_2}(\zeta_i) < \lambda_0(D_1^{(F)}) + \varepsilon.$$

From Proposition 4.14, it follows that $\lambda_0^{\mathrm{ess}}(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$. □

Remark 4.15 In the proof of Theorem 1.2, the only properties of the Friedrichs extension used are self-adjointness and the preservation of the lower bound of D_1. Therefore, this proof establishes the analogous result for any self-adjoint extensions of the operators, as long as the extension of D_1 preserves its lower bound.

The next proposition provides the analogous result in case the operators are symmetric and D_1 is not bounded from below.

Proposition 4.16 *Assume that the operator D_i is symmetric, $i = 1, 2$. If the covering is infinite sheeted and amenable, and D_1 is not bounded from below, then D_2 is not bounded from below.*

Proof Since D_1 is not bounded from below, for any $C \in \mathbb{R}$, there exists a nonzero $\eta \in \mathcal{D}(D_1)$, with $\mathcal{R}_{D_1}(\eta) \leq C$. From Proposition 4.13, it follows that there exists $\zeta \in \mathcal{D}(D_2) \smallsetminus \{0\}$, with $\mathcal{R}_{D_2}(\zeta) < C + 1$. Therefore, D_2 is not bounded from below. □

For sake of completeness, we also present the analogous results for finite-sheeted coverings. It is clear that they cannot be improved in order to obtain as strong statements as in the case of infinite- sheeted amenable coverings.

Proposition 4.17 *Let D_2' be a closed extension of D_2. If p is a finite-sheeted Riemannian covering, then $\sigma_{\mathrm{ap}}(\overline{D}_1) \subset \sigma_{\mathrm{ap}}(D_2')$ and $\sigma_W(\overline{D}_1) \subset \sigma_W(D_2')$.*

Proof If η is in the domain of D_1, then its lift is in the domain of D_2. For $\lambda \in \sigma_W(\overline{D}_1)$, from Lemma 2.2, there exists a Weyl sequence $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(D_1)$ for $\overline{D}_1$ and λ. Then, the sequence consisting of the normalized (in $L^2(E_2, \mu_2)$) lifts of η_k, $k \in \mathbb{N}$, is a Weyl sequence for D_2' and λ. Hence, $\sigma_W(\overline{D}_1) \subset \sigma_W(D_2')$. Similarly, it follows that $\sigma_{\mathrm{ap}}(\overline{D}_1) \subset \sigma_{\mathrm{ap}}(D_2')$. □

Proposition 4.18 *Assume that D_i is symmetric and bounded from below, and denote by $D_i^{(F)}$ its Friedrichs extension, $i = 1, 2$. If p is a finite-sheeted Riemannian covering, then $\lambda_0(D_2^{(F)}) \leq \lambda_0(D_1^{(F)})$.*

Proof If η is in the domain of D_1, then its lift θ is in the domain of D_2. If $\eta \neq 0$, it is easy to see that $\mathcal{R}_{D_1}(\eta) = \mathcal{R}_{D_2}(\theta)$, and the statement follows from (8). □

In the rest of this section, we give applications of our results in the case of Schrödinger operators. Recall that on manifolds with boundary, we are interested in the Dirichlet spectrum of Schrödinger operators. The following proposition characterizes the bottom of the spectrum of a Schrödinger operator as the maximum of its positive spectrum.

Proposition 4.19 *Let S be a Schrödinger operator on a Riemannian manifold M. Then the bottom of the spectrum of S is the maximum of all $\lambda \in \mathbb{R}$, such that there exists $\varphi \in C^\infty(M \smallsetminus \partial M)$ with $S\varphi = \lambda\varphi$, which is positive in $M \smallsetminus \partial M$.*

Proof If M has empty boundary, then the statement may be found in [11, Theorem 7], [14, Theorem 1], and [23, Theorem 2.1]. If M has non-empty boundary, it is clear that $\lambda_0(S) = \lambda_0(S, M \smallsetminus \partial M)$, where $\lambda_0(S, M \smallsetminus \partial M)$ stands for the bottom of the spectrum of S on the interior of M. Hence, in this case, the claim follows from the corresponding statement for manifolds without boundary. □

In particular, there exists $\varphi \in C^\infty(M \smallsetminus \partial M)$ with $S\varphi = \lambda_0(S)\varphi$, which is positive in the interior of M. It is worth to point out that the smooth eigenfunctions of the preceding proposition do not have to be square-integrable. The following corollary is a consequence of Proposition 4.19 (an alternative proof can be found in [2]).

Corollary 4.20 *Let $p\colon M_2 \to M_1$ be a Riemannian covering. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\lambda_0(S_1) \leq \lambda_0(S_2)$.*

Proof Follows immediately from Proposition 4.19, since the lift of an eigenfunction of S_1 is an eigenfunction of S_2. □

Corollary 4.21 *Let $p : M_2 \to M_1$ be an infinite-sheeted, amenable Riemannian covering. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\lambda_0(S_1) = \lambda_0^{\text{ess}}(S_2)$. If, in addition, M_1 is complete, then $\sigma(S_1) \subset \sigma_{\text{ess}}(S_2)$.*

Proof Follows from Theorems 1.1, 1.2 and Corollary 4.20. □

The following results describe the behavior of the spectrum of Schrödinger operators under finite-sheeted coverings.

Corollary 4.22 *Let $p\colon M_2 \to M_1$ be a finite-sheeted Riemannian covering. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then $\lambda_0(S_1) = \lambda_0(S_2)$. If, in addition, M_1 is complete, then $\sigma(S_1) \subset \sigma(S_2)$ and $\sigma_{\text{ess}}(S_1) \subset \sigma_{\text{ess}}(S_2)$.*

Proof Follows from Propositions 2.1, 4.17, 4.18 and Corollary 4.20. □

The following characterization of the bottom of the essential spectrum of a Schrödinger operator follows from the Decomposition Principle ([3, Proposition 1]) and Propositions 2.3 and 4.14. Recall that this quantity is infinite when the spectrum is discrete.

Proposition 4.23 ([5, Proposition 3.2]) *Let S be a Schrödinger operator on a complete manifold M and let $(K_k)_{k\in\mathbb{N}}$ be an exhausting sequence of M. Then*

$$\lambda_0^{\mathrm{ess}}(S) = \lim_k \lambda_0(S, M \smallsetminus K_k),$$

where $\lambda_0(S, M \smallsetminus K_k)$ is the bottom of the spectrum of S on $M \smallsetminus K_k$.

Corollary 4.24 *Let $p\colon M_2 \to M_1$ be a finite-sheeted Riemannian covering of complete manifolds. Consider a Schrödinger operator S_1 on M_1 and its lift S_2 on M_2. Then $\lambda_0^{\mathrm{ess}}(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$ and in particular, $\sigma_{\mathrm{ess}}(S_1) \neq \emptyset$ if and only if $\sigma_{\mathrm{ess}}(S_2) \neq \emptyset$.*

Proof Follows from Corollary 4.22 and Proposition 4.23. □

5 Infinite Deck Transformations Group

Let M be a Riemannian manifold, $E \to M$ a Riemannian or Hermitian vector bundle, endowed with a connection ∇ and $D\colon \Gamma(E) \to \Gamma(E)$ a differential operator on E.

