Addendum to: Explicit Absolute Parallelism for 2-Nondegenerate Real Hypersurfaces $M^5 \subset \mathbb{C}^3$ of Constant Levi Rank 1

Joël Merker¹ · Samuel Pocchiola¹

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Addendum to: J Geom Anal

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1 Introduction

The article [3] and its preprint version [4] (incorporating more calculations) study $\mathcal{C}^\omega$ or $\mathcal{C}^\infty$ 2-nondegenerate graphed hypersurfaces $M^5 \subset \mathbb{C}^3$:

$$u = F(z_1, z_2, \bar{z}_1, \bar{z}_2, v), \quad (w = u + i v)$$

whose Levi form has rank 1 at every point. With no details of proof, the final Sect. 9 of [3, pp. 42–43] states that when Pocchiola's two fundamental differential invariants $J \equiv 0 \equiv W$ of these CR structures vanish identically, a certain collection of ten 1-forms:

$$\{\rho, \kappa, \zeta, \bar{\kappa}, \bar{\zeta}, \pi^1, \pi^2, \bar{\pi}^1, \bar{\pi}^2, \Lambda\}, \quad (\bar{\rho} = \rho, \bar{\Lambda} = \Lambda)$$

enjoy a Lie–Cartan structure having *constant* coefficients (conjugate equations are unwritten):

$$\begin{aligned} d\rho &= \pi^1 \wedge \rho + \bar{\pi}^1 \wedge \rho + i \kappa \wedge \bar{\kappa}, \\ d\kappa &= \pi^1 \wedge \kappa + \pi^2 \wedge \rho + \zeta \wedge \bar{\kappa}, \end{aligned}$$

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✉ Joël Merker
joel.merker@math.u-psud.fr

Samuel Pocchiola
samuel.pocchiola@math.u-psud.fr

¹ Département de Mathématiques, Faculté des sciences d'Orsay, Université Paris-Sud, Bâtiment 425, 91405 Orsay Cedex, France

$$\begin{aligned}
d\zeta &= i\pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \bar{\pi}^1 \wedge \zeta, \\
d\pi^1 &= i\kappa \wedge \bar{\pi}^2 + \zeta \wedge \bar{\zeta} + \Lambda \wedge \rho, \\
d\pi^2 &= \pi^2 \wedge \bar{\pi}^1 + \zeta \wedge \bar{\pi}^2 + \Lambda \wedge \kappa, \\
d\Lambda &= i\pi^2 \wedge \bar{\pi}^2 + \Lambda \wedge \pi^1 + \Lambda \wedge \bar{\pi}^1.
\end{aligned}$$

Furthermore, these equations are exactly equal to the Lie structure (shown in [5]) of the model light cone M_{LC} :

$$u = \frac{z_1 \bar{z}_1 + \frac{1}{2} z_1^2 \bar{z}_2 + \frac{1}{2} \bar{z}_1^2 z_2}{1 - z_2 \bar{z}_2},$$

so that by general Cartan theory, hypersurfaces having $J \equiv 0 \equiv W$ are *all* (locally) biholomorphic to M_{LC} .

Importantly, Pocchiola discovered that *before* prolongation of the equivalence problem, the structure equations are of the form:

$$\begin{aligned}
d\rho &= (\pi^1 + \bar{\pi}^1) \wedge \rho + i\kappa \wedge \bar{\kappa}, \\
d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \bar{\kappa}, \\
d\zeta &= (\pi^1 - \bar{\pi}^1) \wedge \zeta + i\pi^2 \wedge \kappa + \\
&\quad + R\rho \wedge \zeta + \frac{i}{\bar{c}} \bar{J}\rho \wedge \bar{\kappa} + \frac{1}{c} W\kappa \wedge \zeta.
\end{aligned}$$

Furthermore, he showed that the real function $R \equiv 0$ vanishes identically when $W \equiv 0$, so that when $J \equiv 0 \equiv W$, these equations reduce to constant coefficients:

$$\begin{aligned}
d\rho &= (\pi^1 + \bar{\pi}^1) \wedge \rho + i\kappa \wedge \bar{\kappa}, \\
d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \bar{\kappa}, \\
d\zeta &= (\pi^1 - \bar{\pi}^1) \wedge \zeta + i\pi^2 \wedge \kappa.
\end{aligned} \tag{1}$$

At the end of [4], he showed with elementary reasonings that after prolongation, one obtains the last 3 structure equations above for $d\pi^1$, $d\pi^2$, $d\Lambda$, in which *no* nonconstant structure functions appear.