If M has empty boundary, set $\mathcal{D}(D) := \Gamma_c(E)$. If M has non-empty boundary, consider $\nu \in \mathbb{N}$, and for $l = 1, \dots, \nu$, let $k_l \in \mathbb{N}$ and $a_{j,l}$ be real or complex-valued functions (depending on whether E is Riemannian or Hermitian) defined on ∂M, $j = 0, \dots, k_l$. It is worth to point out that we do not impose any assumptions on the functions $a_{j,l}$. Let n be the inward pointing normal to ∂M and consider

$$\mathcal{D}(D) := \{\eta \in \Gamma_c(E) : \sum_{j=0}^{k_l} a_{j,l} \nabla_n^{(j)} \eta = 0 \text{ on } \partial M,\ l = 1, \dots, \nu\}.$$

Let μ be a measure on M expressed via a positive smooth density with respect to the volume element of M; that is, there exists a positive $h \in C^\infty(M)$, such that $d\mu = hd\,\mathrm{Vol}$. Consider D as a densely defined operator

$$D\colon \mathcal{D}(D) \subset L^2(M, \mu) \to L^2(M, \mu). \tag{9}$$

When we refer to closability or symmetry of D, we consider it as in (9). From Lemma 2.4, the operator D is closable and denote by $\overline{D}$ its closure.

Theorem 5.1 *Let Γ be a group of automorphisms of E preserving the metric of E, such that the induced action on M is isometric and $D(g_*\eta) = g_* D\eta$, for any $g \in \Gamma$ and $\eta \in \Gamma(E)$. Moreover, assume that the density function h of μ is Γ-invariant. If M has non-empty boundary, assume that ∇ and the functions $a_{j,l}$ are Γ-invariant along the boundary. If for any compact $K \subset M$, there exists $g \in \Gamma$, such that $gK \cap K = \emptyset$, then $\sigma_{\mathrm{ap}}(\overline{D}) = \sigma_W(\overline{D})$ and $\overline{D}$ does not have eigenvalues of finite multiplicity.*

Proof Let $\lambda \in \sigma_{\mathrm{ap}}(\overline{D})$. From Lemma 2.2, there exists $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(D)$, such that $\|\eta_k\|_{L^2(E,\mu)} = 1$ and $(D - \lambda)\eta_k \to 0$ in $L^2(E, \mu)$. Since η_k is compactly supported, there exists an exhausting sequence $(K_k)_{k\in\mathbb{N}}$ of M, such that $\operatorname{supp} \eta_k \subset K_k$, for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, consider $g_k \in \Gamma$, such that $g_k K_k \cap K_k = \emptyset$, and set $\zeta_k := (g_k)_\star \eta_k$. Then $\zeta_k \in \Gamma_c(E)$ and if M has non-empty boundary, then ζ_k satisfies the same boundary conditions with η_k, since via isometries unit speed geodesics normal to the boundary are mapped to unit speed geodesics normal to the boundary. It follows that $\zeta_k \in \mathcal{D}(D)$, $\|\zeta_k\|_{L^2(E,\mu)} = 1$, and $(D - \lambda)\zeta_k \to 0$ in $L^2(E, \mu)$. It is clear that $\operatorname{supp} \zeta_k = g_k(\operatorname{supp} \eta_k)$, which yields that for any compact $K \subset M$, there exists $k_0 \in \mathbb{N}$, such that $\operatorname{supp} \zeta_k \cap K = \emptyset$, for all $k \geq k_0$. This implies that $\zeta_k \rightharpoonup 0$ in $L^2(E, \mu)$, that is, $(\zeta_k)_{k\in\mathbb{N}}$ is a Weyl sequence for $\overline{D}$ and λ. Hence, $\lambda \in \sigma_W(\overline{D})$.

Assume that there exists an eigenvalue λ of $\overline{D}$ of finite multiplicity, and consider $\theta \in \mathcal{D}(\overline{D})$ with $\|\theta\|_{L^2(E,\mu)} = 1$ and $\overline{D}\theta = \lambda\theta$. Then there exists $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(D)$, such that $\eta_k \to \theta$ and $D\eta_k \to \overline{D}\theta$. It is clear that for any $g \in \Gamma$ and $k \in \mathbb{N}$, we have $g_*\eta_k \in \mathcal{D}(D)$, $g_*\eta_k \to g_*\theta$, and $D(g_*\eta_k) \to g_*(\overline{D}\theta)$, which yields that $g_*\theta \in \mathcal{D}(\overline{D})$ and $\overline{D}(g_*\theta) = \lambda(g_*\theta)$.

Let $(K_k)_{k\in\mathbb{N}}$ be an exhausting sequence of M and consider $(g_k)_{k\in\mathbb{N}} \subset \Gamma$, such that $g_k K_k \cap K_k = \emptyset$, for any $k \in \mathbb{N}$. It is clear that the sections $\theta_k := (g_k)_*\theta$ satisfy $\overline{D}\theta_k = \lambda\theta_k$ and $\|\theta_k\|_{L^2(E,\mu)} = 1$, for all $k \in \mathbb{N}$. Since the eigenspace corresponding to λ is finite-dimensional, after passing to a subsequence, we may assume that $\theta_k \to \theta_0$ in $L^2(E, \mu)$, for some θ_0, with $\|\theta_0\|_{L^2(E,\mu)} = 1$. Consider a non-zero $\zeta \in \Gamma_c(E)$ and set $\zeta_k := (g_k^{-1})_*\zeta$. Then

$$\langle \theta_k, \zeta \rangle^2_{L^2(E,\mu)} = \langle \theta, \zeta_k \rangle^2_{L^2(E,\mu)} \leq \|\zeta\|^2_{L^2(E,\mu)} \int_{\operatorname{supp} \zeta_k} \|\theta\|^2 \mathrm{d}\mu.$$

Let $\varepsilon > 0$ and consider a compact $K \subset M$, such that $\int_{M \smallsetminus K} \|\theta\|^2 \mathrm{d}\mu < \varepsilon^2 / \|\zeta\|^2_{L^2(E,\mu)}$. Since $\operatorname{supp} \zeta$ and K are eventually subsets of K_k, there exists $k_0 \in \mathbb{N}$, such that $\operatorname{supp} \zeta_k \cap K = \emptyset$, for all $k \geq k_0$. Therefore, for $k \geq k_0$, we have $\operatorname{supp} \zeta_k \subset M \smallsetminus K$, and in particular, $|\langle \theta_k, \zeta \rangle_{L^2(E,\mu)}| < \varepsilon$. This yields that $\theta_k \rightharpoonup 0$ in $L^2(E, \mu)$, which is a contradiction, since $\theta_k \to \theta_0$ in $L^2(E, \mu)$ and $\|\theta_0\|_{L^2(E,\mu)} = 1$. □

Theorem 5.2 *Assume that D is symmetric and bounded from below, and denote by $D^{(F)}$ its Friedrichs extension. Under the assumptions of Theorem 5.1, the spectrum of $D^{(F)}$ is essential and $D^{(F)}$ does not have eigenvalues of finite multiplicity.*

Proof Let $\eta \in \mathcal{D}(D^{(F)})$ and $g \in \Gamma$. From the invariance of $\mathcal{D}(D)$ and D under the action of Γ, it follows that $g_*\eta \in \mathcal{D}(D^{(F)})$ and $D^{(F)}(g_*\eta) = g_*(D^{(F)}\eta)$. As in the proof of Theorem 5.1, it follows that $D^{(F)}$ does not have eigenvalues of finite multiplicity. From Proposition 2.1, we obtain that $\sigma(D^{(F)}) = \sigma_{\mathrm{ess}}(D^{(F)})$. □

The above theorems can be applied to Riemannian coverings with infinite deck transformations group. In the context of the previous section, we obtain the following consequences.