This phenomenon is in some sense ‘counterintuitive’ to CR geometers, since for Levi nondegenerate CR structures, and for the corresponding second-order PDE systems, *no* curvatures appear after absorption before prolongation:

$$\begin{aligned}
d\omega &= \sum_{\alpha} \omega^{\alpha} \wedge \omega_{\alpha} + \omega \wedge \varphi, \\
d\omega^{\alpha} &= \sum_{\beta} \omega^{\beta} \wedge \varphi_{\beta}^{\alpha} + \omega \wedge \varphi^{\alpha}, \\
d\omega_{\alpha} &= \sum_{\beta} \varphi_{\alpha}^{\beta} \wedge \omega_{\beta} + \omega_{\alpha} \wedge \varphi + \omega \wedge \varphi_{\alpha},
\end{aligned}$$

while primary and secondary invariants appear afterwards, *e.g.*, like $S_{\beta\rho}^{\alpha\sigma}$ and $R_{\beta\gamma}^{\alpha}$, $T_{\beta}^{\alpha\gamma}$ in:

$$\begin{aligned} d\varphi_\beta^\alpha &= \frac{1}{2} \delta_\beta^\alpha \psi \wedge \omega - \varphi_\beta^\gamma \wedge \varphi_\gamma^\alpha - \varphi_\beta \wedge \omega^\alpha - \varphi^\alpha \wedge \omega_\beta + \delta_\beta^\alpha \omega^\gamma \wedge \varphi_\gamma \\ &\quad + S_{\beta\rho}^{\alpha\sigma} \omega^\rho \wedge \omega_\sigma + R_{\beta\gamma}^\alpha \omega^\gamma \wedge \omega + T_\beta^{\alpha\gamma} \omega_\gamma \wedge \omega. \end{aligned}$$

This addendum is devoted to expose the structure equations for $d\pi^1$, $d\pi^2$, $d\Lambda$, the details of which were completely skipped in [3, Sect. 9], hence asked to appear in print by some experts. At the end of Sect. 4, Theorem 16 summarizes all constructions.

2 Summary of Pocchiola's Constructions Up to the Assumption $J \equiv 0 \equiv W$

This brief technical section is independent of the rest of the present addendum, and serves only as a (nonself-contained) reminder. Once the mentioned equations (1) will be reached at the end of this section, all considerations will be self-contained.

In addition to J and W , a certain derived invariant also occurs in [3,4], which, after absorption of its real part in some modified Maurer–Cartan form, reduces to:

$$W_{\rho\zeta}^\zeta = \frac{1}{c\bar{c}} \left(-\frac{i}{6} \left(\frac{\overline{\mathcal{L}_1}(\overline{\mathcal{L}_1}(k))}{\mathcal{L}_1(k)} - \overline{P} \right) W - \frac{i}{2} \overline{\mathcal{L}_1}(W) \right) + i \frac{e}{c\bar{c}} W,$$

whence $W_{\rho\zeta}^\zeta \equiv 0$ as soon as $W \equiv 0$.

In the product manifold $M^5 \times G^4$ corresponding to the 4-dimensional G -structure obtained in [3, Sect. 6] after several reductions and computations, which is equipped with coordinates:

$$(z_1, z_2, \bar{z}_1, \bar{z}_2, v) \times (c, \bar{c}, e, \bar{e}),$$

the Maurer–Cartan forms are:

$$\delta^1 := \frac{dc}{c}, \quad \delta^2 := i \frac{e dc}{c\bar{c}} - i \frac{e d\bar{c}}{c\bar{c}} - i \frac{de}{c},$$

together with their conjugates $\bar{\delta}^1, \bar{\delta}^2$. Furthermore, the modified Maurer–Cartan forms introduced by Pocchiola are:

$$\begin{aligned} \widehat{\delta}^1 &:= \delta^1 + \frac{1}{2} V_{\rho\zeta}^\zeta \rho - \overline{V}_{\rho\kappa}^\rho \kappa - V_{\rho\zeta}^\rho \zeta - V_{\kappa\bar{\kappa}}^\kappa \bar{\kappa}, \\ \widehat{\delta}^2 &:= \delta^2 - V_{\rho\kappa}^\zeta \rho - (V_{\rho\kappa}^\kappa - \frac{1}{2} V_{\rho\zeta}^\zeta) \kappa - V_{\rho\zeta}^\kappa \zeta - V_{\rho\bar{\kappa}}^\kappa \bar{\kappa} - V_{\rho\zeta}^\kappa \bar{\zeta}, \end{aligned}$$

and they satisfy the neat structure equations:

$$\begin{aligned} d\rho &= (\widehat{\delta}^1 + \widehat{\delta}^1) \wedge \rho + i \kappa \wedge \bar{\kappa}, \\ d\kappa &= \widehat{\delta}^2 \wedge \rho + \widehat{\delta}^1 \wedge \kappa + \zeta \wedge \bar{\kappa}, \\ d\zeta &= i \widehat{\delta}^2 \wedge \kappa + \widehat{\delta}^1 \wedge \zeta - \widehat{\delta}^1 \wedge \zeta. \end{aligned}$$

Lastly, with a single real free variable $\mathbf{t} = -\Re w_\rho^1$ which parametrizes the 1-dimensional prolongation of this G -structure, introducing:

$$\pi^1 := \widehat{\delta}^1 + \mathbf{t} \rho,$$

$$\pi^2 := \widehat{\delta}^2 + \mathbf{t} \kappa,$$

the structure equations under the assumption $J \equiv 0 \equiv W$ become, as was already seen:

$$d\rho = (\pi^1 + \bar{\pi}^1) \wedge \rho + i \kappa \wedge \bar{\kappa}, \quad (2)$$

$$d\kappa = \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \bar{\kappa}, \quad (3)$$

$$d\zeta = i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \bar{\pi}^1 \wedge \bar{\zeta}, \quad (4)$$

together with the conjugate equations $d\bar{\kappa} = \bar{\pi}^2 \wedge \rho + \dots$ and $d\bar{\zeta} = -i \bar{\pi}^2 \wedge \bar{\kappa} + \dots$, while $d\bar{\rho} = d\rho$.

We now proceed to explain why the structure equations for $d\pi^1$ and $d\pi^2$ incorporate only *constant* coefficients, as well as the structure equations for the exterior differential $d\Lambda$, where $\Lambda = \bar{\Lambda}$ will be a final prolonged real 1-form completing an $\{e\}$ -structure on $M^5 \times G^4 \times \mathbb{R}$.

3 The 2-Forms Ω_1 and Ω_2

On the manifold $M^5 \times G^4 \times \mathbb{R}$, we thus start from (2), (3), (4). All reasonings will be elementary, and rely upon Poincaré's relation $d \circ d = 0$ and upon several applications of the famous *Cartan Lemma* which helps to bypass hard explicit computations of Pocchiola's style.

Lemma 5 *The differential 2-forms:*

$$\Omega_1 := d\pi^1 - i \kappa \wedge \bar{\pi}^2 - \zeta \wedge \bar{\zeta},$$

$$\Omega_2 := d\pi^2 - \pi^2 \wedge \bar{\pi}^1 - \zeta \wedge \bar{\pi}^2,$$

satisfy:

$$0 = (\Omega_1 + \bar{\Omega}_1) \wedge \rho, \quad (6)$$

$$0 = \Omega_2 \wedge \rho + \Omega_1 \wedge \kappa, \quad (7)$$

$$0 = (\Omega_1 - \bar{\Omega}_1) \wedge \zeta + i \Omega_2 \wedge \kappa. \quad (8)$$

Proof Using the Poincaré relation $d \circ d = 0$, apply the exterior differentiation operator d to (2), and replace $d\rho$, $d\kappa$, $d\bar{\kappa}$ by means of (2), (3):

$$\begin{aligned} 0 &= d \circ d\rho = (d\pi^1 + d\bar{\pi}^1) \wedge \rho - (\pi^1 + \bar{\pi}^1) \wedge d\rho + i d\kappa \wedge \bar{\kappa} - i \kappa \wedge d\bar{\kappa} \\ &= (d\pi^1 + d\bar{\pi}^1) \wedge \rho - (\pi^1 + \bar{\pi}^1) \wedge \left((\pi^1 + \bar{\pi}^1) \wedge \rho + i \kappa \wedge \bar{\kappa} \right) \end{aligned}$$

$$+i \left(\pi^2 \wedge \rho + \pi^1 \wedge \kappa + \underline{\zeta \wedge \bar{\kappa}}_o \right) \wedge \bar{\kappa} - i \kappa \wedge \left(\bar{\pi}^2 \wedge \rho + \bar{\pi}^1 \wedge \bar{\kappa} + \bar{\zeta} \wedge \kappa \right),$$

which becomes after simplification:

$$0 = \left(d\pi^1 - i \kappa \wedge \bar{\pi}^2 + d\bar{\pi}^1 + i \bar{\kappa} \wedge \pi^2 \right) \wedge \rho,$$

and proves (6), since $-\zeta \wedge \bar{\zeta}$ is purely imaginary.