Corollary 5.3 *If the deck transformations group of the covering is infinite, then $\overline{D}_2$ does not have eigenvalues of finite multiplicity and $\sigma_{\mathrm{ap}}(\overline{D}_2) = \sigma_W(\overline{D}_2)$.*

Proof Follows immediately from Theorem 5.1, for Γ being the deck transformations group of the covering. □

Proof of Corollary 1.3: Follows from Corollary 5.3 and Proposition 2.1. □

Proof of Corollary 1.4: Follows from Theorem 5.2, for Γ being the deck transformations group of the covering. □

Corollary 5.4 *Let Γ be an infinite, discrete group acting properly discontinuously on a complete Riemannian manifold M via isometries. Then there exists no non-zero, square-integrable, $\lambda_0(M)$-harmonic function on M. Moreover, $\lambda_0(M)$ is an accumulation point of $\sigma(M)$.*

Proof For any complete (and connected) Riemannian manifold M, the space of square-integrable, $\lambda_0(M)$-harmonic functions is either trivial or one-dimensional. Therefore, Corollary 1.3 yields the first statement and that $\lambda_0(M) \in \sigma_{\mathrm{ess}}(M)$. The second statement follows from Proposition 2.1, since $\sigma_{\mathrm{ess}}(M)$ consists of eigenvalues of infinite multiplicity and accumulation points of the spectrum. □

Besides Riemannian coverings, the above theorems can be applied to manifolds with high symmetry. For instance, it follows that the spectrum of the Laplacian on a non-compact homogeneous space is essential. Moreover, we obtain the analogous statement, if there exists a non-compact Lie group acting on the manifold properly discontinuously via isometries.

6 Coverings of Closed Manifolds

The *Cheeger's constant* of a Riemannian manifold M is defined by

$$h(M) := \inf_K \frac{\mathrm{Area}(\partial K)}{\mathrm{Vol}(K)},$$

where the infimum is taken over all compact and smoothly bounded domains K of M which do not intersect ∂M. It is related to $\lambda_0(M)$ via Cheeger's inequality (cf. [10]):

$$\lambda_0(M) \geq \frac{1}{4} h(M)^2.$$

Brooks [6] actually proved that a normal Riemannian covering of a closed manifold is amenable if and only if the Cheeger's constant of the covering space is zero. The following result is an extension of that of Brooks, to not necessarily normal coverings.

Theorem 6.1 *Let $p\colon M_2 \to M_1$ be a Riemannian covering of a closed manifold M_1. If $h(M_2) = 0$, then p is amenable.*

In order to prove this theorem, we need the following proposition. In the sequel, for a subset W of M, we denote by $B(W, r)$ the tubular neighborhood

$$B(W, r) := \{z \in M : d(z, W) < r\}.$$

Proposition 6.2 ([9, Lemma 7.2]) *Let M be a non-compact, complete Riemannian manifold, without boundary and with Ricci curvature bounded from below. Then there exists a constant c depending only on the dimension of M, such that for any compact and smoothly bounded domain K of M, with $\operatorname{Area}(\partial K)/\operatorname{Vol}(K) =: H$, and any $0 < r \leq 1/2c \min\{1, 1/H\}$, there exists a bounded, open $U \subset M$, such that*

$$\frac{\operatorname{Vol}(B(\partial U, r))}{\operatorname{Vol}(U)} \leq C(r)H,$$

where $C(r)$ is a constant depending on r, the dimension of M, and the lower bound of the Ricci curvature.

Corollary 6.3 *Let M be a non-compact, complete Riemannian manifold, without boundary and with Ricci curvature bounded from below. If $h(M) = 0$, then for any $\varepsilon, r > 0$, there exists a bounded, open $U \subset M$, such that*

$$\frac{\operatorname{Vol}(B(\partial U, r))}{\operatorname{Vol}(U \smallsetminus B(\partial U, r))} < \varepsilon.$$

Proof Let $r > 0$ and $0 < r_0 \leq 1/2c$, where c is the constant from Proposition 6.2. Denote by $\mathfrak{g}$ the original Riemannian metric and consider the metric $\mathfrak{h} := C\mathfrak{g}$, where $C := r_0/r$. For any compact and smoothly bounded domain K of M, we have

$$\frac{\operatorname{Area}_{\mathfrak{h}}(\partial K)}{\operatorname{Vol}_{\mathfrak{h}}(K)} = C^{-1/2}\frac{\operatorname{Area}_{\mathfrak{g}}(\partial K)}{\operatorname{Vol}_{\mathfrak{g}}(K)}.$$

Since the Cheeger's constant of M with respect to $\mathfrak{g}$ is zero, it follows that so is the Cheeger's constant of M with respect to $\mathfrak{h}$. From Proposition 6.2, for any $\delta > 0$, there exists a bounded, open $U \subset M$, such that

$$\frac{\operatorname{Vol}_{\mathfrak{h}}(B_{\mathfrak{h}}(\partial U, r_0))}{\operatorname{Vol}_{\mathfrak{h}}(U)} < \delta.$$

It follows that

$$\frac{\operatorname{Vol}_{\mathfrak{g}}(B_{\mathfrak{g}}(\partial U, r))}{\operatorname{Vol}_{\mathfrak{g}}(U)} = \frac{\operatorname{Vol}_{\mathfrak{h}}(B_{\mathfrak{g}}(\partial U, r))}{\operatorname{Vol}_{\mathfrak{h}}(U)} = \frac{\operatorname{Vol}_{\mathfrak{h}}(B_{\mathfrak{h}}(\partial U, r_0))}{\operatorname{Vol}_{\mathfrak{h}}(U)} < \delta.$$

This completes the proof, since $\operatorname{Vol}_{\mathfrak{g}}(U) \leq \operatorname{Vol}_{\mathfrak{g}}(U \smallsetminus B_{\mathfrak{g}}(\partial U, r)) + \operatorname{Vol}_{\mathfrak{g}}(B_{\mathfrak{g}}(\partial U, r))$. □

Proof of Theorem 6.1: Evidently, if M_2 is closed, then p is finite sheeted and in particular, amenable. Therefore, it remains to prove the statement for M_2 non-compact. Consider the universal covering $p_1 \colon \tilde{M} \to M_1$, fix $x \in M_1$, $u \in p_1^{-1}(x)$, and identify $\pi_1(M_2)\backslash\pi_1(M_1)$ with $p^{-1}(x)$, as in the beginning of Subsect. 4.2. Denote by D_y

the fundamental domain of p centered at y, with $y \in p^{-1}(x)$. It is clear that for $y \in p^{-1}(x)$ and $z, w \in D_y$, we have

$$d(z, w) \leq d(y, z) + d(y, w) = d(x, p(z)) + d(x, p(w)) \leq 2\text{diam}(M_1),$$

which yields that $\text{diam}(D_y) \leq 2\text{diam}(M_1)$, for all $y \in p^{-1}(x)$. Let $r > 2\text{diam}(M_1)$ and

$$G_r := \{g \in \pi_1(M_1) : d(u, gu) < r\}.$$

From Corollary 6.3, for any $\varepsilon > 0$, there exists a bounded, open $U \subset M_2$, such that

$$\frac{\text{Vol}(B(\partial U, 2r))}{\text{Vol}(U \smallsetminus B(\partial U, 2r))} < \varepsilon. \tag{10}$$

Consider the finite sets

$$F := \{y \in p^{-1}(x) : y \in U \smallsetminus B(\partial U, r)\},$$
$$F' := \{y \in p^{-1}(x) : y \in B(\partial U, r)\}.$$

Recall that $r > 2\text{diam}(M_1) \geq \text{diam}(D_y)$, for all $y \in p^{-1}(x)$, and M_2 is covered by the fundamental domains D_y, with $y \in p^{-1}(x)$. Evidently, $U \smallsetminus B(\partial U, 2r)$ is contained in the union of D_y, with $y \in F$. Furthermore, $B(\partial U, 2r)$ contains the union of D_y, with $y \in F'$. From (10), since the intersection of different fundamental domains is of measure zero, and $\text{Vol}(D_y) = \text{Vol}(M_1)$, for any $y \in p^{-1}(x)$, it follows that

$$\frac{\#(F')}{\#(F)} < \varepsilon.$$

Let $g \in G_r$ and $y \in F \smallsetminus Fg$. Then $y \in U$, $d(y, \partial U) \geq r$ and $y \cdot g^{-1} \notin F$. From Remark 4.7, it follows that $d(y, y \cdot g^{-1}) < r$. Therefore, $y \cdot g^{-1} \in U$ and $d(y \cdot g^{-1}, \partial U) < r$, which yields that $y \cdot g^{-1} \in F'$. Hence, $F \smallsetminus Fg \subset F'g$ and in particular, we obtain that

$$\#(F \smallsetminus Fg) \leq \#(F') < \varepsilon\#(F).$$

For any finite $G \subset \pi_1(M_1)$, there exists $r > 2\text{diam}(M_1)$, such that $G \subset G_r$. The above arguments imply that for any finite $G \subset \pi_1(M_1)$ and $\varepsilon > 0$, there exists a Følner set for G and ε. From Proposition 2.8, it follows that p is amenable. □

7 Applications and Examples

We begin with some examples of operators for which our main results can be applied.

Examples 7.1 In the following examples, we consider a Riemannian manifold M. If M has non-empty boundary, we denote by n the inward pointing normal to ∂M.

(i) *Schrödinger operators.* A Schrödinger operator on M is an operator of the form $S = \Delta + V$, where Δ is the Laplacian and $V \in C^\infty(M)$ is bounded from below. If M has empty boundary, then the operator

$$S\colon C_c^\infty(M) \subset L^2(M) \to L^2(M)$$

is symmetric and bounded from below. If, in addition, M is complete, then this operator is essentially self-adjoint.
If M has non-empty boundary, then one may consider S as a symmetric, densely defined operator in $L^2(M)$, by restricting S on the space of $f \in C_c^\infty(M)$ satisfying Dirichlet ($f = 0$), Neumann ($n(f) = 0$), Robin ($n(f) + bf = 0$), or mixed boundary conditions (that is, $f = 0$ on a subset of ∂M and $n(f) = 0$ on the rest of ∂M).

(ii) *Laplacian with density.* Let μ be a measure on M which is expressed by a positive smooth density in terms of the volume element, $d\mu = h^2 d\,\mathrm{Vol}$. The Laplacian with respect to this density is defined by

$$\Delta_\mu f = \Delta f - \frac{2}{h}\langle \operatorname{grad} h, \operatorname{grad} f\rangle,$$

for any $f \in C^\infty(M)$. If M has empty boundary, then the operator

$$\Delta_\mu\colon C_c^\infty(M) \subset L^2(M,\mu) \to L^2(M,\mu)$$

is symmetric and non-negative definite. If, in addition, M is complete, then this operator is essentially self-adjoint (see, for instance [16, Theorem 2.2]).

(iii) *Higher-order Laplacian.* If M has empty boundary, then for any $k \in \mathbb{N}$, the operator

$$\Delta^k\colon C_c^\infty(M) \subset L^2(M) \to L^2(M)$$

is symmetric and non-negative definite, which yields that it admits Friedrichs extension. If, in addition, M is complete, then this operator is essentially self-adjoint (see for instance, [12]).
If M has non-empty boundary, then we may consider the operator Δ^k on the space of compactly supported, smooth functions satisfying Dirichlet boundary conditions (according to the terminology of [15])

$$\begin{aligned}\Delta^k\colon &\{f \in C_c^\infty(M) : f = n(f) = \dots = n^{(k-1)}(f) = 0 \text{ on } \partial M\}\\ &\subset L^2(M) \to L^2(M).\end{aligned}$$

For $k = 1, 2$, this operator is symmetric and non-negative definite, which yields that it admits Friedrichs extension. For $k \geq 3$, this operator is not symmetric.

However, according to Lemma 2.4, this operator is closable and Theorem 4.1 can be applied.

(iv) *Laplace-type operators*. Let $E \to M$ a Riemannian vector bundle endowed with a metric connection ∇. The (corresponding) connection Laplacian is defined as $\Delta = \nabla^*\nabla$. A Laplace-type operator is an operator of the form $S = \Delta + V$, where $V \in \Gamma(\mathrm{End}E)$ and $V(x)\colon E_x \to E_x$ is symmetric for any $x \in M$. If M has empty boundary, then the operator

$$S\colon \Gamma_c(E) \subset L^2(E) \to L^2(E)$$

is symmetric. If, in addition, the lowest eigenvalue of $V(x)$ is bounded from below, then this operator admits Friedrichs extension. If M is closed, then this operator is essentially self-adjoint (see [18]).

It is worth to point out that the *Hodge Laplacian* $\Delta_k := d^*d + dd^*$ acting on k-forms, for some $0 \le k \le \dim(M)$, is a Laplace-type operator. If we consider it as

$$\Delta_k\colon \Gamma_c(\wedge^k T^*M) \subset L^2(\wedge^k T^*M) \to L^2(\wedge^k T^*M),$$

then it is symmetric and non-negative definite. If, in addition, M is complete, then this operator is essentially self-adjoint (see [12]).

Another example of Laplace-type operator that is of interest in spectral theory is the *Jacobi (stability) operator* of a minimal submanifold. Let $\Phi\colon M^m \to N^k$ be a minimal isometric immersion. The Jacobi operator J is a Laplace-type operator acting on sections of the normal bundle $T^\perp M$. Locally, if $\eta \in \Gamma(T^\perp M)$ and $\{e_1, \ldots, e_m\}$ is a local orthonormal frame of M, the Jacoby operator is given by

$$J\eta = \Delta^\perp \eta - \sum_{i=1}^m \alpha(e_i, A_\eta e_i) - \sum_{i=1}^m (R(\eta, e_i)e_i)^\perp,$$

where $\Delta^\perp$ is the connection Laplacian corresponding to the normal connection, α is the second fundamental form of Φ, A_η is the Weingarten operator with respect to η, and R is the curvature tensor of N. If M is closed, then the operator

$$J\colon \Gamma(T^\perp M) \subset L^2(T^\perp M) \to L^2(T^\perp M)$$

is essentially self-adjoint.