For (7), proceed similarly with (3):

$$\begin{aligned} 0 &= d \circ d\kappa = d\pi^2 \wedge \rho - \pi^2 \wedge d\rho + d\pi^1 \wedge \kappa - \pi^1 \wedge d\kappa + d\zeta \wedge \bar{\kappa} - \zeta \wedge d\bar{\kappa} \\ &= d\pi^2 \wedge \rho - \pi^2 \wedge \left((\pi^1 + \bar{\pi}^1) \wedge \rho + i \kappa \wedge \bar{\kappa} \right) \\ &\quad + d\pi^1 \wedge \kappa - \pi^1 \wedge (\pi^2 \wedge \rho + \zeta \wedge \bar{\kappa}) \\ &\quad + \left(i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \bar{\pi}^1 \wedge \zeta \right) \wedge \bar{\kappa} - \zeta \wedge \left(\bar{\pi}^2 \wedge \rho + \bar{\pi}^1 \wedge \bar{\kappa} + \bar{\zeta} \wedge \kappa \right), \end{aligned}$$

and receive, after simplification and reorganization:

$$0 = \left(d\pi^2 - \pi^2 \wedge \bar{\pi}^1 - \zeta \wedge \bar{\pi}^2 \right) \wedge \rho + \left(d\pi^1 - \zeta \wedge \bar{\zeta} \right) \wedge \kappa,$$

which is (7), since $(-i \kappa \wedge \bar{\pi}^2) \wedge \kappa = 0$.

Lastly, the Poincaré relation applied to (4):

$$\begin{aligned} 0 &= d \circ d\zeta = i d\pi^2 \wedge \kappa - i \pi^2 \wedge d\kappa + d\pi^1 \wedge \zeta - \pi^1 \wedge d\zeta - d\bar{\pi}^1 \wedge \zeta + \bar{\pi}^1 \wedge d\zeta \\ &= i d\pi^2 \wedge \kappa - i \pi^2 \wedge (\pi^1 \wedge \kappa + \zeta \wedge \bar{\kappa}) + d\pi^1 \wedge \zeta \\ &\quad - \pi^1 \wedge (i \pi^2 \wedge \kappa - \bar{\pi}^1 \wedge \zeta) - d\bar{\pi}^1 \wedge \zeta + \bar{\pi}^1 \wedge (i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta), \end{aligned}$$

contracts as and rewrites as (8):

$$\begin{aligned} 0 &= \left(d\pi^1 - d\bar{\pi}^1 - i \bar{\kappa} \wedge \pi^2 \right) \wedge \zeta + i \left(d\pi^2 - \pi^2 \wedge \bar{\pi}^1 \right) \wedge \kappa \\ &= \left(d\pi^1 - i \kappa \wedge \bar{\pi}^2_A - d\bar{\pi}^1 - i \bar{\kappa} \wedge \pi^2 \right) \wedge \zeta + i \left(d\pi^2 - \pi^2 \wedge \bar{\pi}^1 - \underline{\zeta \wedge \bar{\pi}^2}_A \right) \wedge \kappa \\ &= (\Omega - \bar{\Omega}_1) \wedge \zeta + i \Omega_2 \wedge \kappa. \end{aligned}$$

□

Lemma 9 On $M^5 \times G^4 \times \mathbb{R}$, the solutions in 2-forms Ω_1, Ω_2 of the three equations:

$$0 = (\Omega_1 + \bar{\Omega}_1) \wedge \rho, \quad (10)$$

$$0 = \Omega_2 \wedge \rho + \Omega_1 \wedge \kappa, \quad (11)$$

$$0 = (\Omega_1 - \bar{\Omega}_1) \wedge \zeta + i \Omega_2 \wedge \kappa. \quad (12)$$

are:

$$\begin{aligned}\Omega_1 &= \Lambda \wedge \rho, \\ \Omega_2 &= \Lambda \wedge \kappa + h \rho \wedge \kappa,\end{aligned}$$

where $\overline{\Lambda} = \Lambda$ is a certain real 1-form, and where $h = -\overline{h}$ is a certain purely imaginary function.