In the following corollary, we denote by $\lambda_0(\Delta_k, M)$ the bottom of the spectrum of the Friedrichs extension of the Hodge Laplacian (considered as in Examples 7.1 (iv)) acting on k-forms on a Riemannian manifold M.

Corollary 7.2 *Let $p\colon M_2 \to M_1$ be an amenable Riemannian covering of manifolds without boundary. If $\lambda_0(\Delta_k, M_1) = 0$, for some $0 \le k \le \dim(M_1)$, then $\lambda_0(\Delta_k, M_2) = 0$.*

Proof Follows from Theorem 1.2 and Proposition 4.18, since the Hodge Laplacian is non-negative definite. □

We now introduce the notion of renormalized Schrödinger operators, which is required in order to establish Theorem 1.5. This notion was introduced by Brooks in [7] for the Laplacian on complete manifolds without boundary.

Let S be a Schrödinger operator on a possibly non-connected Riemannian manifold M without boundary, and let $\varphi \in C^\infty(M)$ be a positive λ-eigenfunction of S. It is worth to point out that we do not require φ to be square-integrable or M to be complete. Let μ be the measure expressed by $d\mu = \varphi^2 d\,\mathrm{Vol}$ in terms of the volume element of M. Consider the separable Hilbert space $L^2(M, \mu)$. Evidently, the map $m_\varphi \colon L^2(M, \mu) \to L^2(M)$, given by $m_\varphi v := v\varphi$ is an isometric isomorphism.

The *renormalized Schrödinger operator* $S_\varphi \colon \mathcal{D}(S_\varphi) \subset L^2(M, \mu) \to L^2(M, \mu)$ is defined by $S_\varphi v := m_\varphi^{-1}(S^{(F)} - \lambda)(m_\varphi v)$, for all $v \in \mathcal{D}(S_\varphi)$, where $S^{(F)}$ is the Friedrichs extension of S and $\mathcal{D}(S_\varphi) := m_\varphi^{-1}(\mathcal{D}(S^{(F)}))$. Clearly, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{D}(S_\varphi) & \xrightarrow[\simeq]{m_\varphi} & \mathcal{D}(S^{(F)}) \\ S_\varphi \downarrow & & \downarrow S^{(F)} - \lambda \\ L^2(M, \mu) & \xrightarrow[\simeq]{m_\varphi} & L^2(M) \end{array}$$

In particular, S_φ is self-adjoint and $\sigma(S_\varphi) = \sigma(S) - \lambda$. From Proposition 2.3, it follows that

$$\lambda_0(S_\varphi) \leq \inf_{f \in C_c^\infty(M) \smallsetminus \{0\}} \mathcal{R}_{S_\varphi}(f) = \inf_{f \in C_c^\infty(M) \smallsetminus \{0\}} \frac{\langle S_\varphi f, f \rangle_{L^2(M,\mu)}}{\| f \|^2_{L^2(M,\mu)}},$$

Consider $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(M) \smallsetminus \{0\}$, such that $\mathcal{R}_S(f_k) \to \lambda_0(S)$. It is evident that for $h_k := m_\varphi^{-1}(f_k) \in C_c^\infty(M)$, we have $\mathcal{R}_{S_\varphi}(h_k) \to \lambda_0(S_\varphi)$. Hence, the bottom of the spectrum of S_φ can be approximated with Rayleigh quotients of compactly supported smooth functions in M. With a simple computation of the Rayleigh quotient of such a function (as in [7, Sect. 2], using the Divergence Theorem, instead of the $*$-operator), we obtain the following expression for $\lambda_0(S) - \lambda$.

Proposition 7.3 *Let S be a Schrödinger operator on M and let $\varphi \in C^\infty(M)$ be a positive λ-eigenfunction of S. Then*

$$\lambda_0(S) - \lambda = \inf_{f \in C_c^\infty(M) \smallsetminus \{0\}} \frac{\int_M \| \operatorname{grad} f \|^2 \varphi^2}{\int_M f^2 \varphi^2}.$$

The *modified Cheeger's constant* of M is defined by

$$h_\varphi(M) := \inf_K \frac{\int_{\partial K} \varphi^2}{\int_K \varphi^2},$$

where the infimum is taken over all compact and smoothly bounded domains K of M. From the preceding proposition, it is easy to establish an analog of Cheeger's inequality.

Corollary 7.4 *Let S be a Schrödinger operator on M and let $\varphi \in C^\infty(M)$ be a positive λ-eigenfunction of S. Then*

$$\lambda_0(S) - \lambda \geq \frac{1}{4} h_\varphi(M)^2.$$

Proof By virtue of Proposition 7.3, the proof is the same as that of [7, Lemma 3]. □

Moreover, consider the quantity

$$h_\varphi^{\mathrm{ess}}(M) := \sup_K h_\varphi(M \smallsetminus K),$$

where the supremum is taken over all compact and smoothly bounded domains K of M. For $\varphi = 1$, this quantity is denoted by $h^{\mathrm{ess}}(M)$.

Corollary 7.5 *Let S be a Schrödinger operator on a complete manifold M and consider a positive λ-eigenfunction $\varphi \in C^\infty(M)$ of S. Then*

$$\lambda_0^{\mathrm{ess}}(S) - \lambda \geq \frac{1}{4} h_\varphi^{\mathrm{ess}}(M)^2.$$

Proof Let $(K_k)_{k\in\mathbb{N}}$ be an exhausting sequence of M, consisting of smoothly bounded domains. It is easy to see that

$$h_\varphi^{\mathrm{ess}}(M) = \lim_k h_\varphi(M \smallsetminus K_k).$$

From Corollary 7.4, we have that

$$\lambda_0(S, M \smallsetminus K_k) - \lambda \geq \frac{1}{4} h_\varphi(M \smallsetminus K_k),$$

for any $k \in \mathbb{N}$. After taking the limit with respect to k, the statement follows from Proposition 4.23. □

Remark 7.6 The above arguments can be easily modified in order to obtain analogous results for manifolds with boundary. In that case, it suffices to consider a λ-eigenfunction of S which is positive and smooth only in the interior of M. Then, in Proposition 7.3, the infimum is taken over smooth functions with compact support in the interior of M.

Proof of Theorem 1.5: From Corollary 4.21, the first statement implies the second. From Corollary 4.20, the third statement follows from the second.

Assume that $\lambda_0(S_1) = \lambda_0^{\mathrm{ess}}(S_2)$, for some Schrödinger operator S_1 on M_1. From Proposition 4.19, there exists a positive $\lambda_0(S_1)$-eigenfunction $\varphi \in C^\infty(M_1)$ of S_1, and its lift $\hat{\varphi} \in C^\infty(M_2)$ is a positive $\lambda_0(S_1)$-eigenfunction of S_2. From Corollary 7.5, it follows that $h^{ess}_{\hat{\varphi}}(M_2) = 0$. Since φ is positive and M_1 is closed, this yields that $h^{ess}(M_2) = 0$.