At this step, in a passage with no details of proof, Pocchiola missed h in [4], but a bit later we will show that $h = 0$ anyway thanks to some other constraints.

Proof Wedge (11) $\wedge\kappa$:

$$0 = \Omega_2 \wedge \rho \wedge \kappa + 0, \quad (13)$$

hence by the Cartan Lemma, there exist two 1-forms γ and δ such that:

$$\Omega_2 = \gamma \wedge \rho + \delta \wedge \kappa. \quad (14)$$

Next, wedge (10) $\wedge\zeta$ and wedge (12) $\wedge\rho$, taking account of (13):

$$\left. \begin{aligned} 0 &= (\Omega_1 + \overline{\Omega}_1) \wedge \rho \wedge \zeta, \\ 0 &= -(\Omega_1 - \overline{\Omega}_1) \wedge \rho \wedge \zeta + 0 \end{aligned} \right\} \implies 0 = \Omega_1 \wedge \rho \wedge \zeta,$$

hence by the Cartan Lemma again, there exist two 1-forms α and β such that:

$$\Omega_1 = \alpha \wedge \rho + \beta \wedge \zeta.$$

Insert this Ω_1 into (11) $\wedge\rho$:

$$0 = 0 + (\alpha \wedge \rho + \beta \wedge \zeta) \wedge \kappa \wedge \rho = -\beta \wedge \rho \wedge \kappa \wedge \zeta,$$

whence by the Cartan Lemma, there exist certain functions A, B, C such that:

$$\beta = A \rho + B \kappa + C \zeta.$$

We may assume $C = 0$, since this does not change Ω_1 , and we assert that $B = 0$. Indeed, replacing Ω_1 in (10) shows $B = 0 = \overline{B}$:

$$\begin{aligned} 0 &= (\underline{\alpha \wedge \rho}_\circ + \beta \wedge \zeta + \underline{\overline{\alpha} \wedge \rho}_\circ + \overline{\beta} \wedge \overline{\zeta}) \wedge \rho \\ &= (\underline{A \rho \wedge \zeta}_\circ + B \kappa \wedge \zeta + 0 + \underline{\overline{A} \rho \wedge \overline{\zeta}}_\circ + \overline{B} \overline{\kappa} \wedge \overline{\zeta} + 0) \wedge \rho. \end{aligned}$$

Hence $\beta = A \rho$ and we get:

$$\Omega_1 = (\alpha - A \zeta) \wedge \rho =: \tau \wedge \rho,$$

in terms of a new 1-form τ . But by inserting this Ω_1 into (12) $\wedge\kappa$, we get:

$$0 = (-\tau + \bar{\tau}) \wedge \rho \wedge \kappa \wedge \zeta + 0,$$

and the Cartan Lemma (again!) gives:

$$\tau - \bar{\tau} = U\rho + V\kappa + W\zeta \iff \bar{\tau} - \tau = \bar{U}\rho + \bar{V}\kappa + \bar{W}\bar{\zeta},$$

and since $\{\rho, \kappa, \zeta, \bar{\kappa}, \bar{\zeta}\}$ are linearly independent at every point, we see that $V = W = \bar{V} = \bar{W} = 0$ necessarily, while the function $U = -\bar{U}$ must be purely imaginary, and hence we can write:

$$\tau - \bar{\tau} = U\rho = \left(\frac{U-\bar{U}}{2}\right)\rho.$$

Consequently, if we introduce the 1-form:

$$\Lambda := \tau - \frac{1}{2}U\rho = \bar{\Lambda},$$

which is now *real*, we indeed obtain as announced:

$$\Omega_1 = \Lambda \wedge \rho.$$

Next, inserting this into (11):

$$0 = \Omega_2 \wedge \rho + \Lambda \wedge \rho \wedge \kappa = (\Omega_2 - \Lambda \wedge \kappa) \wedge \rho,$$

on application of the Cartan Lemma, we receive a 1-form γ such that:

$$\Omega_2 - \Lambda \wedge \kappa = \gamma \wedge \rho,$$

so that above in (14), we had in fact $\delta = \Lambda$! Next, insert all this into (12):

$$0 = (\underline{\Lambda \wedge \rho - \Lambda \wedge \rho_o}) \wedge \zeta + i(\underline{\Lambda \wedge \kappa_o} + \gamma \wedge \rho) \wedge \kappa = i\gamma \wedge \rho \wedge \kappa.$$