Assume that $h^{ess}(M_2) = 0$. Then $h(M_2) = 0$ and Theorem 6.1 yields that p is amenable. Assume that p is finite sheeted. Then M_2 is closed. Consider a smoothly bounded domain U of M_2, such that $M_2 \setminus U$ is connected. Evidently, $M_2 \setminus U$ is a compact manifold with boundary. It is clear that $h(M_2 \setminus \overline{U}) = h(M_2 \setminus U)$. From [10], it follows that $h^{ess}(M_2) \geq h(M_2 \setminus U) > 0$, which is a contradiction. Hence, p is infinite sheeted. □

Remark 7.7 In Theorem 1.5, if the covering is normal, then $\sigma(S_1) \neq \sigma_{ess}(S_2)$. Indeed, if the equality holds, according to Corollary 1.3, we have $\sigma(S_1) = \sigma(S_2)$. Recall that the space of square-integrable, $\lambda_0(S_2)$-eigenfunctions of S_2 is either trivial or one-dimensional. From Corollary 1.3, it follows that $\lambda_0(S_2)$ is not an eigenvalue of the closure of S_2. From Proposition 2.1, $\sigma_{ess}(S_2)$ consists of eigenvalues of infinite multiplicity and accumulation points of $\sigma(S_2)$. Therefore, it follows that $\lambda_0(S_2)$ is an accumulation point of $\sigma(S_2)$. This is a contradiction, since $\sigma(S_2) = \sigma(S_1)$ is discrete.

For sake of completeness, we also prove the following corollary, describing the analogous properties for finite-sheeted coverings.

Corollary 7.8 *Let $p\colon M_2 \to M_1$ be a Riemannian covering with M_1 closed. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. Then the following are equivalent:*

(i) *p is finite sheeted,*
(ii) *$\sigma(S_1) \subset \sigma(S_2)$ and $\sigma_{ess}(S_2) = \emptyset$,*
(iii) *$\lambda_0(S_1) = \lambda_0(S_2) \notin \sigma_{ess}(S_2)$,*
(iv) *$h(M_2) = 0$ and $h^{ess}(M_2) \neq 0$.*

Proof If the covering is finite sheeted, the inclusion of spectra follows from Corollary 4.22. In this case, M_2 is closed, which yields that the spectrum of S_2 is discrete. From Corollary 4.20, the second statement implies the third.

Assume that the third statement holds. Since $\lambda_0(S_1) = \lambda_0(S_2)$, as in the proof of Theorem 1.5, from Corollary 7.4, it follows that $h(M_2) = 0$. From Theorem 1.5, it is clear that $h^{ess}(M_2) \neq 0$.

Assume that the fourth statement holds. Since $h(M_2) = 0$, from Theorem 6.1, p is amenable. Since $h^{ess}(M_2) \neq 0$, from Theorem 1.5, it follows that p is finite sheeted. □

The following characterization for points of the essential spectrum of a Schrödinger operator is an immediate consequence of the Decomposition Principle.

Proposition 7.9 *Let S be a Schrödinger operator on a complete Riemannian manifold M and let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{ess}(S)$ if and only if there exists $(f_k)_{k\in\mathbb{N}} \subset C^\infty_c(M)$, with $f_k = 0$ on ∂M, $\|f_k\|_{L^2(M)} = 1$, $(S-\lambda)f_k \to 0$ in $L^2(M)$, and for every compact $K \subset M$, there exists $k_0 \in \mathbb{N}$, such that* supp $f_k \cap K = \emptyset$*, for all $k \geq k_0$.*

Our second application is motivated by [1, Corollary 3.8].

Theorem 7.10 *Let $p\colon M_2 \to M_1$ be a Riemannian covering with M_2 simply connected and complete. Let S_1 be a Schrödinger operator on M_1 and S_2 its lift on M_2. If there exists a compact $K \subset M_1$, such that the image of the fundamental group of any connected component of $M_1 \smallsetminus K$ in $\pi_1(M_1)$ is amenable, then $\sigma_{\text{ess}}(S_1) \subset \sigma_{\text{ess}}(S_2)$.*

Proof Let $\lambda \in \sigma_{\text{ess}}(S_1)$. From Proposition 7.9, there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(M)$, such that $f_k = 0$ on ∂M_1, $\|f_k\|_{L^2(M_1)} = 1$, $(S - \lambda)f_k \to 0$ in $L^2(M_1)$, and for every compact $K_0 \subset M_1$, there exists $k_0 \in \mathbb{N}$, such that $\operatorname{supp} f_k \cap K_0 = \emptyset$, for all $k \geq k_0$.

Without loss of generality, we may assume that the supports of f_k are connected, since we may restrict each f_k to a connected component of its support and obtain a sequence with the same properties. Indeed, let K_i, with $1 \leq i \leq \ell(k)$ be the connected components of $\operatorname{supp} f_k$. Since they are disjoint, it is clear that

$$\begin{aligned}\frac{\|(S_1-\lambda)f_k\|^2_{L^2(M_1)}}{\|f_k\|^2_{L^2(M_1)}} &= \frac{\sum_{i=1}^{\ell(k)}\|(S_1-\lambda)(f|_{K_i})\|^2_{L^2(M_1)}}{\sum_{i=1}^{\ell(k)}\|f|_{K_i}\|^2_{L^2(M_1)}}\\ &\geq \min_{1\leq i\leq \ell(k)} \frac{\|(S_1-\lambda)(f|_{K_i})\|^2_{L^2(M_1)}}{\|f|_{K_i}\|^2_{L^2(M_1)}}.\end{aligned}$$

Let i_k be index for which the right-hand side minimum is achieved. Evidently, considering the normalization (in $L^2(M_1)$) of the restriction of f_k on K_{i_k}, instead of f_k, we obtain a sequence with the same properties as $(f_k)_{k\in\mathbb{N}}$, such that the supports are connected.

Consider a compact $K \subset M_1$, such that the image of the fundamental group of any connected component of $M_1 \smallsetminus K$ in $\pi_1(M_1)$ is amenable. Clearly, after passing to a subsequence, we may assume that the functions f_k are supported in $M_1 \smallsetminus K$. Since for any $k \in \mathbb{N}$, the support of f_k is connected, it follows that $\operatorname{supp} f_k \subset U_k$, where U_k is a connected component of $M_1 \smallsetminus K$. From the Lifting Theorem, it follows that the inclusion $U_k \hookrightarrow M_1$ can be lifted to the covering space $M_k' := M_2/\Gamma_k$, where Γ_k is the image of $\pi_1(U_k)$ in $\pi_1(M_1)$. In particular, any f_k can be lifted to some $f_k' \in C_c^\infty(M_k')$.