The Cartan Lemma gives two functions G and K with:

$$\gamma = G\rho + K\kappa.$$

But then:

$$\begin{aligned} \Omega_2 &= \Lambda \wedge \kappa + G\underline{\rho \wedge \rho_o} - (\Re K + i\Im K)\rho \wedge \kappa \\ &= (\Lambda - \Re K \cdot \rho) \wedge \kappa - i\Im K \rho \wedge \kappa, \end{aligned}$$

hence by redefining $\Lambda := \Lambda - \Re K \cdot \rho$ which leaves Ω_1 untouched, and by setting $h := -i\Im K$, we conclude this detailed proof. $\square$

4 Reduction to an $\{e\}$ -Structure with Constant Coefficients

Changing the notation by setting $h =: i H$ with $H = \overline{H}$ being a real-valued function, we have therefore obtained:

$$\begin{aligned} d\pi^1 &= \Lambda \wedge \rho - i \overline{\pi}^2 \wedge \kappa + \zeta \wedge \overline{\zeta}, \\ d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \overline{\pi}^1 - \overline{\pi}^2 \wedge \zeta + i H \rho \wedge \kappa. \end{aligned}$$

Lemma 15 *The function $H = -i h$ vanishes identically.*

Proof Thanks to the Poincaré relation:

$$\begin{aligned} 0 &= d \circ d\pi^1 \\ &= d\Lambda \wedge \rho - \Lambda \wedge d\rho - i d\overline{\pi}^2 \wedge \kappa + i \overline{\pi}^2 \wedge d\kappa + d\zeta \wedge \overline{\zeta} - \zeta \wedge d\overline{\zeta} \\ &= d\Lambda \wedge \rho - \Lambda \wedge \left((\pi^1 + \overline{\pi}^1) \wedge \rho + i \kappa \wedge \overline{\kappa} \right) \\ &\quad - i \left(\Lambda \wedge \overline{\kappa} + \overline{\pi}^2 \wedge \pi^1 - \pi^2 \wedge \overline{\zeta} - i H \rho \wedge \overline{\kappa} \right) \wedge \kappa \\ &\quad + i \overline{\pi}^2 \wedge \left(\pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa} \right) + \left(i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta \right) \wedge \overline{\zeta} \\ &\quad - \zeta \wedge \left(-i \overline{\pi}^2 \wedge \overline{\kappa} + \overline{\pi}^1 \wedge \overline{\zeta} - \pi^1 \wedge \overline{\zeta} \right), \end{aligned}$$

we obtain:

$$0 = \underbrace{\left(d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi}^1 - i \pi^2 \wedge \overline{\pi}^2 \right)}_{=: \Phi} \wedge \rho + H \rho \wedge \kappa \wedge \overline{\kappa}.$$

Similarly,

$$\begin{aligned} 0 &= d \circ d\pi^2 \\ &= d\Lambda \wedge \kappa - \Lambda \wedge d\kappa + d\pi^2 \wedge \overline{\pi}^1 - \pi^2 \wedge d\overline{\pi}^1 - d\overline{\pi}^2 \wedge \zeta + \overline{\pi}^2 \wedge d\zeta \\ &\quad + i dH \wedge \rho \wedge \kappa + i H d\rho \wedge \kappa - i H \rho \wedge d\kappa \\ &= d\Lambda \wedge \kappa - \Lambda \wedge \left(\pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa} \right) + \left(\Lambda \wedge \kappa - \overline{\pi}^2 \wedge \zeta + i H \rho \wedge \kappa \right) \wedge \overline{\pi}^1 \\ &\quad - \pi^2 \wedge \left(\Lambda \wedge \rho + \overline{\zeta} \wedge \zeta \right) - \left(\Lambda \wedge \overline{\kappa} + \overline{\pi}^2 \wedge \pi^1 - \pi^2 \wedge \overline{\zeta} - i H \rho \wedge \overline{\kappa} \right) \wedge \zeta \\ &\quad + \overline{\pi}^2 \wedge \left(i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \overline{\pi}^1 \wedge \zeta \right) \\ &\quad + i dH \wedge \rho \wedge \kappa + i H (\pi^1 + \overline{\pi}^1) \wedge \rho \wedge \kappa - i H \rho \wedge \left(\pi^1 \wedge \kappa + \zeta \wedge \overline{\kappa} \right), \end{aligned}$$