Since the covering $q_k\colon M_2 \to M_k'$ is normal with deck transformations group Γ_k, it follows that it is amenable. If q_k is finite sheeted, let $\tilde{f}_k$ be the normalized (in $L^2(M_2)$) lift of f_k' on M_2. If q_k is infinite sheeted, from Proposition 4.12, there exists $\tilde{f}_k \in C_c^\infty(M_2)$, such that $\|\tilde{f}_k\|_{L^2(M_2)} = 1$, $\operatorname{supp} \tilde{f}_k \subset q_k^{-1}(\operatorname{supp} f_k')$, and

$$\|(S_2-\lambda)\tilde{f}_k\|_{L^2(M_2)} \leq \|(S_k'-\lambda)f_k'\|_{L^2(M_k')} + \frac{1}{k} = \|(S_1-\lambda)f_k\|_{L^2(M_1)} + \frac{1}{k},$$

where S_k' is the lift of S_1 on M_k'. In particular, $(S_2-\lambda)\tilde{f}_k \to 0$ in $L^2(M_2)$ and $\operatorname{supp} \tilde{f}_k$ is contained in $p^{-1}(\operatorname{supp} f_k)$. From Proposition 7.9, it follows that $\lambda \in \sigma_{\text{ess}}(S_2)$. ◻

Remark 7.11 In the proof of Theorem 7.10, the only properties of Schrödinger operators used are essential self-adjointness and Proposition 7.9, which follows from the Decomposition Principle. Therefore, this proof establishes the analogous result for essentially self-adjoint differential operators, for which the Decomposition Principle holds (cf. [3]). For instance, if M_1 has empty boundary, then the statement of Theorem 7.10 holds for any elliptic differential operator D_1, such that D_1 and D_2 are essentially self-adjoint on the spaces of compactly supported smooth sections.

Proof of Corollary 1.6: Follows immediately from Theorem 7.10 and Corollary 4.20. □

Let $p\colon M_2 \to M_1$ be a Riemannian covering of complete manifolds, without boundary. As stated in the Introduction, there are examples where p is non-amenable and $\lambda_0(M_1) = \lambda_0(M_2)$. From Theorem 1.1, Propositions 4.17 and 2.1, if p is amenable, then $\sigma(M_1) \subset \sigma(M_2)$. It is natural to examine if this inclusion implies amenability of the covering. From Theorem 7.10, it is easy to construct an example of a non-amenable, normal Riemannian covering $p\colon M_2 \to M_1$ with M_1 complete, with bounded geometry and of finite topological type (that is, M_1 admits a finite triangulation, where the simplices are defined on the standard simplex with possibly some lower-dimensional faces removed), such that $\sigma(M_1) = \sigma(M_2)$.

Example 7.12 Let M_1 be a 2-dimensional torus with a cusp, endowed with a Riemannian metric, such that M_1 is complete and outside a compact set the metric is the standard metric of the flat cylinder. It is clear that M_1 is of finite topological type and has bounded geometry. From [19, Theorem 1], it follows that $\sigma_{\text{ess}}(M_1) = [0, +\infty)$. Clearly, there exists a compact subset K of M_1, such that $\pi_1(M_1 \smallsetminus K) = \mathbb{Z}$. From Theorem 7.10, it follows that for the simply connected covering space M_2 of M_1, we have $\sigma_{\text{ess}}(M_2) = [0, +\infty)$. However, $\pi_1(M_1)$ is the free group in two generators, which is non-amenable (cf. [4, Sect. 2]).

For our next application, we need the following standard lemma for the spectrum of self-adjoint operators (see, for instance, [17]).

Lemma 7.13 *Let $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Assume that for some $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, there exists $v \in \mathcal{D}(A)$, with $\|v\| = 1$ and $\|(A-\lambda)v\| < \varepsilon$. Then there exists $\lambda' \in \sigma(A)$, such that $|\lambda - \lambda'| < \varepsilon$.*

Proposition 7.14 *Let M be a closed manifold with infinite, amenable, and residually finite fundamental group. Then the spectrum of the Laplacian on the universal covering space $\tilde{M}$ of M is given by*

$$\sigma(\tilde{M}) = \sigma_{\text{ess}}(\tilde{M}) = \overline{\cup_\alpha \sigma(M_\alpha)},$$

where the union is taken over all finite-sheeted covering spaces M_α of M.

Proof Since $\pi_1(M)$ is infinite, from Corollary 1.3, it follows that $\sigma(\tilde{M}) = \sigma_{\text{ess}}(\tilde{M})$. Since $p\colon \tilde{M} \to M$ is infinite sheeted and amenable, from Theorem 1.5, it follows that $\lambda_0^{\text{ess}}(\tilde{M}) = 0$. Let M_α be a finite-sheeted covering space of M. Since M_α is closed and

$\lambda_0^{\mathrm{ess}}(\tilde{M}) = 0$, from Theorem 1.5 (applied to the covering $p_\alpha \colon \tilde{M} \to M_\alpha$), it follows that $\sigma(M_\alpha) \subset \sigma_{\mathrm{ess}}(\tilde{M})$. Since $\sigma(\tilde{M})$ is closed, this yields that

$$\overline{\cup_\alpha \sigma(M_\alpha)} \subset \sigma(M),$$

where the union is taken over all finite-sheeted covering spaces M_α of M.

Consider $\lambda \in \sigma(\tilde{M})$ and $\varepsilon > 0$. From Proposition 2.1 and Lemma 2.2, there exists $f \in C_c^\infty(\tilde{M})$, with $\|f\|_{L^2(\tilde{M})} = 1$ and $\|(\Delta - \lambda)f\|_{L^2(\tilde{M})} < \varepsilon$. Since $\pi_1(M_1)$ is residually finite, for any compact $K \subset \tilde{M}$, there exists a finite-sheeted covering space M_α of M, such that the covering $p_\alpha \colon \tilde{M} \to M_\alpha$ restricted on K is injective (see for instance [8]). In particular, there exists a finite-sheeted covering space M_α of M, such that the covering $p_\alpha \colon \tilde{M} \to M_\alpha$ restricted in a compact neighborhood K of supp f is an isometry onto its image. Consider the function $f_\alpha := f \circ p_\alpha|_K^{-1}$ extended by zero outside $p_\alpha(K)$. Evidently, $f_\alpha \in C^\infty(M_\alpha)$ and satisfies

$$\|f_\alpha\|_{L^2(M_\alpha)} = 1 \text{ and } \|(\Delta - \lambda)f_\alpha\|_{L^2(M_\alpha)} < \varepsilon.$$

From Lemma 7.13, it follows that there exists $\lambda' \in \sigma(M_\alpha)$ with $|\lambda - \lambda'| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this establishes the asserted equality. □

Recall that the spectrum of the Laplacian on a closed Riemannian manifold M is discrete; that is, it consists of isolated eigenvalues

$$0 = \lambda_0(M) < \lambda_1(M) < \lambda_2(M) < \dots,$$

of finite multiplicity. From the above proposition, we can easily recover the following observation of Sunada [24], which was also established by Brooks [8, Theorem 2].

Corollary 7.15 *Let M be a closed manifold with infinite, amenable, and residually finite fundamental group. Then there exists a sequence $(M_i)_{i \in \mathbb{N}}$ of finite-sheeted covering spaces of M, such that $\lambda_1(M_i) \to 0$, as $i \to +\infty$.*

Proof From Proposition 7.14, it follows that $\lambda_0^{\mathrm{ess}}(\tilde{M}) = 0$, where $\tilde{M}$ is the universal covering space of M. Since $p \colon \tilde{M} \to M$ has infinite deck transformations group, from Corollary 5.4, we obtain that zero is an accumulation point of $\sigma(\tilde{M})$. From Proposition 7.14, it follows that there exist finite-sheeted covering spaces M_i of M, with $i \in \mathbb{N}$, and $k_i \in \mathbb{N}$, such that $\lambda_{k_i}(M_i) \to 0$, as $i \to +\infty$. Since $0 < \lambda_1(M_i) \leq \lambda_{k_i}(M_i)$, for any $i \in \mathbb{N}$, this completes the proof. □

We now present some examples of amenable coverings. The following observation, provides a sufficient geometric condition for amenability of coverings.