which simplifies and reorganizes as:

$$\begin{aligned} 0 &= \underbrace{\left(d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \overline{\pi}^1 - i \pi^2 \wedge \overline{\pi}^2 \right)}_{= \Phi \text{ again!}} \wedge \kappa \\ &\quad + i dH \wedge \rho \wedge \kappa + 2i H \rho \wedge \kappa \wedge \pi^1 + 2i H \rho \wedge \kappa \wedge \overline{\pi}^1 - 2i H \rho \wedge \zeta \wedge \overline{\kappa}. \end{aligned}$$

Then wedging this $(\bullet) \wedge \kappa$ gives:

$$0 = 0 + 0 + 0 + 0 - 2i H \rho \wedge \zeta \wedge \bar{\kappa} \wedge \kappa,$$

whence $H = 0$ necessarily. $\square$

We thus have shown that the *real* 2-form:

$$\Phi := d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \bar{\pi}^1 - i \pi^2 \wedge \bar{\pi}^2$$

satisfies:

$$0 = \Phi \wedge \rho = \Phi \wedge \kappa,$$

hence by the Cartan Lemma, there exists a function F such that:

$$\Phi = F \rho \wedge \kappa.$$

But since $\Phi = \bar{\Phi}$ is real, and since $\rho \wedge \kappa$ is linearly independent with its conjugate $\rho \wedge \bar{\kappa}$, we necessarily have $F = 0 = \bar{F}$, and therefore:

$$d\Lambda - \Lambda \wedge \pi^1 - \Lambda \wedge \bar{\pi}^1 - i \pi^2 \wedge \bar{\pi}^2 = 0.$$

We summarized the result obtained under a synthetic form, not formulated at the end of [3,4].

Theorem 16 *After normalizations of the group parameters f, b, d , the equivalence problem for 2-nondegenerate (constant) Levi rank 1 $\mathcal{C}^\omega$ or $\mathcal{C}^\infty$ real hypersurfaces $M^5 \subset \mathbb{C}^3$ conducts to 2 fundamental primary differential invariants denoted J and W by Pocchiola in [4].*

When both $J \equiv 0 \equiv W$ vanish identically, on the 10-dimensional manifold $M^5 \times G^4 \times \mathbb{R}$ equipped with coordinates:

$$(z_1, z_2, \bar{z}_1, \bar{z}_2, v) \times (c, \bar{c}, e, \bar{e}) \times (t),$$

the prolongation of the structure equations with constant coefficients:

$$\begin{aligned} d\rho &= (\pi^1 + \bar{\pi}^1) \wedge \rho + i \kappa \wedge \bar{\kappa}, \\ d\kappa &= \pi^2 \wedge \rho + \pi^1 \wedge \kappa + \zeta \wedge \bar{\kappa}, \\ d\zeta &= i \pi^2 \wedge \kappa + \pi^1 \wedge \zeta - \bar{\pi}^1 \wedge \zeta, \end{aligned}$$

where π^1 and π^2 are modified-prolonged Maurer–Cartan forms, conducts to structure equations also having constant coefficients (no new curvature functions appear):

$$\begin{aligned} d\pi^1 &= \Lambda \wedge \rho - i \bar{\pi}^2 \wedge \kappa + \zeta \wedge \bar{\zeta}, \\ d\pi^2 &= \Lambda \wedge \kappa + \pi^2 \wedge \bar{\pi}^1 - \bar{\pi}^2 \wedge \zeta, \end{aligned}$$

where $\Lambda = \overline{\Lambda} = dt + \cdots$ is a real 1-form also having constant coefficients structure:

$$d\Lambda = \Lambda \wedge \pi^1 + \Lambda \wedge \overline{\pi}^1 + i \pi^2 \wedge \overline{\pi}^2.$$

Lastly, the $\{e\}$ -structure defined by the collection of these 10 structure equations (conjugates are implicit) coincides with the Maurer–Cartan equations [5] of the CR automorphism Lie group of the model light cone M_{LC} . $\square$

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