Proposition 7.16 *Let M_1 be a complete Riemannian manifold, without boundary and with non-negative Ricci curvature. Then any covering $p \colon M_2 \to M_1$ is amenable.*

Proof Let $\tilde{M}$ be the simply connected covering space of M_1. From the Bishop–Gromov Comparison Theorem, it follows that $\tilde{M}$ has polynomial growth and hence, every finitely generated subgroup of $\pi_1(M_1)$ has polynomial growth (cf. [20]). From Corollary 2.9, it follows that every finitely generated subgroup of $\pi_1(M_1)$ is amenable and Corollary 2.10 yields that so is $\pi_1(M_1)$. Therefore, any covering $p\colon M_2 \to M_1$ is amenable. □

Example 7.17 Let M be a Riemannian manifold and denote by $\tilde{M}$ its universal covering space. The homology cover of M is defined by

$$M_H := \tilde{M}/[\pi_1(M), \pi_1(M)].$$

Evidently, the Riemannian covering $p\colon M_H \to M$ is normal with deck transformations group

$$\Gamma = \pi_1(M)/[\pi_1(M), \pi_1(M)] = H_1(M).$$

Since $H_1(M)$ is abelian, from Corollary 2.11, it follows that $p\colon M_H \to M$ is amenable.

Next, we present an example of an infinite-sheeted amenable covering with trivial deck transformations group. In particular, this implies that the results of Sect. 5 cannot be applied to arbitrary infinite-sheeted amenable coverings.

Example 7.18 Let Γ_1 be the countable group of invertible, upper triangular 2×2 matrices with entries in $\mathbb{Q}$ and let M_1 be a Riemannian manifold with $\pi_1(M_1) = \Gamma_1$ (cf. [2, Sect. 5]). Let Γ_2 be the subgroup of Γ_1 consisting of diagonal matrices. Denote by $\tilde{M}$ the simply connected covering space of M_1 and consider $M_2 := \tilde{M}/\Gamma_2$. It is easy to see that the covering $p\colon M_2 \to M_1$ is infinite sheeted and does not have non-trivial deck transformations. However, Γ_1 is solvable and in particular, amenable (from Corollary 2.11), which yields that p is an amenable covering.

Recall that in our main results there are no assumptions on the vector bundles, the connections, and the differential operators. The next example demonstrates that these play a crucial role in the behavior of the spectrum even under finite-sheeted coverings. Namely, this example shows that whether or not the bottom of the spectrum of the connection Laplacian is preserved under a Riemannian covering depends on the corresponding metric connection. Moreover, this example demonstrates that the inequality of Corollary 4.20, which holds for Schrödinger operators, is not true (in general) for the connection Laplacian.

Let M be a complete Riemannian manifold and $E \to M$ a Riemannian vector bundle endowed with a metric connection ∇. The corresponding connection Laplacian Δ (considered as in Examples 7.1(iv)) is essentially self-adjoint (cf. [18]). In the following example, we denote by $\lambda_0(\Delta, E)$ the bottom of the spectrum of its closure. It is worth to point out that if M is closed, then the spectrum of this operator is discrete (cf. [18]).

Example 7.19 Consider $S_1 := \mathbb{R}/\mathbb{Z}$ and the trivial bundle $E_1 := S_1 \times \mathbb{R}^2$ with the standard metric. We can identify smooth sections of E_1 with smooth maps $f\colon \mathbb{R} \to \mathbb{R}^2$

with $f(x) = f(x+1)$, for all $x \in \mathbb{R}$. For $\phi \in \mathbb{R}$, consider the metric connection ∇^{ϕ}, defined by

$$\nabla^{\phi}_{\frac{d}{dx}} f(x) := \begin{pmatrix} \cos(x\phi) & -\sin(x\phi) \\ \sin(x\phi) & \cos(x\phi) \end{pmatrix} \frac{d}{dx} \begin{pmatrix} \cos(x\phi) & \sin(x\phi) \\ -\sin(x\phi) & \cos(x\phi) \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

for any smooth section $f = (f_1, f_2)$ of E_1. Since the spectrum of the connection Laplacian Δ^{ϕ} is discrete for any $\phi \in \mathbb{R}$, it is clear that $\lambda_0(\Delta^{\phi}, E_1) = 0$ if and only if there exists a parallel section of E_1 with respect to ∇^{ϕ}, or equivalently, $\phi = 2k\pi$, for some $k \in \mathbb{Z}$.

For $k \in \mathbb{N} \setminus \{1\}$, consider a k-sheeted Riemannian covering $p_k \colon S_1^{(k)} \to S_1$ and the pullback bundle E_2 of E_1 endowed with the standard metric and the pullback connection ∇^{ϕ}. It is clear that $\lambda_0(\Delta^{2\pi}, E_2) = \lambda_0(\Delta^{2\pi}, E_1) = 0$. However, the above arguments imply that $\lambda_0(\Delta^{2\pi/k}, E_2) = 0 < \lambda_0(\Delta^{2\pi/k}, E_1)$.

This is an example of a finite-sheeted covering which shows that the inequality of Corollary 4.20 does not hold for the connection Laplacian. Based on this example, it is easy to construct an analogous example of an infinite-sheeted covering. Consider the covering $p \colon \mathbb{R} \to S_1^{(k)}$ and the pullback bundle E of E_2 endowed with the standard metric and the pullback connection $\nabla^{2\pi/k}$. Since p is infinite sheeted and amenable, from Theorem 1.2, since the connection Laplacian is non-negative definite, it follows that $\lambda_0(\Delta^{2\pi/k}, E) = \lambda_0^{\mathrm{ess}}(\Delta^{2\pi/k}, E) = 0 < \lambda_0(\Delta^{2\pi/k}, E_1)$.

A natural question arising from our results is whether it is possible to obtain an analog of Theorem 1.1 for Friedrichs extensions of operators (that is, in the context of Theorem 1.2). It is worth to point out that this holds the Laplacian on manifolds which are isometric to the interior of complete manifolds with boundary. Indeed, in such case, the spectrum of the Friedrichs extension of the Laplacian in the interior coincides with the Dirichlet spectrum of the Laplacian on the manifold with boundary. Since the latter one is essentially self-adjoint, the inclusion of the spectra follows from Theorem 1.1.

Moreover, during the last years, there is a lot of progress in the study of the Dirichlet-to-Neumann spectrum. Although the Dirichlet-to-Neumann map is not a differential operator, there are interesting relations between its spectrum and the geometry and topology of the underlying manifold. Therefore, it is natural to ask whether similar results hold for its behavior under Riemannian coverings. This is indeed the case. However, the methods to establish them are quite different. Therefore, we will deal with this in a forthcoming paper.

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